Week 4

Video: https://www.youtube.com/watch?v=gOP_LJqz76E&ab_channel=Euler%27sQuanta

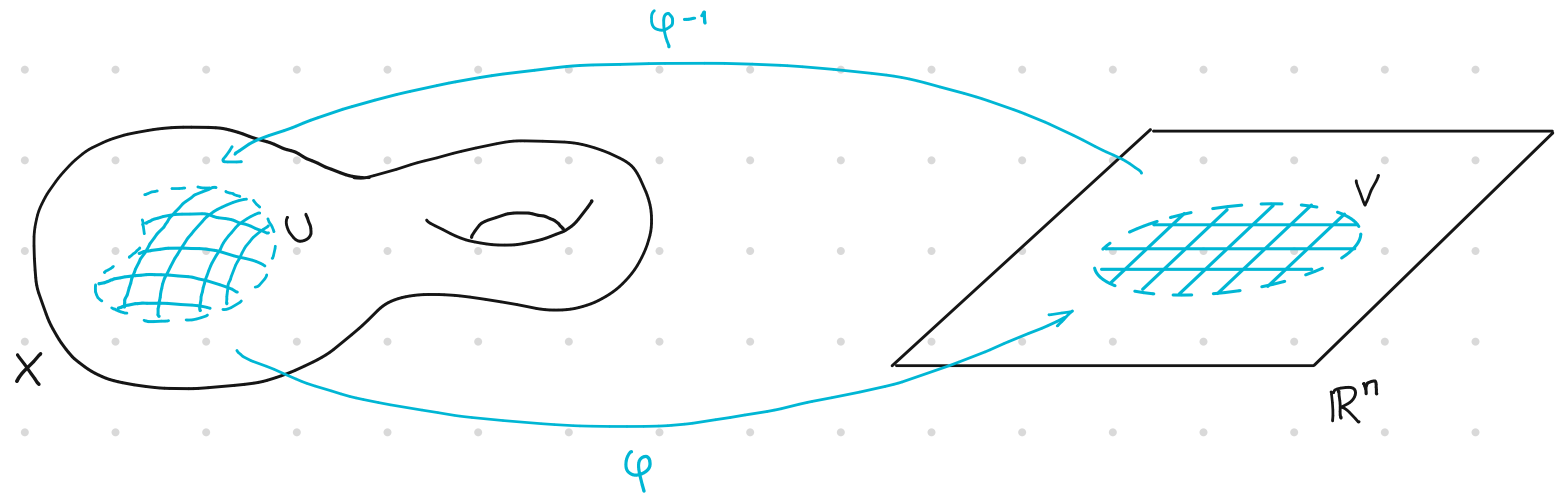
The definition of a topological space is very general; an arbitrary topological space can be extremely complicated and have a very non-geometric flavor.

However, in many applications (e.g., computer vision, medical imaging, physics), particularly nice examples of topological spaces tend to arise; these are spaces which admit Euclidean coordinates, at least locally, and permit the definition of a precise generalization of classical calculus. Such a topological space is called a *manifold*. [7]

But, what do we mean by coordinates?

Def: Let X be a top space. Given an open $\mathcal{U} \subseteq X$, we say that a **chart** is a homeomorphism $\varphi : \mathcal{U} \longrightarrow V$, where $V \subseteq \mathbb{R}^n$ is an open. The inverse φ^{-1} equips U with a coordinate system. We write (U, φ).

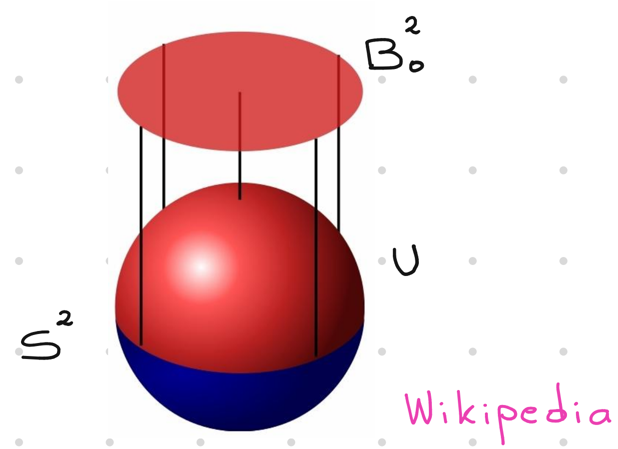
A chart can be thought as a patch on X that looks exactly like $\mathbb{R}^n$.



Ex:

1. All $U \subseteq \mathbb{R}^n$ open are charts with $\varphi = id_U$.

2. Consider the sphere S^2.



$U = \{ (x,y,z) \in S^2 : z > 0 \} \subseteq S^2$ is open.

$\varphi : U \xrightarrow{\cong} B_0^2$. Find θ.

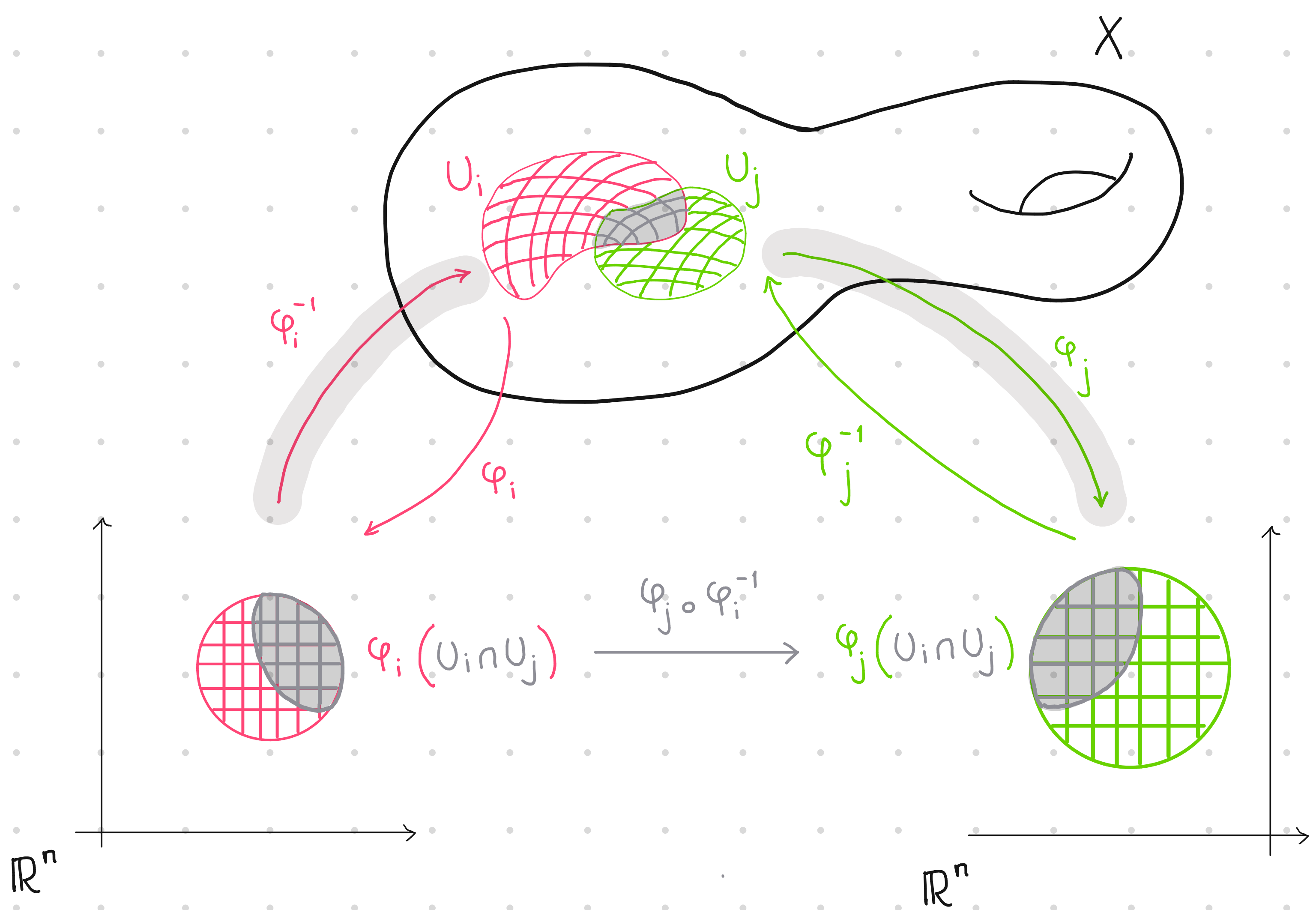
Def: An atlas for X is a collection of charts $\{\mathcal{U}_i\}_{i \in I}$ such that the collection is an open cover of X.

Intersections between charts are important!

Def: If (U_i, φ_i) and (U_j, φ_j) are charts so that $U_i \cap U_j \neq \phi$, the map

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$$

is called a transition map from φ_i to φ_j.



Transition maps describe how coordinates change as we move from chart to chart.

Def: M a top space. We say M is an n-dimensional topological manifold if

1. M is a Hausdorff space. (i.e. M has enough open sets)
2. M is a second-countable space (i.e. there is a countable number of basic open sets)
3. M is locally Euclidean of dim n, i.e. $\forall m \in M \ \exists \mathcal{U}$ open s.t. $m \in \mathcal{U}$ and $(\mathcal{U}, \varphi)$ is a chart. (i.e. there is an atlas for M)

It is often the case that examples have additional smoothness which permits the use of the methods of calculus. Since the transition functions involve maps from subsets of Euclidean space to itself, we can ask about their continuity and derivatives using the standard techniques of multivariable calculus. [3]

Def: M an n-dimensional topological manifold. We say M is an n-dimensional smooth manifold if the transition maps are continuous and infinitely differentiable.

Exercise: Find an example of an n-dim top manifold that is not smooth.

Ex:

1. $\mathbb{R}^n$ is an n-dim smooth manifold, covered by a single chart $(\mathbb{R}^n, id)$.

2. S^1 is a 1-dim smooth manifold, covered by two charts: Let $\epsilon > 0$.

$$\mathcal{U}_+ = \{ (x,y) \in S^1 : y > \epsilon - 1/2 \} \quad \text{and} \quad \mathcal{U}_- = \{ (x,y) \in S^1 : y < \epsilon + 1/2 \}$$

Exercise: Find the transition map and prove it is continuous and infinitely differentiable.

3. The n-sphere $S^n \subseteq \mathbb{R}^{n+1}$ is an n-dim smooth manifold for $n \geq 1$, covered by two charts. Prove it!

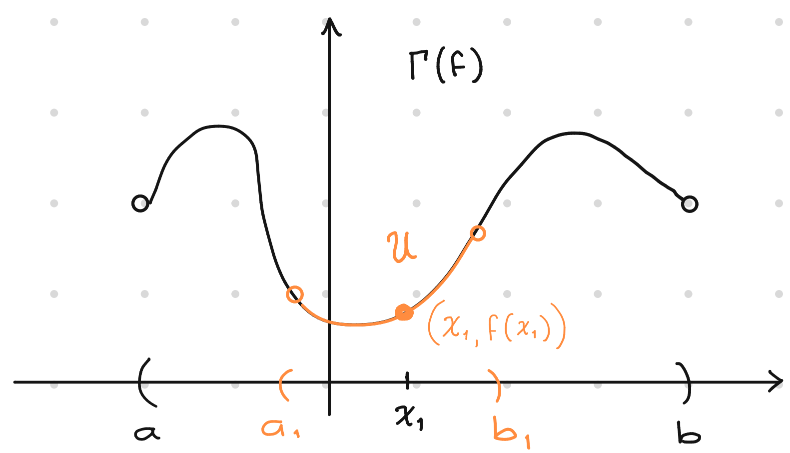
4. The torus $T := \left([0,1] \times [0,1]\right)/\sim$, $(x,0) \sim (x,1)$ and $(0,y) \sim (1,y)$ is a 2-dim smooth manifold. Consider a covering by overlapping squares.

5. The Möbius band $M := \left([0,1] \times [0,1]\right) / \sim$, $(x,0) \sim (1-x,1)$ is a 2-dim smooth manifold.

6. If $f : U \longrightarrow \mathbb{R}$ is a continuous function, $U \subseteq \mathbb{R}$ open, the graph of f

$$\Gamma(f) := \left\{ (x, f(x)) \;:\; x \in U \right\} \subseteq \mathbb{R}^2 \quad \text{is a 1-dim manifold.}$$

literally the graph of f



$$\varphi : \mathcal{U} \longrightarrow (a_1, b_1)$$
$$(x, f(x)) \longmapsto x$$

$$\varphi^{-1} : (a_1, b_1) \longrightarrow \mathcal{U}$$
$$x \longmapsto (x, f(x))$$

7. **(Another Smooth Structure on $\mathbb{R}$).** Consider the homeomorphism $\psi: \mathbb{R} \to \mathbb{R}$ given by

$$\psi(x) = x^3.$$

The atlas consisting of the single chart $(\mathbb{R}, \psi)$ defines a smooth structure on $\mathbb{R}$. This chart is not smoothly compatible with the standard smooth structure, because the transition map $\mathrm{Id}_{\mathbb{R}} \circ \psi^{-1}(y) = y^{1/3}$ is not smooth at the origin. Therefore, the smooth structure defined on $\mathbb{R}$ by ψ is not the same as the standard one. Using similar ideas, it is not hard to construct many distinct smooth structures on any given positive-dimensional topological manifold, as long as it has one smooth structure to begin with .

8. **(Finite-Dimensional Vector Spaces).** Let V be a finite-dimensional real vector space. Any norm on V determines a topology, which is independent of the choice of norm. With this topology, V is a topological n-manifold, and has a natural smooth structure defined as follows. Each (ordered) basis $(E_1, \ldots, E_n)$ for V defines a basis isomorphism $E: \mathbb{R}^n \to V$ by

$$E(x) = \sum_{i=1}^{n} x^i E_i.$$

This map is a homeomorphism, so (V, E^{-1}) is a chart. If $(\widetilde{E}_1, \ldots, \widetilde{E}_n)$ is any other basis and $\widetilde{E}(x) = \sum_j x^j \widetilde{E}_j$ is the corresponding isomorphism, then there is some invertible matrix (A_i^j) such that $E_i = \sum_j A_i^j \widetilde{E}_j$ for each i. The transition map between the two charts is then given by $\widetilde{E}^{-1} \circ E(x) = \widetilde{x}$, where $\widetilde{x} = (\widetilde{x}^1, \ldots, \widetilde{x}^n)$ is determined by

$$\sum_{j=1}^{n} \widetilde{x}^j \widetilde{E}_j = \sum_{i=1}^{n} x^i E_i = \sum_{i,j=1}^{n} x^i A_i^j \widetilde{E}_j.$$

It follows that $\widetilde{x}^j = \sum_i A_i^j x^i$. Thus, the map sending x to $\widetilde{x}$ is an invertible linear map and hence a diffeomorphism, so any two such charts are smoothly compatible. The collection of all such charts thus defines a smooth structure, called the ***standard smooth structure on V***.

[3]

Def: Let M be a smooth m-manifold. A function $f: M \longrightarrow \mathbb{R}^n$ is said to be smooth if

$\forall p \in M$ $\exists$ a chart (U, φ) for M s.t. $p \in U$ and $f \circ \varphi^{-1}: \underset{\mathbb{R}^m}{\underset{\cap}{\varphi(U)}} \longrightarrow \mathbb{R}^n$ is smooth.

All results seen in multivariable calculus can be generalized to smooth manifolds:
Jacobian, gradients, critical points, Hessian, etc...

> Let M be a manifold. Endow M with a "heigth" function $h: M \longrightarrow \mathbb{R}$.
>
> We want to study the info about M encoded in the inverse image $f^{-1}(I)$ for $I \subseteq \mathbb{R}$ an open interval.
>
> The places where the inverse images change in interesting ways turn out to be precisely the critical points of h. That is, M can be characterized by the critical points of suitable continuous functions $M \longrightarrow \mathbb{R}$

Reading: Basic concepts section https://en.m.wikipedia.org/wiki/Morse_theory

Def: Let M be a smooth manifold. A smooth function $h: M \longrightarrow \mathbb{R}$ is a Morse function if and only if

1. None of f's critical points are degenerate.
2. The critical points have distinct function values.

Def: Let $h : M \longrightarrow \mathbb{R}$ be a continuous function and $I \subseteq \mathbb{R}$ a set.

1. The interval levelset of h w.r.t. I is the inverse image of I under h:

$$M_I := h^{-1}(I) = \{ x \in M : h(x) \in I \}$$

2. If $I = (-\infty, a]$, we write $M_{\leq a} := h^{-1}(I)$ and call it the sublevel set $M_{\leq a}$ of I.

3. If $I = [a, \infty)$, we write $M_{\geq a} := h^{-1}(I)$ and call it the superlevel set $M_{\geq a}$ of I.

4. If $I = \{a\}$, we call $h^{-1}(a)$ the levelset of h at $a \in \mathbb{R}$.

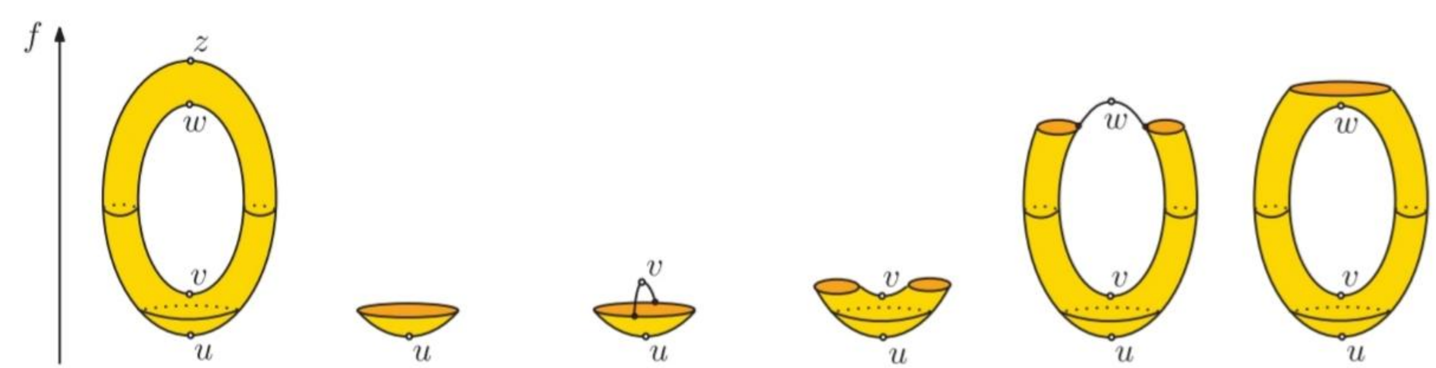
Theorem: Let $h: M \longrightarrow \mathbb{R}$ be a smooth function on a manifold M. Let $a, b \in \mathbb{R}$ s.t. $a < b$. Suppose $M_{[a,b]}$ is compact and contains no critical points of h.

Then

1. $M_{\leq a}$ is diffeomorphic to $M_{\leq b}$.
2. $M_{\leq a}$ is a deformation retract of $M_{\leq b}$
3. The inclusion $i: M_{\leq a} \hookrightarrow M_{\leq b}$ is a homotopy equivalence.

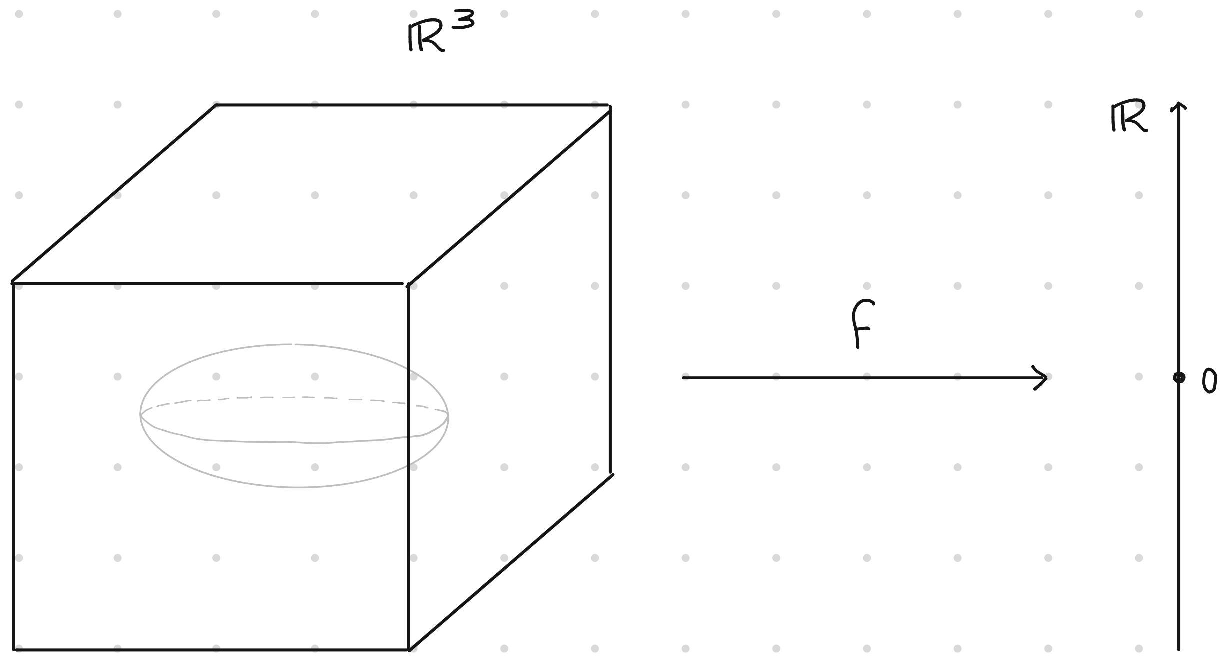
Ex:

As an illustration, consider the example of height function $f: M \to \mathbb{R}$ defined on a vertical torus. There are four critical points for the height function f, u (minimum), v, w (saddles), and z (maximum). We have that $M_{\leq a}$ is: (i) empty for $a < f(u)$; (ii) homeomorphic to a 2-disk for $f(u) < a < f(v)$; (iii) homeomorphic to a cylinder for $f(v) < a < f(w)$; (iv) homeomorphic to a compact genus-one surface with a circle as boundary for $f(w) < a < f(z)$; and (v) a full torus for $a > f(z)$.



[1]

Ex: $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ s.t. $f(x,y,z) = 3(x^2 + y^2 + 3z^2)$. Observe that f is smooth map on a 3-manifold.



* If $r < 0$, $f^{-1}(r) = \phi$.
* If $r = 0$, $f^{-1}(r) = \{(0,0,0)\}$
* If $r > 0$, $f^{-1}(r)$ is the ellipsoid

$$E_r : \left(\frac{x}{\sqrt{r/3}}\right)^2 + \left(\frac{y}{\sqrt{r/3}}\right)^2 + \left(\frac{z}{\sqrt{r/3}}\right)^2 = 1$$

$M_{\leq 0} = \{(0,0,0)\}$ $\qquad$ $M_r = E_r$

$M_{\geq 0} = \mathbb{R}^3$ $\qquad$ $M_{[a,b]} = \bigcup_{a \leq r \leq b} E_r$ $\qquad$ all points outside E_a and inside E_b, including both E_a and E_b.

* $(0,0,0)$ is the only critical point of f because $\nabla f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ only vanishes at $(0,0,0)$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto 6\begin{pmatrix} x \\ y \\ 3z \end{pmatrix}$$

$(0,0,0)$ is nondegenerate because Hessian$(\boldsymbol{x})$ is invertible.

$$H := \text{Hessian}(\boldsymbol{x}) = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

From this f is a Morse function.

* Index of $(0,0,0)$ is the number of negative eigenvalues of H.

The eigenvalues of H are $\{6, 18\}$. Then the index of $(0,0,0)$ is 0.

This means there is a neighborhood $\mathcal{U} \subseteq \mathbb{R}^3$ of $(0,0,0)$ such that for all

$(x,y,z) \in \mathcal{U}$, $F(x,y,z) = x^2 + y^2 + z^2$.

Let's find the intersection between S^2 and E_r.

$$\begin{cases} x^2+y^2+z^2 = 1 \\ 3(x^2+y^2+3z^2) = r \end{cases} \implies 3(1+2z^2) = r \implies z = \pm\sqrt{\frac{r-3}{6}} \text{ providing } r \geq 3.$$

i. $0 < r < 3 \implies S^2 \cap E_r = \phi$

ii. $r = 3 \implies S^2 \cap E_3 = \{(x,y,0) : x^2+y^2 = 1\} \simeq S^1$

iii. $3 \leq r < 9 \implies S^2 \cap E_r = \left\{\left(x, y, \pm\sqrt{\frac{r-3}{6}}\right) : x^2+y^2 = \frac{9-r}{6}\right\} \simeq S^1 \sqcup S^1$

iv. $r = 9 \implies S^2 \cap E_9 = \{(0,0,1), (0,0,-1)\}$

v. $r > 9 \implies S^2 \cap E_r = \phi$

Geo Gebra