The Steenrod Algebra and The Dual Steenrod Algebra

Niny Arcila Maya

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1 Hopf Algebras and Dual Hopf Algebras

Definition 1.1. Let R be a commutative ring with unit.

1. A *graded R-module M* is a sequence $(M_i)_{i\geq 0}$ of unitary R-modules.
2. A *homomorphism of graded R-modules* $f : M \longrightarrow N$ is a sequence $\big(f_i : M_i \longrightarrow N_i\big)_{i\geq 0}$ of homomorphims of R-modules.
3. The *tensor product of graded R-modules* M and N is $\big((M\otimes N)_i\big)_{i\geq 0}$ where

$$(M\otimes N)_i = \bigoplus_{p+q=i} M_p \otimes N_q.$$

Remark 1.2. The ring R is considered as a graded R-algebra by setting $R_0 = R$, $R_i = 0$ $(i \neq 0)$.
Let $\mathbb{1} : A \longrightarrow A$ denote the identity. Let $T : M\otimes N \longrightarrow N\otimes M$ be defined by $T(m\otimes n) = (-1)^{ij}(n\otimes m)$, $i = \deg m$ and $j = \deg n$.

Definition 1.3 (Algebra)**.** Let A denote a graded R-module. We say that A has a *multiplication* if there is a mapping $\nabla : A\otimes A \longrightarrow A$.

1. ∇ is *associative* if the following diagram commutes

$$\begin{tikzcd} A\otimes A\otimes A \arrow[r, "\nabla\otimes\mathbb{1}"] \arrow[d, "\mathbb{1}\otimes\nabla"'] & A\otimes A \arrow[d, "\nabla"] \\ A\otimes A \arrow[r, "\nabla"] & A, \end{tikzcd} \qquad \nabla\big(\nabla(a\otimes b)\otimes c\big) = \nabla\big(a\otimes\nabla(b\otimes c)\big). \tag{1}$$

2. ∇ has a *unit* if there is an R-homomorphism $\eta : R \longrightarrow A$ and in the following diagram both compositions are the identity map

$$\begin{tikzcd} & A\otimes R \arrow[dr, "\mathbb{1}\otimes\eta"] & & \\ A \arrow[ur, "\cong"] \arrow[dr, "\cong"'] & & A\otimes A \arrow[r, "\nabla"] & A, \\ & A\otimes R \arrow[ur, "\eta\otimes\mathbb{1}"'] & & \end{tikzcd} \qquad \nabla\big(a\otimes\eta(1)\big) = a = \nabla\big(\eta(1)\otimes a\big). \tag{2}$$

The triple (A, ∇, η) satisfying 1 and 2, is called an *algebra over R*.

3. An algebra is said to be *commutative* if the following diagram commutes

$$\begin{tikzcd} A\otimes A \arrow[dr, "\nabla"] & \\ & A, \\ A\otimes A \arrow[uu, "T"] \arrow[ur, "\nabla"'] & \end{tikzcd} \qquad \nabla(a\otimes b) = (-1)^{\deg(a)\deg(b)}\nabla(b\otimes a). \tag{3}$$

4. A *homomorphism of graded algebras over R* is a homomorphism of graded R-modules that respects the multiplication and unit.

Definition 1.4. Let A and B be graded R-algebras. The tensor product $A \otimes B$ is given the structure of graded R-algebra by defining $\varphi_{A\otimes B} = (\varphi_A \otimes \varphi_B) \circ (\mathbb{1} \otimes T \otimes \mathbb{1})$. This is,

$$(a_i \otimes b_j)(a_k \otimes b_l) = (-1)^{jk}(a_i a_k \otimes b_j b_l)$$

for $a_i \in A_i$ and $b_j \in B_j$.
Let M be a graded R-module. The *tensor algebra* $\Gamma(M)$ is the graded algebra over R defined by the r-fold tensor product, $\Gamma(M)_r = M^{\otimes r}$, where $M^0 = R$. The product is given by the isomorphism $M^r \otimes M^s \cong M^{r+s}$. The tensor algebra $\Gamma(M)$ is associative, but not commutative.

The homomorphisms multiplication and unit can be dualized by reversing all arrows in the commutative diagrams of Definition 1.3; this defines the structure of a coalgebra.

Definition 1.5 (Coalgebra)**.** Let A denote a graded R-module. We say that A has a *comultiplication* (or *diagonal map*) if there is a mapping $\Delta : A \longrightarrow A \otimes A$.

1. Δ is *coassociative* if the following diagram commutes

$$\begin{tikzcd} A \otimes A \otimes A & A \otimes A \arrow[l, "\Delta \otimes \mathbb{1}"'] \\ A \otimes A \arrow[u, "\mathbb{1} \otimes \Delta"] & A. \arrow[l, "\Delta"'] \arrow[u, "\Delta"'] \end{tikzcd} \tag{4}$$

2. Δ has a *counit* if there is an R-homomorphism $\varepsilon : A \longrightarrow R$ and in the following diagram both compositions are the identity map

$$\begin{tikzcd} & A \otimes R \arrow[dl, "\cong"'] & & \\ A & & A \otimes A \arrow[ul, "\mathbb{1} \otimes \varepsilon"'] \arrow[dl, "\varepsilon \otimes \mathbb{1}"] & A. \arrow[l, "\Delta"'] \\ & A \otimes R \arrow[ul, "\cong"] & & \end{tikzcd} \tag{5}$$

   A counit may also be called an *augmentation* of A.

The triple (A, Δ, φ) satisfying 1 and 2, is called an *coalgebra over R*.

3. An algebra is said to be *cocommutative* if the following diagram commutes

$$\begin{tikzcd} A \otimes A \arrow[dd, "T"'] & \\ & A. \arrow[ul, "\Delta"'] \arrow[dl, "\Delta"] \\ A \otimes A & \end{tikzcd} \tag{6}$$

4. A *homomorphism of graded coalgebras over R* is a homomorphism of graded R-modules that respects the comultiplication and counit.

Definition 1.6 (Hopf Algebra)**.** Let A denote a graded R-module equipped with a multiplication $\nabla : A \otimes A \longrightarrow A$, a comultiplication $\Delta : A \longrightarrow A \otimes A$, a unit $\eta : R \longrightarrow A$, and an augmentation $\varepsilon : A \longrightarrow R$. Then A is a *Hopf algebra over R* if

1. (A, ∇, η) is an algebra over R
2. (A, Δ, ε) is a coalgebra over R
3. The compositions $\eta \circ \varepsilon$ and $\varepsilon \circ \eta$ are the identity on degree zero.
4. The following diagram commutes

$$\begin{tikzcd}
A \otimes A \arrow[r, "\nabla"] \arrow[d, "\Delta\otimes\Delta"'] & A \arrow[r, "\Delta"] & A \otimes A \\
A \otimes A \otimes A \otimes A \arrow[rr, "\mathbb{1}\otimes T\otimes\mathbb{1}"] & & A \otimes A \otimes A \otimes A \arrow[u, "\nabla\otimes\nabla"']
\end{tikzcd} \tag{7}$$

   This is, Δ is a morphism of algebras or; equivalently, ∇ is a morphism of coalgebras.

Definition 1.7 (Dual Hopf Algebra)**.** Let k be a field. Let A be a Hopf algebra over k of finite type (that is, A_i is finite-dimensional over k). The *dual Hopf algebra* A^* is defined by setting $(A^*)_i = (A_i)^*$, that is, the dual of A_i as a vector space over k. The multiplication ∇ and unit η in A gives the comultiplication ∇^* and augmentation η^* in A^*, and the comultiplication Δ and augmentation ε in A gives the multiplication Δ^* and unit ε^* in A^*.

Proposition 1.8. *Let A be a Hopf algebra over k of finite type.*

1. *∇ is associative (commutative) if and only if ∇^* is coassociative (cocommutative).*
2. *Δ is coassociative (cocommutative) if and only if Δ^* is associative (commutative).*
3. *A^* is a Hopf algebra.*
4. *A and A^* are isomorphic as graded vector spaces. (But not as algebras in general).*

Example 1.9. Let k be a field. Let X be an H-space, with product $\mu : X \times X \longrightarrow X$, such that

1. X is path-connected.
2. $\mathrm{H}_n(X;k)$ is a finite-dimensional vector space over k for all n. Therefore, $\mathrm{H}^n(X;k) \cong \mathrm{Hom}(\mathrm{H}_n(X;k),k)$ is also a finite-dimensional vector space over k for all n.

Claim 1.10. The homology ring $\mathrm{H}_*(X;k)$ is a Hopf algebra over R.

Multiplication: The composition of the cross product in homology $\mathrm{H}_*(X;k) \otimes \mathrm{H}_*(X;k) \xrightarrow{\times} \mathrm{H}_*(X \times X;k)$ and the induced map $\mu_* : \mathrm{H}_*(X \times X;k) \longrightarrow \mathrm{H}_*(X;k)$ gives the multiplication,

$$\nabla : \mathrm{H}_*(X;k) \otimes \mathrm{H}_*(X;k) \longrightarrow \mathrm{H}_*(X;k),$$

this product is called the *Pontryagin product.*
Comultiplication: The map induced by the diagonal map $d : X \longrightarrow X \times X$ gives the comultiplication $d_* : \mathrm{H}_*(X;k) \longrightarrow \mathrm{H}_*(X;k) \otimes \mathrm{H}_*(X;k)$.
Unit and counit: By hypothesis, $\mathrm{H}_0(X;k)$ is isomorphic to k. The maps η and ε are this isomorphism in degree zero and the trivial map in higher degrees.

Claim 1.11. The cohomology ring $\mathrm{H}^*(X;k)$ is a Hopf algebra over R.

Multiplication: The cup product $\cup : \mathrm{H}^*(X;k) \otimes \mathrm{H}^*(X;k) \longrightarrow \mathrm{H}^*(X;k)$.
Comultiplication: By Theorem 3.16 [1] and hypothesis, the cross product in cohomology $\mathrm{H}^*(X;k) \otimes \mathrm{H}^*(X;k) \xrightarrow{\times} \mathrm{H}^*(X \times X;k)$ is an isomorphism of rings. The combination of the induced map $\mu^* : \mathrm{H}^*(X;k) \longrightarrow \mathrm{H}^*(X \times X;k)$ with the cross product isomorphism gives the comultiplication,

$$\Delta : \mathrm{H}^*(X;k) \longrightarrow \mathrm{H}^*(X;k) \otimes \mathrm{H}^*(X;k).$$

Unit and counit: Dually to the homology case.

Conclusion: $\left(\mathrm{H}_*(X;k), \nabla, d_*\right)$ and $\left(\mathrm{H}^*(X;k), \cup, \Delta\right)$ are Hopf algebras. Moreover, $\left(\mathrm{H}^*(X;k), \cup, \Delta\right)$ is the dual Hopf algebra of $\left(\mathrm{H}_*(X;k), \nabla, d_*\right)$, and viceversa. This is, $\nabla^* = \Delta$ (the Pontryagin product in homology determines the comultiplication in cohomology) and $(d_*)^* = \cup$ (the comultiplication in homology determines the cup product in cohomology).
To see a more detailed version of this example please go to chapter 3, section C. in [1].

2 The Steenrod Algebra

2.1 Cohomology Operations

Definition 2.1. A *cohomology operation* is a natural transformation $\theta : \mathrm{H}^n(-;G) \longrightarrow \mathrm{H}^n(-;G')$. That is, for all spaces, X and Y, and mappings $f : X \longrightarrow Y$, there are functions θ_X, θ_Y such that the following diagram commutes

$$\begin{tikzcd} \mathrm{H}^n(X;G) \arrow[r, "\theta_X"] & \mathrm{H}^n(X;G') \\ \mathrm{H}^n(Y;G) \arrow[u, "f^*"] \arrow[r, "\theta_Y"] & \mathrm{H}^n(Y;G'). \arrow[u, "f^*"'] \end{tikzcd} \tag{8}$$

Example 2.2. Let R be a ring. We can define a squaring map $\theta : \mathrm{H}^n(X;R) \longrightarrow \mathrm{H}^{2n}(X;R)$ by using the cup product, $\theta(\alpha) = \alpha \cup \alpha =: \alpha^2$.
Note that is generally not a homomorphism. If $R = \mathbb{Z}$, then

$$\theta(\alpha+\beta) = (\alpha+\beta)^2 = \alpha^2 + 2\alpha \cup \beta + \beta^2 \neq \alpha^2 + \beta^2 = \theta(\alpha) + \theta(\beta).$$

But if $R = \mathbb{Z}/2$, then θ is a homomorphism.

Example 2.3. Given a sequence of coefficients $0 \to \mathbb{Z}/p \xrightarrow{\times p} \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$, there is a long exact sequence on cohomology

$$\cdots \to \mathrm{H}^n(X;\mathbb{Z}/p^2) \to \mathrm{H}^n(X;\mathbb{Z}/p) \xrightarrow{\beta_n} \mathrm{H}^{n+1}(X;\mathbb{Z}/p) \to \cdots$$

The connecting homomorphism, β_n, is a cohomology operation called *Bockstein homomorphism.*

Proposition 2.4. *For fixed m, n, G and G' let $\mathcal{O}(G,m;G',n)$ denote the set of all cohomology operations $\theta : \mathrm{H}^m(-;G) \longrightarrow \mathrm{H}^n(-;G')$. Then there is a bijection*

$$\mathcal{O}(G,m;G',n) \approx \mathrm{H}^n(\mathrm{K}(G,m);G')$$
$$\theta \mapsto \theta(\iota_m)$$

where $\iota_m \in \mathrm{H}^m(\mathrm{K}(G,m);G)$ is a fundamental class.

Proof. See Proposition 4L.1. [1]. □

2.2 Steenrod Squares

Theorem 2.5. *Let X be a topological space.There exists a transformation*

$$\mathrm{Sq}^i : \mathrm{H}^*(X;\mathbb{Z}/2) \longrightarrow \mathrm{H}^{*+i}(X;\mathbb{Z}/2)$$

called the ith Steenrod square, with the following properties:

1. $\mathrm{Sq}^i(f^*(\alpha)) = f^*(\mathrm{Sq}^i(\alpha))$ *for* $f : X \longrightarrow Y$.
2. $\mathrm{Sq}^i(\alpha+\beta) = \mathrm{Sq}^i(\alpha) + \mathrm{Sq}^i(\beta)$.
3. $\mathrm{Sq}^0 = \mathbb{1}$, *the identity.*
4. Sq^1 *is the Bockstein homomorphism β associated to the short exact sequence of coefficients*

$$0 \to \mathbb{Z}/2 \xrightarrow{\times 2} \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0.$$

5. *If* $\alpha \in \mathrm{H}^n(X;\mathbb{Z}/2)$, *then* $\mathrm{Sq}^n(\alpha) = \alpha^2 \in \mathrm{H}^{2n}(X;\mathbb{Z}/2)$.
6. *If* $\alpha \in \mathrm{H}^n(X;\mathbb{Z}/2)$ *and* $i > n$, *then* $\mathrm{Sq}^i(\alpha) = 0$.
7. $\mathrm{Sq}^i(\sigma(\alpha)) = \sigma(\mathrm{Sq}^i(\alpha))$ *where* $\sigma : \mathrm{H}^n(X;\mathbb{Z}/2) \longrightarrow \mathrm{H}^{n+1}(\Sigma X;\mathbb{Z}/2)$ *is the suspension isomorphism given by reduced cross product with a generator of* $\mathrm{H}^1(\mathbb{S}^1;\mathbb{Z}/2)$.
8. *The Cartan Formula:*

$$\mathrm{Sq}^i(\alpha \cup \beta) = \sum_{a+b=i} \mathrm{Sq}^a(\alpha) \cup \mathrm{Sq}^b(\beta).$$

9. *The Ádem Relations: For* $a < 2b$

$$\mathrm{Sq}^a\,\mathrm{Sq}^b = \sum_j \binom{b-j-1}{a-2j} \mathrm{Sq}^{a+b-j}\,\mathrm{Sq}^j,$$

where $\mathrm{Sq}^a\,\mathrm{Sq}^b$ *denotes the composition of the Stenrood squares, the binomial coefficient is taken mod 2 and, by convention,* $\binom{m}{n}$ *is taken to be zero if m or n is negative or if $m < n$.*

Proof. See Theorem 1. in chapter 3. [3] or Theorem 4L.12. [1]. □

Example 2.6. Taking $a = 1$ in the Ádem relation we have that

$$\mathrm{Sq}^1\,\mathrm{Sq}^b = (b-1)Sq^{b+1} = \begin{cases} Sq^{b+1}, & \text{if } b \text{ is even} \\ 0, & \text{if } b \text{ is odd} \end{cases}$$

2.3 The Steenrod Algebra $\mathscr{A}$

Let F denote the free $\mathbb{Z}/2$-module generated by the set of symbols $\{\mathrm{Sq}^i \;:\; i = 0,1,2,\ldots\}$. We think of F as a graded $\mathbb{Z}/2$-module with $F_i = \mathbb{Z}/2 \cdot \mathrm{Sq}^i$.
For each pair of integers (a,b) such that $0 < a < 2b$, let

$$R(a,b) = \mathrm{Sq}^a \otimes \mathrm{Sq}^b + \sum_j \binom{b-j-1}{a-2j} \mathrm{Sq}^{a+b-j} \otimes \mathrm{Sq}^j.$$

Let Q denote the two sided ideal of $\Gamma(F)$ generated by all such $R(a,b)$ and $1 + \mathrm{Sq}^0$.

Definition 2.7. The *Steenrod algebra* $\mathscr{A}$ is the quotient algebra $\Gamma(F)/Q$. $\mathscr{A}$ is a graded algebra over $\mathbb{Z}/2$.

In English, the Steenrod algebra is the quotient of the algebra of polynomials with coefficients in $\mathbb{Z}/2$ in the noncommuting variables $\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^3, \dots$ by the two sided ideal generated by the Ádem relations.

Notation 2.8. Given a sequence $\{i_1, \dots, i_r\}$ of positive integers, denote by Sq^I the product $\mathrm{Sq}^{i_1} \cdots \mathrm{Sq}^{i_r}$. For example, $\mathrm{Sq}^{\{2,1,6\}} = \mathrm{Sq}^2 \mathrm{Sq}^1 \mathrm{Sq}^6$.

Definition 2.9. A sequence I as above is *admissible* if $i_k \geq 2i_{k+1}$ for every $k < r$. This condition is vacously satisfied if $r = 1$. In this case we also say that Sq^I is *admissible*.

For example, $\mathrm{Sq}^2 \mathrm{Sq}^1 \mathrm{Sq}^6$ is not admissible, but $\mathrm{Sq}^9 \mathrm{Sq}^4 \mathrm{Sq}^2 \mathrm{Sq}^1$ is admissible. In other words, Sq^I is admissible if no Ádem relation can be applied to it.

Theorem 2.10 (Serre-Cartan basis)**.** *The monomials* Sq^I*, as I runs through all admissible sequences, form a basis for $\mathscr{A}$ as a $\mathbb{Z}/2$-module. This basis is known as Serre-Cartan basis.*

Proof. (Idea) By using the bijection given in Proposition 2.4, the fact that the monomials Sq^I are linearly independent follows from the linear independance of the elements $\mathrm{Sq}^I(\iota_n) \in \mathrm{H}^*\big(\mathrm{K}(\mathbb{Z}/2, n); \mathbb{Z}/2\big)$ for all admissible I of degree less or equal than n. By using the Ádem relations, we can proof that $\{\mathrm{Sq}^I : I \text{ is admissible}\}$ generates $\mathscr{A}$ as a $\mathbb{Z}/2$-module.
See Theorem 1. chapter 6. [3]. □

Example 2.11. By Theorem 2.10, the Steenrod algebra $\mathscr{A} = (\mathscr{A}_n)_{n \geq 0}$ is such that its homogeneous components $\mathscr{A}_6$ and $\mathscr{A}_7$ have as basis

$$\mathrm{Sq}^6, \mathrm{Sq}^5 \mathrm{Sq}^1, \mathrm{Sq}^4 \mathrm{Sq}^2 \text{ and } \mathrm{Sq}^7, \mathrm{Sq}^6 \mathrm{Sq}^1, \mathrm{Sq}^5 \mathrm{Sq}^2, \mathrm{Sq}^4 \mathrm{Sq}^2 \mathrm{Sq}^1, \text{ respectively.}$$

Definition 2.12. An element $a \in A$ a graded algebra over R is *decomposable* if it can be expressed in the form $\sum_i a_i b_i$ with each a_i and b_i having lower degree than a. Otherwise, we say a is *indecomposable*.

As examples, Sq^6 is decomposable because by the Ádem relations $\mathrm{Sq}^2 \mathrm{Sq}^4 = \mathrm{Sq}^6 + \mathrm{Sq}^5 \mathrm{Sq}^1$; Sq^1 is indecomposable and Sq^2 is indecomposable because $\mathrm{Sq}^1 \mathrm{Sq}^1 = 0$.
In [3] we find that an element Sq^i is indecomposable if and only if i is a power of 2.

Theorem 2.13. *The set of indecomposable elements of $\mathscr{A}$, namely $\{\mathrm{Sq}^{2^i} : i \geq 0\}$, generates $\mathscr{A}$ as an algebra.*

Proof. See Theorem 1. chapter 6. [3]. □

Unitl now we know that $\mathscr{A}$ is an algebra over $\mathbb{Z}/2$ with non-commutative multiplication given by the product in the tensor algebra, and the unit given by the identity in degree zero, and trivial otherwise. We will see that the Steenrod algebra possesses additional structure, namely, $\mathscr{A}$ is a Hopf algebra. We must see that there is a diagonal map and a augmentation satisfying Definition 1.6.
Define $\Delta : \Gamma(F) \longrightarrow \Gamma(F) \otimes \Gamma(F)$ by the formulas

$$\Delta(\mathrm{Sq}^i) = \sum_{k=0}^{i} \mathrm{Sq}^{i-k} \otimes \mathrm{Sq}^k \text{ and } \Delta(\mathrm{Sq}^i \otimes \mathrm{Sq}^j) = \Delta(\mathrm{Sq}^i) \otimes \Delta(\mathrm{Sq}^j).$$

Theorem 2.14. *The above Δ induces an algebra homomorphism $\Delta : \mathscr{A} \longrightarrow \mathscr{A} \otimes \mathscr{A}$. The homomorphism Δ is coassociative and cocommutative.*

Proof. See Theorem 2. in chapter 6. [3]. □

2.4 The Dual Steenrod Algebra $\mathscr{A}^*$

Corollary 2.15. *Let $\mathscr{A}^*$ denote the dual to the Steenrod algebra. The dual Steenrod algebra $\mathscr{A}^*$ is a Hopf algebra with commutative multiplication Δ^*.*

Theorem 2.16. *As an algebra, $\mathscr{A}^*$ is isomorphic to the polynomial algebra*

$$\mathbb{Z}/2[\xi_1, \xi_2, ...]$$

where $\deg \xi_i = 2^i - 1$. As a coalgebra, the comultiplication on $\mathscr{A}^*$ is determined by $\nabla^*(\xi_i) = \sum_{k=1}^{i} \xi_{i-k}^{2^k} \otimes \xi_k$.

Proof. See Corollary 2. in chapter 3. [3]. $\square$

Please let me know if there are typos and/or mistakes. Thank you :D

References

[1] Hatcher, A. (2002). *Algebraic topology*. Cambridge: Cambridge University Press. ISBN: 0-521-79160-X; 0-521-79540-0. (Chapter 4. Section 4.L.)

[2] McCleary, J. (2001). *A user's guide to spectral sequences* (Vol. 58). Cambridge University Press, Cambridge. ISBN: 0-521-56759-9. (Chapter 4. Section 4.)

[3] Mosher, Robert E. & Tangora, Martin C. (1968). *Cohomology operations and applications in homotopy theory*. New York : Harper & Row. (Chapter 6.)