Lecture 15

Examples:

1) $\varphi : (\mathbb{Z}, +) \longrightarrow (\mathbb{Z}/n, \oplus)$ defined by $\varphi(a) = [a]$.

✓ Well-defined: $a = b \Rightarrow [a] = [b]$

✓ Homomorphism: $\varphi(a+b) = [a+b] = [a] \oplus [b] = \varphi(a) \oplus \varphi(b)$

✗ Monomorphism: All multiples of n are sent to $[0]$

$$[0] = \varphi(0) = \varphi(n) = \varphi(-n) = \varphi(2n)$$

✓ Epimorphism: For $[a] \in \mathbb{Z}/n$, $\exists\, a \in \mathbb{Z}$ s.t. $\varphi(a) = [a]$

2) $\varphi : (\mathbb{Z}, +) \longrightarrow (n\mathbb{Z}, +)$, $n \neq 0$ by $\varphi(a) = na$ is an isomorphism.

✓ Well-defined

✓ Homomorphism: $\varphi(a+b) = n(a+b) = na + nb = \varphi(a) + \varphi(b)$

✓ Mono: $\varphi(a) = \varphi(b) \Rightarrow na = nb \Rightarrow n(a-b) = 0 \Rightarrow a-b = 0 \Rightarrow a = b$

✓ Epi: Let $x \in n\mathbb{Z}$, $\exists a \in \mathbb{Z}$ s.t. $x = na$. Then $\varphi(a) = na = x$.

Therefore, $\mathbb{Z}$ is isomorphic to $n\mathbb{Z}$ for all $n \in \mathbb{Z} \setminus \{0\}$, $\mathbb{Z} \cong n\mathbb{Z}$.

3) $id_G : G \longrightarrow G$ is an isomorphism.

4) Inner Automorphism of G: Fixed $a \in G$, then $\varphi_a : G \longrightarrow G$ given by

$\varphi_a(g) = aga^{-1}$ is an isomorphism.

5) D_6 is isomorphic to S_3.

$$D_6 \xrightarrow{\rho} S_3$$
$$r \longmapsto (1\ 2\ 3)$$
$$s \longmapsto (2\ 3)$$

In Q2, PS6 you proved that ρ is bijective

You must inspect Cayley tables of D_6 and S_3 to verify ρ is a homomorphism.

6) The Klein group K_4 is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$

$$\begin{aligned} &\xrightarrow{\alpha} \\ 1 &\longmapsto ([0],[0]) \\ a &\longmapsto ([0],[1]) \\ b &\longmapsto ([1],[0]) \\ c &\longmapsto ([1],[1]) \end{aligned}$$

$$\alpha(ab) = \alpha(c) = ([1],[1]) = ([0],[1]) + ([1],[0]) = \alpha(a) + \alpha(b)$$
$$\vdots$$

7) $K_4 \not\cong \mathbb{Z}/4$ because in K_4 all elements have order 2 but the identity, whereas in $\mathbb{Z}/4$ there is one element of order 4.

Let $\mathcal{G}$:= the set of groups. Let G and H in $\mathcal{G}$. Define the relation:

$$G \text{ is related to } H \iff G \cong H$$

e.r.

One goal in group theory: Classify groups up to isomorphism, i.e. understand equivalence classes in $\mathcal{G}$.

Proposition 22: Let $\varphi : G \longrightarrow H$ be an isomorphism.

(1) $|G| = |H|$

(2) G is abelian $\iff$ H is abelian

(3) $|g| = |\varphi(g)|$ for all $g \in G$

(4) G and H have the same number of elements of order n with $n \in \mathbb{Z}^+$.

What do we know about the classification of groups so far?

Order	Abelian	Nonabelian
1	$\{e\}$	
2	$\mathbb{Z}/2 \cong S_2$ Prop: $\lvert G \rvert = 2 \Rightarrow G \cong \mathbb{Z}/2$	
3	$\mathbb{Z}/3$ Prop: $\lvert G \rvert = 3 \Rightarrow G \cong \mathbb{Z}/3$	
4	$\mathbb{Z}/4$, $\underbrace{K_4}_{\text{Klein group}} \cong \mathbb{Z}/2 \times \mathbb{Z}/2$	
5	$\mathbb{Z}/5$ Thm: $\lvert G \rvert = \underbrace{p}_{\text{prime}} \Rightarrow G \cong \mathbb{Z}/p$	
6	$\mathbb{Z}/6 \cong \mathbb{Z}/2 \times \mathbb{Z}/3$	$S_3 \cong D_6$
8	$\mathbb{Z}/8$, $\mathbb{Z}/2 \times \mathbb{Z}/4$, $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$	D_6, $\underbrace{Q_8}_{\text{Quaternion group}}$
⋮		

Theorem 23: Let $\varphi : G \longrightarrow H$ be a homomorphism. Then

(1) $\varphi(e_G) = e_H$.

(2) $\varphi(a^{-1}) = \varphi(a)^{-1}$ for all $a \in G$.

(3) $\varphi(a^n) = \varphi(a)^n$ for all $n \in \mathbb{Z}$.

Proof:

(1) $\varphi(e_G)\,\varphi(e_G) = \varphi(e_G e_G) = \varphi(e_G) \implies \varphi(e_G) = e_H$

(2) $\varphi(a)\,\varphi(a^{-1}) = \varphi(a a^{-1}) = \varphi(e_G) = e_H \implies \varphi(a)^{-1} = \varphi(a^{-1})$

(3) By induction on $n \in \mathbb{N}$.

$n = 0$ ✓ Part (1)

$\varphi(a^{n+1}) = \varphi(a^n a) = \varphi(a^n)\,\varphi(a) = \varphi(a)^n\,\varphi(a) = \varphi(a)^{n+1}$

$n \in \mathbb{Z}^+$ then $\varphi(a^{-n}) = \varphi(a^{-1})^n$
$= (\varphi(a)^{-1})^n$
$= \varphi(a)^{-n}$.

Three Important Isomorphisms

Theorem: Cyclic Infinite Let $G = \langle x \rangle$. If $|x| = \infty$, then $G \cong \mathbb{Z}$.

Cyclic Finite Let $G = \langle x \rangle$. If $|x| = n$, then $G \cong \mathbb{Z}/n$.

Proof: See Thm 7.19 Hungerford's.

Theorem: Cayley's Theorem Every group G is isomorphic to a group of permutations.

$$G \cong H \quad \text{where} \quad H \leq S_G$$

Proof: See Thm 7.21 Hungerford's.