Week 1

Video: https://youtu.be/Zuv_nc7s9mA

If X is a set, the power set of X is $\mathcal{P}(X)$ or 2^X the collection of subsets of X.

Ex: $X = \emptyset \implies \mathcal{P}(X) = \{\emptyset\}$

$X = \{*\} \implies \mathcal{P}(X) = \{\emptyset, \{*\}\}$

$X = \{1,2\} \implies \mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

! If $\#(X) = n$, then $\#(\mathcal{P}(X)) = 2^n$

Def: $X \neq \phi$. A collection $\tau \subseteq \mathcal{P}(X)$ is said to be a topology on X if

1. $\phi, X \in \tau$
2. The union of any (finite or infinite) collection of sets in τ is in τ.
3. The intersection of any finite collection of sets in τ is in τ.

Def: Let (X, τ) be a topological space.

- The elements of τ are called open sets.
- Complements of elementes of τ are called closed sets.

Remark: When $\tau = \{\phi, X\}$ we call τ the indiscrete/trivial topology.

When $\tau = \mathcal{P}(X)$ we call τ the discrete topology.

Intuition

We want the elements of τ to behave like open intervals in $\mathbb{R}$, and the complements of the elements of τ to behave like closed intervals in $\mathbb{R}$. (assuming $\mathbb{R}$ has the usual topology)

An open set is a set where every point has some "wiggle room" without leaving the set. No matter which point in the set you pick there is a little bit of space around that point (in every direction) that is still in the set. In other words, no point in the set is on a boundary (if you were on a boundary, you couldn't move at all in the direction that would take you across the boundary).

Math Stack Exchange

Ex:

1. $(\mathbb{R}, \mu)$ where μ is the collection of all open intervals (a,b) and all unions of open intervals. μ is called the usual topology of $\mathbb{R}$.

2. $X = \{1,2,3,4,5,6\}$ and $\tau = \{\emptyset, X, \{1\}, \{3,4\}, \{1,3,4\}, \{2,3,4,5,6\}\}$

$\{1\}$ is both open and closed.
$\{2,3\}$ is neither open nor closed.
$\{3,4\}$ is open but not closed.
$\{1,2,5,6\}$ is closed but not open.

Warning. The names "open" and "closed" often lead newcomers to the world of topology into error. Despite the names, some open sets are also closed sets! Moreover, some sets are neither open sets nor closed sets! [5]

Def: Let (X, τ) be a topological space and let $A \subseteq X$ (not neccesarily in τ).

other notations

- The interior of A is the union of all open subsets of A: $\mathrm{Int}(A) = \bigcup_{\substack{T \in \tau \\ T \subseteq A}} T$ — A°
- The closure of A is the smallest closed set containing A: $\mathrm{Cl}(A)$ — $\overline{A}$
- The boundary of A is $\mathrm{Bd}(A) := \mathrm{Cl}(A) \setminus \mathrm{Int}(A)$. — ∂A

Remarks: A is open $\iff A = \mathrm{Int}(A)$

A is closed $\iff A = \mathrm{Cl}(A)$

> I think the idea of a boundary ($\partial A = A \setminus A^\circ$) is very helpful.
>
> Essentially, an open set doesn't contain any of its boundary; there is a demarcation *object* (I want to say *point*, but it's the boundary) separating our set, A, from everything that's not in A. Our set is open if this demarcation is completely outside A.
>
> In other words, it's hard to tell (from within A) where A ends. We never reach any point $x \in A$ for which we can say, "Aha, going beyond this point, I would no longer be in A!"
>
> Looking at A from the outside, we would encounter points for which we cannot continue "going toward A", while remaining outside A; we reach the edge of $X \setminus A$, while remaining in $X \setminus A$.
>
> Math Stack Exchange

Def: Let (X, τ) be a topological space, and let $Y \subseteq X$ with $Y \neq \phi$. The collection

$\{ Y \cap A : A \in \tau \}$ is a topology on Y, called the subspace topology, (Y, τ_Y).

Ex: Consider $(\mathbb{R}, \mu)$ with the usual topology and $[0,1] \subseteq \mathbb{R}$.

Some elements of $\mu_{[0,1]}$ are $\{(a,b) : 0 < a < b < 1\}$, $\{[0,b) : 0 < b < 1\}$, and $\{(a,1] : 0 < a < 1\}$.

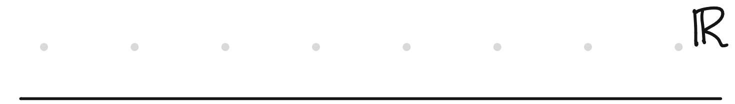
Some surprising things happen:

- $[0, 1/2)$ is not open in $\mathbb{R}$, but it is open in the subspace $[0,1]$ as $[0, 1/2) = [0,1] \cap (-1/2, 1/2)$.
- $[0,1]$ is not open in $\mathbb{R}$, but it is open in $[0,1]$

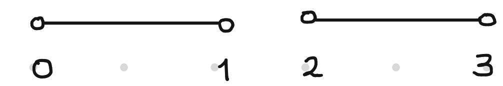
Def: Let (X, τ) be a topological space. We say X is disconnected if there are two non-empty open sets A, B such that $A \cap B = \phi$ and $X = A \cap B$. Otherwise, X is connected.

Ex:

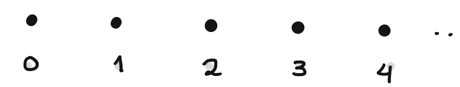
1. $\mathbb{R}$ with the usual topology is connected.



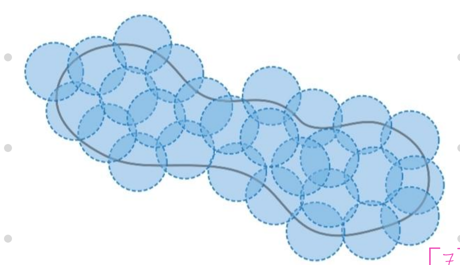
2. $(0,1) \cup (2,3) \subseteq \mathbb{R}$ with the subspace topology is disconnected.



3. $\mathbb{N}$ with the discrete topology is disconnected.



Def: Let (X, τ) be a topological space. An open / closed cover of X is a collection $\mathcal{C}$ of open / closed subsets of X s.t. $X = \bigcup_{A \in \mathcal{C}} A$.



[7]

Ex:

1. $\mathbb{N}$ with the trivial topology. $\mathcal{C} = \{\mathbb{N}\}$ is an open (and closed) cover of $\mathbb{N}$.

2. $\mathbb{R}$ with the usual topology. $\mathcal{C} = \{(-\infty, n) : n \in \mathbb{N}\}$ is an open cover of $\mathbb{R}$.

$Y := \{0, 1, 2, 3\}$ with the subspace topology. $\mathcal{C} = \{\{a\} : a = 0, 1, 2, 3\}$ is an open cover of Y.

Def: Let (X, τ) be a topological space. We say X is compact if for every open cover $\mathscr{C}$ of X there exists a finite subcover $\mathscr{C}'$ of X.

Symbolically: $\forall \mathscr{C}, \ X = \bigcup_{A \in \mathscr{C}} A, \ \exists \mathscr{C}' \subseteq \mathscr{C}, \ \mathscr{C}'$ is finite and $X = \bigcup_{A \in \mathscr{C}'} A$

Ex:

1. $\mathbb{R}$ with the usual top is NOT compact. Why? We must exhibit an infinite cover that can't be reduced to a finite cover.

   $\mathscr{C} = \{ (-n, n) : n \in \mathbb{N} \}$ is a open cover of $\mathbb{R}$ that can't be reduced to a finite cover.

2. $(0,1) \subseteq \mathbb{R}$ with the subspace top is NOT compact because $\mathscr{C} = \{ (1/n, 1) : n \in \mathbb{Z}^+ \}$ doesn't have a finite subcover.

3. $[0,1] \subseteq \mathbb{R}$ is compact. Proof in pg 31, Hatcher.

Compactness is a sort of finiteness property that some spaces have and others do not. The rough idea is that spaces which are 'infinitely large' such as $\mathbb{R}$ or $[0, \infty)$ are not compact. However, we want compactness to depend just on the topology on a space, so it will have to be defined purely in terms of open sets. This means that any space homeomorphic to a noncompact space will also be noncompact, so finite intervals (a, b) and $[a, b)$ will also be noncompact in spite of their 'finiteness'. On the other hand, closed intervals $[a, b]$ will be compact — they cannot be stretched to be 'infinitely large'. [2]

with the usual top

Def: Let (X, τ) be a topological space, let $\sim$ be an equivalence relation on X, and let $\pi : X \longrightarrow X/\sim$ be the quotient map $\pi(x) = [x]$. The quotient set is a topological space where

$$A \subseteq X/\sim \text{ is open} \iff \pi^{-1}(A) \subseteq X \text{ is open}$$

$$A \in \tau_{X/\sim} \iff \pi^{-1}(A) \in \tau$$

We say $X/\sim$ has the quotient topology.

Ex: $X = [0,1] \subseteq \mathbb{R}$. Consider the partition $\mathcal{P} = \{\{0,1\}\} \cup \{\{a\} : a \in (0,1)\}$.

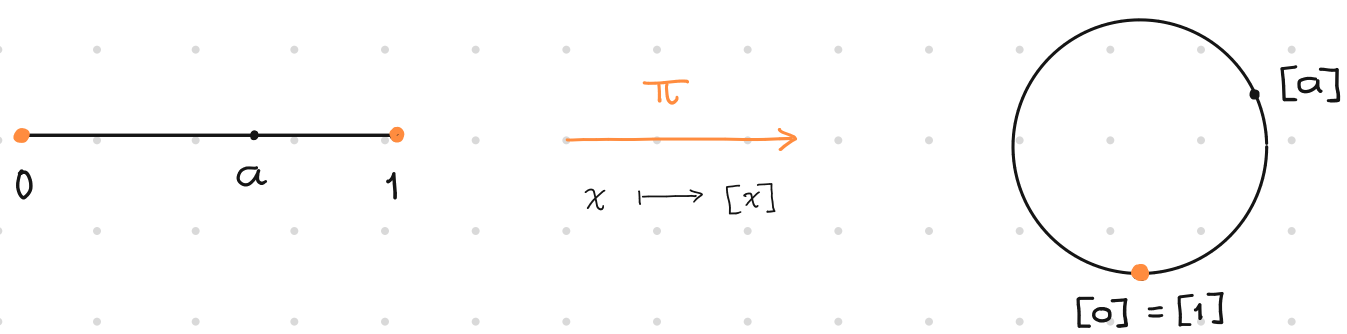
Then we have the following e.r. $\forall a,b \in [0,1]$

$$a \sim b \iff a,b \in \{0,1\} \quad \text{or} \quad a = b$$

Then $X/\sim \; = \{[0] = [1]\} \cup \{[a] = \{a\} : a \in (0,1)\}$

Graphically

$X = [0,1]$ $\qquad$ $X/\sim$ is "equivalent" to S^1



we glued 0 and 1