Lecture 18

Proposition 34: Let $N \trianglelefteq G$, then the natural projection $\pi : G \longrightarrow G/N$ is an epimorphism and $\operatorname{Ker} \pi = N$.

Proof:

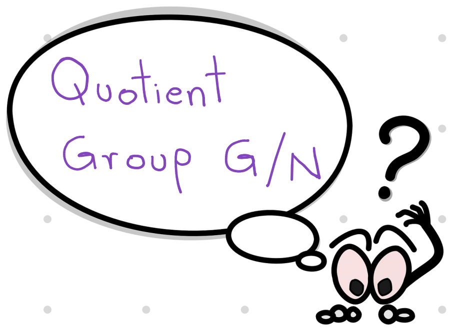
- Let $aN \in G/N$, then $\pi(a) = aN$.

- $(\subseteq)$ Let $a \in \operatorname{Ker} \pi$, then $aN = \pi(a) = N$. By Lemma 27, $a \in N$.

  $(\supseteq)$ Let $n \in N$. By Lemma 27, $nN = N$. Thus, $\pi(n) = nN = N$, i.e. $n \in \operatorname{Ker} \pi$.

! When $N \trianglelefteq G$, it makes sense to talk about the order of G/N, i.e.

$$[G:N] = |G/N|.$$



But, intuitively, what is a quotient group?

$N \trianglelefteq G$

Think that belonging to the subgroup N means you satisfy a special property. For instance,

- $n\mathbb{Z} \subseteq \mathbb{Z}$ means: "You live in $n\mathbb{Z}$ iff you are a multiple of n"
- $\operatorname{Ker}\varphi \subseteq G$ means: "You live in $\operatorname{Ker}\varphi$ iff you get sent to $1 \in H$ by $\varphi : G \longrightarrow H$"
- $Z(G) \subseteq G$ means: "You live in $Z(G)$ iff you commute with every element in G"
- $SL(n,\mathbb{C}) \subseteq GL(n,\mathbb{C})$ means: "You live in $SL(n,\mathbb{C})$ iff you are an $n \times n$ invertible matrix with entries in $\mathbb{C}$ and determinant 1"

When you hear somebody saying ⇝ "Form the quotient G/N"
"mod out G by N"

They mean ⇝ "Consider N and all elements of G that DON'T satisfy the property of belonging to N"

BUT ⇝ There are many ways to fail to satisfy the property to be in N.

SURVEY

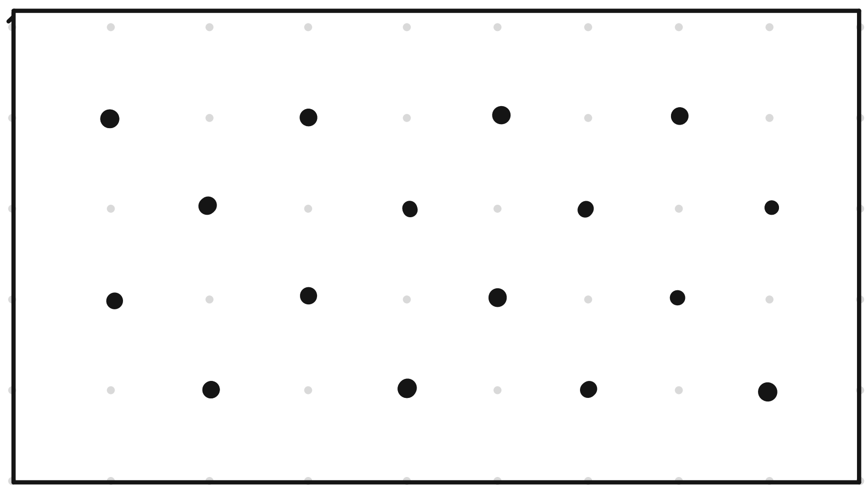
1. Do you satisfy the property to belong to N?

   ○ Yes ○ No

2. If no, how badly do you fail to satisfy the property?

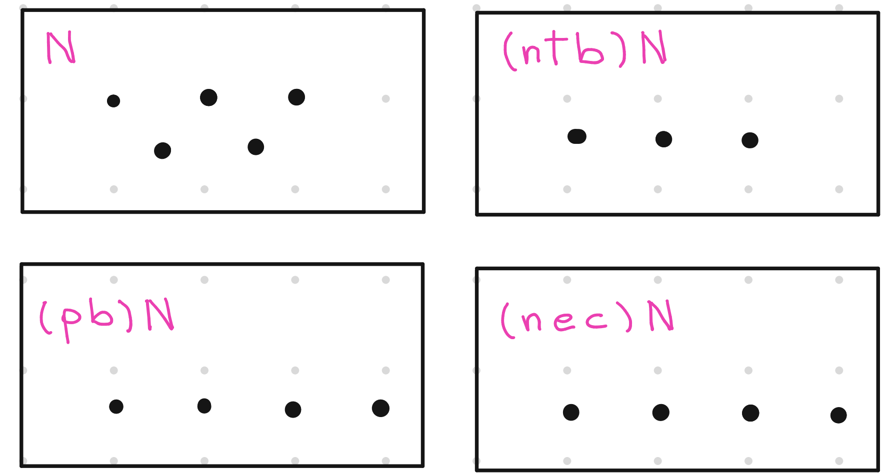
   ○ Not too badly
   ○ Pretty badly
   ○ Not even close

desorganized



G

organized, relative to N



G/N

π

Think of the natural projection as the survey.

We grouped the elements of G according to HOW they answered 1. or 2.

$$G/N = \{N,\ (ntb)N,\ (pb)N,\ (nec)N\}$$

"N and packages of elements that fail to belong to N"

Example:

1) D_6 and $N = \{1, r, r^2\}$

$$\underbrace{\boxed{\begin{matrix} 1 & & r^2 \\ & r & \\ s & & sr^2 \\ & sr & \end{matrix}}}_{D_6} \xrightarrow{\pi} \underbrace{\begin{matrix} \boxed{\begin{matrix} N & & \\ 1 & r & r^2 \end{matrix}} \\ \\ \boxed{\begin{matrix} sN & & \\ s & sr & sr^2 \end{matrix}} \end{matrix}}_{D_6/N}$$

$\rightsquigarrow$ $\boxed{\text{"N and packages of elements not in N"}}$

$D_6/N = \{N, sN\}$ is a group with Cayley table

$\cdot$	N	sN
N	N	sN
sN	sN	N

$|D_6/N| = |D_6| / |N| = 6/3 = 2.$

Then $D_6/N \cong \mathbb{Z}/2$

$N \longmapsto [0]$

$sN \longmapsto [1]$

② $n\mathbb{Z} \leq (\mathbb{Z}, +)$

$n\mathbb{Z}$ ⇜ multiples of n

$$\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z}$$

$1 + n\mathbb{Z}$ ⇜ Folks that aren't in $n\mathbb{Z}$ bc they're off by 1.

$2 + n\mathbb{Z}$ ⇜ Folks that aren't in $n\mathbb{Z}$ bc they're off by 2.

⋮

$(n-1) + n\mathbb{Z}$ ⇜ Folks that aren't in $n\mathbb{Z}$ bc they're off by $n-1$

$\boxed{\begin{array}{l} n\mathbb{Z} \\ 0 \quad \pm n \quad \pm 2n \quad \pm 3n \quad \cdots \end{array}} = [0]$

$\boxed{\begin{array}{l} 1+n\mathbb{Z} \\ 1 \quad 1\pm n \quad 1\pm 2n \quad 1\pm 3n \quad \cdots \end{array}} = [1]$

$\boxed{\begin{array}{l} 2+n\mathbb{Z} \\ 2 \quad 2\pm n \quad 2\pm 2n \quad 2\pm 3n \quad \cdots \end{array}} = [2]$

⋮

$\boxed{\begin{array}{l} (n-1)+n\mathbb{Z} \\ (n-1) \quad (n-1)\pm n \quad \cdots \end{array}} = [n-1]$

❗ $a + n\mathbb{Z} = b + n\mathbb{Z} \iff -b + a \in n\mathbb{Z}$

$\iff a - b \in n\mathbb{Z}$

$\iff n \mid (a-b)$

❗ Observe that

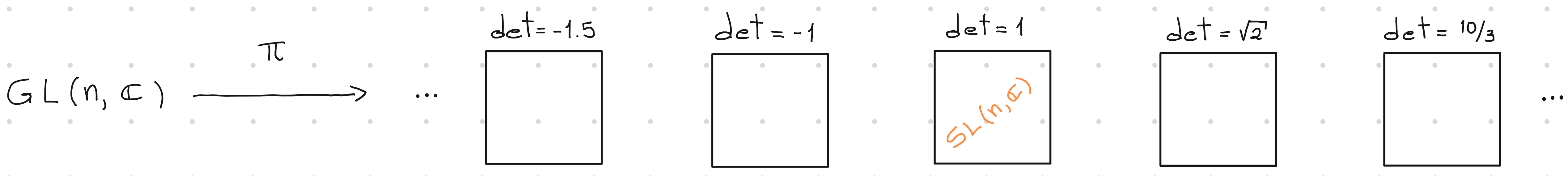
quotient group $\mathbb{Z}/n\mathbb{Z}$ equals quotient set $\mathbb{Z}/\equiv_n$

$\mathbb{Z}/n\mathbb{Z} = \{ n\mathbb{Z}, 1+n\mathbb{Z}, 2+n\mathbb{Z}, \ldots, (n-1)+n\mathbb{Z} \}$ ⇝ "$n\mathbb{Z}$ and packages of integers that fail to belong to $n\mathbb{Z}$"

③ $SL(n,\mathbb{C}) \trianglelefteq GL(n,\mathbb{C})$, $\quad GL(n,\mathbb{C}) / SL(n,\mathbb{C}) = \{ A\,SL(n,\mathbb{C}) \mid A \in GL(n,\mathbb{C}) \}$

$$A\,SL(n,\mathbb{C}) = B\,SL(n,\mathbb{C}) \iff B^{-1}A \in SL(n,\mathbb{C})$$
$$\iff \det(B^{-1})\det(A) = 1$$
$$\iff \det(A) = \det(B)$$

"$SL(n,\mathbb{C})$ and packages of invertible matrices with $\det \neq 1$."



$GL(n,\mathbb{C}) / SL(n,\mathbb{C})$

④ $G/G = \{G\}$ ⇝ "G and packages of elements that fail to belong to G" $\quad G/G \cong \{e\}$

$G/\{e\} = \{ \{g\} : g \in G \}$ ⇝ "$\{e\}$ and packages of elements that fail to belong to $\{e\}$" $\quad G/\{e\} \cong G$

Isomorphism Theorems

The First Isomorphism Theorem: If $\varphi : G \longrightarrow H$ is a homomorphism, then

(1) $\operatorname{Ker}\varphi \trianglelefteq G$.

(2) $G/\operatorname{Ker}\varphi \cong \operatorname{Im}\varphi$.

Proof:

(1) Let $g \in G$. WTS: $g \operatorname{Ker}\varphi \, g^{-1} \subseteq \operatorname{Ker}\varphi$.

Let $k \in \operatorname{Ker}\varphi$, then $\varphi(gkg^{-1}) = \varphi(g)\varphi(k)\varphi(g^{-1}) = \varphi(g)\varphi(g)^{-1} = 1$. Thus, $gkg^{-1} \in \operatorname{Ker}\varphi$.