Examples:

1. \( \varphi: (\mathbb{Z}, +) \rightarrow (\mathbb{Z}/n, \oplus) \) defined by \( \varphi(a) = [a] \).

   - Well-defined: \( a = b \Rightarrow [a] = [b] \)

   - Homomorphism: \( \varphi(a+b) = [a+b] = [a] \oplus [b] = \varphi(a) \oplus \varphi(b) \)

   - Monomorphism: All multiples of \( n \) are sent to \([0]\)

   \[ [0] = \varphi(0) = \varphi(n) = \varphi(-n) = \varphi(2n) \]

   - Epimorphism: For \( [a] \in \mathbb{Z}/n \), \( \exists a \in \mathbb{Z} \) s.t. \( \varphi(a) = [a] \)
2) \( \psi: (\mathbb{Z}, +) \rightarrow (n\mathbb{Z}, +) \), \( n \neq 0 \) by \( \psi(a) = na \) is an isomorphism.

   \checkmark \text{Well-defined}

   \checkmark \text{Homomorphism: } \psi(a + b) = n(a + b) = na + nb = \psi(a) + \psi(b)

   \checkmark \text{Mono: } \psi(a) = \psi(b) \Rightarrow na = nb \Rightarrow n(a - b) = 0 \Rightarrow a - b = 0 \Rightarrow a = b

   \checkmark \text{Epi: Let } x \in n\mathbb{Z}, \exists a \in \mathbb{Z} \text{ s.t. } x = na. \text{ Then } \psi(a) = na = x.

Therefore, \( \mathbb{Z} \) is isomorphic to \( n\mathbb{Z} \) for all \( n \in \mathbb{Z} \setminus \{0\} \), \( \mathbb{Z} \cong n\mathbb{Z} \).

3) \( \text{id}_G: G \rightarrow G \) is an isomorphism.

4) \textbf{Inner Automorphism of } \textbf{G:} \text{ Fixed } a \in G, \text{ then } \varphi_a: G \rightarrow G \text{ given by } \varphi_a(g) = aga^{-1} \text{ is an isomorphism.}
5. $D_6$ is isomorphic to $S_3$.

\[ D_6 \rightarrow S_3 \]

\[ r \rightarrow (1 2 3) \]
\[ s \rightarrow (2 3) \]

In Q2, P56 you proved that $\varphi$ is bijective. You must inspect Cayley tables of $D_6$ and $S_3$ to verify $\varphi$ is a homomorphism.

6. The Klein group $K_4$ is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$

\[ \alpha \]

\[ 1 \rightarrow ([0],[0]) \]
\[ a \rightarrow ([0],[1]) \]
\[ b \rightarrow ([1],[0]) \]
\[ c \rightarrow ([1],[1]) \]

\[ \alpha(ab) = \alpha(c) = ([1],[1]) = ([0],[1]) + ([1],[0]) = \alpha(a) + \alpha(b) \]

\[ \vdots \]

7. $K_4 \neq \mathbb{Z}/4$ because in $K_4$ all elements have order 2 but the identity, whereas in $\mathbb{Z}/4$ there is one element of order 4.
Let $\mathcal{G}$ be the set of groups. Let $G$ and $H$ in $\mathcal{G}$. Define the relation:

$$G \text{ is related to } H \iff G \cong H$$

eq$

One goal in group theory: Classify groups up to isomorphism, i.e. understand equivalence classes in $\mathcal{G}$.

**Proposition 2.2:** Let $\phi: G \longrightarrow H$ be an isomorphism.

1. $|G| = |H|$
2. $G$ is abelian $\iff$ $H$ is abelian
3. $|g| = |\phi(g)|$ for all $g \in G$
4. $G$ and $H$ have the same number of elements of order $n$ with $n \in \mathbb{Z}^+$. 
What do we know about the classification of groups so far?

<table>
<thead>
<tr>
<th>Order</th>
<th>Abelian</th>
<th>Nonabelian</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>{e}</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>\mathbb{Z}/2 \cong S_2</td>
<td>\text{Prop: }</td>
</tr>
<tr>
<td>3</td>
<td>\mathbb{Z}/3</td>
<td>\text{Prop: }</td>
</tr>
<tr>
<td>4</td>
<td>\mathbb{Z}/4, \mathbb{Z}_4 \cong \mathbb{Z}/2 \times \mathbb{Z}/2</td>
<td>\text{Klein group}</td>
</tr>
<tr>
<td>5</td>
<td>\mathbb{Z}/5</td>
<td>\text{Thm: }</td>
</tr>
<tr>
<td>6</td>
<td>\mathbb{Z}/6 \cong \mathbb{Z}/2 \times \mathbb{Z}/3</td>
<td>\text{S}_3 \cong D_6</td>
</tr>
<tr>
<td>8</td>
<td>\mathbb{Z}/8, \mathbb{Z}/2 \times \mathbb{Z}/4, \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2</td>
<td>D_6, Q_8 \text{ Quaternion group}</td>
</tr>
</tbody>
</table>
Theorem 23: Let $\varphi: G \to H$ be a homomorphism. Then

1. $\varphi(e_G) = e_H$.
2. $\varphi(a^{-1}) = \varphi(a)^{-1}$ for all $a \in G$.
3. $\varphi(a^n) = \varphi(a)^n$ for all $n \in \mathbb{Z}$.

Proof:

1. $\varphi(e_G) \cdot \varphi(e_G) = \varphi(e_G e_G) = \varphi(e_G) \Rightarrow \varphi(e_G) = e_H$.
2. $\varphi(a) \cdot \varphi(a^{-1}) = \varphi(a a^{-1}) = \varphi(e_G) = e_H \Rightarrow \varphi(a)^{-1} = \varphi(a^{-1})$.
3. By induction on $n \in \mathbb{N}$.
   - $n = 0 \quad \checkmark$ Part (1)
   - $\varphi(a^{n+1}) = \varphi(a^n a) = \varphi(a^n) \varphi(a) = \varphi(a)^n \varphi(a) = \varphi(a)^{n+1}$
   - $n \in \mathbb{Z}^+$ then $\varphi(a^{-n}) = \varphi(a^{-1})^n = (\varphi(a)^{-1})^n = \varphi(a)^{-n}$. 
Three Important Isomorphisms

**Theorem: Cyclic Infinite**  
Let $G = \langle x \rangle$. If $|x| = \infty$, then $G \cong \mathbb{Z}$.

**Cyclic Finite**  
Let $G = \langle x \rangle$. If $|x| = n$, then $G \cong \mathbb{Z}/n$.

**Proof:** See Thm 7.19 Hungerford's.

**Theorem: Cayley's Theorem**  
Every group $G$ is isomorphic to a group of permutations.  
\[ G \cong H \text{ where } H \leq S_g \]

**Proof:** See Thm 7.21 Hungerford's.