**Def:** Let $(X, τ_X)$ and $(Y, τ_Y)$ be top spaces. A function $f : X \rightarrow Y$ is said to be **continuous** if for all $U \in τ_Y$, $f^{-1}(U) \in τ_X$.

**Theorem:** Let $X$ and $Y$ be topological spaces; let $f : X \rightarrow Y$. Then the following are equivalent:

1. $f$ is continuous.
2. For every subset $A$ of $X$, one has $f(\overline{A}) \subseteq \overline{f(A)}$.
3. For every closed set $B$ of $Y$, the set $f^{-1}(B)$ is closed in $X$.
4. For each $x \in X$ and each neighborhood $V$ of $f(x)$, there is a neighborhood $U$ of $x$ such that $f(U) \subseteq V$.

If the condition in (4) holds for the point $x$ of $X$, we say that $f$ is **continuous at the point** $x$.

**Proof:** Munkres pg 104, 105.
1. Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$, for all $x \in \mathbb{R}$; that is, $f$ is the identity function. Then for any open set $U$ in $\mathbb{R}$, $f^{-1}(U) = U$ and so is open. Hence $f$ is continuous.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = c$, for $c$ a constant, and all $x \in \mathbb{R}$. Then let $U$ be any open set in $\mathbb{R}$. Clearly $f^{-1}(U) = \mathbb{R}$ if $c \in U$ and $\emptyset$ if $c \not\in U$. In both cases, $f^{-1}(U)$ is open. So $f$ is continuous.

3. \[
\chi = \{1, 2, 3\} \quad \tau_1 = \{\emptyset, \chi, \{1, 2\}\} \quad \tau_2 = \{\emptyset, \chi, \{1, 3\}\}
\]

\[
id : (\chi, \tau_1) \longrightarrow (\chi, \tau_1)\]
\[
\begin{array}{c}
1 \\
2 \\
3
\end{array} \quad \begin{array}{c}
1 \\
2 \\
3
\end{array}
\]

$id$ is continuous

\[
f : (\chi, \tau_1) \longrightarrow (\chi, \tau_2)\]
\[
\begin{array}{c}
1 \\
2 \\
3
\end{array} \quad \begin{array}{c}
1 \\
2 \\
3
\end{array}
\]

$f$ is continuous

\[
g : (\chi, \tau_1) \longrightarrow (\chi, \tau_2)\]
\[
\begin{array}{c}
1 \\
2 \\
3
\end{array} \quad \begin{array}{c}
1 \\
2 \\
3
\end{array}
\]

g is not continuous because \(\{1, 3\} \in \tau_2\) but \(\{1, 3\} = g^{-1}(\{1, 3\}) \not\in \tau_1\)
Theorem (Rules for constructing continuous functions). Let $X$, $Y$, and $Z$ be topological spaces.

(a) (Constant function) If $f : X \rightarrow Y$ maps all of $X$ into the single point $y_0$ of $Y$, then $f$ is continuous.

(b) (Inclusion) If $A$ is a subspace of $X$, the inclusion function $j : A \rightarrow X$ is continuous.

(c) (Composites) If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous, then the map $g \circ f : X \rightarrow Z$ is continuous.

(d) (Restricting the domain) If $f : X \rightarrow Y$ is continuous, and if $A$ is a subspace of $X$, then the restricted function $f|A : A \rightarrow Y$ is continuous.

(e) (Restricting or expanding the range) Let $f : X \rightarrow Y$ be continuous. If $Z$ is a subspace of $Y$ containing the image set $f(X)$, then the function $g : X \rightarrow Z$ obtained by restricting the range of $f$ is continuous. If $Z$ is a space having $Y$ as a subspace, then the function $h : X \rightarrow Z$ obtained by expanding the range of $f$ is continuous.

(f) (Local formulation of continuity) The map $f : X \rightarrow Y$ is continuous if $X$ can be written as the union of open sets $U_\alpha$ such that $f|U_\alpha$ is continuous for each $\alpha$.

Proof: Munkres pg 107, 108

Def: A continuous map $i : X \rightarrow Y$ is an embedding if $i$ is injective. Usually denoted by $X \hookrightarrow Y$.

$X$ lives inside of $Y$ and $\dim(X) \leq \dim(Y)$.
**Def:** A continuous map \( f : X \to Y \) is a **homeomorphism** if

1. \( f \) is bijective,
2. \( f^{-1} \) is continuous.

**Remark:**

\[
f : (X, \tau_X) \to (Y, \tau_Y) \iff U \in \tau_X \text{ if and only if } f(U) \in \tau_Y
\]

is a homeomorphism

Homeomorphisms preserve properties that depend on the topology of spaces.

So,

- connectedness
- compactness
- Hausdorff
- dimension
- # of holes
- orientability

are preserved by homeomorphisms.
Two spaces are homeomorphic if there is a continuous bijection between them with continuous inverse. On the top, the circle is deformed into a pentagon. On the bottom, a sphere with the bottom cut off can be stretched onto a plane.

In this situation, we refer to $f$ as a homeomorphism. Intuitively, two spaces are homeomorphic when they are related by a continuous deformation; roughly speaking, this means they are related by stretching and bending without introducing tears or gluing things together.

We write $X \simeq Y$ if $X$ and $Y$ are homeomorphic.

**Ex:**

1. Consider $\mathbb{R}^n$ and $B^n$. Then

$$f : B^n \longrightarrow \mathbb{R}^n$$

$$x \longrightarrow \frac{x}{1 - \|x\|}$$

is a homeomorphism.

To verify this, you must:

- Prove that $f$ is continuous.
- Prove that $f$ is bijective.
- Prove that $f^{-1}$ is continuous.

$$f^{-1}(x) = \frac{x}{1 + \|x\|}.$$ 

Thus, $\mathbb{R} \simeq \mathbb{R}^2 \simeq \ldots$
The $xy$-plane is homeomorphic to a punctured two-dimensional sphere.

$$T: S^3 \setminus \{(0,0,1)\} \rightarrow \mathbb{R}^2$$

$$(x, y, z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

$T$ is called the **stereographic projection** of $S^3 \setminus \{(0,0,1)\}$ onto the plane $z=0$.

$$T^{-1}: \mathbb{R}^2 \rightarrow S^3 \setminus \{(0,0,1)\}$$

$$(x, y) \mapsto \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right)$$

**Video:** https://en.etudes.ru/models/stereographic-projection/

**Reading:** Distinguishing the plane from the puncture plane without homotopy, Frédéric Mynard. (See our folder)
3. \( \mathbb{R}^m \) and \( \mathbb{R}^n \) are not homeomorphic if \( m \neq n \).

\((0,1) \neq [0,1]\) because \((0,1)\) is not compact and \([0,1]\) is compact

\(\{0,1\}, \text{discrete}\) \(\neq\) \(\{1,2,3\}, \text{discrete}\) because they have different cardinality.

4. ! A bijective map \( f: X \rightarrow Y \) can be continuous without being a homeomorphism.
   (i.e. 2nd condition in the definition is necessary).

\( f: [0,2\pi] \rightarrow S^1 \) defined by \( f(\theta) = (\cos \theta, \sin \theta) \) is continuous and bijective, but its inverse is NOT continuous. Why? Exercise.

Also, \([0,2\pi]\) cannot be homeomorphic to \(S^1\)

\[\uparrow\]
not compact

\[\uparrow\]
compact
Prop: Let \( f: (X, \tau_X) \rightarrow (Y, \tau_Y) \) be continuous and bijective. If \((X, \tau_X)\) is compact and \((Y, \tau_Y)\) is Hausdorff, then \( f \) is a homeomorphism.

Proof:
Observe that \((f^{-1})^{-1} = f\). Therefore, we will prove the following:

\[
V \subseteq X \text{ is closed} \Rightarrow f(V) \subseteq Y \text{ is closed}
\]

\(V \subseteq X \text{ closed} \Rightarrow \) since \(X\) is compact, \(V\) is compact \(\Rightarrow f(V)\) is compact \(\Rightarrow f(V)\) is closed

Thus, \(f^{-1}\) is continuous.
Definition: Let $f, g : X \rightarrow Y$ be continuous maps. A homotopy between $f$ and $g$ is a continuous map $H : X \times [0,1] \rightarrow Y$ such that

1. $H(x, 0) = f(x)$ for all $x \in X$
2. $H(x, 1) = g(x)$ for all $x \in X$

In this case, we say $f$ is homotopic to $g$ and write $f \simeq g$.
Def: $X$ and $Y$ are homotopic if there exist maps $f: X \to Y$ and $g: Y \to X$ such that $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$. We write $X \simeq Y$ and say $X$ is homotopy equivalent to $Y$.

connectedness, path-connectedness & # of holes are preserved by homotopies

Ex: A point is homotopic to a disk: $f: B^2 \to \{0\}$ and $g: \{0\} \to B^2_0$

$f \circ g = \text{id}_{\{0\}}$

$g \circ f \simeq \text{id}_{B^2}$

$H: B^2 \times [0,1] \to B^2$

$(x,t) \mapsto tx$

$H(x, 0) = 0 = (g \circ f)(x)$

$H(x, 1) = x$
**Def:** Let $X$ and $Y$ be topological spaces. Let $f, g : X \hookrightarrow Y$ be embeddings.

An isotopy between $f$ and $g$ is a continuous map $I : X \times [0,1] \rightarrow Y$ such that

1. $I(x,0) = f(x)$ \quad $\forall x \in X$
2. $I(x,1) = g(x)$ \quad $\forall x \in X$
3. $i_x : X \rightarrow Y$ $x \mapsto I(x,t)$ is an embedding $\forall t \in [0,1]$

We say $f$ is isotopic to $g$.

**Ex:** Let $f, g : S^1 \rightarrow \mathbb{R}^2$ be given by $f(x,y) = (2x, 2y)$ and $g(x,y) = (3x + 7, 3y)$ for all $(x,y) \in S^1$. 
Prove a) and b).

a) \( f \) and \( g \) are embeddings whose images are \( S((0,0), 2) \) and \( S((\pi,0), 3) \), respectively.

b) \( I : S^1 \times [0,1] \rightarrow \mathbb{R}^2 \) given by \( I((r,y), t) = (1-t)f(x,y) + tg(x,y) \) is an isotopy from \( f \) to \( g \).

Remarks: Among the concepts of "a homeomorphism", "an isotopy", and "a homotopy", the first one is the strictest and the last one is the most flexible.

\[ \text{stronger} \quad \text{Homeomorphism} \quad \text{Isotopy} \quad \text{Homotopy} \quad \text{weaker} \]

This means the equivalence relation induced by \( \ast \) is stronger ("harder") and the equivalence relation induced by \( \triangle \) is weaker ("easier").
If $X$ is homotopic to $Y$, you may think of this as a video where you see how $X$ transforms into $Y$. The continuity condition (required in the def of homotopy) means that the closer two frames are two each other temporally, the more alike they look. In other words, as we watch the video, we should not see any “sudden jumps” in it.

You can think of isotopies and homotopies as follows:

An isotopy is a continuous one-parameter family of homeomorphisms.

A homotopy is a continuous one-parameter family of continuous functions.

The homotopy between $B^2$ and $\{0\}$ is not an isotopy. Why?
**Def:** Let \( A \subseteq X \). A retraction of \( X \) to \( A \) is a map \( r : X \to A \) such that \( r|_A = \text{id}_A \) (\( r \) carries all of \( X \) into \( A \)).

A deformation retraction of \( X \) onto \( A \) is a map \( R : X \times [0,1] \to X \) such that:

1. \( R(x,0) = x \) \( \forall x \in X \)
2. \( R(x,1) \in A \) \( \forall x \in X \)
3. \( R(a,t) = a \) \( \forall a \in A \forall t \in [0,1] \)

\( R \) is a homotopy between the identity on \( X \) and a retraction of \( X \) to \( A \) s.t. each point of \( A \) remains fixed during the homotopy.

We say \( A \) is a deformation retract of \( X \).

**Ex:**

1. \( \mathbb{R}^2 \setminus \{p,q\} \) has the "figure eight" as a deformation retract.
$A = \infty$ and $X = \mathbb{R}^2 \setminus \{p, q\}$. Observe that $A \subseteq X$.

Let $R : (\mathbb{R}^2 \setminus \{p, q\}) \times [0, 1] \longrightarrow \mathbb{R}^2 \setminus \{p, q\}$ be the map defined by steps 1 to 3.

At time $t = 0$ we have $\mathbb{R}^2 \setminus \{p, q\}$.
At time $t = 1$ we have $\infty$.
At any time $t \in (0, 1)$, the points on $\infty$ remain fixed.

2. squash the cylinder down

$S^1 \times [0, 1] \quad \Rightarrow \quad \Rightarrow \quad \Rightarrow \quad S^1$
3. \( \{p\} \subseteq \{p, q\} \) where \( p, q \in \mathbb{R}^2 \), \( p \neq q \) and \( \{p, q\} \) has the discrete topology. 

\( \{p\} \) is not a deformation retract of \( \{p, q\} \). Why? See remarks.

Remarks: 

- If \( A \) is a deformation retract of \( X \), then \( A \simeq X \).
- A deformation retract \( R \) induces a retraction \( r : X \rightarrow A \) by \( r(x) := R(x, 1) \). Prove it!

Reading: https://www.math3ma.com/blog/clever-homotopy-equivalences