Left and Right Cosets

Def: Let $H \leqslant (G, *)$ and $g \in G$

(1) The set $g * H := \{ g * h \mid h \in H \}$ is called the left coset of H in G.

(2) The set $H * g := \{ h * g \mid h \in H \}$ is called the right coset of H in G.

(3) g is called a representative of $g * H$ or $H * g$.

Ex:

1. Consider $H = \{1, (2\ 3)\}$ a subgroup of $S_3 = \{ 1, (1\ 2\ 3), (1\ 3\ 2), (1\ 3), (1\ 2), (2\ 3) \}$.

Left cosets

- $1H = (2\ 3)H = H$
- $(1\ 2\ 3)H = \{ (1\ 2\ 3), (1\ 2\ 3)(2\ 3) \} = \{ (1\ 2\ 3), (1\ 2) \} = (1\ 2)H$
- $(1\ 3\ 2)H = (1\ 3)H = \{ (1\ 3\ 2), (1\ 3) \}$

Right cosets

- $H1 = H(2\ 3) = H$
- $H(1\ 2\ 3) = H(1\ 3) = \{(1\ 2\ 3), (1\ 3)\}$
- $H(1\ 3\ 2) = H(1\ 2) = \{(1\ 3\ 2), (1\ 2)\}$

2. Consider $3\mathbb{Z} \leq \mathbb{Z}$

$0 + 3\mathbb{Z} = \{0 + 3n \mid n \in \mathbb{Z}\} = 3\mathbb{Z} = 3\mathbb{Z} + 0$

$1 + 3\mathbb{Z} = \{1 + 3n \mid n \in \mathbb{Z}\} = 3\mathbb{Z} + 1$

$2 + 3\mathbb{Z} = \{2 + 3n \mid n \in \mathbb{Z}\} = 3\mathbb{Z} + 2$

$3 + 3\mathbb{Z} = 3\mathbb{Z} + 3 = 3\mathbb{Z}$

There are only 3 different left/right cosets of $3\mathbb{Z}$ in $\mathbb{Z}$.

In general, there are n different left/right cosets of $n\mathbb{Z}$ in $\mathbb{Z}$.

Q: Do you notice any similarity between $a + 3\mathbb{Z}$ and $[a] \in \mathbb{Z}/3$?

Remarks:

- Left cosets and right coset are not necessarily equal (see example 1).
- If G is abelian, $g * H = H * g$ for all $g \in G$ and $H \leq G$ (see example 2).
- $g * H$ is not a subgroup of H if $g \neq e$, $eH = H$.

Proposition 26: Let $H \leq G$, and $g \in G$. Then

(1) there is a bijection between H and gH.

(2) there is a bijection between H and Hg.

(3) $|H| = |gH| = |Hg|$.

Proof: Exercise.

Lemma 27: Let $H \leq G$ and $a, b \in G$. Then

(1) $aH = bH \iff b^{-1}a \in H$

(2) $Ha = Hb \iff ab^{-1} \in H$

How do I remember?

$aH = bH \Rightarrow b^{-1}aH = b^{-1}bH$

$Ha = Hb \Rightarrow Hab^{-1} = Hbb^{-1}$

Proof: Exercise.

Lemma 28: Let $H \leq G$. For all $a, b \in G$,

(1) either $aH = bH$ or $aH \cap bH = \phi$. two left cosets are either $=$ or disjoint

(2) either $Ha = Hb$ or $Ha \cap Hb = \phi$. two right cosets are either $=$ or disjoint

Proof: Exercise.

Theorem 29: Let $H \leq G$. Then

(1) $G/H := \{aH : a \in G\}$ forms a partition of G.

set of left cosets

(2) $H \backslash G := \{Ha : a \in G\}$ forms a partition of G.

set of right cosets

(3) There is a bijection $f: G/H \longrightarrow H \backslash G$

Proof: Exercise.

Def: Let $H \leq G$. The number of distinct left (or right) cosets is called the index of H in G.

Notation: $[G : H]$

Example: Find the partitions of $\mathbb{Z}/6$ into cosets of the subgroup $H = \{[0], [3]\}$.

All cosets have 2 elements

$$[0] + H = H$$

$$[1] + H = \{[1], [4]\}$$

$$[2] + H = \{[2], [5]\}$$

$$[3] + H = H \quad \overset{\text{Lemma 27}}{\iff} \quad -[0] + [3] \in H$$

$$[4] + H = [1] + H \quad \iff \quad -[1] + [4] \in H$$

$$[5] + H = [2] + H \quad \iff \quad -[2] + [5] \in H$$

Partition: $(\mathbb{Z}/6)/H = H \backslash (\mathbb{Z}/6) = \{H, [1] + H, [2] + H\}$

$$\mathbb{Z}/6 = H \sqcup ([1] + H) \sqcup ([2] + H)$$

Lagrange's Theorem

Theorem: Let $H \leqslant G$ and G be a finite group.

(1) $|H|$ divides $|G|$.

(2) $[G : H] = \dfrac{|G|}{|H|}$

Proof: Watch "Cosets and Lagrange's Theorem" video. Write a proof using the results in this reading.