Week 5

Topological spaces are too general to be feasible for algorithmic purposes.

Simplicial complexes are spaces constructed from building blocks called simplices, which are points, line segments, filled-in triangles, solid tetrahedra, and their higher dimensional analogues. They provide a highly useful way of constructing topological spaces, and play a fundamental role in geometry and algebraic topology

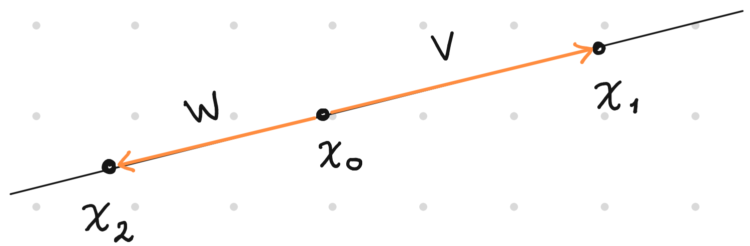
Suppose we are given points $\{x_0, x_1, \dots, x_k\}$ in $\mathbb{R}^n$. We will assume that these points satisfy the condition that the set of vectors $\{x_1 - x_0, x_2 - x_0, \dots, x_k - x_0\}$ in $\mathbb{R}^n$ are linearly independent, i.e. for $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}$

(*) $\boxed{\text{if } \alpha_1(x_1 - x_0) + \alpha_2(x_2 - x_0) + \cdots + \alpha_k(x_k - x_0) = \mathbf{0}, \text{ then } \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0.}$

Ex:

1. $\{x_0, x_1\}$ satisfy $\circledast$ if $x_1 - x_0 \neq 0$

2. $\{x_0, x_1, x_3\}$ satisfy $\circledast$ if $x_0, x_1,$ and x_2 do not lie on the same line.



v and w are not l.i. because $w = -\alpha v$ with $0 < \alpha < 1$.

3. $\{x_0, x_1, x_3, x_4\}$ satisfy $\circledast$ if $x_0, x_1, x_3,$ and x_4 do not lie on the same plane.

Def: Suppose $\{x_0, x_1, \dots, x_k\} \subseteq \mathbb{R}^n$ satisfy $\circledast$. The k-simplex spanned by $\{x_0, x_1, \dots, x_k\}$ is the set of all points

$$z = \sum_{i=0}^{k} a_i x_i \quad \text{such that} \quad a_1, a_2, \dots, a_k \in \mathbb{R}^+ \quad \text{and} \quad \sum_{i=0}^{k} a_i = 1.$$

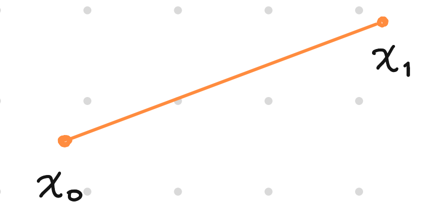
For a given point z, we refer to a_i as the ith barycentric coordinate.

Ex:

1. A 0-simplex is a point. $\{a_0 x_0 : a_0 \in \mathbb{R}^+ \text{ and } a_0 = 1\} = \{x_0\} \subseteq \mathbb{R}^n$

2. A 1-simplex is a line segment with endpoints x_0 and x_1.

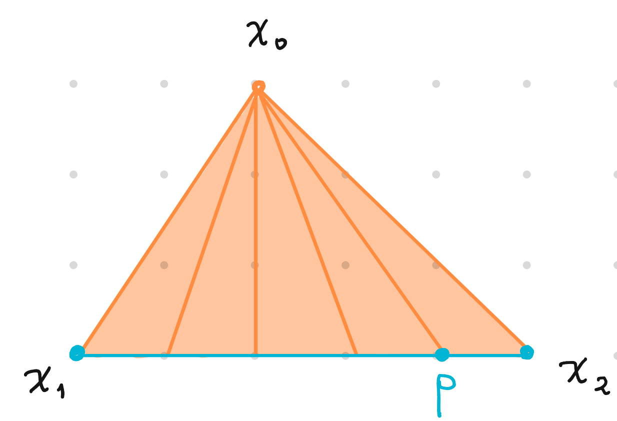
$$\{a_0 x_0 + a_1 x_1 : a_0, a_1 \in \mathbb{R}^+ \text{ and } a_0 + a_1 = 1\} = \{(1-a)x_0 + a x_1 : a \in [0,1]\}$$



3. A 2-simplex is a filled-in triangle with vertices x_0, x_1 and x_2.

$$\{a_0 x_0 + a_1 x_1 + a_2 x_2 : a_0, a_1, a_2 \in \mathbb{R}^+ \text{ and } a_0 + a_1 + a_2 = 1\}$$

$$= \left\{ \underbrace{a_0 x_0 + (1-a_0) \underbrace{\left[\frac{a_1}{1-a_0} x_1 + \frac{a_2}{1-a_0} x_2 \right]}_{\text{a point on the line joining } x_1 \text{ and } x_2}}_{\text{a point on the line joining } x_0 \text{ and } p} : a_0 + a_1 + a_2 = 1 \right\}$$



Remark: We consider a k-simplex as a top space with the subspace topology.

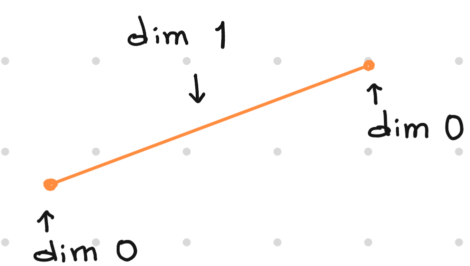
Def: Let S be a k-simplex spanned by $\{x_0, x_1, \dots, x_k\} \subseteq \mathbb{R}^n$.

1. A face of S is any simplex spanned by a subset of $\{x_0, x_1, \dots, x_k\}$.
2. The interior of S is the subset of S where $a_i > 0$ for all barycentric coordinates a_i, we denote it by $\mathrm{Int}(S)$.
3. The boundary of S is $\mathrm{Bd}(S) := S \setminus \mathrm{Int}(S)$.

Ex:

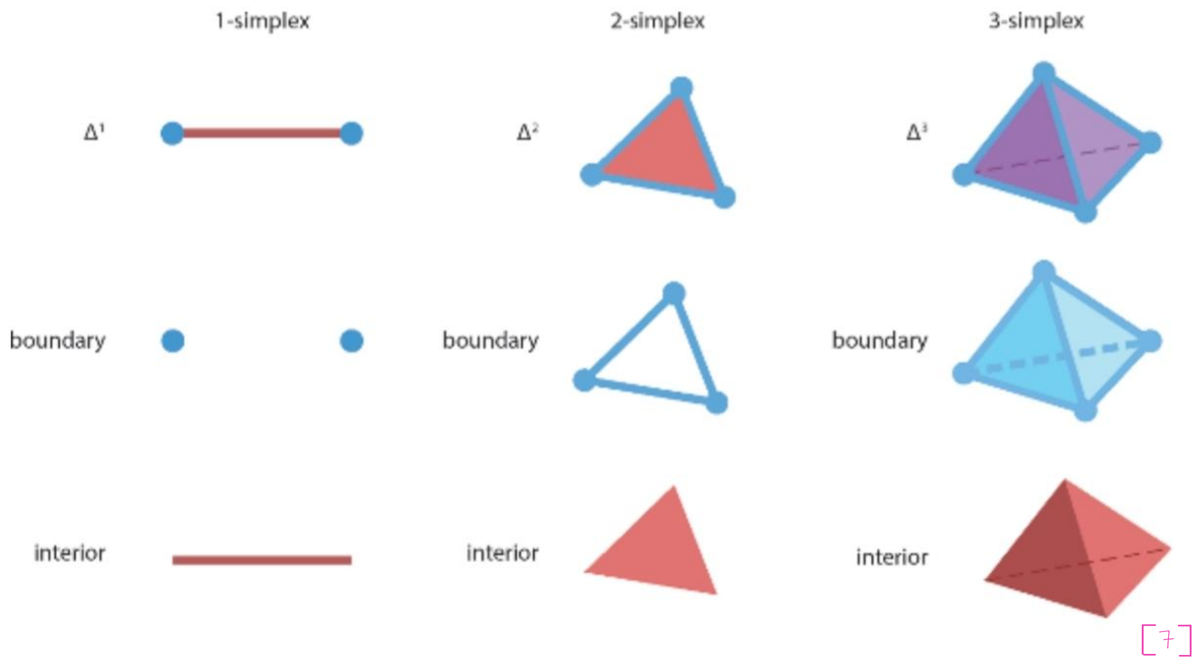
1. A 0-simplex only has one face.
2. A 1-simplex has ✓ two faces of dim 0 : 

   ✓ one face of dim 1 : 



3. Any k-simplex has $k+1$ faces of dimension $(k-1)$.

4.



5. For any n-simplex S, there are homeomorphisms

$$S \cong B^n \quad \text{and} \quad Bd(S) \cong S^{n-1}$$

Def: A simplicial complex X in $\mathbb{R}^n$ is a set of simplices in $\mathbb{R}^n$ such that

① every face of a simplex in X is also a simplex in K, and

② For any two simplices $\sigma, \tau \in X$, their intersection $\sigma \cap \tau$ is either empty or a face of both σ and τ.

We say X has dimension k if k is the maximum dimension among all simplices in X.

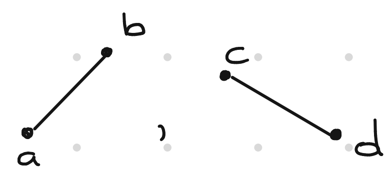
We say X is finite if X has finitely many simplices.

The collection of simplices of dim at most ℓ is referred to as the ℓ-skeleton of the simplicial complex; we denote it by X_ℓ.

The geometric realization $|X|$ of a finite simplicial complex X is the topological space given by the union of simplices in X, given the subspace topology.

Ex:

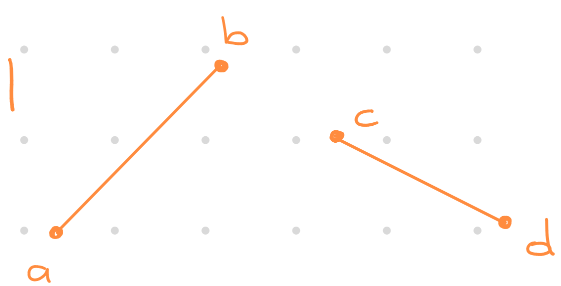
1. Let S be a k-simplex, the collection of all faces of S is a simplicial complex.

2. $X = \{ \{a\}, \{b\}, \{c\}, \{d\},$ 

$\}$

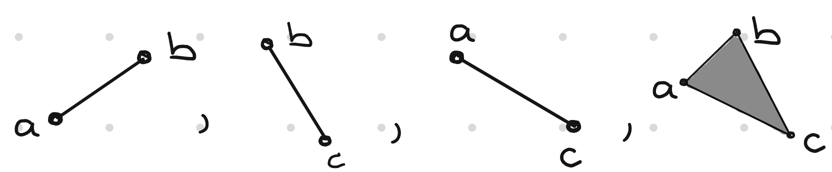
   X is a simplicial 1-complex in $\mathbb{R}^2$

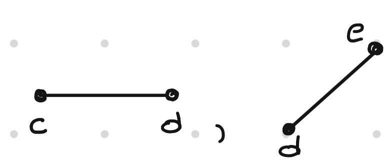
   $|X|$



   0-skeleton $X_0 = \{ \{a\}, \{b\}, \{c\}, \{d\} \}$

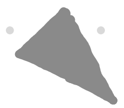
   1-skeleton $X_1 = X$

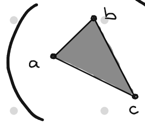
3. $Y = \{ \{a\}, \{b\}, \{c\},$ 

$, \{d\}, \{e\},$ 

$\}$

0-skeleton $\quad Y_0 = \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}\}$

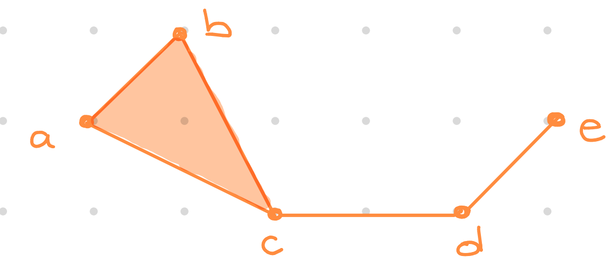
1-skeleton $\quad Y_1 = Y \setminus \{$  $\}$

Int()

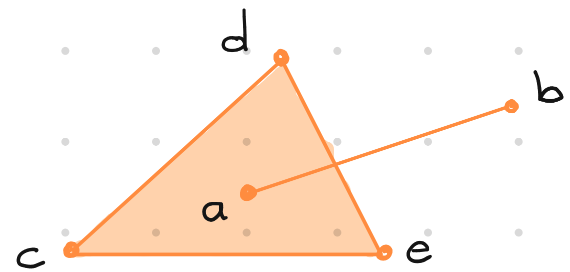
2-skeleton $\quad Y_2 = Y$

Y is a simplicial 2-complex in $\mathbb{R}^2$

$|Y|$



4. $Z = \{\{a\}, \{b\}, \{a,b\}, \{c\}, \{d\}, \{e\}, \{c,d\}, \{c,e\}, \{d,e\}, \{c,d,e\}\}$ is not a simplicial complex because the intersection of the 2-simplex and 1-simplex does not belong to Z.



5. A simplicial complex that has only 0-simplices and 1-simplices represents a graph embedded in $\mathbb{R}^n$.

Def: Let X be a simplicial complex. Any subset $X' \subseteq X$ that is itself a simplicial complex is called a subcomplex of X.

Def: Let X and Y be simplicial complexes. A simplicial map $f: X \longrightarrow Y$ is specified by a map $f_0 : X_0 \longrightarrow Y_0$ such that whenever $\{x_0, \dots, x_k\} \subseteq X_0$ span a simplex of X, $\{f(x_0), \dots, f(x_k)\} \subseteq Y_0$ span a simplex of Y.

The map $f: X \longrightarrow Y$ is an isomorphism of simplicial complexes if

i. f_0 is a bijection, and

ii. $\forall k > 1$ $\{x_0, \dots, x_k\}$ is a simplex of X iff $\{f(x_0), \dots, f(x_k)\}$ is a simplex of Y.

Find examples!

Remarks:

- Given a simplicial k-complex X, there is a chain of simplicial complexes given by its skeleta: $\underset{\text{vertices}}{X_0} \subseteq \underset{\text{edges}}{X_1} \subseteq X_2 \subseteq \cdots \subseteq X_{k-1} \subseteq X_k = X$

- A simplicial map $f: X \longrightarrow Y$ induces a continuous map $f: |X| \longrightarrow |Y|$

- If $f: X \longrightarrow Y$ is an isomorphism, then $f: |X| \longrightarrow |Y|$ is a homeomorphism.

- All the considerations of this section carry over without change if we replace $\mathbb{R}^n$ by an arbitrary finite-dimensional vector space V. We give V the metric topology induced by any norm. We can show that the resulting topology is independent of the norm. The only properties of $\mathbb{R}^n$ that we use are its vector space structure and its topology, and since any choice of basis gives a linear homeomorphism of V with $\mathbb{R}^n$, all the results of this section are true with $\mathbb{R}^n$ replaced by V. We will use this slightly

[4]

! It turns out that the data of a simplicial complex can be abstracted further; all that is really important is the data of how many simplices there are and which faces they are glued along.

Def: An abstract simplicial complex is a collection $\mathcal{K}$ of nonempty finite sets such that if $\sigma \in \mathcal{K}$, then every nonempty subset of σ is in $\mathcal{K}$.

① Elements of $\mathcal{K}$ are called simplices.

② The dimension of $\sigma \in \mathcal{K}$ is $\dim \sigma := \#(\sigma) - 1$ where $\#(\sigma)$ is the number of elements of the set σ

③ Any non-empty subset of a simplex σ is called a face of σ.

④ The vertices of $\mathcal{K}$ are the one-point sets in $\mathcal{K}$

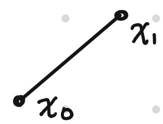
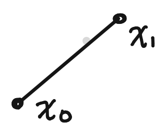
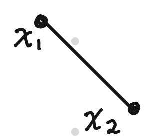
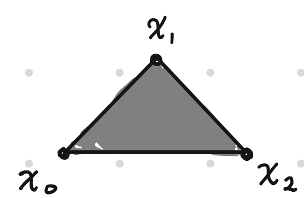
⑤ The n-skeleton of $\mathcal{K}$ is the subset of $\mathcal{K}$ consisting of sets of cardinality $\leq n+1$, we write $\mathcal{K}_\ell$.

⑥ A map $f: \mathcal{K} \longrightarrow \mathcal{L}$ is a map of abstract simplicial complexes if ...

(same as a map of simplicial complexes) (similarly, for iso of abstract simp comp)

Ex:

Let S be a k-simplex spanned by $\{x_0, \dots, x_k\} \subseteq \mathbb{R}^n$.

k	Simplicial complex of S	Abstract simplicial complex of S
0	$K = \{\{x_0\}\}$	$\mathcal{K} = \{\{x_0\}\}$
1	$K = \{\{x_0\}, \{x_1\},$  $\}$	$\mathcal{K} = \{\{x_0\}, \{x_1\}, \{x_0, x_1\}\}$
2	$K = \{\{x_0\}, \{x_1\}, \{x_2\},$  ,  ,  ,  $\}$	$\mathcal{K} = \{\{x_0\}, \{x_1\}, \{x_2\}, \{x_0, x_1\}, \{x_1, x_2\}, \{x_0, x_2\}, \{x_0, x_1, x_2\}\}$
$\vdots$		

The following results explain the relation between simplicial complexes and abstract simplicial complexes.

Lemma: Let K be a simplicial complex spanned by $\{x_0, \dots, x_k\} \subseteq \mathbb{R}^n$. Then there is an associated abstract simplicial complex specified by the collection of subsets of the vertices of K which span a simplex in K.

Theorem: For every abstract simplicial complex $\mathcal{K}$, there exists a simplicial complex $\widetilde{\mathcal{K}}$ such that $\mathcal{K}$ is associated to $\widetilde{\mathcal{K}}$.

- How would you prove these results?
- We can use these results to define the geometric realization of an abstract simplicial complex. How? (see pg 28 and 29, textbook)
- Define the geometric realization of abstract simplicial complex.