Lecture 20

The Fourth Isomorphism Theorem: Let $N \trianglelefteq G$ and $\pi : G \longrightarrow G/N$ be the natural projection. There is a bijection (not an isomorphism)

$$\underbrace{\{A \leq G \text{ such that } A \supseteq N\}}_{\text{subgroups of } G \text{ that contain } N} \xrightarrow{\pi} \underbrace{\{\bar{A} \leq G/N\}}_{\text{subgroups of } G/N}$$

$$A \longmapsto \pi(A)$$

the direct image of A under π

Correspondence Theorem

Π has the following properties: Let $A, B \leq G$ s.t. $N \leq A, B$.

(1) $A \leq B \iff \Pi(A) \leq \Pi(B)$

(2) If $A \leq B$, then $[B : A] = [\Pi(B) : \Pi(A)]$

(3) $\Pi(A \cap B) = \Pi(A) \cap \Pi(B)$

(4) $A \trianglelefteq G \iff \Pi(A) \trianglelefteq G/N$

⚠ Observe that $\Pi(A) = \pi(A) = \{\pi(a) \mid a \in A\} = \{aN \mid a \in A\} = A/N$.

* Hence, this theorem implies that all subgroups of G/N are of the form A/N with $A \leq G$ and $A \supseteq N$.

* Properties (1) - (4) take the form: (1) $A \leq B \iff A/N \leq B/N$

(2) If $A \leq B$, then $[B:A] = [B/N : A/N]$.

(3) $(A \cap B)/N = (A/N) \cap (B/N)$

(4) $A \trianglelefteq G \iff A/N \trianglelefteq G/N$

Three Auxiliary Lemmas

Prove them!

Lemma ★: If $\phi : G \longrightarrow G'$ is a homomorphism and $H \leq G$, then $\phi(H) \leq G'$.

Lemma ▲: If $N \trianglelefteq G$ and $N \leq H \leq G$, then $\pi^{-1}(\pi(H)) = H$, where $\pi : G \longrightarrow G/N$ is the natural projection.

Lemma †: If $f : X \to Y$ is a surjective function and $J \subseteq Y$, then $f(f^{-1}(J)) = J$.

Proof:

Π is well-defined:

i. Is $\Pi(A) = \pi(A)$ a subgroup of G/N? Yes, by Lemma ★.

ii. If $A, B \leq G$, $A, B \supseteq N$ and $A = B$, then $\pi(A) = \pi(B)$, i.e. $\Pi(A) = \Pi(B)$.

Π is injective: Let $A, B \leq G$ and $A, B \supseteq N$. Suppose $\Pi(A) = \Pi(B)$.

WTS: $A = B$.

$$\Pi(A) = \Pi(B) \implies \pi(A) = \pi(B) \implies \pi^{-1}(\pi(A)) = \pi^{-1}(\pi(B)) \implies A = B$$

Lemma ▲

Π is surjective: Let $\mathcal{A} \leq G/N$. WTS: $\exists\, A \leq G$ s.t. $A \supseteq N$ and $\Pi(A) = \mathcal{A}$.

* By Q3(a), PS9, $\pi^{-1}(\mathcal{A})$ is a subgroup of G. Let $A := \pi^{-1}(\mathcal{A})$.

* Observe that $N \subseteq A$ because $\pi(N) = \{ nN \mid n \in N \} = \{N\} \subseteq \mathcal{A}$

  ($N = 1_{G/N} \in \mathcal{A}$ because $\mathcal{A} \leq G/N$)

* $\Pi(A) = \pi(A) = \pi(\pi^{-1}(\mathcal{A})) \underset{\text{Lemma } \dagger}{=} \mathcal{A}$.

Thus, Π is a bijective correspondence.

Proof of properties (1) – (4) : Exercise. ■

Summary

Cancelation Law

3rd IT

$K, H \trianglelefteq G$

$K \leq H \leq G$

$\Longrightarrow$

- $H/K \trianglelefteq G/K$
- $(G/K)/(H/K) \cong G/H$

1st IT

$\varphi : G \longrightarrow H$ $\Longrightarrow$

- $\operatorname{Ker}\varphi \trianglelefteq G$
- $G/\operatorname{Ker}\varphi \cong \operatorname{Im}\varphi$

Isomorphism Theorems

Diamond Theorem

2nd IT

$K, N \leq G$

$N \trianglelefteq G$

$\Longrightarrow$

- $KN \leq G$
- $N \trianglelefteq KN$
- $K \cap N \trianglelefteq K$
- $K/K\cap N \cong KN/N$
- $[N : K\cap N] = [KN/K]$

$$\begin{tikzcd}
 & KN \arrow[dl, dash] \arrow[dr, dash, "\trianglerighteq"'] & \\
K \arrow[dr, dash, "\trianglerighteq"'] & & N \arrow[dl, dash] \\
 & K\cap N &
\end{tikzcd}$$

Correspondence Theorem

4th IT

$N \trianglelefteq G$

$\pi : G \longrightarrow G/N$

$\Longrightarrow$

$$\{A \leq G \ \& \ A \supseteq N\} \xrightarrow{\text{bijective}} \{\mathcal{A} \leq G/N\}$$

$$A \longmapsto \pi(A) = A/G$$

- $A \leq B \iff A/N \leq B/N$
- $A \leq B \Rightarrow [B:A] = [B/N : A/N]$
- $(A\cap B)/N = (A/N) \cap (B/N)$
- $A \trianglelefteq G \iff A/N \trianglelefteq G/N$

Examples:

one quotient per normal subgroup — and — normal subgroups are kernels

① 1st IT Find all quotient groups of S_3.

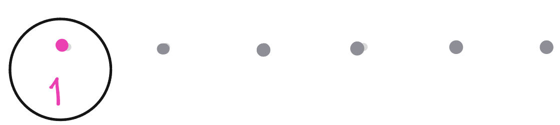
$$t : S_3 \longrightarrow S_3$$
$$\sigma \longmapsto 1$$

$$S_3 / \mathrm{Ker}\, t \cong \mathrm{Im}\, t$$

$$S_3 / S_3 \cong \{1\}$$

$S_3/\mathrm{Ker}\, t$

S_3

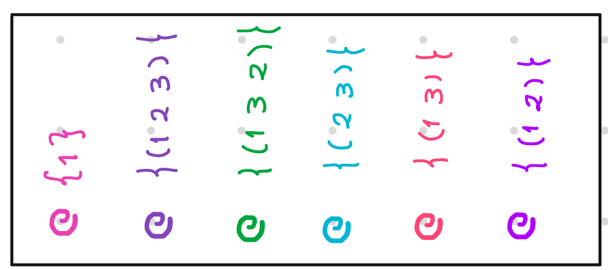


$\mathrm{Im}\, t$

$$\mathrm{id} : S_3 \longrightarrow S_3$$
$$\sigma \longmapsto \sigma$$

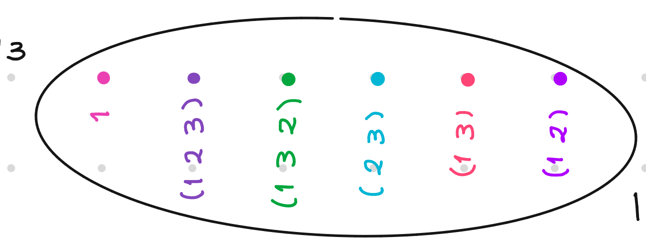
$$S_3 / \mathrm{Ker}\, \mathrm{id} \cong \mathrm{Im}\, \mathrm{id}$$

$$S_3 / \{1\} \cong S_3$$



$S_3/\mathrm{Ker}\, \mathrm{id}$

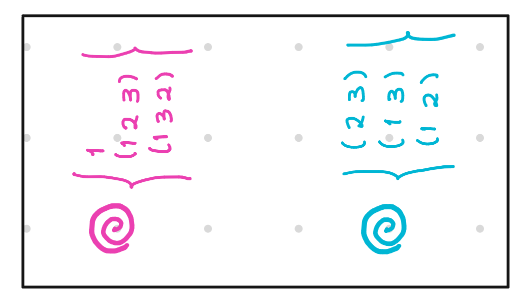
S_3



$\mathrm{Im}\, \mathrm{id}$

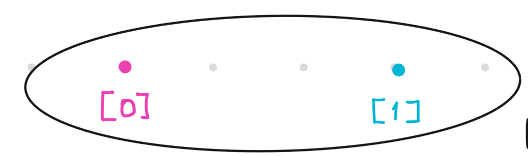
$$\varphi : S_3 \longrightarrow \mathbb{Z}/2$$
$$1 \longmapsto [0]$$
$$(1\,2\,3) \longmapsto [0]$$
$$(1\,3\,2) \longmapsto [0]$$
$$(2\,3) \longmapsto [1]$$
$$(1\,3) \longmapsto [1]$$
$$(1\,2) \longmapsto [1]$$

$$S_3 / \{1, (1\,2\,3), (1\,3\,2)\} \cong \mathbb{Z}/2$$



$S_3/\mathrm{Ker}\, \varphi$

$\mathbb{Z}/2$



$\mathrm{Im}\, \varphi$

② 2nd IT Prove that $2\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/3$.

$(\mathbb{Z}, +)$, $K = 2\mathbb{Z}$, $N = 3\mathbb{Z}$ and $K, N \trianglelefteq \mathbb{Z}$. Then,

- $K + N \leqslant \mathbb{Z}$
- $N \trianglelefteq K + N$
- $K \cap N \trianglelefteq K$
- $K / K \cap N \cong (K+N)/N$

- $2\mathbb{Z} + 3\mathbb{Z} = \mathbb{Z}$ ⟵ because $1 = 2x + 3y$ and $\langle 1 \rangle = \mathbb{Z}$
- $3\mathbb{Z} \trianglelefteq \mathbb{Z}$
- $2\mathbb{Z} \cap 3\mathbb{Z} = 6\mathbb{Z}$, $6\mathbb{Z} \trianglelefteq 3\mathbb{Z}$
- $2\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} = \mathbb{Z}/3$

$$6\mathbb{Z} \to [0]$$
$$2 + 6\mathbb{Z} \to [2]$$
$$4 + 6\mathbb{Z} \to [1]$$

③ 3rd IT Prove that $(\mathbb{Z}/6\mathbb{Z}) \big/ (3\mathbb{Z}/6\mathbb{Z}) \cong \mathbb{Z}/3$.

$(\mathbb{Z}, +)$, $H = 3\mathbb{Z}$, $K = 6\mathbb{Z}$, $H, K \trianglelefteq \mathbb{Z}$ and $K \leq H$. Then,

- $H/K \trianglelefteq \mathbb{Z}/K$
- $(\mathbb{Z}/K) \big/ (H/K) \cong \mathbb{Z}/H$

- $3\mathbb{Z}/6\mathbb{Z} \trianglelefteq \mathbb{Z}/6\mathbb{Z}$

$$\mathbb{Z}/6\mathbb{Z} = \{6\mathbb{Z}, 1+6\mathbb{Z}, 2+6\mathbb{Z}, 3+6\mathbb{Z}, 4+6\mathbb{Z}, 5+6\mathbb{Z}\}$$

$$3\mathbb{Z}/6\mathbb{Z} = \{6\mathbb{Z}, 3+6\mathbb{Z}\}$$

- $(\mathbb{Z}/6\mathbb{Z}) \big/ (3\mathbb{Z}/6\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z} = \mathbb{Z}/3$

$$6\mathbb{Z} + (3\mathbb{Z}/6\mathbb{Z}) \longrightarrow [0]$$
$$1+6\mathbb{Z} + (3\mathbb{Z}/6\mathbb{Z}) \longrightarrow [1]$$
$$2+6\mathbb{Z} + (3\mathbb{Z}/6\mathbb{Z}) \longrightarrow [2]$$

④ 4th IT Consider $\pi : \mathbb{Z} \longrightarrow \mathbb{Z}/12\mathbb{Z}$ the natural projection.

i. Find the subgroups of $\mathbb{Z}$ that contain $12\mathbb{Z}$ and draw a lattice only with these.

ii. Find the subgroups of $\mathbb{Z}/12\mathbb{Z}$ and draw its lattice of subgroups.

iii. Compute the direct image under π for each subgroup in i.

Solution:

i. $\mathcal{D} = \{ \mathbb{Z},\ 2\mathbb{Z},\ 3\mathbb{Z},\ 4\mathbb{Z},\ 6\mathbb{Z},\ 12\mathbb{Z} \}$

ii. $\mathbb{Z}/12\mathbb{Z} = \{ 12\mathbb{Z},\ 1+12\mathbb{Z},\ 2+12\mathbb{Z}, \ldots,\ 11+12\mathbb{Z} \}$, $\mathbb{Z}/12\mathbb{Z} = \langle 1+12\mathbb{Z} \rangle$ and $|\mathbb{Z}/12\mathbb{Z}| = 12$

Then $\mathbb{Z}/12\mathbb{Z}$ has 6 subgroups

$\mathcal{C} = \{ \langle 12\mathbb{Z} \rangle, \langle 6+12\mathbb{Z} \rangle, \langle 4+12\mathbb{Z} \rangle, \langle 3+12\mathbb{Z} \rangle, \langle 2+12\mathbb{Z} \rangle, \mathbb{Z}/12\mathbb{Z} \}$

$$\langle 12 + 12\mathbb{Z} \rangle = \{ 12\mathbb{Z} \} = 12\mathbb{Z}/12\mathbb{Z}$$

$$\langle 6 + 12\mathbb{Z} \rangle = \{ 12\mathbb{Z},\ 6 + 12\mathbb{Z} \} = 6\mathbb{Z}/12\mathbb{Z}$$

$$\langle 4 + 12\mathbb{Z} \rangle = \{ 12\mathbb{Z},\ 4 + 12\mathbb{Z},\ 8 + 12\mathbb{Z} \} = 4\mathbb{Z}/12\mathbb{Z}$$

$$\langle 3 + 12\mathbb{Z} \rangle = \{ 12\mathbb{Z},\ 3 + 12\mathbb{Z},\ 6 + 12\mathbb{Z},\ 9 + 12\mathbb{Z} \} = 3\mathbb{Z}/12\mathbb{Z}$$

$$\langle 2 + 12\mathbb{Z} \rangle = \{ 12\mathbb{Z},\ 2 + 12\mathbb{Z},\ 4 + 12\mathbb{Z},\ 6 + 12\mathbb{Z},\ 8 + 12\mathbb{Z},\ 10 + 12\mathbb{Z} \} = 2\mathbb{Z}/12\mathbb{Z}$$

$$\langle 1 + 12\mathbb{Z} \rangle = \mathbb{Z}/12\mathbb{Z}$$

$$
\begin{tikzcd}
 & \mathbb{Z} \arrow[dl, dash] \arrow[dr, dash] & \\
2\mathbb{Z} \arrow[d, dash] \arrow[drr, dash] & & 3\mathbb{Z} \arrow[d, dash] \\
4\mathbb{Z} \arrow[dr, dash] & & 6\mathbb{Z} \arrow[dl, dash] \\
 & 12\mathbb{Z} &
\end{tikzcd}
\qquad \xrightarrow{\ \pi\ } \qquad
\begin{tikzcd}
 & \mathbb{Z}/12\mathbb{Z} \arrow[dl, dash] \arrow[dr, dash] & \\
2\mathbb{Z}/12\mathbb{Z} \arrow[d, dash] \arrow[drr, dash] & & 3\mathbb{Z}/12\mathbb{Z} \arrow[d, dash] \\
4\mathbb{Z}/12\mathbb{Z} \arrow[dr, dash] & & 6\mathbb{Z}/12\mathbb{Z} \arrow[dl, dash] \\
 & \{12\mathbb{Z}\} &
\end{tikzcd}
$$

$$\pi(\mathbb{Z}) = \{ a + \mathbb{Z}/12\mathbb{Z} \mid a \in \mathbb{Z} \} = \mathbb{Z}/12\mathbb{Z}$$

$$\pi(2\mathbb{Z}) = \{ a + \mathbb{Z}/12\mathbb{Z} \mid a \in 2\mathbb{Z} \} = 2\mathbb{Z}/12\mathbb{Z}$$

$\pi(3\mathbb{Z}) = 3\mathbb{Z}/12\mathbb{Z}$ $\quad$ $\pi(4\mathbb{Z}) = 4\mathbb{Z}/12\mathbb{Z}$ $\quad$ $\pi(6\mathbb{Z}) = 6\mathbb{Z}/12\mathbb{Z}$ $\quad$ $\pi(12\mathbb{Z}) = \{12\mathbb{Z}\}$