Lecture 16

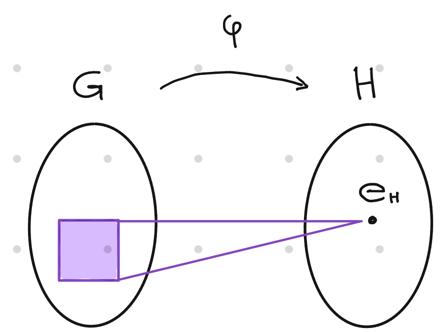
Kernel & Image

Def: Let $\varphi : G \longrightarrow H$ be a homomorphism.

(1) The kernel of φ is the set

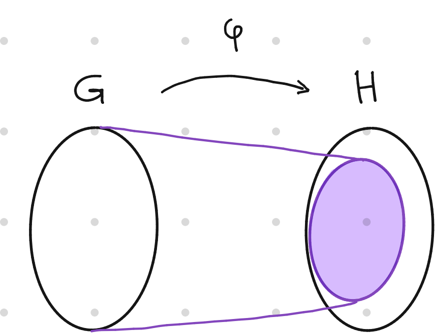
$$\operatorname{Ker}\varphi := \{ a \in G \mid \varphi(a) = e_H \}.$$

measures the degree to which a homomorphism is not injective.



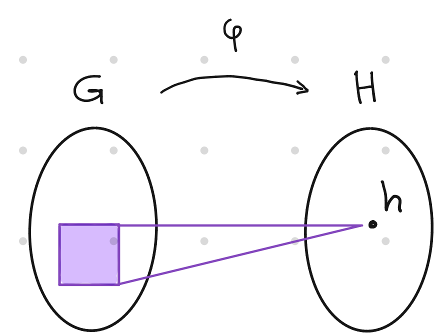
(2) The image of φ is the set

$$\operatorname{Im}\varphi \text{ or } \varphi(G) := \{ \varphi(a) \in H \mid a \in G \}$$



(3) If $h \in H$, the fiber of h under φ is the set

$$\varphi^{-1}(h) := \{ a \in G \mid \varphi(a) = h \}$$



Proposition 24: Let $\varphi : G \longrightarrow H$ be a homomorphism.

(1) $\operatorname{Ker} \varphi \leqslant G$

(2) $\operatorname{Im} \varphi \leqslant H$.

Proof:

(1) ◎ $\operatorname{Ker} \varphi \neq \phi$ because $\varphi(e_G) = e_H$.

◎ Let $a, b \in \operatorname{Ker} \varphi$, then $\varphi(ab^{-1}) = \varphi(a)\varphi(b)^{-1} = e_H \cdot e_H^{-1} = e_H$. Thus $ab^{-1} \in \operatorname{Ker} \varphi$.

(2) ◎ $\operatorname{Im} \varphi \neq \phi$ because $\varphi(e_G) = e_H$.

◎ Let $x, y \in \operatorname{Im} \varphi$, then $\exists\, a, b \in G$ such that $x = \varphi(a)$ and $y = \varphi(b)$.

Then $xy^{-1} = \varphi(a)\varphi(b)^{-1} = \varphi(ab^{-1})$. Thus $xy^{-1} \in \operatorname{Im} \varphi$.

Proposition 25: Let $\varphi : G \longrightarrow H$ be a homomorphism.

(1) φ is injective $\Longleftrightarrow$ $\operatorname{Ker}\varphi = \{e_G\}$.

(2) If φ is injective, then $G \cong \operatorname{Im}\varphi$.

(3) φ is surjective $\Longleftrightarrow$ $\operatorname{Im}\varphi = H$

Proof:

(1) ($\Longrightarrow$) Let $a \in \operatorname{Ker}\varphi$, then $\varphi(a) = e_H = \varphi(e_G)$. Since φ is injective, $a = e_G$.

($\Longleftarrow$) Suppose $a, b \in G$ are so that $\varphi(a) = \varphi(b)$, then

$$\varphi(a)\varphi(b)^{-1} = e_H \Longrightarrow \varphi(ab^{-1}) = e_H \Longrightarrow ab^{-1} \in \operatorname{Ker}\varphi \Longrightarrow ab^{-1} = e_G \Longrightarrow a = b.$$

(2) Observe that $\varphi : G \longrightarrow \operatorname{Im}\varphi$ is an iso.

(3) By def.

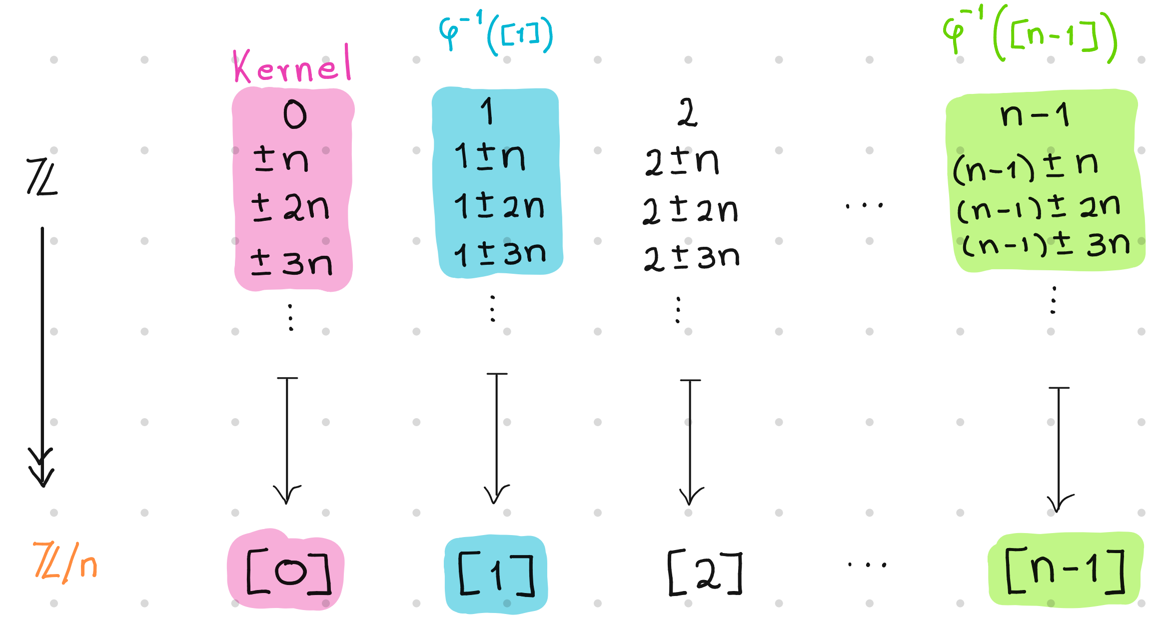
Ex: Find $\operatorname{Ker}\varphi$, $\operatorname{Im}\varphi$, and the fibers.

1. $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}/n$, $\quad a \longmapsto [a]$

$\operatorname{Ker}\varphi = n\mathbb{Z}$ $\qquad$ $\operatorname{Im}\varphi = \mathbb{Z}/n$ $\qquad$ $\varphi^{-1}([a]) = \{\, nk + a \mid k \in \mathbb{Z} \,\}$

Not injective $\quad \operatorname{Ker}\varphi \neq \{0\}$

Surjective $\quad \operatorname{Im}\varphi = \mathbb{Z}/n$

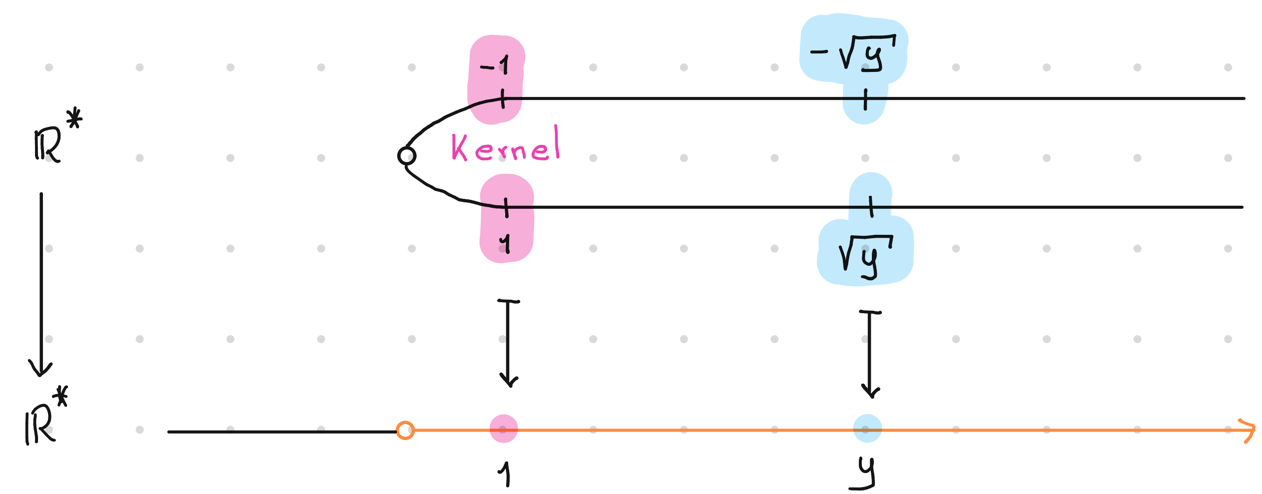


2. $f : \mathbb{R}^* \longrightarrow \mathbb{R}^*$, $\quad x \longmapsto x^2$

$\operatorname{Ker} f = \{-1, 1\}$ $\qquad$ $\operatorname{Im} f = \{\, x^2 \mid x \in \mathbb{R}^* \,\} = \mathbb{R}^+$ $\qquad$ $f^{-1}(y) = \begin{cases} \{-\sqrt{y}, \sqrt{y}\}, & y > 0 \\ \emptyset, & y < 0 \end{cases}$

Not injective $\quad \operatorname{Ker} f \neq \{1\}$

Not surjective $\quad \operatorname{Im} f \subsetneq \mathbb{R}^*$



3. $\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $\pi(x,y) = x + y$.

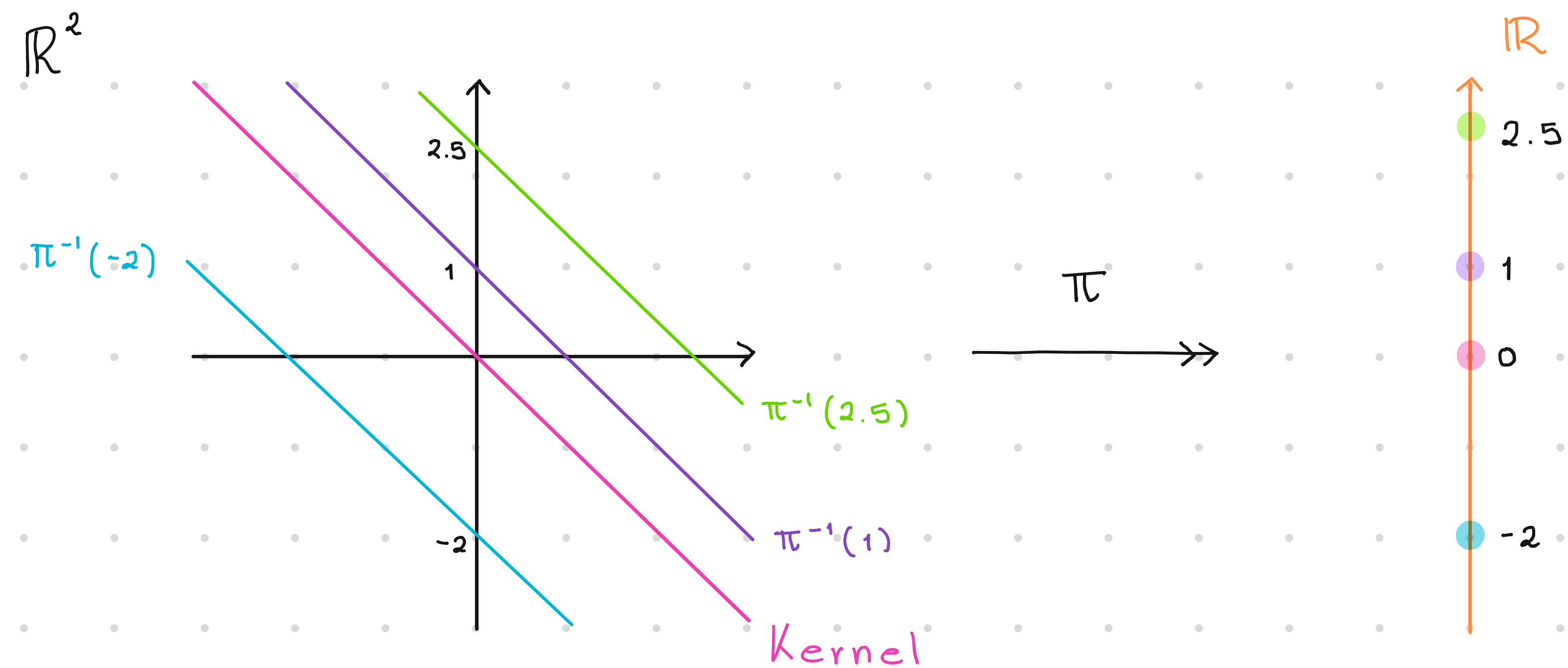
π is an epimorphism, then $\operatorname{Im}\pi = \mathbb{R}$.

$\operatorname{Ker}\pi = \{(x,y) \mid x+y=0\} = \{(x,-x) \mid x \in \mathbb{R}\}$ ← the line $y = -x$

$\pi^{-1}(a) = \{(x,y) \mid x+y=a\} = \{(x,-x+a) \mid x \in \mathbb{R}\}$ ← the line $y = -x + a$

Not injective $\operatorname{Ker}\pi \neq \{(1,1)\}$

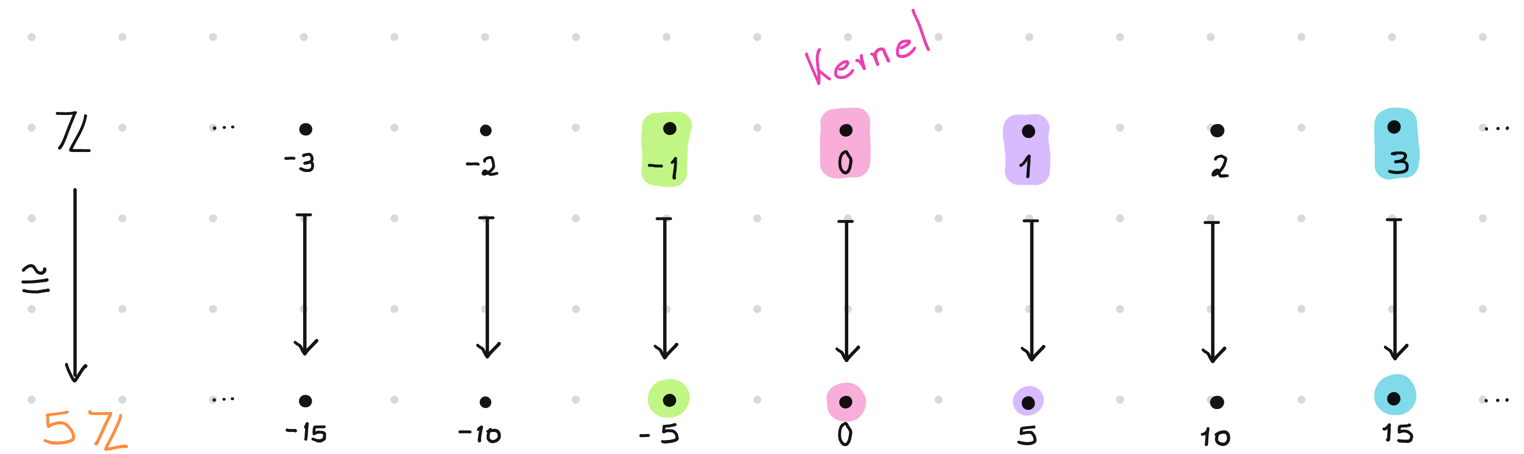
Surjective $\operatorname{Im}\pi = \mathbb{R}$



4. $\psi : \mathbb{Z} \longrightarrow 5\mathbb{Z}$ $\qquad$ $\operatorname{Ker}\psi = \{0\}$ $\qquad$ $\operatorname{Im}\psi = 5\mathbb{Z}$ $\qquad$ $\psi^{-1}(5x) = \{x\}$

$\quad x \longmapsto 5x$

$$\boxed{\begin{array}{ll} \text{Injective} & \operatorname{Ker}\psi = \{0\} \\ \text{Surjective} & \operatorname{Im}\psi = 5\mathbb{Z} \end{array}}$$



5. Let $\gamma : D_8 \longrightarrow S_4$ be defined by $\gamma(r) = \alpha$ and $\gamma(s) = \beta$ where

$\alpha := (1\ 2\ 3\ 4)$ and $\beta = (2\ 4)$.

$\operatorname{Ker}\gamma = \{1\}$ $\qquad$ $\operatorname{Im}\gamma = \{1, \alpha, \alpha^2, \alpha^3, \beta, \beta\alpha, \beta\alpha^2, \beta\alpha^3\}$

$$\gamma^{-1}(\sigma) = \begin{cases} \phi, & \sigma \notin \operatorname{Im}\gamma \\ s^i r^j, & \sigma = \alpha^i \beta^j \end{cases}$$

$i = 0, 1$
$j = 0, 1, 2, 3$

$$\boxed{\begin{array}{ll} \text{Injective} & \operatorname{Ker}\gamma = \{1\} \\ \text{Not surjective} & \operatorname{Im}\gamma \subsetneq S_4 \end{array}}$$

$$\begin{tikzcd}[column sep=small]
 & \text{Kernel} & & & & & & & & & & & \\
D_8 \arrow[d, hook] & 1 \arrow[d, mapsto] & r \arrow[d, mapsto] & r^2 \arrow[d, mapsto] & r^3 \arrow[d, mapsto] & s \arrow[d, mapsto] & sr \arrow[d, mapsto] & sr^2 \arrow[d, mapsto] & sr^3 \arrow[d, mapsto] & & & & \\
S_4 & 1 & \alpha & \alpha^2 & \alpha^3 & \beta & \beta\alpha & \beta\alpha^2 & \beta\alpha^3 & (2\,3\,4) & (2\,4\,3) & \cdots & (1\,2)
\end{tikzcd}$$