Lecture 14

Fundamental Theorem of Cyclic Groups: Let $G = \langle x \rangle$ be a cyclic group.

(1) If $H \leq G$, then $H = \{1\}$ or $H = \langle x^k \rangle$ where $k \in \mathbb{Z}^+$ is the smallest such that $x^k \in H$.

(2) If $|G| = n$, then for each positive divisor a of n, there is a unique subgroup of G of order a. This subgroup is $\langle x^d \rangle$ where $d := n/a$. Furthermore, $\forall m \in \mathbb{Z}$, $\langle x^m \rangle = \langle x^{\gcd(m,n)} \rangle$.

(3) If $|G| = \infty$, then for any $a, b \in \mathbb{Z}^+$, $a \neq b$, $\langle x^a \rangle \neq \langle x^b \rangle$. Furthermore, $\forall m \in \mathbb{Z}$, $\langle x^m \rangle = \langle x^{\text{abs}(m)} \rangle$ where $\text{abs}(m)$ is the absolute value of m.

Proof:

(1) Suppose $G = \langle x \rangle$ and let $H \leq G$.

If $H = \{e\}$, then $H = \langle e \rangle$.

Let $H \neq \{e\}$ and let $S := \{ n \in \mathbb{Z}^+ \mid x^n \in H \}$.

Since H is a nontrivial subgroup, $\exists\, x^i \in H$ s.t. $i \neq 0$ and $x^{-i} \in H$. Observe that either i or $-i$ is positive, then $S \neq \emptyset$. By the WOP, S has a smallest element, namely $k \in \mathbb{Z}^+$.

Claim: $H = \langle x^k \rangle$

($\supseteq$) ✓

($\subseteq$) Let $h \in H$, then $h = x^l$ for some $l \in \mathbb{Z}$ because $h \in G$.

By the Division Algorithm, $l = kq + r$ s.t. $0 \le r < k$. Consequently,

$x^r = x^{l-kq} = x^l (x^{-k})^q = h(x^{-k})^q \in H$ because H is a subgroup, then $r \in S$.

Since k is the smallest in S and $0 \le r < k$, then $r = 0$. Therefore

$h = x^l = (x^k)^q$, i.e. $h \in \langle x^k \rangle$.

(2)

Existence: Let $a \mid n$ and $a \in \mathbb{Z}^+$, then $\exists\, d \in \mathbb{Z}^+$, $n = da$.

Consider the subgroup $\langle x^d \rangle$. Then $|\langle x^d \rangle| \underset{\substack{\uparrow \\ \text{Prop 20}}}{=} |x^d| \underset{\substack{\uparrow \\ \text{Thm 5(c)}}}{=} \dfrac{n}{\gcd(n,d)} = \dfrac{n}{d} = a$.

Uniqueness: Let $H \le G$ s.t. $|H| = a$. By part (1), $H = \langle x^b \rangle$ where $b \in \mathbb{Z}^+$

is the smallest s.t. $x^b \in H$. Observe that $\dfrac{n}{d} = a = |H| = |x^b| = \dfrac{n}{\gcd(n,b)}$, so $d = \gcd(n,b)$. In particular, $d \mid b$, i.e. b is a multiple of d. Thus $H = \langle x^b \rangle \leq \langle x^d \rangle$. Since $|H| = a = |\langle x^d \rangle|$, then $H = \langle x^d \rangle$.

* Let $m \in \mathbb{Z}$ and $d := \gcd(m,n)$. Observe that $\langle x^m \rangle \leq \langle x^d \rangle$. Moreover, $|x^m| = \frac{n}{d}$ and $|x^d| = \frac{n}{\gcd(d,n)} = \frac{n}{d}$. Therefore, $\langle x^m \rangle = \langle x^d \rangle$.

(3) Exercise. $\blacksquare$

Summary: Let $G = \langle x \rangle$

Exercise: Write this summary in additive notation.

1. G is abelian

2. $|\langle x \rangle| = |x|$

$|\langle x \rangle| = n$

- $|x| = n$
- $x^i = x^j \iff i \equiv j \pmod{n}$
- $\langle x^a \rangle = \langle x \rangle \iff \gcd(a,n) = 1$
- $a \mid n \implies \langle x^{n/a} \rangle$ is the only subgroup of order a
- $\forall m \in \mathbb{Z}, \quad \langle x^m \rangle = \langle x^{\gcd(m,n)} \rangle$

$|\langle x \rangle| = \infty$

- $|x| = \infty$
- $x^i = x^j \iff i = j$
- Only x and x^{-1} generate $\langle x \rangle$.
- Each $\langle x^a \rangle$, $a \in \mathbb{Z}^+$ is a different subgroup
- $\forall m \in \mathbb{Z}, \quad \langle x^m \rangle = \langle x^{\mathrm{abs}(m)} \rangle$

Examples:

1) Let $G = \langle a \rangle$ with $|a| = 30$. Find $\langle a^{26} \rangle$ and $\langle a^{17} \rangle$

$$\langle a^{26} \rangle = \langle a^{\gcd(30,26)} \rangle = \langle a^2 \rangle = \{e, a^2, a^4, a^6, \ldots, a^{26}, a^{28}\}$$

$$|\langle a^{26} \rangle| = |a^{26}| = \frac{30}{2} = 15$$

$$\langle a^{17} \rangle = \langle a^{\gcd(30,17)} \rangle = \langle a \rangle = \{e, a, a^2, \ldots, a^{29}\}$$

2) Find all the generators of $Rot := \{1, r, r^2, \ldots, r^{11}\} \leq D_{24}$

Since $|r| = 12$ then we have a generator r^a if $\gcd(a, 12) = 1$.

Thus, r^5, r^7 and r^{11} are generators of Rot.

3) Let $G = \langle b \rangle$ and $|b| = 20$. Find all the subgroups of G.

We have a subgroup per each divisor of 20.

$1 \mid 20 \Rightarrow H_1 = \langle b^{20/1} \rangle = \{e\}$ order 1

$2 \mid 20 \Rightarrow H_2 = \langle b^{20/2} \rangle = \langle b^{10} \rangle = \{e, b^{10}\}$ order 2

$4 \mid 20 \Rightarrow H_4 = \langle b^{20/4} \rangle = \langle b^{5} \rangle = \{e, b^{5}, b^{10}, b^{15}\}$ order 4

$5 \mid 20 \Rightarrow H_5 = \langle b^{4} \rangle = \{e, b^{4}, b^{8}, b^{12}, b^{16}\}$ order 5

$10 \mid 20 \Rightarrow H_{10} = \langle b^{2} \rangle = \{e, b^{2}, b^{4}, b^{6}, b^{8}, b^{10}, b^{12}, b^{14}, b^{16}, b^{18}\}$ order 10

$20 \mid 20 \Rightarrow H_{20} = \langle b \rangle = G$ order 20

Lattice of subgroups of a group

order 20

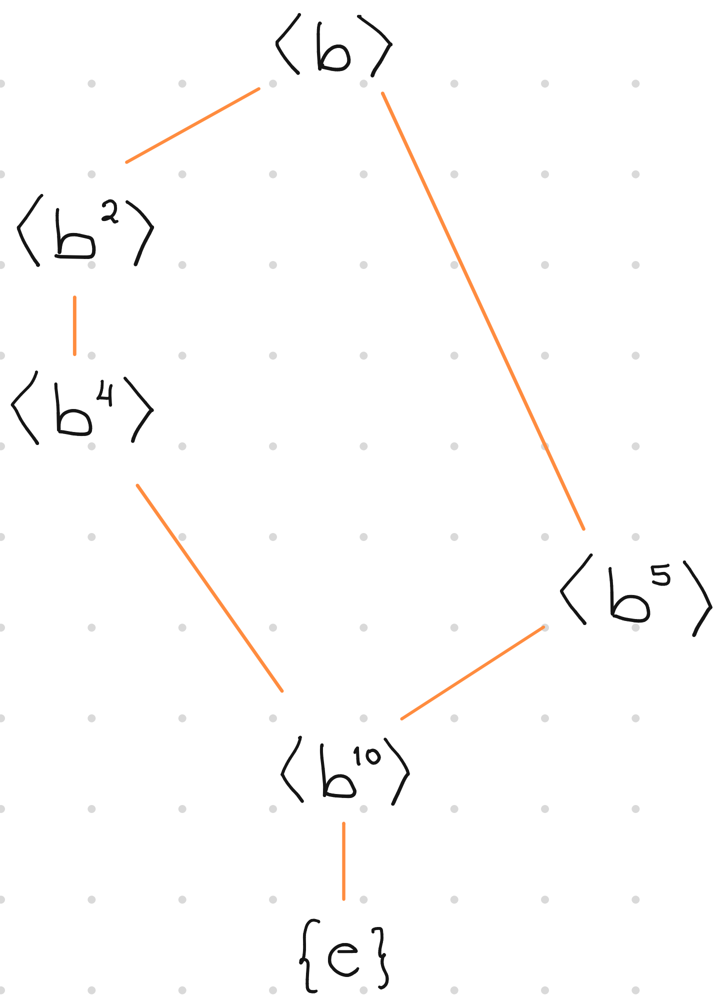
order 10

order 5

order 4

order 2

order 1



$\langle b^4 \rangle$ is a subgroup of $\langle b^2 \rangle$

$\langle b^5 \rangle$ is a subgroup of $\langle b \rangle$

$\langle b^{10} \rangle$ is a subgroup of $\langle b^4 \rangle$ and $\langle b^5 \rangle$

4) Find all the subgroups of $\mathbb{Z}/12$ and draw the lattice diagram.

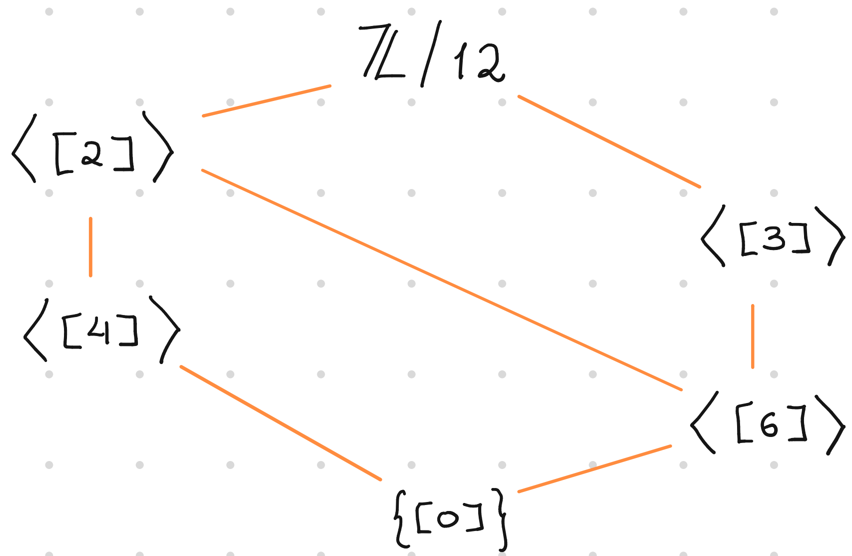
⚠ $\mathbb{Z}/12$ is an additive group. Watch out!!!

$2 \mid 12 \implies H_2 = \langle (12/2)[1] \rangle = \langle 6[1] \rangle = \langle [6] \rangle = \{ [0], [6] \}$

$3 \mid 12 \implies H_3 = \langle (12/3)[1] \rangle = \langle [4] \rangle = \{ [0], [4], [8] \}$

$4 \mid 12 \implies H_4 = \langle (12/4)[1] \rangle = \langle [3] \rangle = \{ [0], [3], [6], [9] \}$

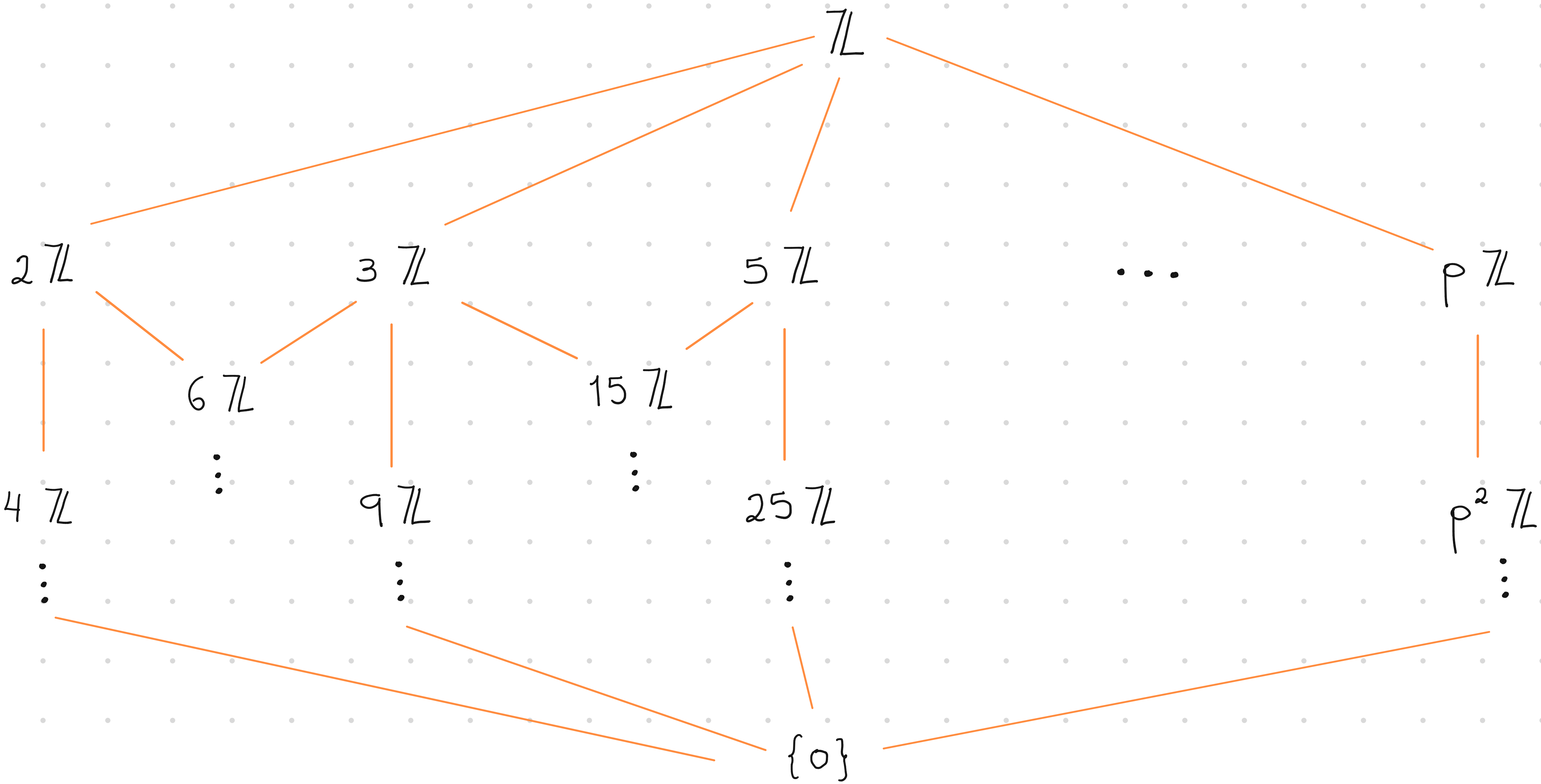
$6 \mid 12 \implies H_6 = \langle [2] \rangle = \{ [0], [2], [4], [6], [8], [10] \}$



5) Find all subgroups of $\mathbb{Z}$.

All subgroups of $(\mathbb{Z}, +)$ are of the form $n\mathbb{Z} = \langle n \rangle = \langle -n \rangle$ with $n \in \mathbb{Z}^+$.

Also, $n\mathbb{Z} \leq m\mathbb{Z} \iff m \mid n$. For instance, $\{0\} \leq \cdots \leq 2^n\mathbb{Z} \leq \cdots \leq 8\mathbb{Z} \leq 4\mathbb{Z} \leq 2\mathbb{Z}$.



Group Homomorphisms and Isomorphisms

Def: A map of sets $f: A \longrightarrow B$ is well-defined if $\forall a, a' \in A$ $\quad a = a' \Rightarrow f(a) = f(a')$.

Def: Let $(G, *)$ and $(H, \bullet)$ be groups. A well-defined map $\varphi: G \longrightarrow H$ is called a homomorphism if

$$\varphi(a * b) = \varphi(a) \bullet \varphi(b) \qquad \forall a, b \in G.$$

Def: A homomorphism $\varphi: G \longrightarrow H$ is called

(1) a monomorphism if φ is injective, i.e. the following condition is satisfied:

$$\forall a, b \in G, \qquad \varphi(a) = \varphi(b) \implies a = b$$

(2) an epimorphism if φ is surjective, i.e. the following condition is satisfied:

$$\forall h \in H \quad \exists a \in G \qquad \varphi(a) = b.$$

(3) an isomorphism if φ is injective and surjective. We say G and H are isomorphic.

Notation:

$\varphi : G \hookrightarrow H$ monomorphism

$\varphi : G \twoheadrightarrow H$ epimorphism

$\varphi : G \xrightarrow{\cong} H$ isomorphism or $G \cong H$