Endogenous Timing of Moves in 2x2 Games

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Abstract

Suppose that, prior to playing a game, each player first commits whether to move "early" or "late". If both move early or both move late, the game then has simultaneous moves; otherwise, it has sequential moves. In all 2x2 games having a unique Nash equilibrium, the equilibrium outcome of this meta-game is unique and does not depend on the timing of players' commitments to move early or late.

Keywords: endogenous timing of moves; 2x2 games.

JEL Codes: C70, A22, A23

1. Introduction

The timing of moves in a game is typically treated as an exogenous feature of the strategic environment. This paper provides a novel approach to endogenize the timing of moves in games, and applies that approach to all 2x2 games having a unique Nash equilibrium. In particular, the timing of moves in the "underlying game" is viewed as the *equilibrium* outcome of a "timing game" played prior to the underlying game. In this timing game, each player first observably commits to move

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in the underlying game at one of finitely many times. (Without loss, I focus on the case in which only two times are available, "early" and "late".) If both move early or both move late, the underlying game then proceeds with simultaneous moves, with some Nash equilibrium (NE) of the underlying game being played. Otherwise, if one player moves early and the other moves late, the game proceeds with sequential moves, with the early-mover as first-mover and a subgame-perfect equilibrium (SPE) of the underlying game then being played.

	Early	Late
Early	*Simultaneous* *(NE played)*	*Row First* *(SPE played)*
Late	*Column First* *(SPE played)*	*Simultaneous* *(NE played)*

Figure 1: Timing Game

The model is agnostic about the timing of moves in the timing game, allowing for either simultaneous or sequential moves in the timing game. However, as I will show, the timing of moves in the timing game is irrelevant in all 2x2 games that possess a unique Nash equilibrium, a fairly broad but tractable class of some independent interest.[2] Indeed, these games all possess what I will call a *unique equilibrium outcome*, meaning that all equilibria of the meta-game induce the same distribution over outcomes in the underlying 2x2 game, no matter what the timing of

[2] Rapoport et al (1976) catalogued all seventy-eight strategically distinct 2x2 games (with strictly ordered payoffs), accounting for symmetries associated with re-labeling players and strategies. Sixty of these possess a unique Nash equilibrium. For a taste of the literature on "taxonomies" of 2x2 games, see Rapoport and Guyer (1966), Barany, Lee, and Shubik (1992) and Walliser (1988).

moves in the timing game.

Example [Research-funding game]. Consider a research-funding game in which two government agencies, say the US Department of Energy (DoE) and the Defense Advanced Research Projects Agency (DARPA), each decide which of two research projects ("batteries" or "solar") to fund. DoE has a dominant strategy to invest in batteries, but wants most for DARPA to invest in solar. DARPA prefers to invest in the same project as DoE, but wants most for DoE to invest in batteries. (Each agency's *ordinal* payoffs are shown in Figure 2, with "4" best and "1" worst.)

		DARPA Batteries	*DARPA* Solar
DoE	Batteries	**2 , 4** *Nash / DARPA first*	**4 , 3**
DoE	Solar	**1 , 1**	**3 , 2** *DoE first*

Figure 2: Research-funding game

Each agency can influence the timing of moves in this game by how it requests funding from Congress. In particular, suppose that each agency can request funds *now* or *next year*. An agency whose funds arrive now must immediately commit to a project, while one whose funds arrive next year cannot credibly commit to its choice until next year. Their funding decisions will be sequential if one agency requests funds now and the other requests funds next year; otherwise, their funding decisions will be simultaneous.

Each agency's incentive to request funds now or next year depends on how those decisions will impact the timing of moves and hence subsequent equilibrium play. If both move simultaneously or if DARPA moves first, the unique equilibrium outcome is DARPA's best outcome (Batteries, Batteries). Otherwise, if DoE moves first, the unique equilibrium outcome is (Solar, Solar), which is better for DoE. So, DARPA's ranking is Simultaneous = DARPA first > DoE first, while DoE has the opposite ranking DoE first > Simultaneous = DARPA first.

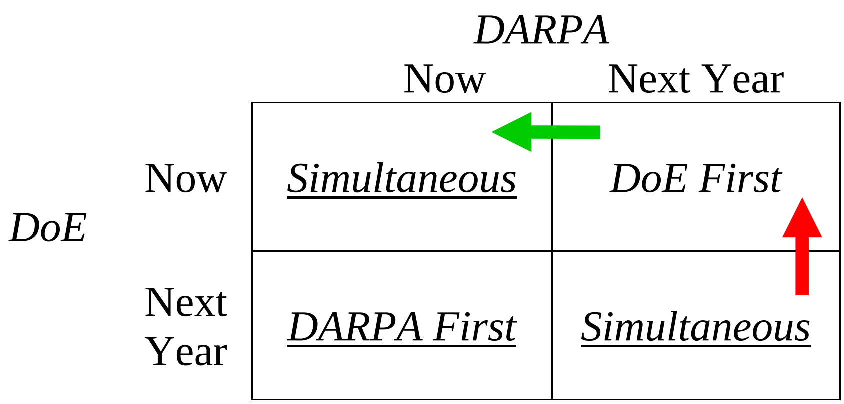

Figure 3: Timing in the research-funding game

Regardless of the timing of DoE and DARPA's budgetary requests, the equilibrium timing of moves will be either "Simultaneous moves" or "DARPA moves first", each of which induces (Batteries, Batteries) as the unique equilibrium outcome of the underlying game. To see why, note that the only other possible equilibrium outcome is (Solar, Solar), if DoE moves first. However, DARPA can block this timing of moves and ensure its preferred outcome of (Batteries, Batteries) by requesting funds now.

The remainder of the paper is organized as follows. The introduction continues with a discussion of some related literature. Section 2 presents the model. Section 3 characterizes the unique equilibrium outcome in every 2x2 game in which a pure-strategy Nash equilibrium (PSNE) exists and is the unique NE. Section 4 then characterizes the unique equilibrium outcome in every 2x2 game in which no PSNE exists but a mixed-strategy Nash equilibrium (MSNE) exists. Section 6 offers concluding remarks.

Related literature: The literature on dynamic games endogenizes the timing of moves in games by allowing players to choose when to move *during* the game. See e.g. Cramton (1991) on bargaining with incomplete information, where players decide when to make the first offer, Bulow (1999) on the war of attrition, and Chassang (2010) on exit games.[3] This paper departs from the dynamic games literature by allowing players to *commit* to when they will move, prior to playing the game. This alternative assumption captures the idea that each player's ability to move at various times depends on its capabilities, and that players can influence the timing of moves by making observable investments in such capabilities.

2. Model

The overall meta-game has two phases. In the first phase, called the "timing game", each of two players decides whether to move "early" or "late". (These decisions may be simultaneous or sequential.) If both move early or both move late, a Nash equilibrium (NE) of the underlying 2x2 game is then played. Otherwise, if one player moves early and the other moves late, the underlying 2x2 game is then played with sequential moves, with the early-mover moving first.

The equilibrium solution concept for the meta-game is subgame-perfect equilibrium in weakly undominated strategies.

[3] Much of the interest in these dynamic-game models arises from the fact that players' payoffs depend on when they move. For simplicity, I abstract here from such direct timing effects.

Definition **[Outcome-equivalence]** Meta-game equilibria are *"outcome-equivalent"* if they induce the same distribution over outcomes in the underlying 2x2 game.

Definition **[Unique equilibrium outcome]** The underlying 2x2 game has a *"unique equilibrium outcome"* if all equilibria of the meta-game are outcome-equivalent, regardless of the timing of moves in the timing game.

3. Games with dominant strategies

Theorem 1. *In all 2x2 games in which at least one player has a dominant strategy, there is a unique equilibrium outcome.*

Proof: If both players have a dominant strategy, the unique equilibrium outcome is the Nash equilibrium (NE) outcome regardless of the timing of moves. So, suppose that just one player (Row) has a dominant strategy and, without loss, that the unique NE is (Up,Left) as in Figure 3. Since Row has a dominant strategy, both players are indifferent between simultaneous moves and Column moving first. What if Row moves first? If A > B, then Row will choose Up as first-mover; thus, (Up,Left) is the unique equilibrium outcome. Otherwise, if B > A, Row will choose Down as first-mover, inducing (Down,Right).

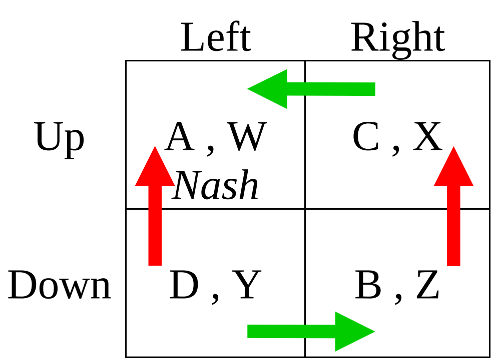

Figure 3: Only Row has dominant strategy (A > D, C > B, W > X, Z > Y)

Games in which B > A and Z > W. If B > A in Figure 3, then Row's preference ranking is Row first > Column first = Simultaneous moves. If Z > W, Column shares this preference ranking. In that case, players' preferences in the timing game are as depicted in Figure 4. If the timing game has simultaneous moves, all Nash equilibrium in weakly undominated strategies induce "Row first" in the subsequent 2x2 game. Further, "Row first" is the unique SPE outcome of the timing game if Row moves first or Column moves first. Thus, Row moves first and **(Down,Right) is the unique equilibrium outcome**.

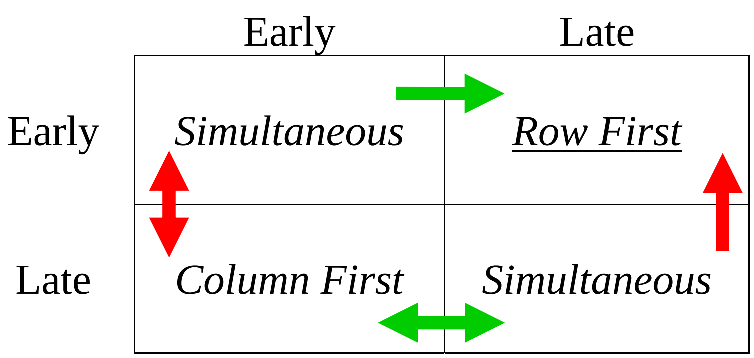

Figure 4: Timing Game if B > A and Z > W

Games in which B > A and W > Z. On the other hand, if Z < W, Column has the opposite preferences Column first = Simultaneous moves > Row first. In that case, players' preferences in the timing game are as depicted in Figure 5. If the timing game has simultaneous moves, the unique Nash equilibrium in weakly undominated strategies induces "Simultaneous moves" in the subsequent 2x2 game. Further, if Row moves first or Column moves first in the timing game, all SPE induce either "Simultaneous moves" or "Column first", both of which induce (Up,Left) in equilibrium. Thus, **(Up,Left) is the unique equilibrium outcome**.

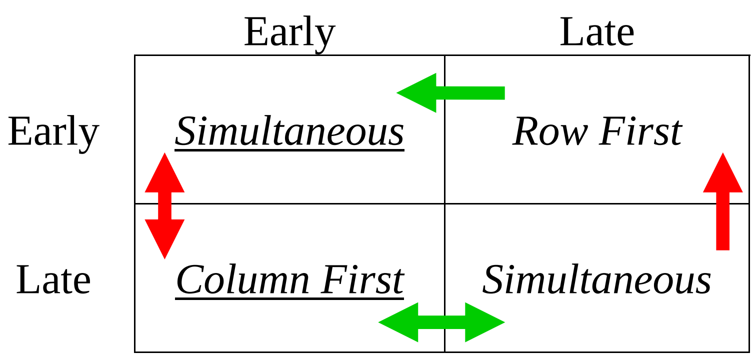

Figure 5: Timing Game if B > A but Z < W

4. Games with no pure-strategy Nash equilibrium

<u>Theorem 2.</u> *In all 2x2 games in which a pure-strategy Nash equilibrium (PSNE) does not exist, there is a unique equilibrium outcome.*

4.1. Preliminaries

<u>MSNE payoffs</u>. Any 2x2 game with no PSNE possesses a unique mixed-strategy Nash equilibrium (MSNE). A key observation is that each player's expected payoff in this MSNE is always better than his second-worst outcome but worse than his second-best outcome (payoff is "between 2 and 3").

<u>MSNE Lemma:</u> *In any 2x2 game that possesses a mixed-strategy Nash equilibrium (MSNE), each player's expected payoff in the MSNE is better than his second-worst outcome but worse than his second-best outcome.*

Proof: Consider Row player. (A similar argument applies to Column.) Without loss, suppose that (Up,Left) is Row's best outcome. First, note that (Up,Right) cannot be Row's second-best outcome, since then Up would be Row's dominant strategy and the game's only Nash equilibrium would be one in pure strategies. Thus, Row's payoff in (Up,Right) must be “1” or “2”. These two cases are illustrated in Figures 6A and 6B. (All payoffs are expressed *ordinally*, with “4” best and “1” worst.)

	Left	Right
Up	4 , ??	1 , ??
Down	2 or 3 , ??	2 or 3 , ??

Figure 6A: Case #1

	Left	Right
Up	4 , ??	2 , ??
Down	1 or 3 , ??	1 or 3 , ??

Figure 6B: Case #2

<u>Case #1: (Up,Right) is Row’s worst outcome.</u> When playing Down, Row’s realized payoff is sometimes “2” and sometimes “3”. Thus, Row’s expected payoff when playing Down is “between 2 and 3”. In any MSNE, Row must be indifferent between playing Up and Down. In particular, Row’s expected payoff in any MSNE must be “between 2 and 3”.

<u>Case #2: (Up,Right) is Row’s second-worst outcome.</u> When playing Down, Row’s realized payoff is sometimes “1” and sometimes “3”. Thus, Row’s expected payoff when playing Down is “less than 3”. On the other hand, when playing Up, Row’s realized payoff is sometimes “2” and sometimes “4”. Thus, Row’s expected payoff when playing Down is “greater than 2”. So, again, Row’s expected payoff in any MSNE must be “between 2 and 3”. QED

<u>*Dependence*</u>. As we shall see, the equilibrium timing of moves depends crucially on which players “depend on the other player”.

<u>Definition</u> [Dependence]. "*Row depends on Column*" if Row's two best outcomes are in the same column. Similarly, "*Column depends on Row*" if Column's two best outcomes are in the same row.

<u>Dependence Lemma</u>: *In any 2x2 game having no PSNE, any player who is <u>not</u> dependent on the other player prefers moving last over moving simultaneously.*

Proof: Suppose that Row is not dependent. (The proof is symmetric for Column player.) Since Row's two best outcomes are in different columns, playing his best response will allow Row to achieve *at least* his second-best outcome, no matter what Column plays as first-mover. By contrast, if moves are simultaneous, Row's expected payoff is worse than his second-best outcome by the MSNE Lemma. QED

4.2. When neither player is dependent

Suppose first that neither player is dependent. Without loss, suppose that (Up,Left) is one of Row's two best outcomes (payoff "3" or "4"). (Up,Left) must then be one of Column's two worst outcomes (payoff "1" or "2"). To see why, suppose that (Up,Left) is one of Column's two best outcomes. Since neither player is dependent, (Down,Right) must also be one of both players' two best outcomes. However, both (Up,Left) and (Down,Right) are then PSNE. We conclude that, since the game has no PSNE, each of Row's two best outcomes must be one of Column's two worst outcomes, and vice versa, as illustrated in Figure 7.

	Left	Right
Up	**3 or 4, 1 or 2**	**1 or 2 , 3 or 4**
Down	**1 or 2 , 3 or 4**	**3 or 4, 1 or 2**

Figure 7: Zero-PSNE games in which neither player is dependent

Each player gets his second-worst outcome (payoff "2") when moving first, one of his two best outcomes (payoff "3" or "4") when moving last, and payoff "between 2 and 3" when moving simultaneously. Thus, Row's ranking is Column first > Simultaneous > Row first while Column has the opposite ranking Row first > Simultaneous > Column first. See Figure 8.

	Early	Late
Early	*Simultaneous*	*Row First*
Late	*Column First*	*Simultaneous*

Figure 8: Timing Game corresponding to Figure 7

Since both players have a dominant strategy to move late, the unique NE of the timing game induces "Simultaneous moves" in the underlying 2x2 game, as does the unique SPE when Row moves first or Column moves first in the timing game. The unique (random) equilibrium outcome is that which arises in the unique MSNE.

4.3. When both players are dependent

Suppose next that both players are dependent. Without loss, suppose that (Up,Left) is Row's best outcome ("4"). Since Row depends on Column, (Down,Left) must be his second-best outcome ("3"). Further, since Row does not have a dominant strategy (else PSNE would exist), Row must prefer (Down,Right) over (Up,Right). In particular, (Up,Right) must be Row's worst outcome ("1") while (Down,Right) is his second-worst ("2"). What about Column? Since Column is also dependent, Column's two best outcomes ("4" and "3") are in one row while its two worst outcomes ("2" and "1") are in another row. Further, since no PSNE exists, Column must prefer (Up,Right) over (Up,Left) and (Down,Left) over (Down,Right). Overall, this leaves just two possibilities for payoffs in the game, as shown in Figure 9 below.

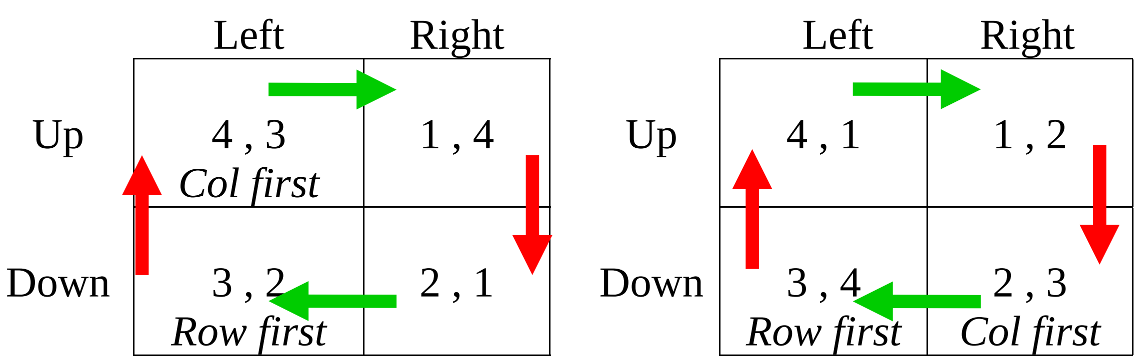

Figure 9: Zero-PSNE games in which both players are dependent

<u>The game on the left-hand-side of Figure 9</u>. Recalling that each player gets expected payoff "between 2 and 3" given simultaneous moves (MSNE Lemma), Row's ranking is Column first > Row first > Simultaneous moves, while Column's ranking is Column first > Simultaneous moves > Row first. The induced timing game is illustrated on the left-hand-side of Figure 10 below. The unique NE of this timing game generates timing "Column first" in the underlying 2x2 game, as does the SPE

when Row moves first or Column moves first in the timing game. The SPE outcome when Column moves first is the unique equilibrium outcome.

The game on the right-hand-side of Figure 9. Row's ranking is Row first > Simultaneous moves > Column first, while Column's ranking is Row first > Column first > Simultaneous moves. The induced timing game is illustrated on the right-hand-side of Figure 10. The unique NE of this timing game generates timing "Row first" in the underlying 2x2 game, as does the SPE when Row moves first or Column moves first in the timing game. The SPE outcome when Row moves first is the unique equilibrium outcome.

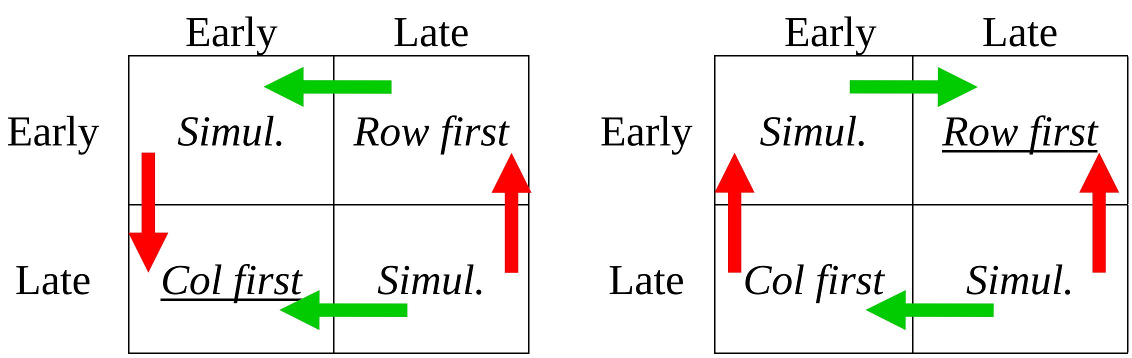

Figure 10: Timing Games corresponding to Figure 9

4.4. When only one player (Column) is dependent

As a final case, consider those games with no PSNE in which only one player (say Column) is dependent. Without loss, suppose that (Up,Left) is Row's best outcome ("4"). Since Row is not dependent and does not have a dominant strategy, (Down,Right) must be Row's second-best outcome ("3"). What about Column? Since Column depends on Row, Column's two best outcomes are in the same row. Further, since there is no PSNE, Column must prefer (Up,Right) over (Up,Left) and (Down,Left) over (Down,Right). This leaves four possibilities, shown in Figure 11.

Row's preference over timing of moves. In the two games at the top of Figure 11, Row gets payoff "2" when moving first and "3" when moving last, compared to expected payoff "between 2 and 3" under simultaneous moves (by the MSNE Lemma). Thus, Row's ranking in these games is Column first > Simultaneous > Row first. Similarly, since Row's payoff is "2" when moving first and "4" when moving last in the two games at the bottom of Figure 11, Row's ranking in these games is again Column first > Simultaneous > Row first.

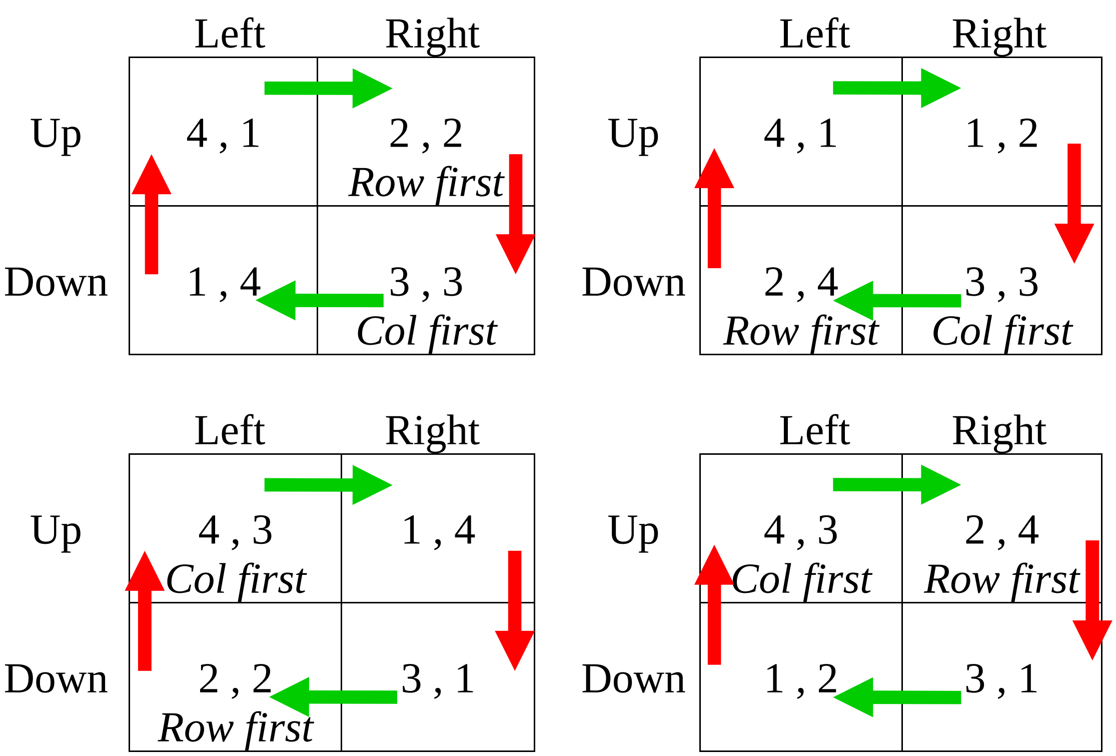

Figure 11: Zero-PSNE games in which only Column is dependent

Column's preference over timing of moves. In the two games on the left side of Figure 11, Column gets payoff "3" when moving first and "2" when moving last, compared to expected payoff "between 2 and 3" under simultaneous moves (again by the MSNE Lemma). Thus, Column's ranking in these games is Column first > Simultaneous > Row first. By contrast, in the games on the right side of Figure 11, Column's payoffs are "3" when moving first, "4" when moving last, and "between 2 and 3" when moving simultaneously, leading to a different ranking of Row first > Column first > Simultaneous.

All together, there are two possible timing games arising from these four 2x2 games, corresponding to the left and right pairs of games in Figure 11. In each timing game illustrated in Figure 12, the unique NE generates "Column first" timing in the underlying 2x2 game, as does the SPE when Row or Column moves first. Thus, <u>the SPE outcome when Column moves first is the unique equilibrium outcome</u>.

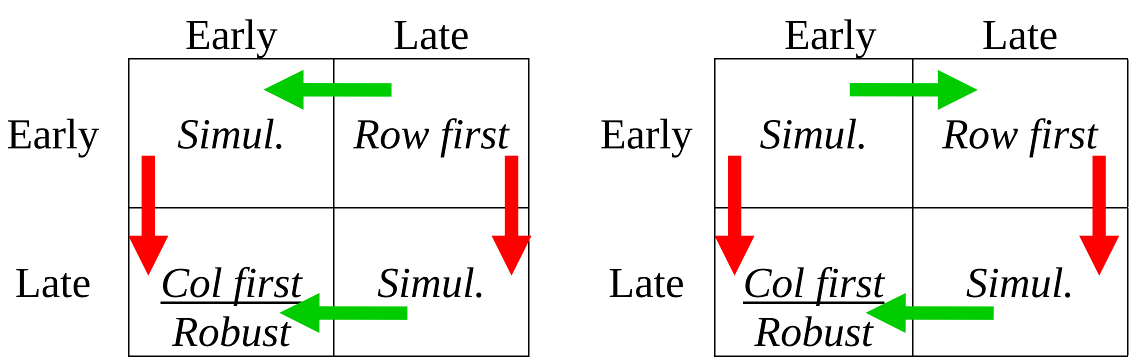

Figure 12: Timing Games corresponding to Figure 11

5. Concluding remarks

Most game-theoretic analysis takes the timing of moves as given, but players often have an incentive to influence the timing of moves to their advantage. This paper considers a simple model of endogenous timing, in which players first observably commit *when* they will make their move ("early" or "late"), with simultaneous moves iff both choose to move at the same time.

The main finding is that, at least for all 2x2 games having a unique Nash equilibrium, the resulting meta-game has a unique equilibrium outcome, independent of the timing of moves at the timing phase. The assumption that players can pre-commit to the time of their move is essential for this uniqueness result. Consider an alternative model without pre-commitment, in which (i) each player simultaneously decides whether to move early, (ii) all early-moves are observed, and then (iii) all those who did not move early then simultaneously move late. Many 2x2 games having a unique Nash equilibrium do not have a unique equilibrium timing of moves in this model.

For example, consider the "research-funding game" illustrated in Figure 2. In one dynamic-game equilibrium, both agencies move early and support battery research. In another, DoE (Row) moves early and supports solar, while DARPA moves late and also supports solar. DoE is willing to support solar early in this latter equilibrium, since it believes that DARPA will observe its choice and respond by also supporting solar. This sort of equilibrium does not arise in my model, since DARPA can observably pre-commit to be incapable of responding to DoE's choice.

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