Week 7

Def: Let G be an abelian group.

We say G is free if $\exists A \subseteq G$ s.t. every element of G can be written uniquely as a linear combination $\sum_{i=1}^{r} n_i a_i$ for some $r \in \mathbb{N}$, $n_i \in \mathbb{Z}$ and $a_i \in A$. We call A a basis of G, and its cardinality is called the rank of G. We write $G = \mathbb{Z}[A]$ or $G = \langle A \rangle$.

When uniqueness is dropped, we say A generates G.

this means $\boxed{\sum_{i=1}^{r} n_i a_i = 0 \implies n_i = 0 \quad \forall i = 1, \dots, r}$

Ex:

1. $\mathbb{Z} = \mathbb{Z}[1]$ because for all $n \in \mathbb{Z}$, $n = n \cdot 1$

$A = \{1\}$ is a basis for $\mathbb{Z}$ and $\operatorname{rank}(\mathbb{Z}) = 1$.

2. $n\mathbb{Z} = \{ n \cdot x : x \in \mathbb{Z} \} = \mathbb{Z}[n]$

3. $\mathbb{Z}^n := \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{n\text{-times}}$ has as a basis $A = \{ (1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1) \}$.

4. Let $A = \{a, b, c\}$, then $\mathbb{Z}[A] = \{ n_a \cdot a + n_b \cdot b + n_c \cdot c \mid n_a, n_b, n_c \in \mathbb{Z} \}$

Observe that $\mathbb{Z}[A] \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

5. Let $A = \mathbb{N} \setminus \{0\}$, then $\mathbb{Z}[A] \cong \mathbb{Z} \times 2\mathbb{Z} \times \cdots \times n\mathbb{Z} \times \cdots \cong \underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z} \times \cdots}_{\displaystyle \overset{!!}{\prod_{n=1}^{\infty} \mathbb{Z}}}$

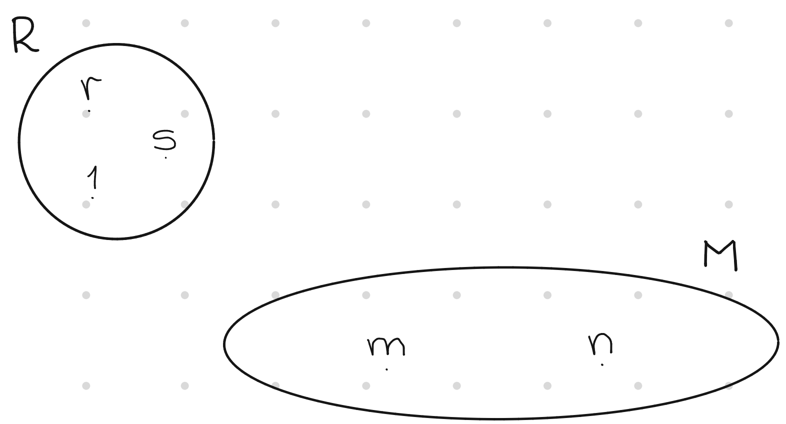
The following definition generalizes the concept of vector space.

Def: Let R be a commutative ring with identity, and let M be an abelian group.

We say M is an *R-module* if there is an operation $\cdot : R \times M \longrightarrow M$

"scalar multiplication"

s.t. $\forall r, s \in R \ \forall m, n \in M$



(i) $r \cdot (m + n) = r \cdot m + r \cdot n$

(ii) $(r + s) \cdot m = r \cdot m + s \cdot m$

(iii) $(rs) \cdot m = r \cdot (s \cdot m)$

(iv) $1_R \cdot m = m$

! When R is a field, we say M is an *R-vector space*.

Def: Let M and N be R-modules. A function $\varphi : M \longrightarrow N$ is an *R-homomorphism*

if $\forall r \in R$ and $\forall m, n \in M$

(i) $\varphi(m+n) = \varphi(m) + \varphi(n)$

(ii) $\varphi(r \cdot m) = r \cdot \varphi(m)$

Ex:

1. Every abelian group G is a $\mathbb{Z}$-module, where $\forall n \in \mathbb{N}$ $\forall g \in G$

   - $n \cdot g = \underbrace{g + \cdots + g}_{n\text{-times}}$
   - $0 \cdot g = 0_G$
   - $(-n) \cdot g = \underbrace{(-g) + \cdots + (-g)}_{n\text{-times}}$

2. $\mathbb{R}^n$ is an $\mathbb{R}$-vector space with $r \cdot (x_1, \dots, x_n) := (r x_1, \dots, r x_n)$, $\forall r, x_i \in \mathbb{R}$

3. Let R be a commutative ring with identity.

- $R[x]$ is an R-module where $r \cdot \sum_{i=0}^{n} a_i x^i := \sum_{i=0}^{n} (ra_i) x^i \quad \forall r, a_i \in R \ \forall 0 \leq i \leq n$

- $M_{m \times n}(R)$ is an R-module where $r(a_{ij}) := (ra_{ij}) \quad \forall r, a_{ij} \in R$
  $\forall 1 \leq i \leq m,\ 1 \leq j \leq n$

4. Let M be a smooth manifold.

$C^\infty(M) := \{ f : M \longrightarrow \mathbb{R} \mid f \text{ is a smooth map} \}$ is a ring. Prove it!

- $C^\infty(M)$ is an $\mathbb{R}$-vector space. Prove it!

- $$\mathcal{V} := \left\{ X : C^\infty(M) \longrightarrow C^\infty(M) \;\middle|\; \begin{array}{l} X \text{ is a linear transformation over } \mathbb{R}, \text{ and} \\ X(fg) = f X(g) + X(f) g \\ \text{for all } r \in \mathbb{R}, \text{ and } f, g \in C^\infty(M) \end{array} \right\}$$

is a $C^\infty(M)$-module. Prove it!

5. Find examples of R-homomorphisms.

Chains

Def: Let R be a commutative ring with identity. Let K be a k-simplicial complex, and $0 \leq p < k$.

(i) Let $S_p \subseteq K$ be the set of p-simplices of K, and $m_p := \#(S_p)$.

(ii) An element of $C_p(K;R) := \left\{ \sum_{i=1}^{m_p} r_i \sigma_i \,:\, r_i \in R,\ \sigma_i \in S_p \right\}$ is called a *p-chain* in K. We call r_i *coefficients* for all $1 \leq i \leq m_p$.

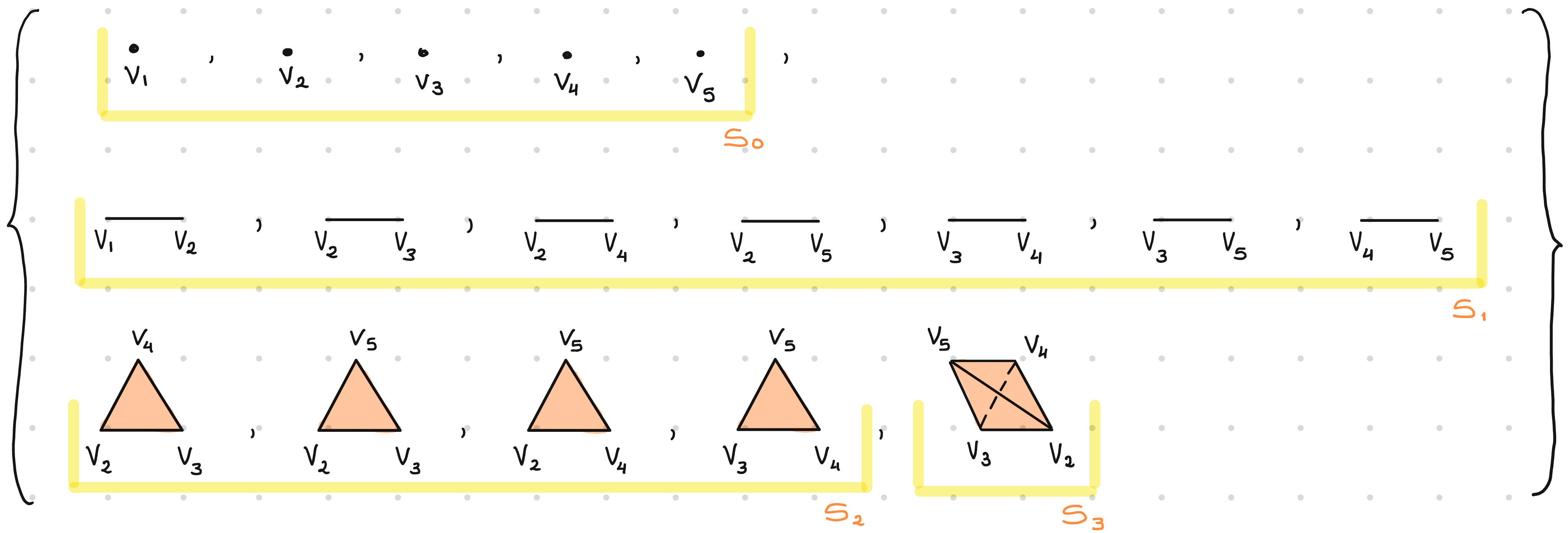
Convention: When writing a chain, we omit simplices with coefficient 0_R.

Prop: For all $0 \le p \le k$, $C_p(K;R)$ is an R-module where $r \cdot \sum_{i=1}^{m_p} r_i \sigma_i := \sum_{i=1}^{m_p} (r r_i)\sigma_i$

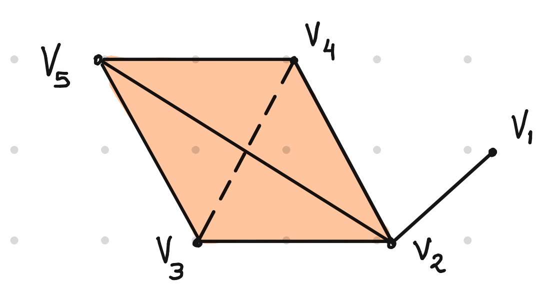
for all $r, r_i \in R$ and $\sigma_i \in S_p$.

Proof: Exercise.

Ex: Consider the 3-simplicial complex K:

$K =$ 

$|K|$



$m_0 = 5$ $\quad m_1 = 7$ $\quad m_2 = 4$ $\quad m_3 = 1$

$S_0 = \{ v_1, v_2, v_3, v_4, v_5 \}$ $\quad S_1 = \{ e_1, \dots, e_7 \}$ $\quad S_2 = \{ f_1, \dots, f_5 \}$ $\quad S_3 = \{ \sigma \}$

vertices edges faces tetrahedrons

Let $R = \mathbb{Z}$, then the following are examples of p-chains in K:

0-chains $C_0(K;\mathbb{Z})$ $\quad 20 v_5, \quad -3v_1 - 6v_4$

1-chains $C_1(K;\mathbb{Z})$ $\quad 6e_2 + 5e_4, \quad \sum_{i=1}^{7} e_i$

2-chains $C_2(K;\mathbb{Z})$ $\quad f_2 - 3f_4$

3-chains $C_3(K;\mathbb{Z})$ $\quad 11\sigma, \quad -63\sigma$

Let $R = \mathbb{F}_2$, then p-chains have coefficients 0 or 1:

In $C_0(K;\mathbb{F}_2)$ $\quad (v_1 + v_2) + (v_2 + v_3) = v_1 + \overset{0}{\cancel{2v_2}} + v_3 = v_1 + v_3$

Remark: From now on, we will assume $R = \mathbb{F}_2$, and simply write $C_p(K)$. In this case:

- The inverse of every $\sigma \in C_p(K)$ is itself.
- A p-chain $\sigma_1 + \sigma_2 + \cdots + \sigma_n$ can be treated as a set $\{\sigma_1, \sigma_2, \dots, \sigma_n\}$, where addition of two p-chains A and B is given by $(A \cup B) - (A \cap B)$, and the zero chain is the empty set.

Ex: In example ★, we have the following

$C_1(K)$ $\quad \{e_1, e_2, e_3\} + \{e_4, e_5\} = \{e_1, e_2, e_3, e_4, e_5\}$

$C_2(K)$ $\quad \{f_3, f_4, f_5\} + \{f_1, f_4, f_5\} = \{f_1 + f_3\}$

$C_3(K)$ $\quad \{\sigma\} + \{\sigma\} = \phi$

Cycles and Boundaries

Def: Let R be a commutative ring with identity. Let $(C_i, \partial_i)_{i=0}^{\infty}$ be a sequence of R-modules connected by R-homomorphisms as follows

$$\cdots \longrightarrow C_i \xrightarrow{\partial_i} C_{i-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

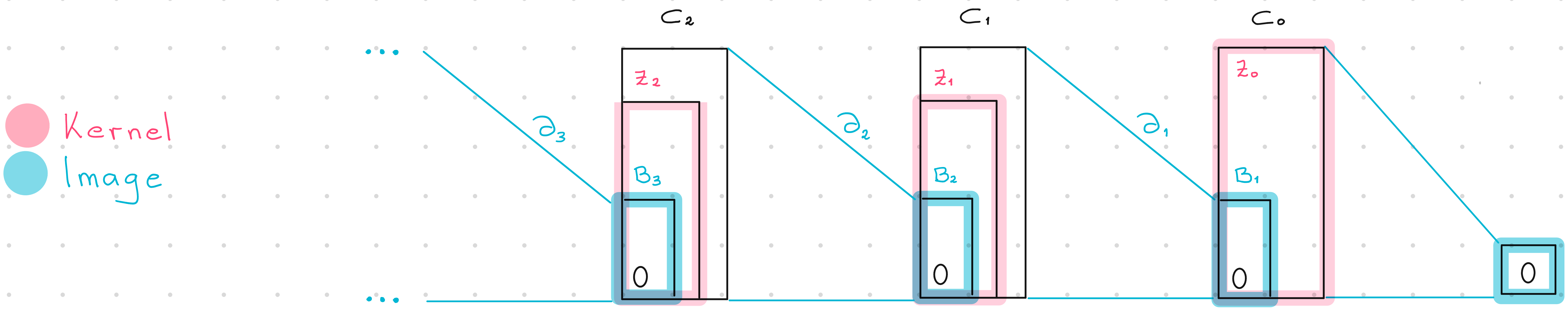
We say $(C_i, \partial_i)_{i=0}^{\infty}$ is a chain of R-modules (or chain complex over R) if the composition $\partial_{i-1} \circ \partial_i$ equals zero for all $i \geq 1$.

The R-homomorphisms are called boundary homomorphisms (or differentials).

! The condition $\partial_{i-1} \circ \partial_i = 0 \ \forall i \geq 1$ has important consequences on $\operatorname{Ker} \partial_{i-1}$ and $\operatorname{Im} \partial_i$. More concretely, $\forall i \geq 1 \ \operatorname{Im} \partial_i \subseteq \operatorname{Ker} \partial_{i-1}$.

$\operatorname{Ker} \partial_{i-1}$ is denoted by Z_{i-1} and its elements are called $(i-1)$-cycles.

$\operatorname{Im} \partial_i$ is denoted by B_i and its elements are called i-boundaries.



$$\begin{array}{ccc} C_2 & C_1 & C_0 \\ \cup & \cup & \| \\ Z_2 & Z_1 & Z_0 \\ \cup & \cup & \cup \\ B_3 & B_2 & B_1 \end{array}$$

❗ B_i and Z_{i-1} are R-modules.

Chain complex associated to a simplicial complex (over $\mathbb{F}_2$)

Let K be a k-simplicial complex. Let us define a chain complex $(C_p(K), \partial_p)_{p \geq 0}$.

- If $p > k$, then $C_p(K) = 0$ and $\partial_p = 0$.
- Let $v_i \in K_0$ and let $v_0 v_1 \cdots v_p$ be a p-simplex with $0 < p \leq k$. Define

$$\partial_p (v_0 v_1 \cdots v_p) = \sum_{i=0}^{p} v_0 v_1 \cdots \hat{v}_i \cdots v_p$$ where $\hat{v}_i$ indicates that the vertex v_i is omitted, i.e.

$$\partial_p (v_0 v_1 \cdots v_p) = \{ v_1 v_2 \cdots v_p,\ v_0 v_2 \cdots v_p,\ \cdots,\ v_0 \cdots v_{p-2} v_{p-1} \}$$

or $v_1 v_2 \cdots v_p + v_0 v_2 \cdots v_p + \cdots + v_0 \cdots v_{p-2} v_{p-1}$

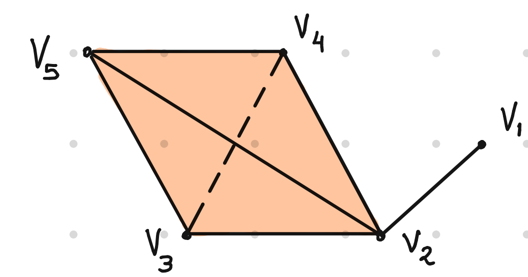
Now, we can define $\partial_p : C_p(K) \longrightarrow C_{p-1}(K)$ on an arbitrary p-chain $\{\sigma_1, \dots, \sigma_{m_p}\}$ as follows: $\partial_p \{\sigma_1, \dots, \sigma_{m_p}\} = \partial_p \sigma_1 + \cdots + \partial_p \sigma_{m_p}$. ∂_p is a linear transformation.

Prop: $(C_p(K), \partial_p)$ is a chain complex.

Proof: Exercise.

Ex: Let's compute $(C_p(K), \partial_p)$ for K in example ⭐.

$|K|$



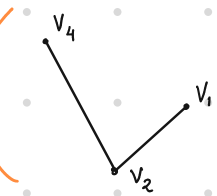
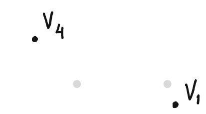
$$0 \longrightarrow C_3(K) \xrightarrow{\partial_3} C_2(K) \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \longrightarrow 0$$

Notation:

- V_1, V_2, V_3, V_4, V_5
- $e_1 = V_1V_2$ $\quad e_2 = V_2V_3$ $\quad e_3 = V_2V_4$ $\quad e_4 = V_2V_5$ $\quad e_5 = V_3V_4$ $\quad e_6 = V_3V_5$ $\quad e_7 = V_4V_5$
- $f_1 = V_2V_3V_4$ $\quad f_2 = V_2V_3V_5$ $\quad f_3 = V_2V_4V_5$ $\quad f_4 = V_3V_4V_5$
- $\sigma = V_2V_3V_4V_5$

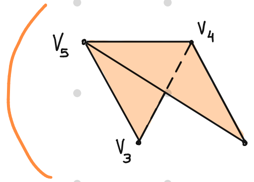
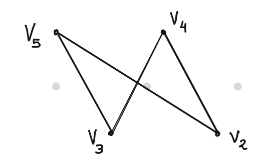
Observe:

$$\partial_1\left(\{v_1v_2,\ v_2v_4\}\right) = \{v_2, v_1\} + \{v_4, v_2\} = \{v_1, v_4\}$$

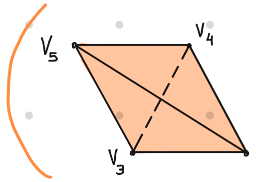
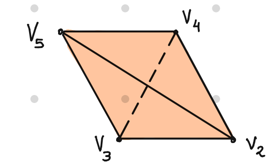
i.e. boundary $\Big($  $\Big)$ = 

$$\partial_2\left(\{v_2v_4v_5,\ v_3v_4v_5\}\right) = \{v_4v_5,\ v_2v_5,\ v_2v_4\} + \{v_4v_5,\ v_3v_5,\ v_3v_4\}$$

$$= \{v_2v_4,\ v_2v_5,\ v_3v_4,\ v_3v_5\}$$

i.e boundary $\Big($  $\Big)$ = 

$$\partial_3\left(v_2v_3v_4v_5\right) = \{v_3v_4v_5,\ v_2v_4v_5,\ v_2v_3v_5,\ v_2v_3v_4\}$$

i.e. boundary $\Big($  $\Big)$ = 

solid tetrahedron

hollow tetrahedron

These calculations give us some examples of boundaries:

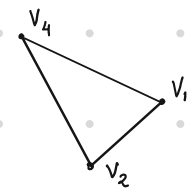
1-boundary $\{V_1, V_4\} \in B_1$

2-boundary $\{V_2V_4,\ V_2V_5,\ V_3V_4,\ V_3V_5\} \in B_2$

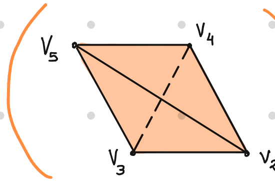
3-boundary $\{V_3V_4V_5,\ V_2V_4V_5,\ V_2V_3V_5,\ V_2V_3V_4\} \in B_3$

The following are some examples of cycles:

0-cycle Every vertex V_i

1-cycle $\{V_1V_2,\ V_1V_2,\ V_2V_4\} \in Z_1$ $\partial_1 \Big($ 

$\Big) = \phi$

2-cycle $\{V_3V_4V_5,\ V_2V_4V_5,\ V_2V_3V_5,\ V_2V_3V_4\} \in Z_2$ $\partial_2 \Big($ 

$\Big) = \phi$

hollow tetrahedron

3-cycle Only $0 \in C_3(K)$.

Let's describe $(C_p(K), \partial_p)$ algebraically:

$$C_3(K) = \{0, \sigma\} \cong \mathbb{F}_2 \longleftarrow m_3$$

$$C_2(K) = \left\{ \sum_{i=1}^{4} r_i f_i \; : \; r_i \in \mathbb{F}_2 \right\} \cong \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2 =: \mathbb{F}_2^{4} \longleftarrow m_2$$

$$C_1(K) \cong \mathbb{F}_2^{7} \longleftarrow m_1$$

$$C_0(K) \cong \mathbb{F}_2^{5} \longleftarrow m_0$$

Thus we have the chain complex

$$0 \longrightarrow \mathbb{F}_2 \xrightarrow{\partial_3} \mathbb{F}_2^4 \xrightarrow{\partial_2} \mathbb{F}_2^7 \xrightarrow{\partial_1} \mathbb{F}_2^5 \longrightarrow 0$$

In order to write ∂_p algebraically, we need to study what ∂_p does to the basis of $C_p(K)$ and translate that into the language of $\mathbb{F}_2$-vector spaces.

p	S_p geometrically	S_p algebraically
3	σ	1
2	f_1, f_2, f_3, f_4	$(1,0,0,0),\ (0,1,0,0),\ (0,0,1,0),\ (0,0,0,1)$
1	$e_1, e_2, \dots, e_7$	$(1,0,\dots,0),\ (0,1,0,\dots,0),\ \dots,\ (0,\dots,0,1)$
0	$v_1, v_2, \dots, v_5$	$(1,0,0,0,0),\ (0,1,0,0,0),\ \dots,\ (0,0,0,0,1)$

p	$\partial_p : C_p(K) \longrightarrow C_{p-1}(K)$	$\partial_p : \mathbb{F}_2^{m_p} \longrightarrow \mathbb{F}_2^{m_{p-1}}$
3	$\partial_3(\sigma) = f_1 + f_2 + f_3 + f_4$	$\partial_3(1) = (1,1,1,1)$
2	$\partial_2(f_1) = e_2 + e_3 + e_5$ $\partial_2(f_2) = e_2 + e_4 + e_6$	$\partial_2(1,0,0,0) = (0,1,1,0,1,0,0)$ $\partial_2(0,1,0,0) = (0,1,0,1,0,1,0)$

	$\partial_2(f_3) = \cdots$ exercise $\partial_2(f_4) = \cdots$ exercise	$\partial_2(0,0,1,0) = \cdots$ exercise $\partial_2(0,0,0,1) = \cdots$ exercise
1	$\partial_1(e_1) = V_1 + V_2$ $\partial_1(e_2) = \cdots$ exercise $\vdots$ $\partial_1(e_7) = \cdots$ exercise	$\partial_1(1,0,\ldots,0) = (1,1,0,0,0)$ $\partial_1(0,1,\ldots,0) = \cdots$ exercise $\vdots$ $\partial_1(0,\ldots,1) = \cdots$ exercise

From this, we can represent ∂_p as an $m_{p-1} \times m_p$ matrix A_{∂_p} (or simply A_p):

$$0 \longrightarrow \mathbb{F}_2 \xrightarrow[\partial_3]{\overset{A_3}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}} \mathbb{F}_2^4 \xrightarrow[\partial_2]{\overset{A_2}{\begin{bmatrix} 0 & 0 & \vert & \vert \\ 1 & 1 & \vert & \vert \\ 1 & 0 & \vdots & \vdots \\ 0 & 1 & \vdots & \vdots \\ 1 & 0 & \vdots & \vdots \\ 0 & 1 & \vert & \vert \\ 0 & 0 & \vert & \vert \end{bmatrix}}} \mathbb{F}_2^7 \xrightarrow[\partial_1]{\overset{A_1}{\begin{bmatrix} 1 & \vert & \vert & \vert & \vert & \vert & \vert \\ 1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \vert & \vert & \vert & \vert & \vert & \vert \\ 0 & \vert & \vert & \vert & \vert & \vert & \vert \end{bmatrix}}} \mathbb{F}_2^5 \longrightarrow 0$$

Finally,

$$B_1 = \operatorname{Im} \partial_1 = \operatorname{Col}(A_1) \subseteq Z_0 \cong \mathbb{F}_2^5$$

$$B_2 = \operatorname{Col}(A_2) \subseteq Z_1 = \operatorname{Ker} \partial_1 = \operatorname{Null}(A_1)$$

$$B_3 \cong \mathbb{F}_2 \subseteq Z_2 = \operatorname{Null}(A_2)$$

$$B_i = 0 \quad \forall i \geq 4 \subseteq Z_3 = \operatorname{Null}(A_3)$$

$$Z_i = 0 \quad \forall i \geq 4$$

Exercise:

* Find A_2 and A_1.
* Find all boundary and cycle groups.
* Find basis for all B_i and Z_i.

Here $\operatorname{Null}(A)$ and $\operatorname{Col}(A)$ are the null space and column space from Linear Algebra.

Exercise: Problems 1 and 10 textbook.

Remark: We can define $(C_p(K;R), \partial_p)_{p \geq 0}$ for an arbitrary commutative ring with identity:

- If $p > k$, then $C_p(K) = 0$ and $\partial_p = 0$.
- If $v_i \in K_0$ and $v_0 v_1 \cdots v_p$ is a p-simplex with $0 < p \leq k$, then

$$\partial_p(v_0 v_1 \cdots v_p) = \sum_{i=0}^{p} (-1)^i \, v_0 v_1 \cdots \hat{v}_i \cdots v_p .$$

If $r_1 \sigma_1 + \cdots + r_{m_p} \sigma_{m_p} \in C_p(K)$, then

$$\partial_p(r_1 \sigma_1 + \cdots + r_{m_p} \sigma_{m_p}) = r_1 \partial_p(\sigma_1) + \cdots + r_{m_p} \partial_p(\sigma_{m_p}).$$