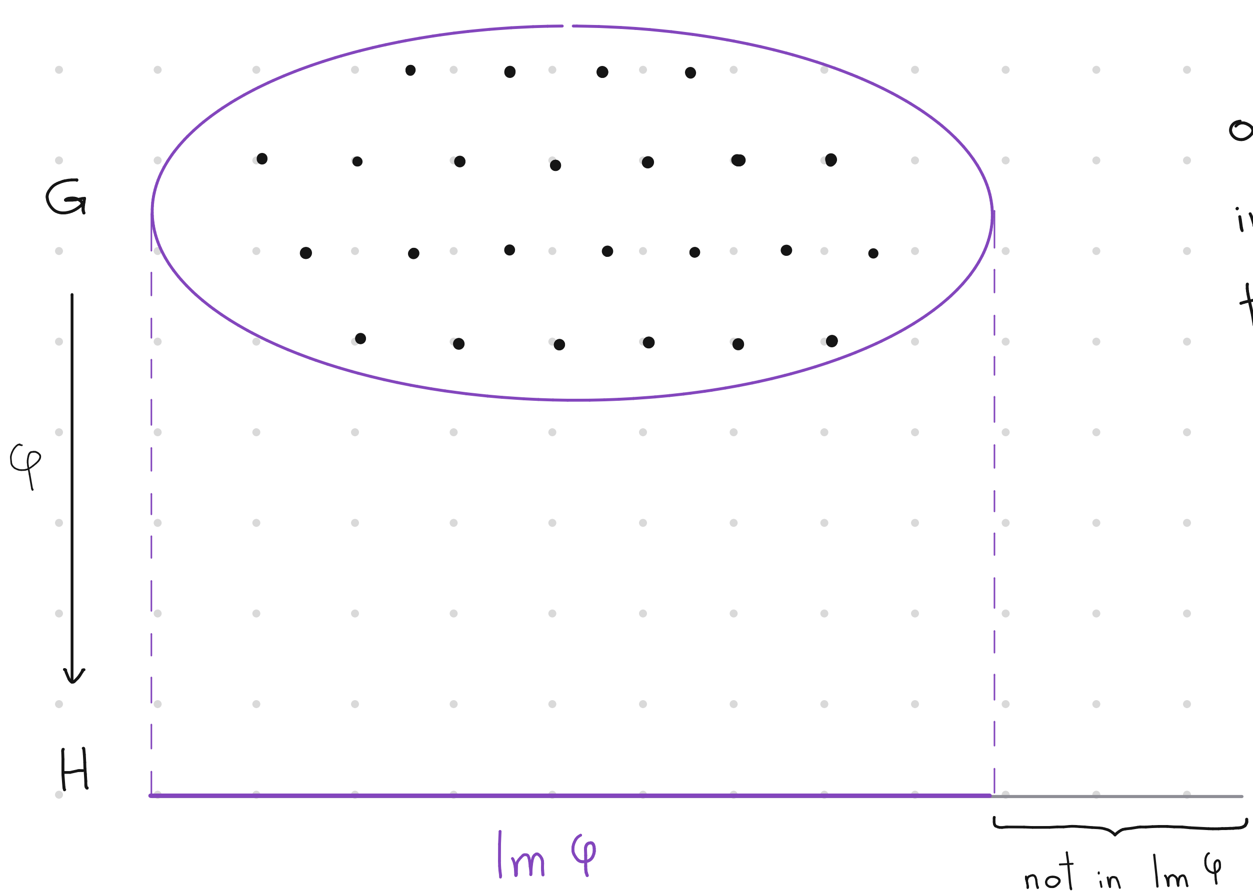
Lecture 19

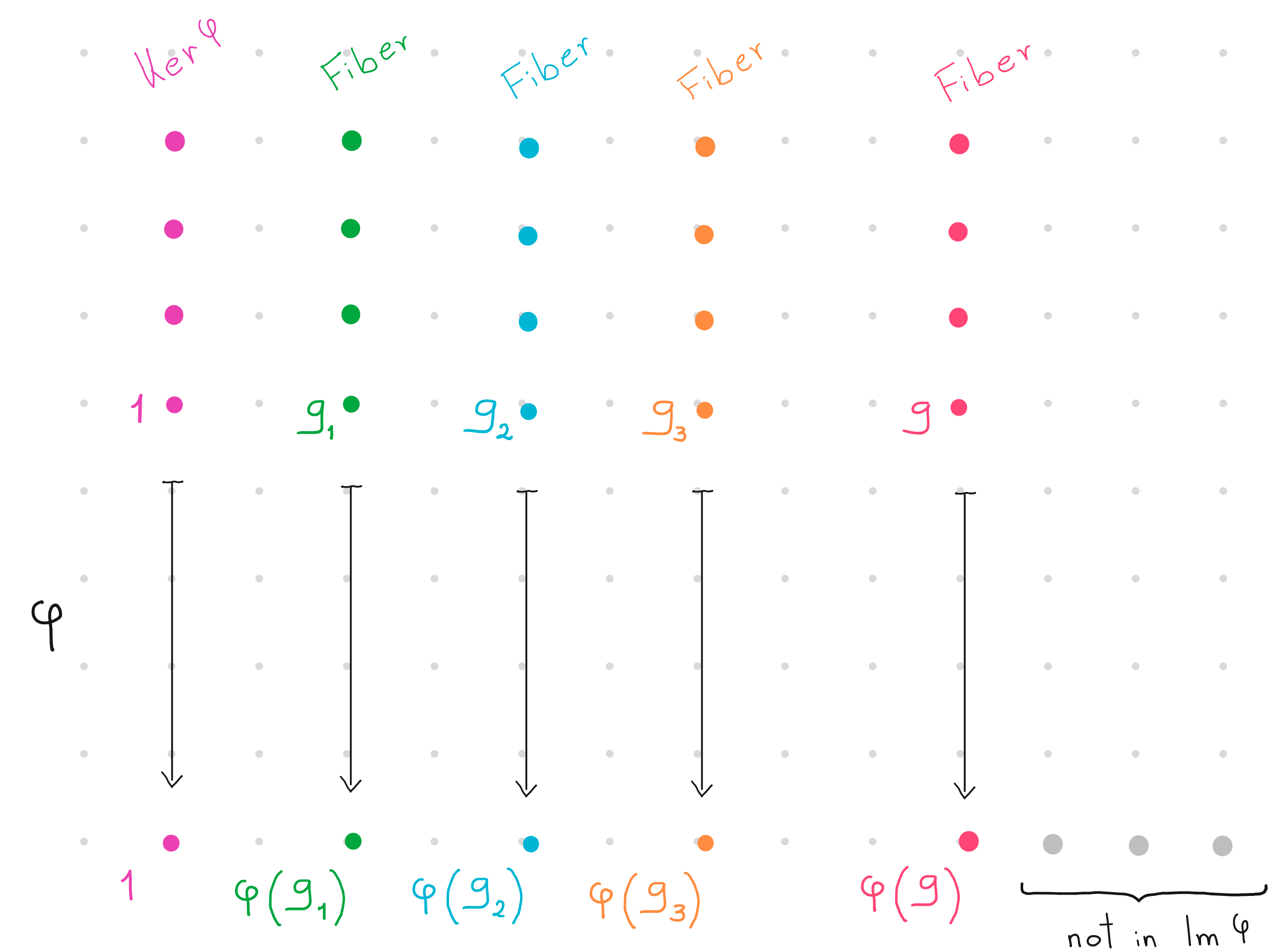
IDEA

$\varphi : G \longrightarrow H$.

$G/\operatorname{Ker}\varphi$ $\rightsquigarrow$ "$\operatorname{Ker}\varphi$ and packages of elements in G that are not sent to $1 \in H$ by φ"



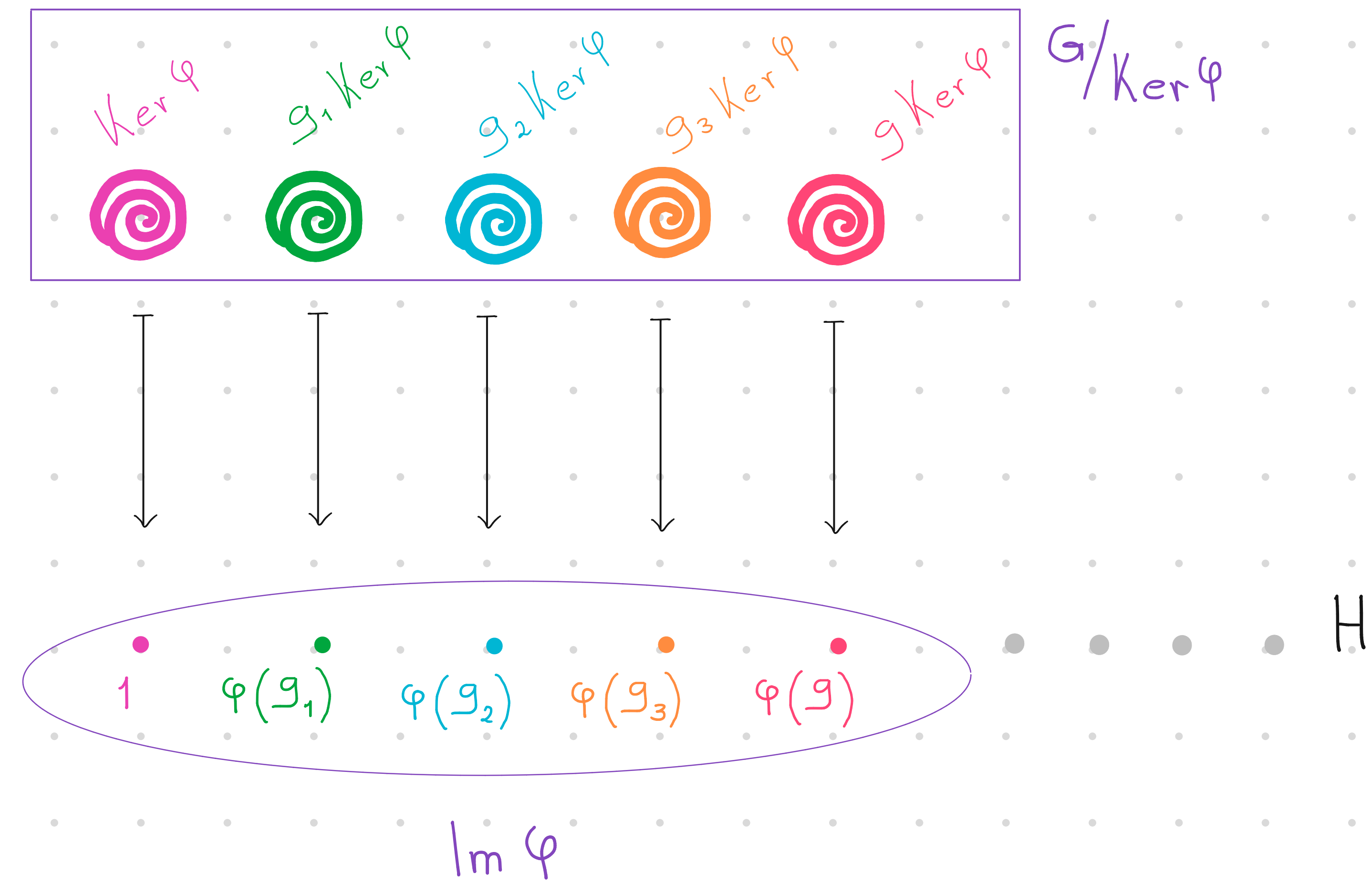
organize elements in G according to their image



We have that $\boxed{\text{Fiber over } \varphi(g) = g\operatorname{Ker}\varphi}$

Let $h \in \operatorname{Im}\varphi$, then $\exists g \in G$ s.t. $\varphi(g) = h$.

$$\begin{aligned}
\varphi^{-1}(h) &= \{ a \in G \mid \varphi(a) = h \} \\
&= \{ a \in G \mid \varphi(a) = \varphi(g) \} \\
&= \{ a \in G \mid g^{-1}a = k,\ k \in \operatorname{Ker}\varphi \} \\
&= \{ gk \mid k \in \operatorname{Ker}\varphi \} = g\operatorname{Ker}\varphi
\end{aligned}$$



Proof:

(2) Define $\Phi : G/\operatorname{Ker}\varphi \longrightarrow \operatorname{Im}\varphi$ by $\Phi(g\operatorname{Ker}\varphi) = \varphi(g) \in \operatorname{Im}\varphi$.

* Well-defined: Suppose $a\operatorname{Ker}\varphi = b\operatorname{Ker}\varphi$. WTS: $\Phi(a\operatorname{Ker}\varphi) = \Phi(b\operatorname{Ker}\varphi)$

By Lemma 27, $b^{-1}a \in \operatorname{Ker}\varphi$, then $1 = \varphi(b^{-1}a) = \varphi(b)^{-1}\varphi(a)$. Thus, $\varphi(a) = \varphi(b)$.

* Homomorphism: True bc φ is a homomorphism.

* Injective: Let $a\operatorname{Ker}\varphi \in \operatorname{Ker}\Phi$. WTS: $a\operatorname{Ker}\varphi = \underbrace{\operatorname{Ker}\varphi}_{1_{G/\operatorname{Ker}\varphi}}$ the identity of $G/\operatorname{Ker}\varphi$

$$
\begin{aligned}
a\operatorname{Ker}\varphi \in \operatorname{Ker}\Phi \quad &\iff \quad \Phi(a\operatorname{Ker}\varphi) = 1 \in H \\
&\iff \quad \varphi(a) = 1 \\
&\iff \quad a \in \operatorname{Ker}\varphi \\
&\iff \quad a\operatorname{Ker}\varphi = \operatorname{Ker}\varphi
\end{aligned}
$$

Thus $\operatorname{Ker}\Phi = \{1_{G/\operatorname{Ker}\varphi}\}$. From Prop 25 (a), Φ is injective.

* Surjective: Let $h \in \operatorname{Im}\varphi$, then $\exists g \in G$ s.t. $h = \varphi(g)$. Then $\Phi(g\operatorname{Ker}\varphi) = \varphi(g) = h$.

$\blacksquare$

The Second Isomorphism Theorem: Let K and N be subgroups of G such that $N \trianglelefteq G$, then

(1) KN is a subgroup of G.

(2) $K \cap N \trianglelefteq K$

(3) $N \trianglelefteq KN$

(4) $\underbrace{K / K\cap N}_{\text{group}} \cong \underbrace{KN / N}_{\text{group}}$ $\left(\text{or } \underbrace{K / K\cap N \cong (K+N)/N}_{\text{additive notation}} \right)$

$$\begin{tikzcd}
& G \arrow[d, dash] & \\
& KN \arrow[dl, dash] \arrow[dr, dash, "\trianglerighteq"'] & \\
K \arrow[dr, dash, "\trianglerighteq"'] & & N \arrow[dl, dash] \\
& K\cap N \arrow[d, dash] & \\
& \{e\} &
\end{tikzcd}$$

Diamond Theorem

⚠ In addition to (1) – (4) we also have

$[KN : K] = [N : K\cap N]$.

Watch out! KN/K and $N/K\cap N$ are not groups necessarily.

Proof: (1) See Q1(e) PS9

(2) and (3) Exercise

(4) From (1) - (3), the quotients $K/K\cap N$ and KN/N are groups. Define the map $\eta : K \longrightarrow KN/N$ by $\eta(k) = kN$.

- η is well-defined: Clearly, $k = k' \implies kN = k'N$.
- η is a homomorphism: $\eta(kk') = (kk')N \underset{\uparrow}{=} (kN)(k'N) = \eta(k)\eta(k')$.

  definition of operation in KN/N

- η is surjective: Let $aN \in KN/N$, then $\exists\, k \in K, n \in N$ s.t. $a = kn$. Observe that $(kn)N = kN$ because $k^{-1}(kn) = n \in N$. Thus, $\eta(k) = kN = (kn)N = aN$.

Claim: $\operatorname{Ker}\eta = K \cap N$

$$k \in \operatorname{Ker}\eta \subseteq K \iff k \in K \text{ and } \eta(k) = N$$

$$\iff k \in K \text{ and } kN = N$$

$$\iff k \in K \text{ and } k \in N$$

$$\iff k \in K \cap N$$

By the 1st isomorphism theorem, $K/\operatorname{Ker}\eta \cong \operatorname{Im}\eta$, i.e. $K/\underbrace{K\cap N}_{\operatorname{Ker}\eta} \cong \underbrace{KN/N}_{\operatorname{Im}\eta}$.

$\blacksquare$

The Third Isomorphism Theorem: Let H and K be normal subgroups of G with $K \leq H$, then

(1) $H/K \trianglelefteq G/K$

(2) $(G/K)/(H/K) \cong G/H$ "Cancelation Law"

This shows that quotients of quotients don't provide new structural information.

Proof: (1) Exercise.

(2) Define $\Gamma : G/K \longrightarrow G/H$ by $\Gamma(gK) = gH$.

- Γ is well-defined: If $aK = bK$ in G/K, then $b^{-1}a \in K$. Since $K \subseteq H$, then $b^{-1}a \in H$. Thus $aH = bH$, i.e. $\Gamma(aK) = \Gamma(bK)$.

- Γ is a homomorphism: Clear from definition
- Γ is surjective: Clear from definition

Claim: $\operatorname{Ker}\Gamma = H/K$

$$
\begin{aligned}
gK \in \operatorname{Ker}\Gamma &\iff \Gamma(gK) = H \\
&\iff gH = H \\
&\iff g \in H \\
&\iff gK \in H/K
\end{aligned}
$$

By the 1st isomorphism theorem, $(G/K)/\operatorname{Ker}\Gamma \cong \operatorname{Im}\Gamma$, i.e. $(G/K)\big/(H/K) \cong G/H$. ∎