Adoption Epidemics and Viral Marketing

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Abstract

An innovation (e.g., new product or idea) spreads like a virus, transmitted by those who have previously adopted it. We characterize equilibrium adoption dynamics and the resulting lifecycle of virally-spread innovations. Herding on adoption can occur but only early in the innovation lifecycle, and adoption eventually ceases for all virally-spread innovations. A producer capable of advertising directly to consumers finds it optimal to wait and allow awareness to grow virally initially after launch. When most innovations would otherwise be mostly high (or low) quality absent any viral spread, running an optimal-length viral campaign decreases (or increases) equilibrium investment in innovation quality.

Keywords: SIR model, economic epidemic, innovation lifecycle

JEL Classification: C72, D62, D83

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When a novel virus enters a population, infected hosts expose others who, if successfully infected, will start spreading the virus as well. In such an *infectious-disease epidemic*, virus strains that are more successful at causing infection spread more quickly through the population. In the same way, when a new product is launched, a new idea espoused, or a new method developed, an epidemic diffusion process ensues in which those who have purchased the product, accepted the idea, or adopted the method spread awareness and cause others to consider it as well. Those exposed later during such an *adoption epidemic* can make inferences about quality based on how long it took for them to be exposed. For example, in a local political election, seeing a lawn sign promoting a candidate relatively early during the campaign season may prompt people who see it to learn about that candidate and potentially post their own sign, spreading awareness in epidemic fashion and ultimately increasing that candidate's share of the vote (Green et al. (2016)). Similarly, hearing a friend recommend an anime series (Ameri et al. (2019)), Netflix show, or TikTok recipe may prompt you to try it out yourself and potentially recommend it to others as well.

Our economic-epidemic model adapts the susceptible-infected-recovered (SIR) model of viral epidemiology[1] to an economic context in which consumers receive informative private signals about quality and decide whether to adopt a new innovation. There is a unit-mass population of consumers and an "innovation" that is "good" with probability α and "bad" with probability $1 - \alpha$. When first exposed to the innovation, each consumer i receives a conditionally independent private signal $s_i \in \{G, B\}$ that matches the true state with probability $\rho \in (1/2, 1)$. Consumer i then decides whether to adopt the innovation, preferring to adopt whenever she believes that the innovation is more likely to be good than bad. Those who adopt are "infected" and subsequently expose others, while those who choose not to adopt are "immune/recovered" and do not expose anyone else to the innovation.

A consumer who *adopts* in our model is making a choice to spread awareness of the innovation, which in some applications may be distinct from the decision to *use or act* on it. For example, in our local-election example, "adopting" does not correspond to voting for a candidate but rather putting up a lawn sign or volunteering for their campaign. A key assumption of our model is that consumers only want to make that decision to spread awareness when the innovation in question is sufficiently likely to be good.[2] An

[1] The SIR model was formulated in 1908 by Ronald Ross (who also famously discovered that malaria is transmitted by mosquito) and developed further by Kermack and McKendrick (1927). It remains the workhorse model of the field; see Anderson (1991) and Blackwood and Childs (2018).

[2] In a simpler alternative model, consumers get a direct benefit from sharing things that they like (i.e., when they get a favorable private signal) but ultimately don't care whether the innovation is good or bad.

implication of this assumption is that consumers find it optimal to account for the time it has taken for them to hear about the innovation, in addition to their own private signal. Indeed, we show that consumers find it optimal during some phases of the epidemic to ignore their own private signals, "herding" on adoption or non-adoption.

Our first main finding is that the adoption epidemic has a unique equilibrium epidemic trajectory, which depends on (i) consumers' ex ante belief $\alpha \in [0, 1]$ about the likelihood that the innovation is good, (ii) the precision $\rho \in (1/2, 1)$ of consumers' private signals, and (iii) the fraction L of the consumer population that learns about the innovation at "launch" at time $t = 0$. The case with $L = 1$ is relatively trivial since all consumers are exposed to the innovation at time $t = 0$ and simultaneously decide whether to adopt; we refer to this as a "traditional ad campaign." By contrast, when $L \approx 0$, almost all consumers encounter the innovation socially; we refer to this case as a "viral campaign."

The qualitative features of the equilibrium trajectory of a viral campaign depend on whether or not the innovation is more likely to be good than bad, i.e., is $\alpha > 1/2$ or $\alpha < 1/2$? When $1/2 < \alpha < \rho$,[3] we show that consumers herd on adoption immediately after launch, but this herding phase eventually ends and is followed by subsequent phases in which newly-exposed consumers are less and less likely to adopt—until eventually all adoption ceases, an endogenous obsolescence. By contrast, when $1 - \rho < \alpha < 1/2$, consumers do not herd on adoption immediately after launch and newly-exposed consumers' belief about innovation quality initially rises over time. However, as when $1/2 < \alpha < \rho$, newly-exposed consumers eventually become sufficiently pessimistic about quality that all adoption ceases.

In an extension, we allow the producer of the innovation to launch the innovation virally but then end the viral campaign at any time $T \in [0, \infty)$ with an ad that reaches all still-unexposed consumers. Our main finding in this extension is that a traditional ad campaign (corresponding to $T = 0$) leads to strictly less overall adoption than an optimal-length viral campaign.[4] On the other hand, we also show that it is never optimal to run a viral campaign forever.

In a further extension, we model the producer as deciding whether to make a costly investment to increase the likelihood that their innovation is good. We find that equilibrium investment is *more moderate* when the producer uses an optimal-length viral campaign

Innovation infectivity remains constant over time in such a model, whereas in our model infectivity is endogoneous and eventually falls to zero in equilibrium.

[3] If $\alpha > \rho$ (or $\alpha < 1 - \rho$), then consumer behavior is trivial with everyone (or no one) adopting.

[4] In our analysis, we characterize the minimal campaign length T^* that maximizes the mass of consumers who adopt the innovation. If the producer prefers quicker adoption, such as when adoption corresponds to purchasing a new product and the producer is a firm that discounts profits, then the producer may prefer running a traditional ad campaign even though doing so leads to less overall adoption.

than in a non-viral benchmark in which the producer's only option is to run a traditional ad campaign. In particular, let $\hat{\alpha}$ and α^* denote the equilibrium likelihood of a good innovation in the non-viral benchmark and when the producer runs an optimal-length viral campaign, respectively. We show that $\hat{\alpha} < \alpha^* < 1/2$ if $\hat{\alpha} < 1/2$ but that $\hat{\alpha} > \alpha^* > 1/2$ if $\hat{\alpha} > 1/2$.

Intuitively, whether optimal-length viral marketing increases or decreases the producer's incentive to invest in innovation quality depends on the relative strength of two competing effects. First, because those exposed early on during a viral campaign experience positive word of mouth, they are more likely to adopt when the innovation is good. This "word-of-mouth effect" gives the producer more incentive to invest in quality, to drive early adoption and cause more consumers to encounter the innovation while word of mouth remains positive. On the other hand, we show that viral campaigns can also cause some consumers to adopt even after a negative private signal. This "herding effect" reduces the importance of quality in driving adoption and hence reduces the producer's incentive to invest.

A key simplifying assumption of our main analysis is that consumers receive equally-informative binary private signals. This allows us to characterize the equilibrium trajectory of the adoption epidemic in an especially simple way, in terms of a sequence of several clearly-delineated phases (Figure 2). However, this is not essential. In Appendix B, we extend the model to a setting in which consumers differ in the precision of their private signals, characterize the unique equilibrium epidemic trajectory, and show how the basic qualitative features of the equilibrium trajectory in our binary-signal model carry over to this richer context.

Relation to the literature. The idea that ideas can spread like a virus is widely appreciated[5] and well-studied, with some going even further to explore how ideas mutate as they circulate through a population; see e.g., Simmons et al. (2011), Adamic et al. (2016), and Jackson et al. (2022). We abstract from the possibility of mutation, but push the literature forward by modeling becoming infected as an economic choice. In doing so, we characterize the equilibrium dynamics of the epidemic and show how these dynamics change over time, passing through several phases with distinctive patterns of adoption.

Most closely related is Banerjee (1993), who pioneered the study of adoption epidemics in the context of *rumors*, when only those exposed at launch have informative private signals about quality. Our model differs from Banerjee (1993)'s by allowing all

[5]See e.g., "The Age of the Viral Idea" by Bill Davidow, *The Atlantic*, Nov 17, 2011 and "The Internet Catches a Viral Epidemic" by Bill Wasik, *Wired*, April 16, 2013.

consumers to receive informative private signals. To highlight the significance of this difference, we compare our "innovation model" (studied in the main text) with a much simpler "rumor model" variation in which only those exposed at launch are privately informed (Appendix A). We find that several qualitative features of equilibrium adoption dynamics differ fundamentally between these cases:

- In the rumor model, more delay in hearing about a rumor is always bad news about its quality. By contrast, in the innovation model, consumer beliefs about the likelihood that an innovation is good can be non-monotone in the time that it takes to hear about the innovation.

- In the rumor model, either a traditional ad campaign or a viral campaign that lasts forever is always optimal. By contrast, in the innovation model, neither a traditional ad campaign nor a viral campaign that lasts forever is ever optimal.

- In the rumor model, optimal-length viral marketing never reduces equilibrium rumor quality. By contrast, in the innovation model, allowing the producer to run an optimal-length viral campaign reduces equilibrium innovation quality whenever equilibrium quality would otherwise be high.

Because awareness of the innovation in our model spreads by word of mouth, the paper connects with the broader economic literature on diffusion; see e.g., Campbell (2013), Campbell et al. (2017), Leduc et al. (2017), and Sadler (2020). The main difference is that this literature mostly focuses on consumers' search technology and social network, whereas we focus on the impact of consumers' private information about quality. There is also a literature in marketing and consumer behavior on the diffusion of new products through influentials, e.g. Dodson Jr. and Muller (1978) and Van den Bulte and Joshi (2007). This literature also develops compartmental models where consumers transit between different states marking their awareness of the product and/or their adoption behavior. However, consumers in these models typically make decisions according to rules governed by exogenous parameters; see Watts and Dodds (2007) for a comprehensive survey. By contrast, the consumers in our analysis are Bayesian utility maximizers.

An extensive literature endogenizes the diffusion dynamics of an infectious pathogen; see e.g., Newman (2002) on disease spread over a social network, Lipsitch et al. (2007) and Toxvaerd and Rowthorn (2022) on when best to deploy antiviral treatment during a viral epidemic, Bauch and Bhattacharyya (2012) on the dynamics of vaccine scares, Laxminarayan and Brown (2001) and McAdams (2017) on when to switch to a new antibiotic in the face of rising resistance, and Farboodi et al. (2021) and McAdams et al. (2023) on the impact of social distancing during the outbreak and endemic phases of an epidemic. The basic difference with this literature is that agents in an infectious-disease epidemic prefer to avoid infection, whereas being "infected" in our model may or may not benefit consumers depending on whether the innovation is good or bad.

Finally, the paper relates indirectly to the literature on social learning. In the classic social learning model (Bikhchandani et al. (1992), Banerjee (1992), Smith and Sorensen (2000)), infinitely-many agents are arrayed in a line and sequentially decide whether to adopt, based on their own private signal and all decisions made by those before them. By contrast, in our model, only those who have chosen to adopt expose others to the innovation and, when deciding whether to adopt, consumers do not know the length of the chain of exposures that led to their own exposure.[6]

The rest of the paper is organized as follows. Section 1 presents the model. Section 2 characterizes the equilibrium epidemic trajectory of innovation adoption over time. We then present two extensions, allowing the producer to choose when to end a viral campaign with an ad that reaches all remaining consumers (Section 3) and whether to make a costly investment in innovation quality (Section 4). Concluding remarks are in Section 5. Appendix A analyzes a variation of our model in which only those exposed at time $t = 0$ get informative private signals about the innovation, referred to in this case as a "rumor." Appendix B extends the equilibrium-trajectory analysis of Section 2 to a richer setting in which consumers' private signals are drawn from a continuum with differing precision. Appendix C contains formal proofs omitted in the main text.

1 Model

There is an "innovation" which may be either "good" or "bad." Each consumer i gets payoff $u_g > 0$ when adopting a good innovation, $-u_b < 0$ when adopting a bad innovation, or zero when not adopting, and seeks to maximize their own expected payoff. To simplify equations, suppose that $u_g = u_b$ so that each consumer strictly prefers to adopt if and only if they believe that the innovation's likelihood of being good exceeds $1/2$.[7] Let $\alpha \in [0, 1]$ be the ex ante probability that the innovation is good.

[6] Classic social learning reemerges within a variation of our model if one instead assumes (i) all infected and recovered consumers expose others at the same rate and (ii) each consumer is able to observe the history of decisions made along the entire chain of consumers leading to their exposure. In that context, consumers along each exposure chain behave exactly as in the classic model.

[7] This normalization is without loss of generality. If $u_g \neq u_b$, then consumers will adopt based on belief threshold $u_b/(u_b + u_g)$ rather than $1/2$, but otherwise our analysis carries through directly.

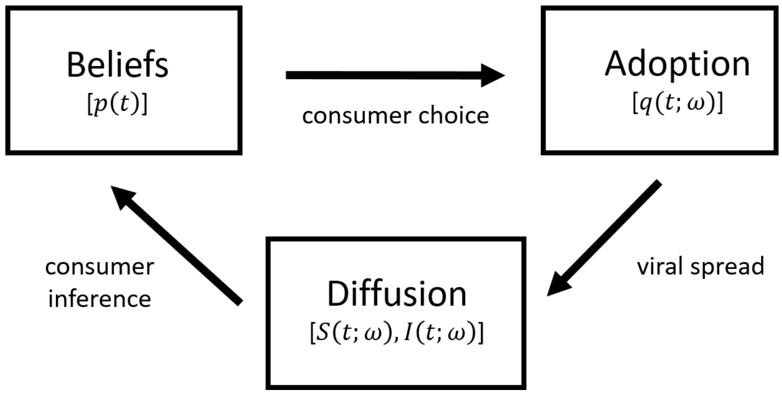

Figure 1: Illustration of "viral social learning" during an adoption epidemic, whereby the dynamics of innovation adoption drive the epidemiological dynamics of innovation awareness, which in turn determine the dynamics of consumer beliefs about innovation quality.

Epidemiological dynamics. Innovation awareness spreads through the consumer population like a virus, according to a Susceptible-Infected-Recovered (SIR) model (Kermack and McKendrick (1927)). At each point in time $t \geq 0$, each consumer is in one of three epidemiological states: *susceptible*, if not yet exposed to the innovation; *infected*, if previously exposed and chose to adopt; or *recovered*, if previously exposed and chose not to adopt. We assume that mass $L > 0$ of consumers are exposed to the innovation at time $t = 0$ regardless of innovation quality. Those who adopt then become infected and spread innovation awareness virally, meeting another randomly-selected consumer at rate $\beta > 0$ and exposing that other consumer to the innovation. If susceptible, that other consumer receives a private signal and decides whether or not to adopt, then transitions immediately either to the infected state (if adopting) or to the recovered state (if not adopting).

Let $S_\omega(t)$, $I_\omega(t)$, and $R_\omega(t)$ denote the mass of susceptible, infected, and recovered consumers at time t, conditional on the unobserved innovation-quality state $\omega \in \{g, b\}$. Since the population has unit mass, $R_\omega(t) = 1 - S_\omega(t) - I_\omega(t)$ and the overall epidemiological process is described by $(S_\omega(t), I_\omega(t) : t \geq 0, \omega = g, b)$. Let $q_\omega(t)$ denote time-t consumers' likelihood of adopting when the state is $\omega \in \{g, b\}$.

Epidemiological dynamics are characterized by the system of differential equations

$$S'_\omega(t) = -\beta I_\omega(t) S_\omega(t) \tag{1}$$

$$I'_\omega(t) = q_\omega(t) \beta I_\omega(t) S_\omega(t) \tag{2}$$

Equation (1) follows from the fact that each infected consumer meets another consumer at rate $\beta > 0$ and fraction $S_\omega(t)$ of others remain suspectible, generating a state-dependent flow $\beta I_\omega(t)S_\omega(t)$ of newly-exposed consumers who are then no longer susceptible. Equation (2) follows from the fact that fraction $q_\omega(t)$ of these newly-exposed consumers choose to adopt. Note that epidemiological dynamics are completely determined by the adoption process $(q_\omega(t) : t \geq 0, \omega = g, b)$ and the mass L of consumers exposed at time $t = 0$.

To simplify equations, we will henceforth normalize $\beta = 1$. This is without loss of generality since, with any $\hat{\beta} \neq 1$, equilibrium epidemiological dynamics are exactly the same but happen $\hat{\beta}$ times faster than when $\beta = 1$.

Consumer belief formation. Let $p(t)$ be the probability that the innovation is good conditional on encountering it socially at time t, what we refer to as the "interim belief" of consumers exposed socially at time t. Let $f(t|\omega)$ denote the endogenous[8] p.d.f. of consumers' time of exposure conditional on the state $\omega \in \{g, b\}$. By Bayes' Rule, $p(t) = \frac{\alpha f(t|\omega=g)}{\alpha f(t|\omega=g)+(1-\alpha)f(t|\omega=b)}$ or, equivalently,

$$\frac{p(t)}{1-p(t)} = \frac{\alpha}{1-\alpha} \times \frac{f(t|\omega = g)}{f(t|\omega = b)} \tag{3}$$

Once exposed to the innovation, each consumer i observes private signal $s_i \in \{G, B\}$. These signals are conditionally i.i.d. with $\Pr(s_i = G|\omega = g) = \Pr(s_i = B|\omega = b) = \rho \in (1/2, 1)$. A consumer i exposed at launch ($t_i = 0$) with signal $s_i \in \{G, B\}$ updates to "ex post belief" $p(0; s_i)$, where

$$\frac{p(0;G)}{1-p(0;G)} = \frac{\alpha}{1-\alpha} \times \frac{\rho}{1-\rho} \quad \text{and} \quad \frac{p(0;B)}{1-p(0;B)} = \frac{\alpha}{1-\alpha} \times \frac{1-\rho}{\rho}. \tag{4}$$

A consumer i exposed socially at time t_i updates her belief based on both her own private signal $s_i \in \{G, B\}$ and when she is exposed, forming "ex post belief" $p(t_i; s_i)$. Again by Bayes Rule,

$$\frac{p(t_i;G)}{1-p(t_i;G)} = \frac{p(t_i)}{1-p(t_i)} \times \frac{\rho}{1-\rho} \quad \text{and} \quad \frac{p(t_i;B)}{1-p(t_i;B)} = \frac{p(t_i)}{1-p(t_i)} \times \frac{1-\rho}{\rho}. \tag{5}$$

By assumption, all consumers receive equally-informative private signals, regardless of whether they encountered the innovation direction at launch or indirectly through a social interaction.

In Appendix A, we analyze a simpler alternative setting (motivated by Banerjee (1993)'s model of rumors) in which only those exposed at launch receive informative private sig-

[8]We will characterize the equilibrium distribution of $t|\omega$, showing that $f(t|\omega)$ exists and is continuous in t at all but finitely-many points when the innovation lifecycle transitions from one phase to the next.

nals. In that context, a consumer exposed socially at time t_i receives an uninformative private signal and hence has ex post belief $p(t_i)$.

In Appendix B, we analyze a richer alternative setting in which consumers receive conditionally i.i.d. private signals $s_i \in [-1, 1]$ drawn from a distribution having p.d.f. $h(\cdot; \omega)$ and satisfying the monotone likelihood ratio property. In that context, a consumer exposed socially at time t_i has ex post belief $p(t_i; s_i)$ satisfying $\frac{p(t_i;s_i)}{1-p(t_i;s_i)} = \frac{p(t_i)}{1-p(t_i)} \frac{h(s_i;\omega=g)}{h(s_i;\omega=b)}$.

Belief dynamics. Since the consumer population has unit mass, the flow of newly-exposed consumers can be interpreted as the density of the time-until-exposure t, i.e., $f(t|\omega) = |S'_\omega(t)| = S_\omega(t) I_\omega(t)$, where $|S'_\omega(t)|$ is the flow of consumers exposed at time t ("time-t consumers") when the innovation is good ($\omega = g$) or bad ($\omega = b$). Thus, time-t consumers' interim belief is given by

$$\frac{p(t)}{1-p(t)} = \frac{\alpha}{1-\alpha} \times \frac{S_g(t) I_g(t)}{S_b(t) I_b(t)}. \tag{6}$$

Adoption dynamics. Let $a_{s_i}(t)$ denote the likelihood that each time-t consumer chooses to adopt given private signal $s_i \in \{G, B\}$. Time-t consumers are said to "herd on adoption" if $a_G(t) = a_B(t) = 1$ and to "herd on non-adoption" if $a_G(t) = a_B(t) = 0$. On the other hand, they are said to be "sensitive to signals" if $a_G(t) = 1$ but $a_B(t) = 0$. Note that time-t consumers find it optimal to herd on adoption whenever $p(t) > \rho$, to herd on non-adoption when $p(t) < 1 - \rho$, and to be sensitive to signals when $1 - \rho < p(t) < \rho$. Time-t consumers are indifferent whether to adopt after a bad private signal if $p(t) = \rho$ and indifferent whether to adopt after a good signal if $p(t) = 1 - \rho$.

Equilibrium. Our solution concept is Bayesian Nash equilibrium (or simply "equilibrium"). We will show by construction that an equilibrium exists and that generically this equilibrium is essentially unique, in the sense that all equilibria generate the same population-wide epidemiological dynamics $(S_\omega(t), I_\omega(t) : t \geq 0; \omega \in \{g, b\})$.

Discussion: observability of time since launch. We assume that, when consumers encounter the innovation, they are able to observe how much time has elapsed since launch. However, our analysis can be easily extended to a setting in which fraction $\eta \in [0, 1]$ of consumers are unable to observe the time since launch. In particular, because all consumers are eventually exposed to the innovation, a consumer who is unable to observe the time will not make any inference about innovation quality and so will decide whether to adopt *as if* encountering the innovation at launch. The overall likelihood that a consumer exposed at time $t > 0$ will adopt in innovation-quality state $\omega \in \{g, b\}$ is therefore $\tilde{q}_\omega(t) = \eta q_\omega(0) + (1-\eta)q_\omega(t)$, where $q_\omega(0)$ and $q_\omega(t)$ are the likelihoods that consumers who *can* observe the time will adopt, respectively, at time 0 and time t. The rest of our analysis then carries over, with more complex formulas but little additional insight.

2 Adoption Epidemic Dynamics

This section characterizes the unique equilibrium trajectory of the adoption epidemic during a viral campaign from launch through endogenous obsolescence, what we refer to as the "innovation lifecycle." We begin with some preliminary analysis, then complete our characterization of the equilibrium epidemic trajectory in Section 2.1.

Trivial cases: $\alpha > \rho$ or $\alpha < 1-\rho$. Suppose first that $\alpha > \rho$. The innovation is sufficiently likely to be good that consumers exposed at launch choose to adopt even after a bad private signal; thus, $I_g(0) = I_b(0) = L$ and $S_g(0) = S_b(0) = 1 - L$. By equation (6), consumers exposed socially immediately after launch have interim belief $p(0) = \alpha$ and so must also herd on adoption. So long as consumers continue to herd on adoption, $I_g(t) = I_b(t)$ and $S_g(t) = S_b(t)$; hence, $p(t) = \alpha$ and consumers continue to herd on adoption. We conclude, in the unique equilibrium trajectory, all consumers eventually adopt regardless of innovation quality.

Suppose next that $\alpha < 1-\rho$. The innovation is sufficiently unlikely to be good that consumers exposed at launch choose not to adopt even after a good signal. And with no one adopting at launch, no one is "infected" to spread awareness virally and so no one is ever exposed socially. We conclude that, in the unique equilibrium trajectory, no consumers adopt regardless of quality.

The rest of this section characterizes the unique equilibrium trajectory in the most interesting case when $1-\rho < \alpha < \rho$.[9]

Consumer behavior at and immediately after launch. Since $1-\rho < \alpha < \rho$, we have $p(0; B) < 1/2 < p(0; G)$ and any consumer exposed at launch finds it optimal to adopt

[9] The cases when $\alpha = \rho$ and $\alpha = 1-\rho$ are more complex because consumers have multiple best responses at launch, but this extra complexity does not lead to any additional insight. For example, if $\alpha = 1-\rho$, consumers exposed at launch will adopt with some probability $a_G \in [0, 1]$ after a good signal but not adopt after a bad signal, resulting in initial infected mass $I_g(0+) = a_G \rho L$ when the innovation is good and $I_b(0+) = a_G(1-\rho)L$ when it is bad and initial belief $p(0+) = 1/2$. For each $a_G \in (0, 1]$, subsequent equilibrium dynamics are then uniquely determined by similar arguments as used here for the case when $\alpha \in (1-\rho, 1/2)$.

after getting a good signal but not after a bad signal, i.e., they are sensitive to signals. Since good signals are more likely for good innovations, more consumers adopt at launch and word of mouth spreads more rapidly for good innovations. Hearing quickly about an innovation is therefore good news about its quality. More precisely, $I_g(t) \approx \rho L$, $I_b(t) \approx (1-\rho)L$, and $S_g(t) \approx S_b(t) \approx 1-L$ for all $t \approx 0$, where L is the mass of consumers exposed at launch. By equation (6), we conclude that

$$\frac{p(t)}{1-p(t)} \approx \frac{\alpha}{1-\alpha} \times \frac{\rho}{1-\rho} \text{ for all } t \approx 0, \tag{7}$$

Consumers' interim belief shortly after launch is the same as if they have gotten a good private signal of precision ρ. Because $\alpha > 1-\rho$, equation (7) implies that (i) $p(0+) \equiv \lim_{t\to 0} p(t) > 1/2$ and (ii) $p(0+) > \rho$ if and only if $\alpha > 1/2$. Consumers exposed immediately after launch will therefore herd on adoption if $\alpha > 1/2$ but remain sensitive to signals if $\alpha < 1/2$.

Interim belief dynamics after launch. Equation (6) characterizes consumers' interim belief $p(t)$ at time t, depending on the ex ante likelihood α that the innovation is good and the ratio $\frac{S_g(t)I_g(t)}{S_b(t)I_b(t)}$. Rather than focusing on $p(t)$ directly, we find it convenient to consider the percentage rate of change of the likelihood ratio $\frac{p(t)}{1-p(t)}$, gotten by taking the log of both sides of (6) and differentiating:

$$\begin{aligned} X(t) \equiv \frac{\mathrm{d}\log\left(\frac{p(t)}{1-p(t)}\right)}{\mathrm{d}t} &= \frac{S_g'(t)}{S_g(t)} - \frac{S_b'(t)}{S_b(t)} + \frac{I_g'(t)}{I_g(t)} - \frac{I_b'(t)}{I_b(t)} \\ &= -I_g(t) + I_b(t) + q_g(t)S_g(t) - q_b(t)S_b(t) \end{aligned} \tag{8}$$

where $\frac{S_\omega'(t)}{S_\omega(t)} = -I_\omega(t)$ and $\frac{I_\omega'(t)}{I_\omega(t)} = q_\omega(t)S_\omega(t)$ by equations (1-2). Since $\frac{p(t)}{1-p(t)}$ grows exponentially at rate $X(t)$, we have $p'(t) \gtreqless 0$ iff $X(t) \gtreqless 0$.

Lemma 1 summarizes some implications of equation (8), depending on whether consumers herd on adoption, are sensitive to signals, or herd on non-adoption. (Formal proofs are provided in Appendix C.)

Lemma 1. *(i) Suppose that consumers herd on adoption at time t. $p'(t) < 0$ if $I_g(t) > I_b(t)$ and $S_g(t) < S_b(t)$. (ii) Suppose that consumers are sensitive to signals at time t. $p'(t) > 0$ if and only if the following inequality holds:*

$$\rho S_g(t) - (1-\rho)S_b(t) > I_g(t) - I_b(t). \tag{SS}$$

(We refer to this as "Condition SS," mnemonic for "sensitive to signal.") (iii) Suppose that consumers herd on non-adoption at time t. $p'(t) < 0$ if $I_g(t) > I_b(t)$.

2.1 Equilibrium Lifecycle of an Innovation

This section characterizes equilibrium economic-epidemiological dynamics. Our main finding is that consumer behavior transitions over time through up to[10] four distinct phases, what we refer to collectively as the "innovation lifecycle"; see Figure 2. Behavior immediately after launch (Phase I) depends on whether the innovation is more likely to be good ($\alpha > 1/2$) or bad ($\alpha < 1/2$). Subsequent behavior then passes through a period of partial herding (Phase II), a period in which consumers are sensitive to signals (Phase III), and a final period with zero adoption (Phase IV).

Theorem 1. *Suppose that $\alpha \in (1-\rho, \rho)$ and $L \approx 0$. Equilibrium epidemiological dynamics $(S_\omega(t), I_\omega(t) : t \geq 0; \omega \in \{g, b\})$ are uniquely determined, with consumers' post-launch equilibrium behavior transitioning through four phases.*

> Phase I: *(i) If $\alpha \in (1/2, \rho)$, then consumers herd on adoption and interim belief $p(t) > \rho$ decreases until time $t_1 > 0$ at which $p(t_1) = \rho$. (ii) If $\alpha \in (1-\rho, 1/2)$, then consumers are sensitive to signals and $p(t) \in (1/2, \rho)$ increases until time $t_1 > 0$ at which $p(t_1) = \rho$. (iii) If $\alpha = 1/2$, then $p(0+) \equiv \lim_{\epsilon \to 0} p(\epsilon) = \rho$ and Phase I does not occur, i.e., $t_1 = 0$.*
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> Phase II: *Consumers partially herd on adoption, adopting always after a good signal and with probability $a_B(t) \in (0, 1)$ after a bad signal, where $a_B(t)$ is decreasing in t, until time $t_2 > t_1$ at which $a_B(t_2) = 0$. Consumers' interim belief $p(t) = \rho$ for all $t \in [t_1, t_2]$.*
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> Phase III: *Consumers are sensitive to signals and interim belief $p(t) \in (1-\rho, \rho)$ is decreasing in t until time $t_3 > t_2$ is reached at which $p(t_3) = 1 - \rho$.*
>
> Phase IV: *Consumers herd on non-adoption, what we refer to as "viral obsolescence," and consumers' interim belief $p(t) < 1 - \rho$ continues to decline with $\lim_{t \to \infty} p(t) = 0$.*

The rest of this section establishes Theorem 1 through a series of five propositions.

Phase I: herding on adoption case. Suppose first that $\alpha \in (1/2, \rho)$, so that consumers herd on adoption immediately after launch. We show that consumers' interim belief $p(t)$ declines until time $t_1 > 0$ is reached at which $p(t_1) = \rho$.

[10]Depending on model parameters, some of these phases may have zero length.

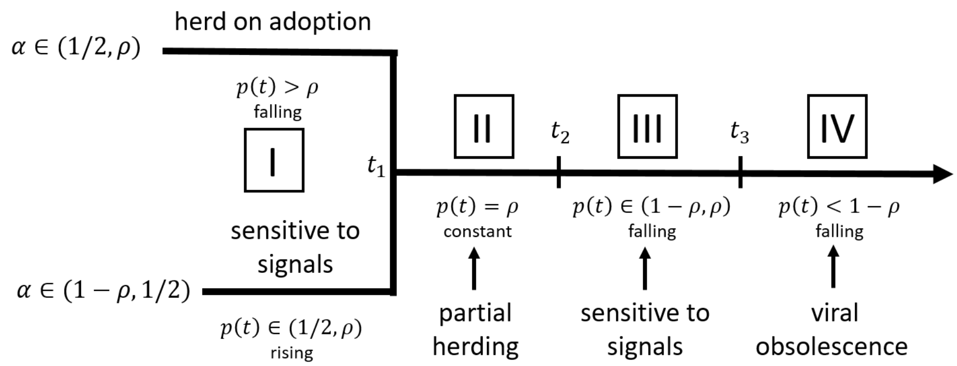


Figure 2: Visual summary of equilibrium adoption behavior and interim beliefs over the innovation lifecycle, when consumers' ex ante belief $\alpha \in (1-\rho, \rho)$.

Proposition 1 (Phase I: herding on adoption)**.** *Suppose that $\alpha \in (1/2, \rho)$. There exists $t_1 > 0$ such that, in any equilibrium trajectory, (i) consumers herd on adoption for all $t \in (0, t_1)$, (ii) $p(t)$ is strictly decreasing over $t \in (0, t_1)$, and (iii) $p(t_1) = \rho$.*

Discussion: downward pressure on consumer beliefs. Because more people adopt good innovations at launch, the fact that consumers herd on adoption for a period of time after launch causes more people to be exposed and infected when the innovation is good than when it is bad. Consequently, there are fewer susceptible people when the innovation is good, causing each infected person to "meet" susceptible people at a slower rate. The mass of infected people therefore increases at a slower percentage rate when the innovation is good; i.e., the ratio $\frac{I_g(t)}{I_b(t)}$ falls over time. Similarly, because there are more infected people when the innovation is good, each susceptible person is exposed at a faster rate, causing the mass of susceptible people to decrease at a faster percentage rate, i.e., the ratio $\frac{S_g(t)}{S_b(t)}$ falls over time. By equation (6), these observations imply that newly-exposed consumers' belief $p(t)$ falls over time so long as consumers continue to herd on adoption. This effect, which arises from the epidemiological dynamics of innovation diffusion whenever consumers' adoption decisions do not depend on their private information, puts a downward pressure on consumer beliefs.

Phase I: sensitive to signals case. Suppose next that $\alpha \in (1-\rho, 1/2)$, so that consumers are sensitive to signals after launch. In this case, we show that consumers' interim belief $p(t)$ increases until a finite time $t_1 > 0$ is reached at which $p(t_1) = \rho$.

Proposition 2 (Phase I: sensitive to signals)**.** *Suppose that $\alpha \in (1-\rho, 1/2)$ and $L \approx 0$. There exists $t_1 > 0$ such that, in any equilibrium trajectory, (i) consumers are sensitive to signals for all*

$t \in (0, t_1)$, (ii) $p(t)$ is strictly increasing over $t \in (0, t_1)$, and (iii) $p(t_1) = \rho$.

Discussion: upward pressure on consumer beliefs. When newly-exposed consumers are sensitive to signals, new exposures are more likely to translate into new infections when the innovation is good. This increases the *relative* growth rate of the mass of infected people when the innovation is good, putting an upward pressure on consumer beliefs. This upward pressure on beliefs weighs against the downward pressure discussed earlier, with the downward pressure growing stronger as the epidemic progresses. Our small-launch assumption here ($L \approx 0$) ensures that, during Phase I of the epidemic, the upward pressure when consumers are sensitive to signals dominates the downward pressure. However, as we discuss next, the balance between these forces shifts over the course of the epidemic until, at the end of Phase II, enough consumers have been exposed that the downward pressure exactly counterbalances the upward pressure associated with consumers being sensitive to signals.

Phase II: Partial herding. For a non-empty interval of time after time t_1, we find that consumers randomize whether to adopt after a bad private signal (and always adopt after a good signal), what we call "partial herding." Over this period of time, consumers' interim belief remains equal to ρ and the likelihood $a_B(t)$ that consumers adopt after a bad signal declines continuously until, at some time t_2, $a_B(t) = 0$ and consumers become sensitive to signals. We refer to the period from t_1 until t_2 as "Phase II".

Proposition 3 (Phase II)**.** *Suppose that $\alpha \in (1-\rho, \rho)$ and $L \approx 0$. There exists $t_2 > t_1$ such that, in any equilibrium trajectory, (i) consumers partially herd with adoption probability $a_B(t) \in (0,1)$ after a bad signal for all $t \in (t_1, t_2)$, where*

$$a_B(t) = \frac{\rho S_g(t) - (1-\rho)S_b(t) - \big(I_g(t) - I_b(t)\big)}{\rho S_b(t) - (1-\rho)S_g(t)} \tag{9}$$

and (ii) $p(t) = \rho$ for all $t \in (t_1, t_2)$. Moreover, $a_B(t)$ is continuously decreasing over $t \in (t_1, t_2)$ with $a_B(t_2) = 0$.

To gain intuition why consumers must randomize[11] whether to adopt after getting a bad private signal, consider the following simple contradiction argument. First, suppose

[11]The equilibrium mixed strategies here can be "purified" by augmenting the model so that consumers' private signals have differing precision. Suppose that each socially-exposed consumer observes the *random* precision ρ_i of their own signal as well as its realization s_i, with (ρ_i, s_i) conditionally i.i.d. across consumers and ρ_i drawn from atomless support $(\rho - \epsilon, \rho + \epsilon)$ for $\epsilon \approx 0$. This enriched model has a unique equilibrium that is approximately outcome equivalent to the equilibrium in our basic model, but with pure strategies. In particular, during "Phase II," consumers adopt after any good private signal and after any sufficiently-imprecise bad signal, and interim beliefs fall continuously from $\rho + \epsilon$ at time t_1 to $\rho - \epsilon$ at time t_2.

that consumers at time t_1 always adopt after a bad signal. Consumers' interim beliefs would then fall (as discussed after Proposition 1), causing consumers never to adopt after a bad signal, a contradiction. Similarly, if consumers were to never adopt after a bad signal, interim beliefs would rise (as discussed after Proposition 2, since condition SS is satisfied at time t_1), causing consumers always to adopt after a bad signal, another contradiction. So, it must be that consumers sometimes adopt and sometimes do not adopt when getting a bad private signal. This in turn requires that consumers' interim belief remain equal to ρ after time t_1. In the Appendix, we show further that the unique equilibrium mixing probability decreases over time and reaches zero at finite time t_2.

Discussion: how Phase II ends. The time t_2 at which Phase II ends is the first time at which $\rho S_g(t) - (1-\rho)S_b(t) = I_g(t) - I_b(t)$, i.e., the first time that Condition SS is satisfied with equality. Intuitively speaking, partial herding is needed during Phase II to temper the upward pressure associated with consumers being sensitive to signals, to keep beliefs from rising above ρ. However, this upward pressure declines over time until, eventually, beliefs begin to fall even if consumers are sensitive to signals. Time t_2 is the transition point after which consumer beliefs do not rise over time *even if* consumers are sensitive to signals.

Phases III and IV: End of the innovation lifecycle. We have two main findings about consumer behavior after time t_2. First, consumers remain sensitive to signals for a period of time but, even though newly-exposed consumers are only adopting after a good private signal, consumers' interim belief falls until a time t_3 is reached at which $p(t_3) = 1-\rho$ (Proposition 4). Second, consumers herd on non-adoption after time t_3, what we refer to as "viral obsolescence," and their interim beliefs continue to decline toward zero (Proposition 5). We refer to the sensitive-to-signal period from t_2 to t_3 as "Phase III" and the obsolescent period after t_3 as "Phase IV".

Proposition 4 (Phase III)**.** *Suppose that $\alpha \in (1-\rho, \rho)$ and $L \approx 0$. There exists $t_3 > t_2$ such that, in any equilibrium trajectory, (i) consumers are sensitive to signals for all $t \in (t_2, t_3)$, (ii) $p(t)$ is strictly decreasing over $t \in (t_2, t_3)$, and (iii) $p(t_3) = 1-\rho$. Moreover, $S_g(t) < S_b(t)$ and $I_g(t) > I_b(t)$ for all $t \in [0, t_3]$.*

Proposition 5 (Phase IV)**.** *Suppose that $\alpha \in (1-\rho, \rho)$ and $L \approx 0$. After time t_3 in any equilibrium trajectory, consumers herd on non-adoption and $p(t)$ declines with* $\lim_{t\to\infty} p(t) = 0$.

The proofs of Propositions 4-5 are the most technically challenging in the paper, but the intuition underlying these results is easy to explain. At the end of Phase II, the epidemic is sufficiently mature that the downward pressure on consumer beliefs is so large that beliefs must fall over time even if newly-exposed consumers are sensitive to signals. Consumers' interim beliefs therefore fall throughout Phase III, from ρ at time t_2 to $1-\rho$ at time t_3, giving newly-exposed consumers an incentive to be sensitive to signals throughout this phase of the epidemic. Once consumers' interim belief hits $1-\rho$ at time t_3, consumers then lose their incentive to adopt after a good private signal, causing an endogenous obsolescence as no one adopts after time t_3.

3 Stopping the Viral Campaign

Here we extend the analysis to allow the producer to decide *how long* to continue the viral campaign. Suppose that, at any time $T \geq 0$, the producer can stop the viral campaign by running a "broadcast advertisement" (or simply "broadcast") that reaches all still-unexposed consumers. In this section, we characterize the optimal time at which to stop the viral campaign.

To keep the analysis as simple as possible, we assume that the producer must choose the broadcast time $T \in [0, \infty]$ before launch and before knowing whether its innovation will be good or bad; running the broadcast is costless; and the producer's objective is to maximize the expected mass of consumers who adopt the innovation.[12]

As in Section 2, we focus on the case when $\alpha \in (1-\rho, \rho)$,[13] so that consumers are sensitive to signals at launch, and assume a small initial launch ($L \approx 0$).

We show that the producer finds it optimal to run a viral campaign of limited duration, i.e., $T = 0$ and $T = \infty$ are both sub-optimal. In Appendix A, we show the opposite is true in the "rumor model" in which only those exposed at launch are privately informed; in that case, either $T = 0$ or $T = \infty$ is always optimal.

Broadcast-updated beliefs. Consumers who see the broadcast at time T update their belief about innovation quality based on the fact that they did not encounter the innovation during the preceding viral campaign. Let $p_{BR}(T)$ denote consumers' updated belief after seeing the broadcast at time T. Conditional on the innovation being good or bad, each consumer will encounter the innovation via broadcast with ex ante probability $S_g(T)$

[12]For simplicity, we assume that the producer does not care about the timing of adoption. Introducing discounting complicates the analysis but does not generate any additional insight.

[13]The other main cases are trivial. In the high-quality case when $\alpha > \rho$, $I_g(T) = I_b(T) > 0$ for all T since consumers herd on adoption; thus, consumers do not update their beliefs and all consumers adopt no matter when (or whether) the producer decides to run the broadcast. Similarly, in the low-quality case when $\alpha < 1-\rho$, $I_g(T) = I_b(T) = 0$ for all T and no consumers ever adopt.

or $S_b(T)$, respectively. By Bayes' Rule:

$$\frac{p_{BR}(T)}{1-p_{BR}(T)} = \frac{\alpha}{1-\alpha} \times \frac{S_g(T)}{S_b(T)}. \tag{10}$$

As shown below in Lemma 2, $p_{BR}(T)$ is strictly decreasing and continuous in T, with $p_{BR}(0) = \alpha$ and $\lim_{T\to\infty} p_{BR}(T) = 0$. Let $\overline{T}$ denote the time at which consumers' broadcast-updated belief equals $1-\rho$, i.e., $p_{BR}(\overline{T}) = 1-\rho$. We refer to $\overline{T}$ as the time of "broadcast obsolescence" since, at any time after $\overline{T}$, all consumers exposed via broadcast will choose not to adopt.

Optimal-length viral campaigns. Here we characterize the optimal length for a viral campaign, in terms of the threshold times t_1, t_2, and t_3 characterized in Theorem 1.

Theorem 2. *Suppose that $\alpha \in (1-\rho, \rho)$ and $L \approx 0$, and let T^* denote the earliest optimal stopping time. (i) If $p_{BR}(t_2) \geq 1-\rho$, then $T^* = t_2$. (ii) If $p_{BR}(t_2) < 1-\rho$, then* either *$T^* = \overline{T}$ (broadcast obsolescence)* or *$T^* = t_3$ (viral obsolescence). Moreover, $T^* = t_3$ if and only if $p_{BR}(t_2) < 1-\rho$ and*

$$\alpha\left(\int_{\overline{T}}^{t_2} a_B(t)(1-\rho)|S_g'(t)|\mathrm{d}t - S_g(t_3)\right) + (1-\alpha)\left(\int_{\overline{T}}^{t_2} a_B(t)\rho|S_b'(t)|\mathrm{d}t - S_b(t_3)\right) \geq 0 \tag{11}$$

where $\left(S_g(t), S_b(t), a_B(t) : t \geq 0\right)$ were derived in Section 2.

An implication of Theorem 2 is that a producer seeking to maximize adoption will always choose to run a viral campaign for at least some period of time. When is it optimal to stop the viral campaign? Theorem 2 lays out three possibilities:

(a) if $p_{BR}(t_2) \geq 1-\rho$, then it is optimal to stop at time t_2;

(b) if $p_{BR}(t_2) < 1-\rho$ and inequality (11) is satisfied, then it is optimal to run the viral campaign until viral obsolescence (time t_3) or, equivalently, to run the viral campaign forever; or

(c) if $p_{BR}(t_2) < 1-\rho$ and inequality (11) is not satisfied, then it is optimal to run the viral campaign until broadcast obsolescence (time $\overline{T}$).

Interestingly, only possibilities (a) and (c) arise in practice. Lemma 2 establishes some facts about broadcast-updated beliefs that are useful in establishing Theorem 2.

Lemma 2. *Suppose that $\alpha \in (1-\rho, \rho)$ and $L \approx 0$. (i) $p_{BR}(T) < p(T)$ for all $T > 0$. (ii) $p_{BR}(0+) = \alpha$ and $\frac{p_{BR}(T)}{1-p_{BR}(T)}$ falls exponentially at rate $I_g(T) - I_b(T) > 0$ for all T. Define $\overline{T}$ implicitly by $p_{BR}(\overline{T}) = 1-\rho$. (iii) $\overline{T} \in (t_1, t_3)$. (iv) If $\alpha \in (1/2, \rho)$, then $\overline{T} \in (t_2, t_3)$.*

Because awareness grows faster during the viral campaign when the innovation is good, encountering the innovation via broadcast is bad news about innovation quality. Moreover, consumers' negative inference when seeing a broadcast gets worse as time goes on (Lemma 2(ii)) and is worse than the inference they would make if encountering the innovation socially at the same time (Lemma 2(i)).

When $\alpha \in (1/2, \rho)$, Lemma 2(iv) implies $\overline{T} \geq t_2$, meaning that only possibility (a) arises in this case. When $\alpha \in (1-\rho, 1/2]$, the theory is unclear but numerical simulation reveals that stopping time $\overline{T}$ is always better than never stopping. Over the relevant parameter space $\{(\alpha, \rho) : \rho \in (1/2, 1), \alpha \in (1-\rho, 1/2]\}$, we solved the system of differential equations (1,2) to determine the equilibrium epidemiological dynamics and computed the mass of consumers who adopt good and bad innovations when the producer uses stopping time $\overline{T}$ (which depends on α and ρ) versus never stopping the viral campaign. As illustrated in Figure 3, stopping the campaign at time $\overline{T}$ is strictly better across the entire parameter space, increasing adoption by as much as 72% for some parameter values.

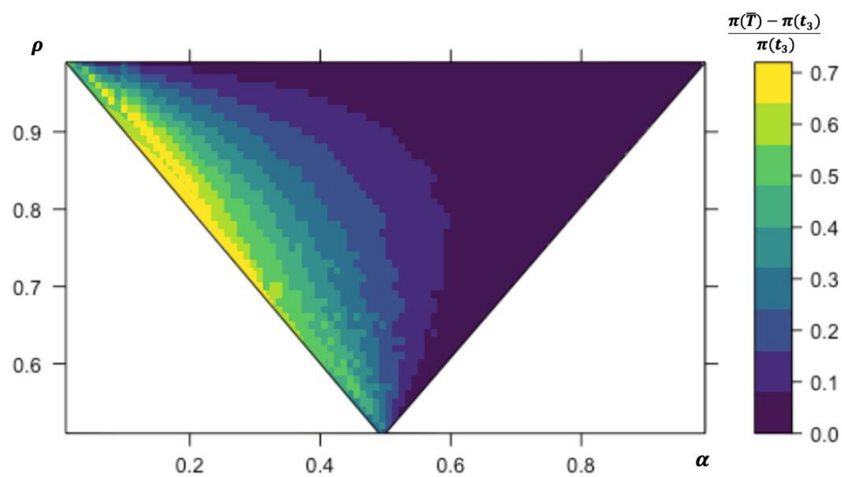


Figure 3: Percentage increase in the producer's expected measure of adopting consumers, denoted $\pi(T)$, when stopping the viral campaign at broadcast obsolescence ($T = \overline{T}$) versus viral obsolescence ($T = t_3$). For all combinations of (α, ρ), stopping at $T = \overline{T}$ is more profitable.

The rest of this section proves Theorem 2.

Go/no-go threshold for broadcast. Should the viral campaign continue until time $\overline{T}$, the producer must decide whether to run the broadcast right at that moment, so that still-unexposed consumers are willing to adopt after a good signal ("go"), or never run the broadcast at all, allowing the campaign to continue until viral obsolescence ("no-go").

Case #1: when $\overline{T} \geq t_2$, always "go". Suppose first that $\overline{T} \geq t_2$ so that the go/no-go threshold is in Phase III; this occurs if and only if $p_{BR}(t_2) \geq 1-\rho$. In this case, the producer unambiguously prefers to "go". To see why, consider what would happen if the viral campaign continued after time $\overline{T}$. The consumers who remain unexposed at time $\overline{T}$ will encounter the innovation later either (i) during the remainder of Phase III and be sensitive to signals or (ii) during Phase IV and herd on non-adoption. By comparison, should the producer run the broadcast at (or infinitesimally before) time $\overline{T}$, all of these consumers will be sensitive to signals—leading to strictly more adoption, whether the innovation is good or bad.

Case #2: when $\overline{T} < t_2$, "no go" if and only if inequality (11) holds. Suppose next that $\overline{T} < t_2$ so that the go/no-go threshold is in Phase II; this occurs if and only if $p_{BR}(t_2) < 1-\rho$. As before, running the broadcast at time $\overline{T}$ ensures that all still-unexposed consumers will be sensitive to signals, avoiding the downside that consumers exposed in Phase IV never adopt. However, there is also a benefit associated with continuing to run the viral campaign, that consumers exposed in the remainder of Phase II (at times $t \in (\overline{T}, t_2)$) will sometimes adopt after getting a negative private signal as well as after a positive signal.[14] Whether the producer prefers to continue the viral campaign past time $\overline{T}$ depends on the magnitudes of these countervailing effects.

The downside of continuing the viral campaign is that all consumers who get a positive signal and would have been exposed during Phase IV choose to adopt under the time-$\overline{T}$ broadcast but not under the continued viral campaign. These consumers have mass $\rho S_g(t_3)$ when the innovation is good and mass $(1-\rho)S_b(t_3)$ when it is bad. Overall, then, the "viral downside" equals $\alpha\rho S_g(t_3) + (1-\alpha)(1-\rho)S_b(t_3)$.

The upside of continuing the viral campaign is that some consumers who get a negative signal and would have been exposed during the remainder of Phase II choose to adopt under the continued viral campaign but not under the time-$\overline{T}$ broadcast. These consumers have mass $\int_{\overline{T}}^{t_2} a_B(t)(1-\rho)|S_g'(t)|\mathrm{d}t$ when the innovation is good and mass $\int_{\overline{T}}^{t_2} a_B(t)\rho|S_b'(t)|\mathrm{d}t$ when it is bad, where $a_B(t)$ is consumers' equilibrium likelihood of adopting after a bad signal during Phase II. Overall, then, the "viral upside" equals $\alpha \int_{\overline{T}}^{t_2} a_B(t)(1-\rho)|S_g'(t)|\mathrm{d}t + (1-\alpha)\int_{\overline{T}}^{t_2} a_B(t)\rho|S_b'(t)|\mathrm{d}t$, and the upside exceeds the downside if and only if inequality (11) holds.

Summarizing our progress thus far: (i) If $p_{BR}(t_2) \geq 1-\rho$, then $\overline{T} \geq t_2$ and the optimal stopping time is during Phase III prior to broadcast obsolescence. (ii) If $p_{BR}(t_2) < 1-\rho$, then $\overline{T} < t_2$ and it is optimal *either* to allow the viral campaign to continue until viral

[14] We can ignore the consumers exposed in Phase III, since they are sensitive to signals and hence adopt exactly as they would have under a time-$\overline{T}$ broadcast.

obsolescence, if (11) holds, *or* to stop at time $\overline{T}$ during Phase II, if (11) does not hold.

Stopping the viral campaign prior to time $\min\{t_2, \overline{T}\}$ **is suboptimal.** The producer is strictly better off extending the viral campaign until time $\min\{t_2, \overline{T}\}$. Suppose that some time $T' < \min\{t_2, \overline{T}\}$ has been reached and the producer is considering whether to run the broadcast at that moment or wait until (just before) time $\min\{t_2, \overline{T}\}$. Either way, all consumers who see the broadcast and get a positive private signal will adopt. However, under the longer viral campaign, consumers who encounter the innovation between $\max\{T', t_1\}$ and $\min\{t_2, \overline{T}\}$ (the portion of Phase II that is after T' and before $\overline{T}$) also sometimes adopt after a negative private signal, due to partial herding. Thus, lengthening the viral campaign until time $\min\{t_2, \overline{T}\}$ unambiguously increases overall adoption.

Putting these pieces together, we can now complete the proof.

First, consider the case when $p_{BR}(t_2) \geq 1 - \rho$ so that $\overline{T} \geq t_2$. We have shown that it is suboptimal to stop the viral campaign prior to time t_2 *and* suboptimal to allow it to continue beyond time $\overline{T}$. What about the remaining time interval from t_2 to $\overline{T}$? Consumers who encounter the innovation during the interval $[t_2, \overline{T}]$ are sensitive to signals, regardless of whether they encounter the innovation virally or through the broadcast. Thus, all broadcast times within this time interval generate exactly the same pattern of consumer adoption and hence must all be optimal for the producer. This completes the proof of Theorem 2(i).

Second, consider the case when $p_{BR}(t_2) < 1 - \rho$ so that $\overline{T} < t_2$. As argued earlier, any stopping time prior to $\overline{T}$ is suboptimal and waiting until viral obsolescence (any stopping time $T \geq t_3$) is better than $\overline{T}$ if and only if inequality (11) holds. Finally, because no one adopts after seeing a broadcast later than $\overline{T}$, stopping at any time between $\overline{T}$ and t_3 is strictly worse than stopping at $\overline{T}$. We conclude that stopping at viral obsolescence (time t_3) is optimal if inequality (11) holds and that stopping at broadcast obsolescence (time $\overline{T}$) is optimal if inequality (11) does not hold. This completes the proof of Theorem 2(ii). □

4 Investment in Innovation Quality

This section considers an extension in which the producer decides whether to invest in innovation quality. We find that equilibrium investment is higher (or lower) when the producer runs an optimal-length viral campaign compared to the non-epidemic benchmark in which the producer is constrained to reach consumers through a traditional ad campaign, so long as innovations would be mostly good (or mostly bad) in that non-epidemic benchmark.

Model: investment in innovation quality. During a development phase prior to launching the innovation, the producer makes an investment decision that impacts the innovation's likelihood of being good, what we refer to as its average quality. In particular, the innovation is good with probability $\alpha_P \in [0,1]$ if the producer incurs quality-investment cost $C(\alpha_P)$, where we assume that $C''(\alpha_P) > 0$ and $C'(\alpha_P) \geq 0$ for all $\alpha_P \in [0,1]$. Consumers do not observe producer investment but form a "market belief" $\alpha \in [0,1]$ about average quality. (For expositional clarity, we denote the producer's chosen average quality by α_P and the market belief by α. In equilibrium, it must be that $\alpha = \alpha_P$.)

We consider both a "traditional marketing game" and an "optimal-length viral campaign game." In the traditional marketing game which serves as our non-epidemic benchmark, all consumers are exposed at launch and the producer's only choice is how much to invest in innovation quality. In the optimal-length viral campaign game, by contrast, the producer also chooses the time $T \geq 0$ at which the viral campaign will end. As in Section 3, we assume in this case that the producer chooses T prior to launching the innovation and without knowing whether the innovation will ultimately be good or bad.

The producer's objective is to maximize its expected profit, with revenue equal to the mass of consumers who adopt. Let $R_g(T)$ and $R_b(T)$ denote the revenue earned by good and bad innovations, respectively, for any given viral-campaign stopping time $T \geq 0$. For any given market belief α held by consumers, the producer's problem is as follows.

1. Traditional marketing game: Given market belief α, the producer chooses α_P to maximize
$$\alpha_P R_g(0;\alpha) + (1-\alpha_P) R_b(0;\alpha) - C(\alpha_P).$$
2. Optimal-length viral campaign: (i) Given α and α_P, the producer chooses T to maximize
$$\alpha_P R_g(T;\alpha) + (1-\alpha_P) R_b(T;\alpha)$$
and (ii) given α and T, the producer chooses α_P to maximize
$$\alpha_P R_g(T;\alpha) + (1-\alpha_P) R_b(T;\alpha) - C(\alpha_P).$$

For ease of exposition, it is helpful to transform the producer's problem here into an equivalent alternative form. Rather than choosing α_P directly at cost $C(\alpha_P)$, suppose that the producer has a random cost $c \in [0,\infty)$ of producing a good innovation versus zero cost of producing a bad innovation. Moreover, suppose that c is drawn from a distribution with c.d.f. $F(\cdot)$ defined implicitly by the condition $F^{-1}(\alpha_P) = C'(\alpha_P)$. Under this interpretation, the producer minimizes the expected cost of achieving expected quality

α_P by choosing to invest whenever $c < C'(\alpha_P)$, and this leads to the same expected cost $C(\alpha_P) = \int_0^{C'(\alpha_P)} c\,\mathrm{d}F(c)$. Note that the random cost c has interval support $[C'(0), C'(1)]$, where $C'(0) \geq 0$. Define shorthand $\bar{c} \equiv C'(1)$.

In an equilibrium with expected innovation quality α^*, consumers have market belief α^* and the producer finds it optimal to invest so that innovations are good with probability α^*. Specifically:

– In a traditional marketing campaign, equilibrium requires that

$$R_g(0;\alpha^*) - R_b(0;\alpha^*) = C'(\alpha^*) = F^{-1}(\alpha^*).$$

– In an optimal-length viral campaign, a set of necessary equilibrium conditions is

$$\alpha^* \in \arg\max_{\alpha_P} \left(\alpha_P R_g(T(\alpha^*);\alpha^*) + (1-\alpha_P) R_b(T(\alpha^*);\alpha^*) - C(\alpha_P)\right)$$
$$T(\alpha^*) \in \arg\max_{T} \left(\alpha^* R_g(T;\alpha^*) + (1-\alpha^*) R_b(T;\alpha^*)\right)$$

Given the characterization of $T(\alpha^*)$ in Theorem 2, the above conditions can be simplified to

$$R_g(T(\alpha^*);\alpha^*) - R_b(T(\alpha^*);\alpha^*) = C'(\alpha^*) = F^{-1}(\alpha^*).$$

In this context, suppose that consumers have ex ante belief $\alpha \in (1-\rho, \rho)$, so that those who encounter the innovation at launch are sensitive to signals. (For the moment, we view α as a fixed parameter and "α notation" is suppressed to shorten equations; later, we will endogenize α.) We characterized $R_g(T)$ and $R_b(T)$ in Section 3. If $T > \overline{T}$, then those exposed at the time-T broadcast do not adopt and $R_g(T) = I_g(T)$ and $R_b(T) = I_b(T)$. If $T \leq \overline{T}$,[15] then broadcast-exposed consumers are sensitive to signals and

$$R_g(T) = I_g(T) + \rho S_g(T) \text{ and } R_b(T) = I_b(T) + (1-\rho) S_b(T). \tag{12}$$

Let $\Delta R(T) = R_g(T) - R_b(T)$ denote the extra revenue earned by good innovations; we refer to $\Delta R(T)$ as "the incentive to invest."

[15]When $T = \overline{T}$, broadcast-exposed consumers are indifferent whether to adopt and may mix after a good private signal. We ignore such mixing here for ease of exposition and because it can never arise in any equilibrium of the viral-marketing game (analyzed later); if an equilibrium existed in which $T = \overline{T}$ and some fraction of broadcast-exposed consumers did not adopt after a good private signal, the producer would prefer to deviate and stop the campaign at time $\overline{T} - \epsilon$ for $\epsilon \approx 0$.

Incentive to invest prior to a traditional ad campaign. Suppose that the producer runs a traditional ad campaign. Given any $\alpha \in (1-\rho, \rho)$, consumers are sensitive to signals; so, $R_g(0) = \rho$ and $R_b(0) = 1-\rho$. The producer finds it optimal to invest in quality whenever $c < \Delta R(0) = 2\rho - 1$.

Incentive to invest prior to a viral campaign. Suppose next that the producer runs a viral campaign that stops at time $T > 0$. Anticipating how the viral social learning process will unfold, the producer finds it optimal to invest whenever $c < \Delta R(T)$. For all $T \leq \overline{T}$, equations (1-2,12) imply

$$R_g'(T) = I_g'(T) + \rho S_g'(T) = S_g(T)I_g(T)\left(\rho a_G(T) + (1-\rho)a_B(T) - \rho\right)$$
$$R_b'(T) = I_b'(T) + (1-\rho)S_b'(T) = S_b(T)I_b(T)\left((1-\rho)a_G(T) + \rho a_B(T) - (1-\rho)\right)$$

where $a_G(T)$ and $a_B(T)$ are the likelihoods that virally-exposed consumers would adopt at time T after a good or bad signal, respectively. Since virally-exposed consumers with good signals adopt until viral obsolescence at time t_3 and since $\overline{T} < t_3$ (Lemma 2), it must be that $a_G(T) = 1$ for all $T \leq \overline{T}$. We conclude that

$$\Delta R'(T) = a_B(T)\left((1-\rho)S_g(T)I_g(T) - \rho S_b(T)I_b(T)\right) \tag{13}$$

for all $T \leq \overline{T}$. In particular, over this range, (i) $\Delta R'(T) > 0$ if and only if $a_B(T) > 0$ and $\frac{S_g(T)I_g(T)}{S_b(T)I_b(T)} > \frac{\rho}{1-\rho}$ and (ii) $\Delta R'(T) < 0$ if and only if $a_B(T) > 0$ and $\frac{S_g(T)I_g(T)}{S_b(T)I_b(T)} < \frac{\rho}{1-\rho}$.

By Bayes' Rule, consumers' interim belief $p(t)$ at any time $t > 0$ during the viral campaign satisfies $\frac{p(t)}{1-p(t)} = \frac{\alpha}{1-\alpha} \times \frac{S_g(t)I_g(t)}{S_b(t)I_b(t)}$; in particular, $\frac{p(0+)}{1-p(0+)} = \frac{\alpha}{1-\alpha} \times \frac{\rho}{1-\rho}$. Thus, the condition $\frac{S_g(T)I_g(T)}{S_b(T)I_b(T)} \gtrless \frac{\rho}{1-\rho}$ is equivalent to $p(T) \gtrless p(0+)$ or, in words, "whether virally-exposed consumers' interim belief at stopping time T is higher or lower than it would be after observing a good private signal and nothing else." Proposition 6 summarizes our findings on the producer's incentive to invest, in light of our previous results on adoption dynamics during the viral campaign (Theorem 1) and optimal stopping time T^* (Theorem 2).

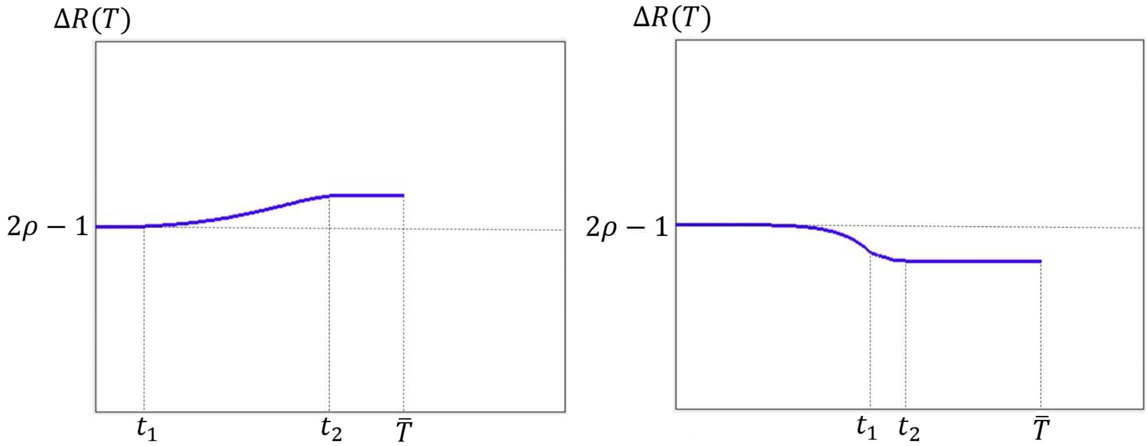
Proposition 6. *Suppose that consumers believe that fraction $\alpha \in (1-\rho, \rho)$ of innovations are good and behave as characterized in Theorem 1. (i) If $\alpha \in (1/2, \rho)$, then $\Delta R(T^*) < \Delta R(0)$. (ii) If $\alpha \in (1-\rho, 1/2)$, then $\Delta R(T^*) > \Delta R(0)$. (iii) If $\alpha = 1/2$, then $\Delta R(T^*) = \Delta R(0)$.*

Proof. The proof is in the Appendix. □

Intuition for Proposition 6: Consumers' ex ante belief, α, plays a key role in determining qualitative features of the producer's incentive to invest, $\Delta R(T^*)$, prior to an optimal-length viral campaign. When consumers believe that most innovations are good ($\alpha \in (\frac{1}{2}, \rho)$), their behavior when exposed during a viral campaign transitions from an initial phase with herding on adoption to a phase with partial herding, and then a phase in which consumers are sensitive to signals (Theorem 1). By contrast, in a traditional ad campaign, all consumers are sensitive to signals. Relative to that sensitive-to-signals benchmark, herding and partial herding benefit a bad innovation more than a good one, since more consumers with bad signals who would otherwise not adopt now do adopt. Moreover, the ratio between the measures of adopters of a good innovation and a bad one shrinks during the herding phase, making the partial-herding phase even more favorable to a bad innovation. The producer thus has *less* incentive to invest than in a traditional ad campaign.

When consumers believe that most innovations are bad ($\alpha \in (1-\rho, \frac{1}{2})$), they remain sensitive to signals throughout the viral campaign except during Phase II, when they partially herd on adoption. From the producer's perspective, whether any given consumer is sensitive to signals in a viral campaign or in an ad campaign makes no difference. However, the fact that consumers are sensitive to signals during the first phase of the viral campaign causes the measure of adopters to grow faster when the innovation is good. And with more consumers "infected" by good innovations at the end of the first phase, more are then exposed to a good innovation during the second phase when consumers partially herd on adoption. This is favorable for good innovations, so much so that the producer has *more* incentive to invest than in a traditional ad campaign.

Figure 4 illustrates how the producer's incentive to invest varies with the length of the viral campaign, depending on whether $\alpha \in (1/2, \rho)$ or $\alpha \in (1-\rho, 1/2)$.



(a) $\alpha \in (\frac{1}{2}, \rho)$

(b) $\alpha \in (1-\rho, \frac{1}{2})$

Figure 4: Incentive to invest $\Delta R(T)$, depending on the viral campaign's stopping time T, up until broadcast obsolescence at time $\overline{T}$. (At time $\overline{T}$, there is a discontinuity in $\Delta R(T)$, not shown to keep the figure as simple as possible.) Panel (a) illustrates the case with parameters $\alpha = 0.55$ and $\rho = 0.65$, and panel (b) the case when $\alpha = 0.45$ and $\rho = 0.65$.

Equilibrium investment in the traditional marketing game. We now begin investigating the producer's equilibrium investment in innovation quality, first in the traditional marketing game. Recall that the producer observes its private cost c and decides whether to produce a good innovation ("invest"). Then, the innovation is advertised to all consumers at time $t = 0$, who simultaneously decide whether to adopt.

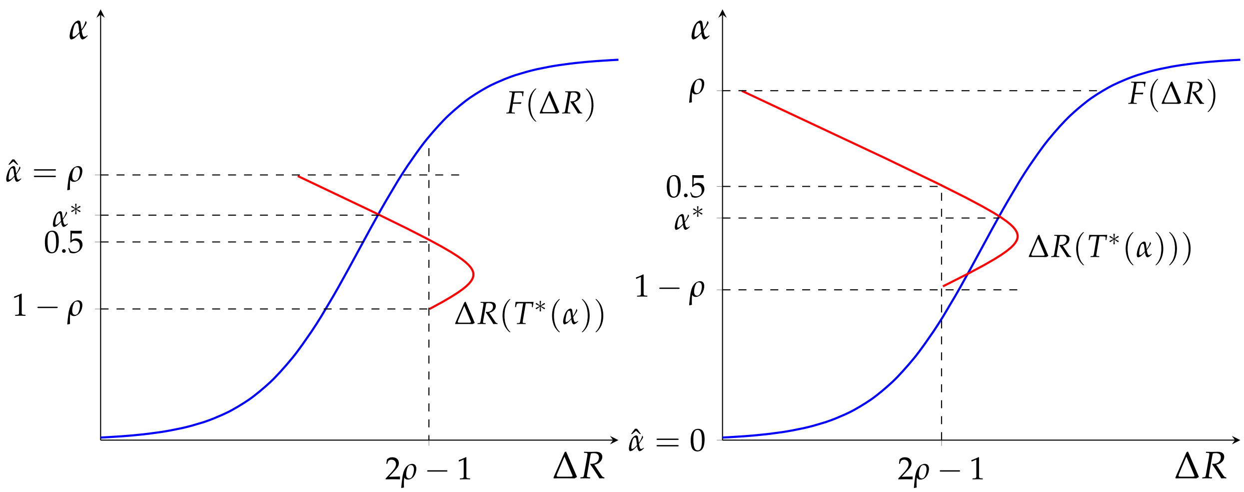
An equilibrium always exists in which all innovations are bad,[16] but equilibria with positive investment may also exist. Proposition 7 characterizes the maximal amount of investment in any equilibrium, for any given c.d.f. $F(\cdot)$ of producer investment cost.

Proposition 7. *Let $\hat{\alpha}$ denote the maximal equilibrium likelihood of a good innovation in the traditional marketing game. (i) If $F(2\rho - 1) < 1 - \rho$, then $\hat{\alpha} = 0$. (ii) If $F(2\rho - 1) \in [1 - \rho, \rho]$, then $\hat{\alpha} = F(2\rho - 1)$. (iii) If $F(2\rho - 1) > \rho$, then $\hat{\alpha} = \rho$.*

Proof. The proof is in the Appendix. □

To gain intuition, consider the case when $F(2\rho - 1) \in [1 - \rho, \rho]$. In the equilibrium with ex ante belief $\hat{\alpha} = F(2\rho - 1)$, all consumers adopt after a good signal but not after a bad one, causing mass ρ of consumers to adopt good innovations while mass $1 - \rho$ adopt bad ones. Anticipating this, the producer invests whenever its cost $c < 2\rho - 1$, which occurs with probability $\hat{\alpha}$.

[16]Consumers never adopt if they believe that all innovations are bad, and producers never invest (and hence all innovations are bad) if they believe that consumers will never adopt.



(a) $F(2\rho - 1) > \rho$ (b) $F(2\rho - 1) < 1 - \rho$

Figure 5: The maximal equilibrium likelihood that innovations are good when the producer uses an optimal-length viral campaign, α^*, compared to a traditional ad campaign, $\hat{\alpha}$. The blue curve shows how the fraction α of good innovations varies with the producer's incentive to invest ΔR, which is equal to the probability of the producer having a investment cost $c \leq \Delta R$, i.e. $F(\Delta R)$. The red curve shows how the producer's incentive to invest varies with α, assuming that the producer uses an optimal-length viral campaign. Panel (a) shows a scenario in which $F(2\rho - 1) > \rho$, so that $\hat{\alpha} = \rho$, and $\alpha^* \in (1/2, \rho)$. Panel (b) shows a scenario in which $F(2\rho - 1) < 1 - \rho$, so that $\hat{\alpha} = 0$, and $\alpha^* \in (1 - \rho, 1/2)$.

Equilibrium investment in the viral marketing game. Consider next the viral marketing game. Recall that the producer first commits to its marketing strategy, choosing the stopping time T for the viral campaign. The producer then observes its cost c and privately decides whether to produce a good innovation. Finally, the viral campaign unfolds as in Section 3, with consumers adopting optimally and innovation awareness spreading virally until time T, when all still-susceptible consumers are exposed non-socially.

Any equilibrium of this game must satisfy three conditions. First, the producer must invest optimally, whenever the cost c is less than the extra revenue earned by good innovations. Second, consumers' ex ante belief α must be correct, equal to the true likelihood of good innovations. Finally, the stopping time T must be optimal for the producer given α, as characterized in Theorem 2.

Optimal-length viral marketing may increase or decrease equilibrium investment, depending on how much equilibrium investment can be supported in the traditional marketing game. In particular, viral marketing leads to less investment if $\hat{\alpha} > 1/2$, more investment if $\hat{\alpha} < 1/2$, and equal investment if $\hat{\alpha} = 1/2$.

Proposition 8. *Let α^* denote the maximal equilibrium likelihood of a good innovation in the viral* *marketing game. (i) If $\hat{\alpha} > 1/2$, then $\alpha^* \in (1/2, \hat{\alpha}]$ with $\alpha^* < \hat{\alpha}$ if $\hat{\alpha} \in (1/2, \rho)$. (ii) If $\hat{\alpha} < 1/2$, then $\alpha^* \in [\hat{\alpha}, 1/2)$ with $\alpha^* > \hat{\alpha}$ if $\hat{\alpha} \in (1 - \rho, 1/2)$. (iii) If $\hat{\alpha} = 1/2$, then $\alpha^* = 1/2$.*

Proof. The proof is in the Appendix. □

Proposition 8, illustrated in Figure 5, is the equilibrium analogue of Proposition 6. In that result, we showed that optimal consumer behavior gives a producer who is running an optimal-length campaign *more* incentive to invest than under a traditional ad campaign if most innovations are bad, i.e., $\alpha < 1/2$, but *less* incentive to invest if most innovations are good, i.e., $\alpha > 1/2$. Similarly, Proposition 8 establishes that the equilibrium likelihood of good innovations is always *closer to 1/2* when the producer uses an optimal-length viral campaign than under a traditional ad campaign. Thus, the practice of optimal-length viral marketing has a moderating impact on equilibrium innovation quality.

5 Concluding remarks

This paper introduces and analyzes an economic-epidemiological model of innovation diffusion and adoption, in which awareness of an innovation (e.g., new product or practice, scientific finding, etc.) spreads by word of mouth from those who have already adopted it. In this context, consumers learn about the quality of an innovation based on when they first encounter it. We characterize the equilibrium trajectory of the resulting "adoption epidemic" and determine the lifecycle of virally-spread innovations. Moreover, we show that using a viral campaign to launch an innovation increases total adoption relative to a traditional advertisement campaign, but that using an optimal-length viral campaign may increase or decrease producers' incentive to invest in innovation quality.

The model captures a rich interplay between the epidemiological dynamics of diffusion and the economic dynamics of consumer beliefs. As such, the paper serves to bridge the economic literature on social learning and the epidemiological literature on social transmission, combining ideas and methods from both fields.

The paper's economic contribution builds on the pioneering work of Banerjee (1993), the first to bring infectious-disease insights to social learning. The main difference is that agents in our model all receive a private signal about innovation quality whereas, in Banerjee (1993)'s rumor model, only those who encounter the innovation at launch are privately informed. This distinction generates several novel features that only arise when socially-exposed consumers receive informative private signals. Most notably, we show that it is optimal to run a viral campaign of limited length (ending before viral obsolescence) and that enabling producers to conduct an optimal viral campaign can reduce equilibrium investment in innovation quality.

From an epidemiological point of view, the paper expands the scope of the classic Susceptible-Infected-Recovered (SIR) model to an economic setting in which the parameters of the diffusion model depend on equilibrium economic incentives. In particular, we endogenize *infectivity*, the likelihood that a newly-exposed host will become infected, and show how infectivity changes throughout an economic epidemic. In future work, our methodology could be extended in several interesting directions to endogenize other key parameters, most notably, the transmission rate and the informativeness of agents' private signals. Such work could also explore the consumer-welfare implications of viral social learning.[17]

Several other natural directions for future research could build on our analysis, a few of which we highlight here.

Pricing. Suppose that a new product generates value v_g or v_b when of good or bad quality, respectively, that the product is launched virally, and that "adoption" corresponds to buying the product at price $p \in (v_b, v_g)$. This fits our model with gain $u_g = v_g - p$ when buying a good product and loss $u_b = p - v_b$ from a bad product. For any given price p,[18] our analysis characterizes the proportion of consumers who will buy good products, $D_g(p) \equiv \lim_{t\to\infty} I_g(t;p)$, and bad products, $D_b(p) \equiv \lim_{t\to\infty} I_b(t;p)$. Computing the fixed price p^* that maximizes ex ante expected revenue, $R(p) = p\left(\alpha D_g(p) + (1-\alpha)D_b(p)\right)$, is then a straightforward numerical exercise. But other important pricing issues remain for future research, including (i) how prices change over time throughout a product's adoption lifecycle, if the producer is able to set prices dynamically, and (ii) how much consumers are able to learn from prices, if the producer has information about quality when setting prices.

Temporary infectiousness. Suppose that consumers who adopt only remain infectious for a limited period of time. For example, people who choose to play a new game may eventually get bored and stop introducing it to others. In this context, how widely an innovation spreads through the consumer population depends on innovation quality.

The option to wait. An important simplifying feature of this paper's model is that con-

[17]Being exposed virally to an innovation provides a valuable "social signal" about its quality. Holding innovation quality fixed, consumers are therefore clearly better off (ex ante) when an innovation is marketed virally. As we have shown, however, viral marketing can in some cases lead to less investment in quality.

[18]In the trivial cases when price $p \geq u_g$ (or $p \leq u_b$), the unique equilibrium has no one (or everyone) buying the product.

sumers decide whether or not to adopt the innovation when they first encounter it. This assumption could be appropriate in some contexts but, in many situations, consumers have the option to wait and learn more. For instance, in our political-campaign example, a person might wait until she has seen multiple lawn signs for a candidate before posting a sign herself. Similarly, a consumer with a bad personal impression of a new product might still wind up buying it at a later date if, over time, she encounters enough others who have also done so.

Adding the option to wait complicates the analysis, but our methodology can be generalized to this more complex setting. A consumer i who has not yet adopted by time t is in a susceptible state with history h_{it}, where h_{it} is a vector encoding the previous times at which consumer i encountered someone else who was infected. For each susceptible history h_{it}, the likelihood ratio of that history occurring when the innovation is good versus bad determines consumer i's updated belief about innovation quality, and the dynamics of belief evolution determine the option value of waiting. At each point of time, then, there will be a subset of susceptible histories from which consumers will choose to adopt. Overall, as in this paper, the dynamics of adoption determine the dynamics of consumer beliefs, which in turn determine the dynamics of adoption. However, there are important differences and complications. For instance, once consumers have the option to wait, there might in some cases be multiple equilibria with different time-paths of adoption.

Reversible adoption decisions. Another important simplifying assumption is that consumers' adoption decisions are irreversible. This assumption also can be relaxed within our basic analytical framework. Suppose that, rather than "buying" the innovation irreversibly, each consumer decides at each instant whether or not to "rent" it.[19] Each consumer will choose to rent whenever their epidemiological state is such that their belief about innovation quality exceeds the threshold for adoption. Because consumers' decisions at each point in time depend only on their current beliefs, equilibria in this context can be derived relatively simply, much as in this paper, with current incentives only depending on patterns of past behavior. An interesting open question in this context is whether consumers will over time successfully aggregate their initially-dispersed information. That is, as time $t \to \infty$, will the fraction of consumers renting a good innovation converge to 100% while the fraction renting a bad innovation converges to 0%?

[19] This alternative model corresponds to the Susceptible-Infected-Susceptible (SIS) model, used in infectious-disease epidemiology to model infection dynamics when recovery from infection does *not* provide immunity from re-infection. Here, the decision to stop renting corresponds to "recovery" while the decision to (re-)start renting corresponds to (re-)infection.

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A The Rumor Model

A key difference between our model of virally-spread innovations and Banerjee (1993)'s model of rumors is that, in our model, all consumers receive informative private signals. To highlight the significance of this difference, we analyze in this Appendix a variation of our model in which only those exposed at launch are privately informed. In particular, those exposed at time $t = 0$ get conditionally i.i.d. signals exactly as in the main text, while those exposed after time $t = 0$ do not receive a private signal.

We identify three main qualitative differences between equilibrium outcomes in this "rumor model" versus the "innovation model" analyzed in the main text:

- Appendix A.1: In the rumor model, more delay in hearing about a rumor is always bad news about its quality (Theorem 3). By contrast, in the innovation model, consumer beliefs can be non-monotone in the time that it takes to hear about the innovation (Phase I(ii) of Theorem 1 and Lemma 1(ii)).

- Appendix A.2: In the rumor model, *either* a traditional ad campaign *or* a purely-viral campaign is always optimal (Proposition 10). By contrast, in the innovation model, *neither* a traditional ad campaign *nor* a purely-viral campaign is ever optimal (Theorem 2).

- Appendix A.3: In the rumor model, optimal viral marketing never reduces equilibrium rumor quality (Proposition 11). By contrast, in the innovation model, allowing the producer to run an optimal-length viral campaign reduces equilibrium innovation quality whenever producer costs are sufficiently low (Proposition 8).

To avoid confusion, we use the term "rumor" to refer to innovations in the rumor model. As in the main text, we focus on the most interesting case when α, the rumor's ex ante likelihood of being good, lies between $1-\rho$ and ρ.[20]

A.1 Equilibrium Lifecycle

We begin by characterizing equilibrium economic-epidemiological dynamics in the rumor model. Our main finding is that consumer behavior transitions over time through two distinct phases, (i) a "Herding Phase" in which all socially-exposed consumers choose

[20]Consumer behavior is trivial when $\alpha < 1-\rho$ (no one adopts) or when $\alpha > \rho$ (everyone adopts). When $\alpha = 1-\rho$ and $\alpha = \rho$, launch-exposed consumers may randomize whether to adopt after a good or bad private signal, respectively, but no additional insight emerges from this extra complication.

to adopt followed by (ii) an "Obsolescence Phase" in which no one adopts.Moreover, unlike in the innovation model, consumers' interim beliefs are always strictly decreasing in the time it takes to encounter the rumor.

Theorem 3. *Suppose that $\alpha \in (1-\rho, \rho)$ in the rumor model. Equilibrium epidemiological dynamics $(S_\omega(t), I_\omega(t) : t \geq 0; \omega \in \{g, b\})$ are uniquely determined. In equilibrium, consumers' post-launch behavior transitions through two phases.*

> Herding Phase: *Consumers herd on adoption and interim belief $p(t) > \rho$ decreases until time $t_0 > 0$ is reached at which $p(t_0) = 1/2$.*
>
> Obsolescence Phase: *Consumers herd on non-adoption and interim belief $p(t) < 1/2$ continues to decline with $\lim_{t\to\infty} p(t) = 0$.*

Proof. Because $\alpha \in (1-\rho, \rho)$, consumers exposed at launch are sensitive to signals; so, $I_g(0) = \rho L$, $I_b(0) = (1-\rho)L$, and $S_g(0) = S_b(0) = 1 - L$, where L is the mass of launch-exposed consumers. By equation (6), $\frac{p(0+)}{1-p(0+)} = \frac{\alpha}{1-\alpha}\frac{\rho}{1-\rho}$; so, $p(0+) > 1/2$ and all consumers exposed immediately after launch herd on adoption. The following Lemma, which is also useful in subsequent proofs, establishes that good rumors must be more widely seen and more widely adopted so long as socially-exposed consumers continue to herd on adoption.

Lemma 3. *Suppose that consumers are sensitive to signals at launch and herd on adoption after launch until time t. Then $S_g(t) < S_b(t)$ and $I_g(t) > I_b(t) + L(2\rho - 1)$.*

Proof. Because consumers are sensitive to signals at launch, $S_g(0) = S_b(0) = 1 - L$, $I_g(0) = \rho L$, and $I_b(0) = (1-\rho)L$. Let $\Delta I(0) \equiv I_g(0) - I_b(0) = L(2\rho - 1)$. We show next that $I_g(t) - I_b(t) > \Delta I(0)$ for all $t \in (0, t_0)$. Observe first that $I_g'(0) = S_g(0)I_g(0) > S_b(0)I_b(0) = I_b'(0)$; thus, $I_g(t) - I_b(t) > \Delta I(0)$ for all $t \approx 0$. Now suppose for the sake of contradiction that there exists $\hat{t} \in (0, t)$ such that $I_g(\hat{t}) - I_b(\hat{t}) = \Delta I(0)$, and without loss let $\hat{t}$ be the smallest such time. This requires $I_g'(\hat{t}) \leq I_b'(\hat{t})$. Since $I_\omega'(\hat{t}) = S_\omega(\hat{t})I_\omega(\hat{t})$ for $\omega \in \{g, b\}$ and $I_g(\hat{t}) - I_b(\hat{t}) = \Delta I(0) > 0$, this in turn requires $S_g(\hat{t}) < S_b(\hat{t})$. But because all socially-exposed consumers adopt up until time t, $I_\omega(t) - I_\omega(0) = S_\omega(0) - S_\omega(t)$ for all $t \in (0, t_0)$. Since $I_g(\hat{t}) - I_b(\hat{t}) = I_g(0) - I_b(0) = \Delta I(0)$, this implies that $S_g(\hat{t}) - S_b(\hat{t}) = S_g(0) - S_b(0) = 0$, contradicting the requirement that $S_g(\hat{t}) < S_b(\hat{t})$. □

Returning to the proof of Theorem 3, we have shown that consumers are sensitive to signals at launch and that $p(0+) > 1/2$. Let t_0 be the first time at which $p(t_0) = 1/2$, or $t_0 = \infty$ if $p(t) > 1/2$ for all t. Because socially-exposed consumers herd on adoption in the rumor model whenever consumers' interim belief $p(t) > 1/2$, Lemma 1 implies that

$p(t)$ is strictly decreasing from time 0 to time t_0. By equation (8), $\frac{p(t)}{1-p(t)}$ decreases at rate $-X(t) = I_g(t) - I_b(t) + S_b(t) - S_g(t)$ so long as consumers herd on adoption. By Lemma 3, $-X(t) > \Delta I(0) > 0$. Thus, $p(t)$ falls monotonically throughout the Herding Phase and eventually reaches $1/2$ at a finite time $t_0 > 0$, as desired.

Consumers exposed at time t_0 are indifferent whether to adopt. Let $q(t_0) \in [0,1]$ be the likelihood that such consumers adopt. (Because these consumers do not receive informative private signals, their likelihood of adopting conditional on exposure at time t_0 must be the same for good and bad rumors.) By equation (8) and Lemma 3, $-X(t_0) = (I_g(t_0) - I_b(t_0)) + q(t_0)(S_b(t_0) - S_g(t_0)) > L(2\rho - 1) > 0$. Thus, consumers' interim belief continues to decline and consumers exposed immediately after time t_0 must herd on non-adoption.

Because consumers stop adopting after time t_0, $I_g(t) = I_g(t_0)$ and $I_b(t) = I_b(t_0)$ at times $t > t_0$. Again by equation (8) and Lemma 3, $-X(t) > L(2\rho - 1) > 0$, implying that $p(t)$ to continue to fall. Consumers therefore find it optimal to continue to herd on non-adoption, causing $\frac{p(t)}{1-p(t)}$ to continue to fall at a constant rate; so, $\lim_{t\to\infty} p(t) = 0$, as desired. □

Let $I_\omega(\infty) \equiv \lim_{t\to\infty} I_\omega(t) = I_\omega(t_0)$ be the overall measure of consumers who adopt in state $\omega \in \{g, b\}$, and let $\Delta I(\infty) = I_g(\infty) - I_b(\infty)$ be the extra overall adoption of good rumors. Proposition 9 establishes several basic facts about how $I_g(\infty)$, $I_b(\infty)$, and $\Delta I(\infty)$ vary with the likelihood that rumors are good, α, and the mass of privately-informed consumers exposed at launch, L. These facts will be useful later in Appendix A.2-A.3.

Proposition 9. *(i) $I_g(\infty)$ and $I_b(\infty)$ are strictly increasing in α for all $\alpha \in (1-\rho, \rho)$. (ii) $\Delta I(\infty)$ is strictly increasing in α for all $\alpha \in (1-\rho, 1/2)$ and strictly decreasing in α for all $\alpha \in (1/2, \rho)$.*

Proof of (i). Let $I_\omega(t;\alpha)$ and $S_\omega(t;\alpha)$ be the mass of infected and susceptible consumers in state $\omega \in \{g, b\}$ in the unique equilibrium characterized in Theorem 3, viewed as a function of $\alpha \in (1-\rho, \rho)$. Launch-exposed consumers are sensitive to signals and socially-exposed consumers herd on adoption until time $t_0(\alpha)$ at which consumers' interim belief equals $1/2$. Increasing $\Pr(\omega = g)$ from α to $\alpha_P \in (\alpha, \rho)$ therefore has no effect on the equilibrium trajectory prior to time $t_0(\alpha)$ but extends the window of time during which socially-exposed consumers adopt to $t_0(\alpha_P) > t_0(\alpha)$. It follows immediately that overall adoption $I_\omega(\infty;\alpha) = I_\omega(t_0(\alpha);\alpha)$ is strictly increasing in α for both good rumors ($\omega = g$) and bad rumors ($\omega = b$).

Proof of (ii). For any $\alpha \in (1-\rho, \rho)$, the time $t_0(\alpha)$ at which adoption ends is characterized

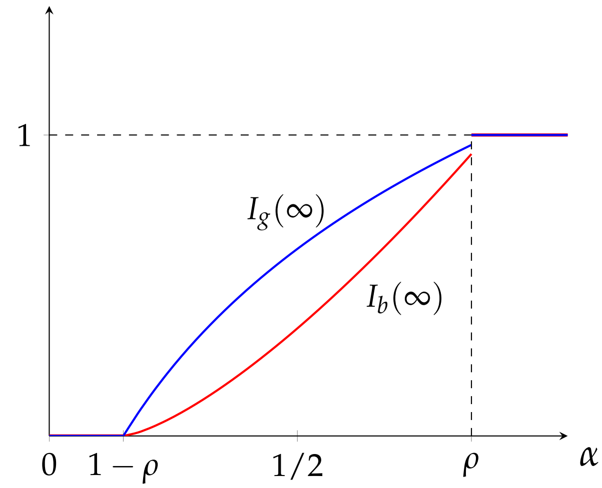


Figure 6: Measure of adopting consumers given quality, as a function of prior α. The figure shows the case when $\rho = 0.65$ and $L = 0.0001$.

by the condition

$$\frac{1-\alpha}{\alpha} = \frac{S_g(t_0(\alpha);\alpha)I_g(t_0(\alpha);\alpha)}{S_b(t_0(\alpha);\alpha)I_b(t_0(\alpha);\alpha)} = \frac{I_g'(t_0(\alpha);\alpha)}{I_b'(t_0(\alpha);\alpha)}.$$

If $\alpha \in (1-\rho, 1/2)$, then $\frac{1-\alpha}{\alpha} > 1$ and hence $I_g'(t_0(\alpha);\alpha) > I_b'(t_0(\alpha);\alpha)$; so, increasing α over this range increases good-rumor adoption more than bad-rumor adoption. By contrast, if $\alpha \in (1/2, \rho)$, then $\frac{1-\alpha}{\alpha} < 1$ and hence $I_g'(t_0(\alpha);\alpha) < I_b'(t_0(\alpha);\alpha)$; so, increasing α over this range increases bad-rumor adoption more than good-rumor adoption, as desired.

□

Figure 6 illustrates how overall adoption of good and bad rumors varies with α. When $\alpha \in [0, 1-\rho)$, no one adopts and hence $I_g(\infty) = I_b(\infty) = 0$. As α increases from $1-\rho$ to $1/2$, adoption of good and bad rumors both increase, but at a faster rate for good rumors. As α increases from $1/2$ to ρ, good and bad rumor adoption continues to increase, but now at a faster rate for bad rumors. Note that the extra overall adoption of good rumors, $\Delta I(\infty)$, equals zero at $\alpha = 1-\rho$, is maximized at $\alpha = 1/2$, remains positive to the left of $\alpha = \rho$, and equals zero to the right of $\alpha = \rho$.

A.2 Optimal Length of Viral Campaign

Here we show that, for a rumor, it is always optimal *either* to stop the viral campaign immediately ($T = 0$) *or* to allow it to continue forever ($T = \infty$).

Proposition 10. *Suppose that $\alpha \in (1-\rho, \rho)$ in the rumor model. The producer finds it optimal to conduct either a traditional ad campaign ($T = 0$) or a purely-viral campaign ($T = \infty$).*

Proof. Let T_0 denote the time at which $p_{BR}(T_0) = 1/2$, or $T_0 \equiv -\infty$ if $p_{BR}(T) < 1/2$ for all $T \geq 0$. Uninformed consumers who see the broadcast all adopt if the broadcast occurs at time $T < T_0$, but none adopt if $T > T_0$. Similarly, define t_0 by the condition $p(t_0) = 1/2$. Uninformed consumers who encounter the rumor socially at time t adopt if $t < t_0$ but not if $t > t_0$. Because $p(t) > p_{BR}(t)$ for all $t \geq 0$ (Lemma 2(ii)), we have $t_0 > T_0$.

Suppose first that $\alpha \in (1/2, \rho)$. Since $\alpha > 1/2$, we have $T_0 > 0$. In any viral campaign that stops at time $T < T_0$, all uninformed consumers adopt, whether exposed socially prior to T or non-socially at the time-T broadcast. By contrast, in a viral campaign that stops at time $T > T_0$, all those exposed at the time-T broadcast (and, if $T > t_0$, any exposed socially from t_0 until T) will choose not to adopt. We conclude that any viral campaign of length $T < T_0$ is optimal, while any campaign of length $T > T_0$ is suboptimal. In particular, a traditional ad campaign ($T = 0$) is optimal.

Suppose next that $\alpha \in (1-\rho, 1/2)$. Because $p_{BR}(T) \leq \alpha < 1/2$ for all $T \geq 0$, uninformed consumers exposed via the broadcast never adopt. On the other hand, because $\alpha > 1 - \rho$, we have $p(0+) > 1/2$ and hence $t_0 > 0$. Consumers exposed prior to time t_0 will therefore adopt in a viral campaign. We conclude that any viral campaign of length $T > t_0$ is optimal, while any campaign of length $T < t_0$ is suboptimal. In particular, a purely-viral campaign ($T = \infty$) is optimal.

Suppose finally that $\alpha = 1/2$. In a purely-viral campaign, adoption is $I_g(t_0)$ and $I_b(t_0)$ for good and bad rumors, respectively. In a traditional ad campaign, uninformed consumers are indifferent and adopt with some probability $\psi \in [0, 1]$, leading to adoption $L\rho + (1-L)\psi$ when the rumor is good and $L(1-\rho) + (1-L)\psi$ when it is bad. $T = 0$ is optimal if $\psi \geq \psi^* \equiv \frac{I_g(t_0)+I_b(t_0)-(2\rho-1)L}{1-L}$ and $T = \infty$ is optimal if $\psi \leq \psi^*$. We conclude as desired that *either* a traditional ad campaign *or* a purely-viral campaign is optimal. □

Consider the special case when $\alpha \in (1-\rho, 1/2)$. In a traditional ad campaign, those with a favorable private signal choose to adopt, while all those who are uninformed or receive an unfavorable signal choose not to adopt. The overall mass of consumers who adopt is ρL, where L is the mass of informed consumers. In a purely-viral campaign, on the other hand, informed consumers who get favorable signals *and* uninformed consumers who are exposed during the Herding Phase choose to adopt. We conclude that, for rumors, a purely-viral campaign dominates a traditional ad campaign regardless of the mass $L > 0$ of informed consumers. By Proposition 10, this in turn implies that a purely-viral campaign is optimal.

Corollary to Proposition 10. *Consider the rumor model. If $\alpha \in (1-\rho, 1/2)$, then a purely-viral campaign is optimal for the rumor producer.*

A.3 Investment in Quality

Here we show that, for a rumor, optimal viral marketing never leads to less equilibrium investment than in the non-epidemic benchmark in which the producer is constrained to use a traditional ad campaign.

Proposition 11. *Consider the rumor model. Let $\hat{\alpha}$ and α^* denote the maximal equilibrium likelihood of a good innovation in the traditional marketing game and the viral marketing game, respectively. $\alpha^* \geq \hat{\alpha}$, with $\alpha^* > \hat{\alpha}$ whenever $\hat{\alpha} \in (1-\rho, 1/2)$.*

Proof. $\hat{\alpha}$, the maximal equilibrium likelihood of good rumors in the traditional marketing game, can be characterized as in the proof of Proposition 9, the only difference being that now only mass L of consumers are privately informed. In particular: $\hat{\alpha} = \rho$ if $F(L(2\rho - 1)) > \rho$; $\hat{\alpha} = F(L(2\rho-1))$ if $F(L(2\rho-1)) \in [1-\rho, \rho]$; and $\hat{\alpha} = 0$ if $F(L(2\rho-1)) < 1-\rho$.

Let α^* be the maximal equilibrium likelihood of good rumors in the viral marketing game. We need to show that $\alpha^* \geq \hat{\alpha}$, with $\alpha^* > \hat{\alpha}$ if $\hat{\alpha} \in (1-\rho, 1/2)$. When $\hat{\alpha} = 0$, there is nothing to prove. We can therefore focus on the remaining cases when $\hat{\alpha} \in [1-\rho, \rho]$.

Suppose $\hat{\alpha} = 1-\rho$. If consumers believe that $\alpha = 1-\rho$, then no socially-exposed consumers ever adopt ($t_0 = 0$) and the producer finds it optimal to use a traditional ad campaign. Thus, an equilibrium exists with $\alpha = \hat{\alpha}$, and $\alpha^* \geq \hat{\alpha}$ as desired.

Suppose $\hat{\alpha} \in (1-\rho, 1/2)$. If consumers believe that $\alpha = \hat{\alpha}$, the producer finds it optimal to use a purely-viral campaign, resulting in extra adoption $\Delta I(\infty)$ of good rumors. By Lemma 3, $\Delta I(\infty) > L(2\rho - 1)$ and hence the producer optimally invests with likelihood $F(\Delta I(\infty)) > F(L(2\rho-1)) = \hat{\alpha}$; so, there is *too much* investment to support an equilibrium with $\alpha = \hat{\alpha}$. On the other hand, the producer has zero incentive to invest when $\alpha > \rho$; thus, there is *too little* investment to support any $\alpha > \rho$.

To complete the proof for this case, we need to show that an equilibrium exists with $\alpha \in (\hat{\alpha}, \rho]$. Let $\Delta I(\infty; \alpha)$ be the extra overall adoption of good rumors, viewed now as a function of α. Note by inspection of the system of differential equations (1,2) that $\Delta I(\infty; \alpha)$ is uniquely determined and continuous in α for all $\alpha \in (1-\rho, \rho)$. If $F(\Delta I(\infty; \rho-)) < \rho$, continuity implies that an equilibrium exists with $\alpha \in (\hat{\alpha}, \rho)$. But what if $F(\Delta I(\infty; \rho-)) \geq \rho$? At the threshold ρ, launch-exposed consumers are indifferent whether to adopt after a bad signal, generating an epidemic trajectory that varies continuously with their probability $a_B \in [0,1]$ of adopting after a bad signal. Thus, $\Delta I(\infty; \rho)$ is an interval containing $[0, \Delta I(\infty; \rho-)]$. In particular, there exists a_B such that good rumors have extra adoption $F^{-1}(\rho)$, inducing the producer to invest just enough that fraction ρ of rumors are good; so, an equilibrium exists with $\alpha = \rho$. No matter what, we conclude that $\alpha^* > \hat{\alpha}$, as desired.

Suppose $\hat{\alpha} = 1/2$. Uninformed consumers are indifferent whether to adopt in a traditional ad campaign. If all uninformed consumers would adopt, then the producer finds it optimal to use a traditional ad campaign, leading to the same equilibrium outcome as in the traditional marketing game, with $\alpha = \hat{\alpha}$; thus, $\alpha^* \geq \hat{\alpha}$, as desired.

Finally, suppose $\hat{\alpha} > 1/2$. If consumers believe that $\alpha = \hat{\alpha}$, then the producer finds it optimal to use a traditional ad campaign, and the equilibrium with $\alpha = \hat{\alpha}$ in the traditional marketing game remains an equilibrium in the viral marketing game. $\square$

To gain intuition, consider how uninformed consumers' adoption choices impact the producer's incentive to invest in the rumor model. In the traditional marketing game, uninformed consumers are necessarily equally likely to adopt any rumor, giving the producer no additional incentive to invest. By contrast, in the viral marketing game, good rumors are more adopted by uninformed consumers because they diffuse more widely during the Herding Phase while these consumers are still willing to adopt, increasing the producer's incentive to invest.

B Extension: Continuous Private Signals

In the main text, we assumed that each consumer i receives a conditionally i.i.d. binary private signal $s_i \in \{G, B\}$ with likelihood ratios $\frac{\Pr(s_i=G|\omega=g)}{\Pr(s_i=G|\omega=b)} = \frac{\rho}{1-\rho}$ and $\frac{\Pr(s_i=B|\omega=g)}{\Pr(s_i=B|\omega=b)} = \frac{1-\rho}{\rho}$. Here we extend the analysis to a setting in which consumers' signals are drawn from a continuous distribution. In particular, suppose that each consumer receives a conditionally i.i.d. signal $s_i \in [-1, 1]$ satisfying the monotone likelihood ratio property (MLRP). Let $H(\cdot|\omega)$ and p.d.f. $h(\cdot|\omega)$ be the conditional c.d.f. and conditional p.d.f. of s_i, with likelihood ratio $l(s_i) \equiv \frac{h(s_i|\omega=g)}{h(s_i|\omega=b)}$ that is strictly increasing and continuous in s_i.

Consumer behavior at and immediately after launch. Consumer i who is exposed at launch with private signal s_i has ex post belief $p(0; s_i)$ determined by

$$\frac{p(0; s_i)}{1 - p(0; s_i)} = \frac{\alpha}{1 - \alpha} \times l(s_i).$$

For each $p \in (0, 1)$, define $s(p)$ implicitly by the condition that $l(s(p)) = \frac{1-p}{p}$. (If $l(-1) > \frac{1-p}{p}$, then set $s(p) = -1$. If $l(1) < \frac{1-p}{p}$, then set $s(p) = 1$.) $s(\alpha)$ serves as a private-signal threshold for adoption at time $t = 0$. If $s_i > s(\alpha)$, then $p(0; s_i) > 1/2$ and consumer i strictly prefers to adopt. On the other hand, if $s_i < s(\alpha)$, then $p(0; s_i) < 1/2$ and consumer i strictly prefers not to adopt.

$S_g(0) = S_b(0) = 1 - L$ where $L > 0$ is the mass of consumers exposed at launch. Mass $I_g(0) = L(1 - H(s(\alpha)|\omega = g))$ of consumers adopt when the innovation is good while mass $I_b(0) = L(1 - H(s(\alpha)|\omega = b))$ adopt when the innovation is bad. By equation (3) (which applies generally regardless of the private-signal structure), consumers' interim belief immediately after launch, $p(0+)$, is given by

$$\frac{p(0+)}{1 - p(0+)} = \frac{\alpha}{1 - \alpha} \times \frac{S_g(0) I_g(0)}{S_b(0) I_b(0)} = \frac{\alpha}{1 - \alpha} \times \frac{1 - H(s(\alpha)|\omega = g)}{1 - H(s(\alpha)|\omega = b)}.$$

Because s_i satisfies MLRP, $H(s|g) < H(s|b)$ for all signal-levels s. Thus, much as in the benchmark analysis with binary signals, good innovations are more widely adopted at launch and being exposed socially shortly after launch is "good news" about innovation quality in the sense that $p(0+) > \alpha$.

Equilibrium economic-epidemiological dynamics. At each point in time $t > 0$, the epidemiological dynamics (1-2) are determined by the current stock of susceptible and infected agents, as well as the likelihoods $q_g(t)$ and $q_b(t)$ that time-t-exposed consumers will adopt conditional on the innovation being good or bad, respectively. $q_g(t)$ and $q_b(t)$ in turn depend only on consumers' time-t interim belief $p(t)$ and the private-signal distribution. Consumers exposed at time t adopt if and only if s_i exceeds the threshold $s(p(t))$ defined above. Thus,

$$q_g(t) = 1 - H(s(p(t))|\omega = g) \quad \text{and} \quad q_b(t) = 1 - H(s(p(t))|\omega = b), \tag{14}$$

where by equation (3) we have

$$p(t) = \frac{\alpha S_g(t) I_g(t)}{\alpha S_g(t) I_g(t) + (1-\alpha) S_b(t) I_b(t)}. \tag{15}$$

Together, equations (1,2,14,15) determine the equilibrium epidemic trajectory, which is unique and easily computable for any given signal distribution.

Numerical examples. For expositional convenience, define the random variable

$$x_i \equiv \ln l(s_i)$$

and let $x(s_i)$ be the function mapping realizations of s_i into corresponding realizations of x_i. Let $F(x_i|\omega)$ and $f(x_i|\omega)$ denote the c.d.f. and p.d.f. of x_i conditional on the state $\omega \in \{g, b\}$. By construction, $F(x(s_i)|\omega) = H(s_i|\omega)$ and $f(x(s_i)|\omega) = \frac{h(s_i|\omega)}{x'(s_i)}$ for $\omega = g, b$, where $x'(s_i) > 0$ because s_i satisfies MLRP. This implies that

$$\frac{f(x(s_i)|g)}{f(x(s_i)|b)} = \frac{h(s_i|g)}{h(s_i|b)} = l(s_i) = e^{x(s_i)}. \tag{16}$$

In the binary-signal case analyzed in the main text, x_i is a binary variable with support $x_i \in \{-\ln \frac{\rho}{1-\rho}, \ln \frac{\rho}{1-\rho}\}$. Let $K \equiv \ln \frac{\rho}{1-\rho}$. In our numerical explorations, we have considered a family of distributions indexed by $J \in (-1, \infty)$, defined as follows:

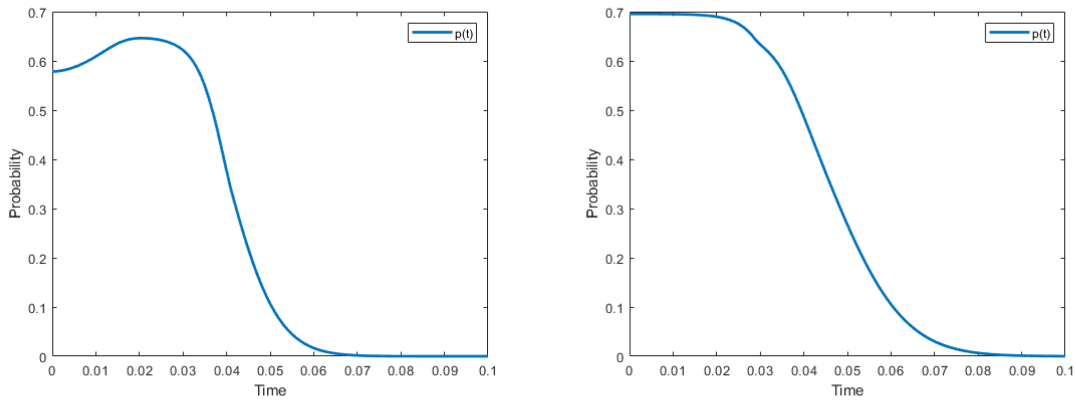
$$\begin{cases} f_J(x|g) = \dfrac{e^x}{1+e^x} \dfrac{J+1}{K^{J+1}} x^J \text{ for all } x \in [0, K] & (17) \\ f_J(x|g) = \dfrac{1}{1+e^{-x}} \dfrac{J+1}{K^{J+1}} (-x)^J \text{ for all } x \in [-K, 0] & (18) \end{cases}$$

and

$$f_J(x|b) = f_J(-x|g) \text{ for all } x \in [-K, K]. \tag{19}$$

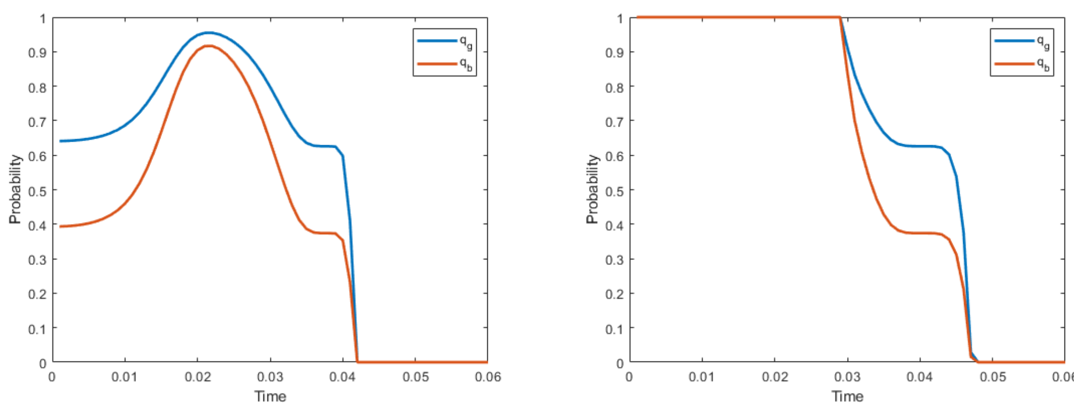
In the section below titled "Details about $f_J(x|\omega)$," we verify that (17-19) specify a valid conditional p.d.f. for x_i and discuss the limiting cases $J \to \infty$ and $J \to -1$.

Figures 7-8 illustrate the equilibrium epidemic trajectory in two numerical examples with shared parameters $J = 4$ and $\rho = 0.65$ (so that $K = \ln(\frac{\rho}{1-\rho}) \approx 0.62$), but different levels of innovation quality: $\alpha = 0.45$ in Example #1 illustrated in Figures 7a and 8a, while $\alpha = 0.6$ in Example #2 illustrated in Figures 7b and 8b.



(a) Example #1: $\alpha = 0.45$ (b) Example #2: $\alpha = 0.6$

Figure 7: Interim belief over time



(a) Example #1: $\alpha = 0.45$ (b) Example #2: $\alpha = 0.6$

Figure 8: Adoption probability by state over time

In each example, a consumer exposed at launch with private signal s_i has ex post belief $p(0; s_i)$ given by $\frac{p(0;s_i)}{1-p(0;s_i)} = \frac{\alpha}{1-\alpha} l(s_i)$; $p(0; s_i)$ exceeds $1/2$ iff $x_i = \ln l(s_i) > -\ln\left(\frac{\alpha}{1-\alpha}\right)$.

Thus,

$$q_g(0) = 1 - F_g\left(-\ln\left(\frac{\alpha}{1-\alpha}\right)\right) \text{ and } q_b(0) = 1 - F_b\left(-\ln\left(\frac{\alpha}{1-\alpha}\right)\right).$$

In Example #1 with $\alpha = 0.45$, we have $q_g(0) \approx 0.624$ and $q_b(0) \approx 0.372$. In Example #2 with $\alpha = 0.6$, we have $q_g(0) \approx 0.676$ and $q_b(0) \approx 0.444$. Consumers exposed immediately after launch take into account the fact that more people are "infected" when the innovation is good, forming interim belief $p(0+)$ according to $\frac{p(0+)}{1-p(0+)} = \frac{\alpha}{1-\alpha}\frac{q_g(0)}{q_b(0)}$.

In Example #1, we have $p(0+) \approx 0.578$. This is low enough (below $\rho = 0.65$) that some consumers exposed immediately after launch choose not to adopt, i.e., consumers are "sensitive to signals" as in the binary-signal model when $\alpha \in (1-\rho, 1/2)$. As in that case of the binary-signal model, consumers' interim belief initially rises over time (due to the "upward pressure" from consumers being sensitive to signals), but eventually interim belief begins falling and continues falling until no one adopts the innovation any more ("obsolescence").

In Example #2, we have $p(0+) \approx 0.695$. This is high enough (above $\rho = 0.65$) that consumers "herd on adoption" immediately after launch, as in the binary-signal model when $\alpha \in (1/2, \rho)$. As in that case of the binary-signal model, this puts immediate downward pressure on consumers' interim belief, which continues to decline monotonically until eventual obsolescence.

Details about $f_J(x|\omega)$. To check that (17-19) specify a valid conditional p.d.f. for $x_i = \ln l(s_i)$, we need to verify that (a) $\int_{-K}^{K} f_J(x|g)\mathrm{d}x = 1$, (b) $\int_{-K}^{K} f_J(x|b)\mathrm{d}x = 1$, and (c) $\frac{f(x|g)}{f(x|b)} = e^x$ for all $x \in [-K, K]$.

Verifying (a): For expositional convenience, define $z(x) = \frac{J+1}{2K^{J+1}}x^J$ for all $x \in [0, K]$ and $z(x) = \frac{J+1}{2K^{J+1}}(-x)^J$ for all $x \in [-K, 0]$. As can be easily checked, $\int_0^K z(x)\mathrm{d}x = 1/2$ and $z(x) = z(-x)$ for all $x \in [-K, K]$. Thus, $\int_{-K}^{K} z(x)\mathrm{d}x = 1$. For any $x \in [0, K]$, we have

$$f_J(x|g) + f_J(-x|g) = \frac{2e^x}{1+e^x}z(x) + \frac{2}{1+e^x}z(x) = 2z(x).$$

Thus, $\int_{-K}^{K} f_J(x|g)\mathrm{d}x = 2\int_0^K z(x)\mathrm{d}x = 1$, as desired.

Verifying (b-c): Because $f_J(x|b) = f_J(-x|g)$, we have both $\int_{-K}^{K} f_J(x|b)\mathrm{d}x = \int_{-K}^{K} f_J(x|g)\mathrm{d}x = 1$ and $\frac{f_J(x|g)}{f_J(x|b)} = \frac{f_J(x|g)}{f_J(-x|g)} = e^x$, as desired.

We chose this parametric family because the limiting case as $J \to \infty$ corresponds to our binary-signal model, while the other limiting case as $J \to -1$ corresponds to a degenerate special case in which consumers' private signals are uninformative.

Limit as $J \to \infty$: For all $x \in (-K, K)$, $\lim_{J\to\infty} f_J(x|g) = 0$ because $\lim_{J\to\infty}(x/K)^J = 0$. At the same time,

$$\frac{f_J(K|g)}{f_J(-K|g)} = \frac{h_J(s_i = 1|g)}{h_J(s_i = -1|g)} = e^K = \frac{\rho}{1-\rho} \text{ for all } J \in (-1, \infty).$$

So, in the limit as $J \to \infty$, the distribution of each consumer's private signal conditional on $\omega = g$ converges to a binary distribution with mass ρ on $s_i = 1$ and mass $1-\rho$ on $s_i = -1$, exactly as in the binary-signal model analyzed in the main text. (Similarly, the distribution of $s_i|b$ converges to a binary distribution with mass $1-\rho$ on $s_i = 1$ and mass ρ on $s_i = -1$.)

Limit as $J \to -1$: For all $x \neq 0$, $\lim_{J\to-1} f_J(x|g) = 0$ because $\lim_{J\to-1}(J+1) = 0$. The distributions of $x|g$ and $x|b$ therefore both converge to a Dirac delta function on $x = 0$, meaning that consumers receive uninformative private signals.

C Omitted Proofs

The following lemma is useful in several of the proofs that follow.

Lemma 4. *Suppose that $p(t) > \alpha$ for all $t \in (0, \hat{t})$ for some $\hat{t}$. Then $I_g(t) > I_b(t)$, $I_g'(t) > I_b'(t)$, $S_g(t) < S_b(t)$, and $S_g'(t) < S_b'(t)$ for all $t \in (0, \hat{t})$.*

Proof. Suppose that $p(t) > \alpha$ for all $t \in (0, \hat{t})$. Because $p(0+) > \alpha$, consumers exposed at launch must be strictly more likely to adopt when the innovation is good, i.e., $q_g(0) > q_b(0)$ and hence $I_g(0) = q_g(0)L > q_b(0)L = I_b(0)$. By equation (6), $\frac{p(t)}{1-p(t)} = \frac{\alpha}{1-\alpha} \times \frac{I_g(t)S_g(t)}{I_b(t)S_b(t)} = \frac{\alpha}{1-\alpha} \times \frac{|S_g'(t)|}{|S_b'(t)|}$; so, $S_g'(t) < S_b'(t) < 0$ for all $t \in (0, \hat{t})$. Because $S_\omega'(t) < 0$, $I_\omega'(t) = -q_\omega(t)S_\omega'(t)$ (by equations 1-2), and $q_g(t) \geq q_b(t)$, this in turn implies that $I_g'(t) > I_b'(t)$ for all $t \in (0, \hat{t})$. Finally, since $S_g(0) = S_b(0) = 1 - L$ and $I_g(0) > I_b(0)$, we conclude that $S_g(t) < S_b(t)$ and $I_g(t) > I_b'(t)$ for all $t \in (0, \hat{t})$, as desired. □

Proof of Lemma 1. The desired results follow immediately from equation (8) since: (i) $q_g(t) = q_b(t) = 1$ when consumers herd on adoption; (ii) $q_g(t) = \rho$ and $q_b(t) = 1 - \rho$ when consumers are sensitive to signals; and (iii) $q_g(t) = q_b(t) = 0$ when consumers herd on non-adoption. □

Proof of Proposition 1. Because $\alpha \in (1/2, \rho)$, consumers are sensitive to signals at launch and so $I_g(0) = \rho L$, $I_b(0) = (1-\rho)L$, and $S_g(0) = S_b(0) = 1 - L$. Consumers' exposed immediately after launch have interim belief $p(0+) > \rho$ and herd on adoption, i.e., $q_g(0+) = q_b(0+) = 1$. By Lemma 3 and equation (8), the likelihood ratio $\frac{p(t)}{1-p(t)}$ falls at rate $-X(t) > (2\rho - 1)L > 0$ as long as consumers continue to herd on adoption. Thus, consumers' interim belief $p(t)$ decreases over time and reaches ρ in finite time, as desired. □

Proof of Proposition 2. Suppose $\alpha \in (1 - \rho, \hat{\alpha})$. Since $\alpha \in (1 - \rho, \rho)$ and $p(0+) < \rho$, consumers are sensitive to signals at launch and immediately after launch. So long as consumers remain sensitive to signals, interim belief is increasing during Phase I if and only if Condition SS is satisfied. Recall that, at times $t \approx 0$, $S_g(t) \approx S_b(t) \approx 1 - L$, $I_g(t) \approx \rho L$, and $I_b(t) \approx (1 - \rho)L$. Thus, $X(t) \approx 2\rho - 1 - L(2\rho - 1)) \approx 2\rho - 1 > 0$; so, interim beliefs are initially increasing.

However, the rate at which $\frac{p(t)}{1-p(t)}$ increases itself decreases over time. Why? By equations (1-2), $S_g'(t) = -I_g(t)S_g(t)$, $S_b'(t) = -I_b(t)S_b(t)$, $I_g'(t) = \rho I_g(t)S_g(t)$, and $I_b'(t) = (1-\rho)I_b(t)S_b(t)$; thus,

$$X'(t) = -2\left(\rho I_g(t)S_g(t) - (1-\rho)I_b(t)S_b(t)\right). \tag{20}$$

Note that $X'(t) \geq 0$ iff $\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)} \leq \frac{1-\rho}{\rho}$. By equation (6), that is only possible at times when $\frac{p(t)}{1-p(t)} \leq \frac{\alpha(1-\rho)}{(1-\alpha)\rho}$ which, since $\alpha < 1/2$, implies that $p(t) < 1-\rho$. We conclude that, so long as $p(t) > 1-\rho$ and consumers are sensitive to signal, $X'(t) < 0$.

Because $X'(t) < 0$, consumers' interim beliefs may begin to decline if consumers are sensitive to signal for long enough. Given our assumption that L is small,[21] however, this does not happen for a long time. To see why, note that the total mass of consumers exposed by any given time $\tilde{t}$ can be made arbitrarily small by beginning with a sufficiently small initial mass L of consumers exposed at launch. In particular, for any time $\tilde{t}$ and any small $\epsilon > 0$, we can find L sufficiently small so that (i) $S_g(t), S_b(t) \in (1-\epsilon, 1)$ for all $t < \tilde{t}$ and (ii) $I_g(t), I_b(t) \in (0, \epsilon)$ for all $t < \tilde{t}$, implying that $X(t) > (\rho(1-\epsilon)-\epsilon) - (1-\rho-0) = 2\rho - 1 - \epsilon(1+\rho) > 0$ for all $t \in (0, \tilde{t})$. Recall by Lemma 1 that the likelihood ratio $\frac{p(t)}{1-p(t)}$ rises exponentially at rate $X(t)$; so, for small L, $\frac{p(t)}{1-p(t)}$ rises exponentially at approximate rate $2\rho - 1$ until a time t_1 is reached at which consumers' interim belief equals ρ.

We conclude that, in any equilibrium when $L \approx 0$, consumers are sensitive to signal and interim belief $p(t)$ is strictly increasing until a finite time t_1 at which $p(t_1) = \rho$. □

Proof of Proposition 3. We begin by showing that consumers' interim belief must remain constant immediately after time t_1. First, suppose that $p(t)$ were to rise after time t_1, causing consumers to herd on adoption. Since $p(t) > \alpha$ for all $t \in (0, t_1)$ (shown previously[22]), Lemma 4 implies that, at time t_1, good innovations must be more widely seen ($S_g(t_1) < S_b(t_1)$) and more widely adopted ($I_g(t_1) > I_b(t_1)$). Thus, at all times shortly after t_1, $S_g(t) - I_g(t) < S_b(t) - I_b(t)$ and consumers' interim belief must decline over time by Lemma 1(i), a contradiction.

Next, suppose that $p(t)$ were to fall after time t_1, causing consumers to be sensitive to signals. As discussed in the proof of Proposition 2, the assumption here of a small launch ($L \approx 0$) implies that only a small mass of consumers are exposed to the innovation prior to Phase II; in particular, $S_g(t_1), S_b(t_1) \in (1-\epsilon, 1)$ and $I_g(t_1), I_b(t_1) \in (0, \epsilon)$ for some small ϵ. Consequently, for all times t shortly after t_1, Condition SS holds. Were consumers to be sensitive to signals immediately after time t_1, consumers' interim belief would therefore increase over time by Lemma 1(ii), a contradiction. We conclude that, in any equilibrium,

[21] When L is sufficiently large, $p(t)$ never reaches ρ and Phase I proceeds directly to Phase III.

[22] When $\alpha = 1/2$, $t_1 = 0$. When $\alpha \in (1/2, \rho)$, $p(t) > \rho > \alpha$ for all $t \in (0, t_1)$ by Proposition 1. Finally, when $\alpha \in (1-\rho, 1/2)$, $p(t) > 1/2 > \alpha$ for all $t \in (0, t_1)$ by Proposition 2.

consumers' interim belief must remain $p(t) = \rho$ for some period of time after t_1.

By equation (6), interim belief $p(t) = \rho$ requires that $\frac{\rho}{1-\rho} = \frac{\alpha I_g(t) S_g(t)}{(1-\alpha) I_b(t) S_b(t)}$ or, equivalently, $\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)} = \frac{(1-\alpha)\rho}{\alpha(1-\rho)}$. In order for this ratio not to change over time, the ratio of derivatives $\frac{(I_g(t)S_g(t))'}{(I_b(t)S_b(t))'}$ must also equal $\frac{(1-\alpha)\rho}{\alpha(1-\rho)}$. Taking derivatives, using equations (1-2), and re-arranging yields

$$\frac{(1-\alpha)\rho}{\alpha(1-\rho)} = \frac{I_g'(t)S_g(t) + I_g(t)S_g'(t)}{I_b'(t)S_b(t) + I_b(t)S_b'(t)} = \frac{I_g(t)S_g^2(t)q_g(t) - I_g^2(t)S_g(t)}{I_b(t)S_b^2(t)q_b(t) - I_b^2(t)S_b(t)}$$
$$= \frac{I_g(t)S_g(t)(S_g(t)q_g(t) - I_g(t))}{I_b(t)S_b(t)(S_b(t)q_b(t) - I_b(t))}$$

and so it must be that

$$S_g(t)q_g(t) - I_g(t) = S_b(t)q_b(t) - I_b(t). \tag{21}$$

Let $a_B(t)$ denote the likelihood that consumers exposed at time t adopt the innovation after getting a bad signal. The overall likelihood that a good innovation is adopted equals $q_g(t) = \rho + (1-\rho)a_B(t)$; similarly, a bad innovation is adopted with likelihood $q_b(t) = 1 - \rho + \rho a_B(t)$. Equation (21) can now be re-written as

$$\big(\rho S_g(t) - (1-\rho)S_b(t)\big) - \big(I_g(t) - I_b(t)\big) + a_B(t)\left((1-\rho)S_g(t) - \rho S_b(t)\right) = 0 \tag{22}$$

or, equivalently,

$$a_B(t) = \frac{\rho S_g(t) - (1-\rho)S_b(t) - \big(I_g(t) - I_b(t)\big)}{\rho S_b(t) - (1-\rho)S_g(t)} \tag{23}$$

The proofs of Propositions 1-2 characterized the time t_1 at which Phase II begins and the initial conditions $\big(I_g(t_1), I_b(t_1), S_g(t_1), S_b(t_1)\big)$. Equation (23) then uniquely determines $a_B(t_1+)$, consumers' equilibrium likelihood of adopting after a bad signal immediately after time t_1. Note that, since $I_g(t_1) > I_b(t_1)$ and $S_b(t_1) > S_g(t_1)$ (by Lemma 4), $a_B(t_1+) < 1$. Moreover, because Condition SS holds at time t_1 (discussed earlier), the numerator in (23) is positive; so, $a_B(t_1+) > 0$.

Equations (1,2,23) now uniquely determine the path of $\big(a_B(t), S_g(t), S_b(t), I_g(t), I_b(t)\big)$, starting at time t_1 and so long as $a_B(t) \in [0,1]$.

Let t_2 be the first time after t_1 at which $a_B(t) \in \{0,1\}$, or $t_2 = \infty$ if $a_B(t)$ remains forever between zero and one. To complete the proof, we need to show that $a_B(t)$ is strictly decreasing after t_1 and reaches zero in finite time.

By Lemma 4, $S_b(t) > S_g(t)$ so long as consumers' interim belief continues to exceed the initial belief α.

Let t_2 denote the first time after t_1 at which consumers no longer partially herd, i.e., $a_B(t_2) = 0$ and $a_B(t) > 0$ for all $t \in (t_1, t_2)$, or $t_2 = \infty$ if consumers partially herd forever. Since $p(t) > \alpha$ throughout Phase I and $\rho > \alpha$, Lemma 4 implies that $S_b(t) > S_g(t)$, ensuring that the denominator of (23) remains positive. Moreover, t_2 is the first time as which the numerator of (23) equals zero, i.e., when Condition SS holds with equality.

Next, note that

$$a'_B(t) = \frac{\begin{array}{c}(\rho S'_g(t) - (1-\rho)S'_b(t) - (I'_g(t) - I'_b(t))(\rho S_b(t) - (1-\rho)S_g(t)) \\ -(\rho S_g(t) - (1-\rho)S_b(t) - (I_g(t) - I_b(t)))(\rho S'_b(t) - (1-\rho)S'_g(t))\end{array}}{(\rho S_b(t) - (1-\rho)S_g(t))^2}.$$

Rearranging and simplifying the numerator, we have

$$\begin{aligned}\text{numerator} =& (\rho^2 - (1-\rho)^2)(S'_g(t)S_b(t) - S'_b(t)S_g(t)) \\ &- (I'_g(t) - I'_b(t))(\rho S_b(t) - (1-\rho)S_g(t)) \\ &+ (I_g(t) - I_b(t))(\rho S'_b(t) - (1-\rho)S'_g(t)).\end{aligned}$$

By (1-2), the second term above can be re-written as

$$\begin{aligned}&- (I'_g(t) - I'_b(t))(\rho S_b(t) - (1-\rho)S_g(t)) \\ =& - (I_g(t)S_g(t)(\rho + (1-\rho)a_B(t)) - I_b(t)S_b(t)(1-\rho+\rho a_B(t)))(\rho S_b(t) - (1-\rho)S_g(t)) \\ =& - I_b(t)(I_g(t) - I_b(t))(\rho S_b(t) - (1-\rho)S_g(t)) \\ &- (I_g(t) - I_b(t))S_g(t)(\rho + (1-\rho)a_B(t))(\rho S_b(t) - (1-\rho)S_g(t))\end{aligned} \tag{24}$$

Similarly, the third term above can be re-written as

$$\begin{aligned}&(I_g(t) - I_b(t))(\rho S'_b(t) - (1-\rho)S'_g(t)) \\ =& - (I_g(t) - I_b(t))(\rho I_b(t)S_b(t) - (1-\rho)I_g(t)S_g(t)) \\ =& - I_b(t)(I_g(t) - I_b(t))(\rho S_b(t) - (1-\rho)S_g(t)) \\ &+ (I_g(t) - I_b(t))S_g(t)(1-\rho)(I_g(t) - I_b(t))\end{aligned} \tag{25}$$

To establish that the entire numerator is negative, we will show that the first term is negative and that the sum of the second term (24) and third term (25) is negative. To that end, recall that $I_g(t) > I_b(t)$, $I'_g(t) > I'_b(t)$, $S_g(t) < S_b(t)$, and $S'_g(t) < S'_b(t)$ at all times $t < t_2$ (Lem 4). The fact that the first term is negative now follows immediately from (1-2), since $S_g'(t)S_b(t) - S_b'(t)S_g(t) = -S_g(t)S_b(t)(I_g(t) - I_b(t)) < 0$. Moreover, $\rho S_b(t) > (1-\rho)S_g(t)$ because $S_b(t) > S_g(t)$ and $\rho > 1/2$; so, the first part of (24) and the first part of (25) are negative. To show that the sum of (24) and (25) is negative, it therefore suffices to show that $(\rho + (1-\rho)a_B(t))(\rho S_b(t) - (1-\rho)S_g(t)) > (1-\rho)(I_g(t) - I_b(t))$. But this follows immediately from the fact that $\rho S_b(t) - (1-\rho)S_g(t) > I_g(t) - I_b(t)$ (since Condition SS remains satisfied) and $\rho + (1-\rho)a_B(t) > 1-\rho$ (since $\rho > 1/2$ and $a_B(t) \geq 0$).

Overall, we conclude that $a_B(t) > 0$ but that $a_B'(t) < 0$ so long as the numerator of equation (23) continues to be positive, i.e., so long as Condition SS continues to be satisfied. Moreover, there is a finite time t_2 at which partial herding ceases. To see why, suppose for the sake of contradiction that consumers were to partially herd forever. Because all consumers are eventually exposed to the innovation, $\lim_{t\to\infty} S_g(t) = \lim_{t\to\infty} S_b(t) = 0$. On the other hand, because $I_g'(t) > I_b'(t)$ so long as $a_B(t) > 0$, $\lim_{t\to\infty}(I_g(t) - I_b(t)) > I_g(t_1) - I_g(t_1) > 0$. All together, then, the numerator of (23) must eventually become negative, a contradiction. $\square$

Proofs of Propositions 4-5. We prove Propositions 4-5 together, dividing the proof into four main steps.

Step 1: After time t_2, $\frac{p(t)}{1-p(t)}$ declines exponentially at an increasing rate until some time $\tilde{t}$ at which $p(\tilde{t}) = \max\{1-\rho, \underline{\alpha}\}$, where $\underline{\alpha} \equiv \frac{(1-\rho)\alpha}{(1-\rho)\alpha+\rho(1-\alpha)} \in \left(\frac{(1-\rho)^2}{(1-\rho)^2+\rho^2}, \frac{1}{2}\right)$.

By Lemma 1, $\frac{p(t)}{1-p(t)}$ declines exponentially at rate $X(t)$. So, it suffices to show that $X(t) < 0$ and $X'(t) < 0$ at all times after t_2 until a time $\tilde{t}$ is reached at which $p(\tilde{t}) = \max\{1-\rho, \underline{\alpha}\}$. By the proof of Proposition 3: $p(t_2) = \rho$; consumers are sensitive to signal at time t_2 (because $a_B(t_2) = 0$); and $X(t_2) = \left(\rho S_g(t_2) - I_g(t_2)\right) - ((1-\rho)S_b(t_2) - I_b(t_2)) = 0$. It suffices to show that $X'(t) < 0$ at all times $t \in [t_2, \tilde{t})$, since then it must also be that $X(t) < 0$ at all times $t \in (t_2, \tilde{t})$.

By equation (20), $X'(t) = -2\left(\rho S_g(t)I_g(t) - (1-\rho)S_b(t)I_b(t)\right)$ while consumers are sensitive to signals. Thus, $X'(t) < 0$ so long as $\frac{S_g(t_2)I_g(t_2)}{S_b(t_2)I_b(t_2)} > \frac{1-\rho}{\rho}$. By equation (6), $\frac{p(t)}{1-p(t)} = \frac{\alpha S_g(t_2)I_g(t_2)}{(1-\alpha)S_b(t_2)I_b(t_2)}$; so, $\frac{S_g(t_2)I_g(t_2)}{S_b(t_2)I_b(t_2)} > \frac{1-\rho}{\rho}$ if and only if $p(t) > \underline{\alpha}$ or, equivalently, $\frac{p(t)}{1-p(t)} > \frac{\alpha(1-\rho)}{(1-\alpha)\rho} = \frac{\underline{\alpha}}{1-\underline{\alpha}}$. In other words:

$$\text{when consumers are sensitive to signal, } X'(t) \gtrless 0 \text{ iff } p(t) \gtrless \underline{\alpha} \tag{26}$$

At time t_2, consumers are sensitive to signal and $p(t_2) = \rho > \underline{\alpha}$; so, $X'(t_2) < 0$. Moreover, $X'(t) < 0$ at times $t \in (t_2, \tilde{t})$ since (i) consumers remain sensitive to signal (because

$p(t) \in (1-\rho, \rho)$) and (ii) $p(t) > \underline{\alpha}$. We conclude that $\frac{p(t)}{1-p(t)}$ decreases exponentially at an increasing rate from time t_2 until time $\tilde{t}$.

What about after time $\tilde{t}$? There are two relevant cases. First, suppose that $\alpha \in (1-\rho, 1/2]$, so that $\underline{\alpha} \leq 1-\rho$. In this case, $p(\tilde{t}) = 1-\rho$ and Phase III ends at time $\tilde{t}$, i.e., $t_3 = \tilde{t}$. Second, suppose that $\alpha \in (1/2, \rho)$. In this more challenging case, $\underline{\alpha} \in (1-\rho, 1/2)$ and the argument so far shows that $\frac{p(t)}{1-p(t)}$ declines at an increasing rate until time $\tilde{t}$, when consumers' interim belief hits $\underline{\alpha}$. However, we still need to show that consumers' interim belief *continues* falling long enough after time $\tilde{t}$ to reach $1-\rho$.

Step 2: In the case when $\alpha \in (1/2, \rho)$, $\frac{p(t)}{1-p(t)}$ declines exponentially at a decreasing rate from time $\tilde{t}$ until time t_3 at which $p(t_3) = 1-\rho$.

The argument in Step 1 established that $p(\tilde{t}) = \underline{\alpha} \in (1-\rho, 1/2)$ and $X(\tilde{t}) < 0$; thus, consumers' interim belief continues to fall below $\underline{\alpha}$ right after time $\tilde{t}$. By condition (26), we conclude that $X'(t) > 0$ right after $\tilde{t}$ and at all times $t > \tilde{t}$ so long as consumers' interim belief remains between $1-\rho$ and $\underline{\alpha}$.

This leaves three possibilities for what happens after time $\tilde{t}$: (i) $p(t)$ decreases until a time t_3 at which point $p(t_3) = 1-\rho$ and Phase III ends; (ii) $p(t)$ decreases forever but never reaches $1-\rho$; or (iii) $p(t)$ stops decreasing (and starts increasing) at some time $\hat{t}$ before reaching $1-\rho$.

We will prove that possibility (i) always occurs, by ruling out (ii) and (iii).

As shorthand, define $X(\infty) = \lim_{t\to\infty} X(t)$, $I_g(\infty) = \lim_{t\to\infty} I_g(t)$, and so on.

"Possibility (ii)" cannot occur.

Suppose for the sake of contradiction that consumers' interim belief continues falling forever after time t_2 but never reaches $1-\rho$. This is only possible if $X(\infty) = 0$, which in turn requires that $I_g(\infty) - \rho S_g(\infty) = I_b(\infty) - (1-\rho) S_b(\infty)$. Since all consumers eventually encounter the innovation, $S_g(\infty) = S_b(\infty) = 0$. Thus, it must be that $I_g(\infty) = I_b(\infty)$. We will reach a contradiction by showing that $I_g(\infty) > I_b(\infty)$.

Recall that we are focusing here on the case in which $\alpha \in (1/2, \rho)$. We have shown: consumers are sensitive to signals at launch ($t = 0$), adopting good innovations with probability ρ and bad ones with probability $1-\rho$; consumers herd on adoption in Phase I ($t \in (0, t_1)$), adopting all innovations with probability one; and consumers partially herd on adoption in Phase II ($t \in (t_1, t_2)$), adopting good innovations with probability $\rho + a_B(t)(1-\rho)$ and bad ones with probability $1-\rho + a_B(t)\rho$. Moreover, given the presumption that possibility (ii) is occurring, consumers are again sensitive to signals at all times $t > t_2$. Overall, the mass of consumers who adopt a good innovation therefore takes the form:

$$I_g(\infty) = \rho L + \int_0^{t_1} |S_g'(t)|dt + \int_{t_1}^{t_2} (\rho + (1-\rho)a_B(t))|S_g'(t)|dt + \int_{t_2}^{\infty} \rho |S_g'(t)|dt$$
$$= \rho + \int_0^{t_1} (1-\rho)|S_g'(t)|dt + \int_{t_1}^{t_2} (1-\rho)a_B(t)|S_g'(t)|dt \tag{27}$$

where $|S_g'(t)|$ is the flow of consumers being exposed at time t and $L + \int_0^\infty |S_g'(t)|dt = 1$ because the consumer population has unit mass. Similarly, the overall share of consumers who adopt a bad innovation takes the form:

$$I_b(\infty) = (1-\rho)L + \int_0^{t_1} |S_b'(t)|dt + \int_{t_1}^{t_2} (1-\rho + \rho a_B(t))|S_b'(t)|dt + \int_{t_2}^{\infty} (1-\rho)|S_b'(t)|dt$$
$$= (1-\rho) + \int_0^{t_1} \rho |S_b'(t)|dt + \int_{t_1}^{t_2} \rho a_B(t)|S_b'(t)|dt \tag{28}$$

Since consumers' interim belief exceeds ρ throughout Phase I and equals ρ throughout Phase II, $|S_g'(t)| > |S_b'(t)|$ for all $t \in (0, t_2)$ by Lemma 4. Thus,

$$I_b(\infty) < (1-\rho) + \int_0^{t_1} \rho |S_g'(t)|dt + \int_{t_1}^{t_2} \rho a_B(t)|S_g'(t)|dt \tag{29}$$

(27, 29) together imply

$$I_g(\infty) - I_b(\infty) > (2\rho - 1)\left(1 - \int_0^{t_1} |S_g'(t)|dt - \int_{t_1}^{t_2} a_B(t)|S_g'(t)|dt\right). \tag{30}$$

Finally, note that $\int_0^{t_1} |S_g'(t)|dt = (1-L) - S(t_1)$ and, since $a_B(t) < 1$ for all $t \in (t_1, t_2)$, $\int_{t_1}^{t_2} a_B(t)|S_g'(t)|dt < S(t_1) - S(t_2)$. We conclude that $I_g(\infty) - I_b(\infty) > (2\rho - 1)(L + S(t_2)) > 0$; so, $I_g(\infty) > I_b(\infty)$, completing the desired contradiction.

"Possibility (iii)" cannot occur.

Suppose for the sake of contradiction that there exists $t' > t_2$ such that $X(t) < 0$ for all $t \in (t_2, t')$, $X(t') = 0$, and $p(t') > 1 - \rho$. For future reference, note that $X(t') = 0$ requires that $\rho S_g(t') - I_g(t') = (1-\rho)S_b(t') - I_b(t')$. Also recall that, since $X(t_1) = 0$ and $p(t_1) = \rho > \underline{\alpha}$, condition (26) implies that $X'(t_1) < 0$ and that $X(t)$ grows more negative until time $\tilde{t}$ at which $p(\tilde{t}) = \underline{\alpha}$. Thus, it must be that $t' > \tilde{t}$ and that $p(t') \in (1-\rho, \underline{\alpha})$ or equivalently, given equation (6), $\frac{(1-\rho)(1-\alpha)}{\rho\alpha} < \frac{I_g(t')S_g(t')}{I_b(t')S_b(t')} < \frac{1-\rho}{\rho}$.

Several equations that follow are quite complex, so we introduce the following shorthand: $a = S_g(t_2)$; $b = S_b(t_2)$; $c = \rho S_g(t_2) - I_g(t_2) = (1-\rho)S_b(t_2) - I_b(t_2)$; and $d = -(\rho S_g(t') - I_g(t')) = -((1-\rho)S_b(t') - I_b(t'))$.

We know that

$$\begin{aligned}
c+d &= (\rho S_g(t_2) - I_g(t_2)) - (\rho S_g(t') - I_g(t')) \\
&= \int_{t_2}^{t'} 2\rho I_g(t) S_g(t)\mathrm{d}t = 2(I_g(t') - I_g(t_2)) = -2\rho(S_g(t') - S_g(t_2)) \\
&= \int_{t_2}^{t'} 2(1-\rho) I_b(t) S_b(t)\mathrm{d}t = 2(I_b(t') - I_b(t_2)) = -2(1-\rho)(S_b(t') - S_b(t_2)),
\end{aligned} \tag{31}$$

which implies that

$$\begin{aligned}
I_g(t') - I_g(t_2) = I_b(t') - I_b(t_2) &= \frac{c+d}{2} \\
S_g(t') - S_g(t_2) &= -\frac{c+d}{2\rho} \\
S_b(t') - S_b(t_2) &= -\frac{c+d}{2(1-\rho)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{I_g(t')S_g(t')}{I_b(t')S_b(t')} &= \frac{(I_g(t_2) + I_g(t') - I_g(t_2))(S_g(t_2) + S_g(t') - S_g(t_2))}{(I_b(t_2) + I_b(t') - I_b(t_2))(S_b(t_2) + S_b(t') - S_b(t_2))} \\
&= \frac{(a - \frac{c+d}{2\rho})((a\rho - c) + \frac{c+d}{2})}{(b - \frac{c+d}{2(1-\rho)})((b(1-\rho) - c) + \frac{c+d}{2})} \\
&= \frac{a(a\rho - c) + \frac{c^2-d^2}{4\rho}}{b(b(1-\rho) - c) + \frac{c^2-d^2}{4(1-\rho)}}
\end{aligned}$$

We already know that $\frac{a(a\rho-c)}{b(b(1-\rho)-c)} = \frac{(1-\alpha)\rho}{\alpha(1-\rho)} > 1$. Hence, no matter whether $c^2 - d^2 \geq 0$ or $c^2 - d^2 < 0$, $\frac{I_g(t')S_g(t')}{I_b(t')S_b(t')} > \frac{1-\rho}{\rho}$, a contradiction.

Step 3: At all times $t \leq t_3$, $S_g(t) < S_b(t)$ and $I_g(t) > I_b(t)$.

Let $LS(t) \equiv S_b(t) - S_g(t)$ denote the "exposure gap," the extra share of consumers who have been exposed to good innovations by time t, and let $LI(t) \equiv I_g(t) - I_b(t)$ denote the "adoption gap," the extra share who have adopted. At launch, $S_g(0) = S_b(0) = L$, $I_g(t) = \rho L$, and $I_b(t) = (1-\rho)L$; so, $LS(0) = 0$ and $LI(0) = (2\rho - 1)L > 0$. Here we will show that $LS(t) > 0$ and $LI(t) > 0$ at all times $t \in (0, t_3)$.

$LS(t) > 0$ and $LI(t) > 0$ for all $t \leq t_2$.

By Steps 1-2, consumers' interim belief $p(t)$ declines throughout Phase III, from ρ at time t_2 to $1-\rho$ at time t_3; so, there is a unique time $\hat{t} \in (t_2, t_3)$ at which $p(\hat{t}) = \alpha$. Note that $p(t)$ exceeds consumers' ex ante belief α at all times $t \in (0, t_2]$ by Propositions 1-3 and that $p(t) > \alpha$ for all $t \in (t_2, \hat{t})$ by definition of $\hat{t}$. Lemma 4 therefore implies that $LS'(t) = S'_b(t) - S'_g(t) > 0$ and $LI'(t) = I'_g(t) - I'_b(t) > 0$ for all $t \in (0, \hat{t})$. Since $LS(0) = 0$ and $LI(0) > 0$, we conclude that $LS(t) > 0$ and $LI(t) > 0$ for all $t \in (0, \hat{t})$, and thus for all $t \leq t_2$.

$LS(t) > 0$ and $LI(t) > 0$ for all $t \in [t_2, t_3]$.

We begin by showing that the "adoption gap" $LI(t)$ exceeds $LI(t_2)$ during all of Phase III. Fix any $t' \in (t_2, t_3)$. Recall that $X(t_2) = 0$ (shown in the proof of Proposition 3), $X(t') < 0$ (proven in Step Two), and $X(t) = (\rho S_g(t) - I_g(t)) - ((1-\rho)S_b(t) - I_b(t))$ for all $t \in [t_2, t_3)$ (by Lemma 1, because consumers are sensitive to signals). Thus,

$$(\rho S_g(t') - I_g(t')) - ((1-\rho)S_b(t') - I_b(t')) < (\rho S_g(t_2) - I_g(t_2)) - ((1-\rho)S_b(t_2) - I_b(t_2)).$$

Rearranging and reformulating terms as in equation (31) yields

$$\int_{t_2}^{t'} -2\rho I_g(t) S_g(t) \mathrm{d}t < \int_{t_2}^{t'} -2(1-\rho) I_b(t) S_b(t) \mathrm{d}t. \tag{32}$$

Since $I'_g(t) = \rho I_g(t) S_g(t)$ and $I'_b(t) = (1-\rho) I_b(t) S_b(t)$, inequality (32) implies that $I_g(t') - I_g(t_2) > I_b(t') - I_b(t_2)$, which in turn implies that $LI(t') > LI(t_2)$. Since $LI(t_2) > 0$, we conclude that $LI(t') > 0$ for all $t' \in (t_2, t_3]$, as desired.

The "exposure gap" $LS(t) = S_b(t) - S_g(t)$ is non-monotone during Phase III, but we can show that $LS(t) > 0$ for all $t \in (t_2, t_3]$. Recall that $p(t) > \alpha$ for all $t \in [t_2, \hat{t})$ and $p(t) < \alpha$ for all $t \in (\hat{t}, t_3]$, where $\hat{t} \in (t_2, t_3)$ is the unique time during Phase III at which consumers' interim belief $p(t)$ equals their ex ante belief α. Also recall that, by equation (6),

$$p(t) \gtreqless \alpha \text{ iff } S_g(t) I_g(t) \gtreqless S_b(t) I_b(t) \text{ iff } -S'_g(t) \gtreqless -S'_b(t). \tag{33}$$

Prior to time $\hat{t}$, $p(t) > \alpha$ and condition (33) implies that $LS'(t) > 0$, i.e., the exposure gap is increasing and hence obviously still positive. After time $\hat{t}$, $p(t) < \alpha$ and condition (33) implies that $S_g(t) I_g(t) < S_b(t) I_b(t)$; since $I_g(t) > I_b(t)$, this is only possible if $S_b(t) > S_g(t)$. Thus, even though the exposure gap tightens after time $\hat{t}$, it must remain positive throughout Phase III.

Step 4: After time t_3, $\frac{p(t)}{1-p(t)}$ declines exponentially at a constant rate, $LI(t)$ is constant, and $LS(t)$ is decreasing but positive.

Consumers' interim belief at time t_3 equals $1-\rho$, making them indifferent whether to adopt after a good private signal. Let $a_G(t_3) \in [0, 1]$ be the probability with which consumers exposed to the innovation at time t_3 adopt after a good signal. Note that

$$X(t_3) = a_G(t_3)\left(\rho S_g(t_3) - (1-\rho)S_b(t_3)\right) - \left(I_g(t_3) - I_b(t_3)\right).$$

To establish that consumers' interim belief continues declining below $1-\rho$, it suffices to show that $X(t_3) < 0$. However, this follows immediately from the facts that $X(t_3-) < 0$ (proven in Step 2), $I_g(t_3) > I_b(t_3)$ (proven in Step 3), and $a_B(t_3) \in [0,1]$.

Once consumers' interim belief falls below $1-\rho$, immediately after time t_3, consumers herd on non-adoption; so, $X(t_3+) = -\left(I_g(t_3) - I_b(t_3)\right) < 0$ by Step 3 and beliefs continue to fall. Consumers therefore still herd on non-adoption, meaning that $I_g(t) = I_g(t_3)$, $I_b(t) = I_b(t_3)$, and hence $X(t) = X(t_3)$ and $LI(t) = LI(t_3)$ for all $t > t_3$. We conclude that all adoption ceases after time t_3 and that $\frac{p(t)}{1-p(t)}$ forevermore declines exponentially at the constant rate $|X(t_3)|$. In particular, $\lim_{t\to\infty} p(t) = 0$.

Finally, as discussed in Step 3, the fact that $p(t) < \alpha$ implies that $S_b(t)I_b(t) > S_g(t)I_g(t)$; hence, the exposure gap must shrink during obsolescence, i.e., $LS'(t) < 0$ for all $t > t_3$. At the same time, because $I_g(t) > I_b(t)$, the condition $S_b(t)I_b(t) > S_g(t)I_g(t)$ is only possible if $S_b(t) > S_g(t)$; thus, $LS(t) > 0$ for all $t > t_3$. □

Proof of Lemma 2. (i) In the proof of Theorem 1, we showed when $\alpha \in (1-\rho, \rho)$ that $I_g(t) > I_b(t)$ at all times $t \geq 0$ during a purely-viral campaign. Comparing equations (6,10), this implies $p_{BR}(T) < p(T)$ for all $T \geq 0$.

(ii) $p_{BR}(0+) = \alpha$ by equation (10) and $S_g(0+) = S_b(0+) = 1 - L$. $\frac{\mathrm{d}\log\left(S_g(T)/S_b(T)\right)}{\mathrm{d}T} = \frac{S_g'(T)}{S_g(T)} - \frac{S_b'(T)}{S_b(T)} = -(I_g(T) - I_b(T))$ by equation (1); thus $\frac{p_{BR}(T)}{1-p_{BR}(T)}$ falls exponentially at rate $I_g(T) - I_b(T)$.

(iii) By part (i) and Proposition 5, $\lim_{T\to\infty} p_{BR}(T) \leq \lim_{T\to\infty} p(T) = 0$. By part (ii), $p_{BR}(0+) = \alpha > 1-\rho$ and $p_{BR}(T)$ is strictly decreasing and continuous. $\overline{T}$ is therefore well-defined as the unique time at which $p_{BR}(\overline{T}) = 1-\rho$. Moreover, $\overline{T} > t_1$ because $p_{BR}(t_1) \approx \alpha > 1-\rho$ and $\overline{T} < t_3$ because $p_{BR}(t_3) < p(t_3) = 1-\rho$.

(iv) So far, we have shown that $\overline{T}$ must occur during Phase II or Phase III. To complete the proof, we need to show that $\overline{T}$ occurs during Phase III when $\alpha \in (\frac{1}{2}, \rho)$. In Section 2.1, we showed that condition SS holds throughout Phase II (corresponding to the intuition that there is "upward pressure" on beliefs when consumers are sensitive to signals) but fails to hold throughout Phase III. When $\alpha \in (\frac{1}{2}, \rho)$, the fact that $\frac{\alpha S_g(\overline{T})}{(1-\alpha)S_b(\overline{T})} = \frac{1-\rho}{\rho}$ (by definition of $\overline{T}$) implies $\frac{S_g(\overline{T})}{S_b(\overline{T})} < \frac{1-\rho}{\rho}$ (because $\alpha > 1/2$) and hence $\rho S_g(\overline{T}) - (1-\rho)S_b(\overline{T}) < 0$. Because $I_g(\overline{T}) - I_b(\overline{T}) > 0$, we conclude that condition SS must fail at time $\overline{T}$ and hence $\overline{T} \in (t_2, t_3)$. □

Proof of Proposition 6. (i) When $\alpha \in (1/2, \rho)$, the earliest optimal stopping time $T^* = t_2 < \overline{T}$ (Lemma 2, Theorem 2) and, as shown earlier, $\Delta R(T) < \Delta R(0)$ for all $T \leq \overline{T}$; so, $\Delta R(T^*) < \Delta R(0)$. (ii) When $\alpha \in (1-\rho, 1/2)$, $T^* = \overline{T}$ (Fig 3), $\overline{T} \in (t_1, t_3)$ and, as shown earlier, $\Delta R(T) > \Delta R(0)$ for all $T \in (t_1, t_3)$; so, $\Delta R(T^*) > \Delta R(0)$. (iii) When $\alpha = 1/2$, we have $t_1 = 0$ (Theorem 1), $p(0+) = \rho$, and $T^* \in (0, t_3)$. In this case: Phase I does not occur; $\frac{S_g(t)I_g(t)}{S_b(t)I_b(t)} = \frac{\rho}{1-\rho}$ for all $t \in (0, t_2]$ (Phase II); and $a_B(t) = 0$ for all $t \in [t_2, t_3]$ (Phase III). By equation (13), we conclude that $\Delta R'(T) = 0$ for all $T \in (0, t_3)$; so, $\Delta R(T^*) = \Delta R(0)$. □

Proof of Proposition 7. Let $\mathbf{a} = (a_G, a_B)$ denote consumers' "adoption rule" in the traditional marketing game, with a_G and a_B being, respectively, their likelihood of adopting after a good or bad private signal. Let $\mathcal{A}(\alpha)$ denote the set of optimal adoption rules, depending on the likelihood α of innovation goodness: if $\alpha > \rho$, then $\mathcal{A}(\alpha) = (1,1)$; if $\alpha = \rho$, then $\mathcal{A}(\alpha) = \{(1, a_B) : a_B \in [0,1]\}$; if $\alpha \in (1-\rho, \rho)$, then $\mathcal{A}(\alpha) = (1,0)$; if $\alpha = 1-\rho$, then $\mathcal{A}(\alpha) = \{(a_G, 0) : a_G \in [0,1]\}$; and if $\alpha < 1-\rho$, then $\mathcal{A}(\alpha) = (0,0)$. Good innovations earn revenue $R_g(\mathbf{a}) = \rho a_G + (1-\rho)a_B$, bad innovations earn revenue $R_b(\mathbf{a}) = (1-\rho)a_G + \rho a_B$, and good innovations earn extra revenue $\Delta R(\mathbf{a}) = (2\rho - 1)(a_G - a_B)$.

Let $\alpha(\mathbf{a}) = F(\Delta R(\mathbf{a}))$ denote the likelihood that the producer finds it optimal to invest when consumers use adoption rule $\mathbf{a}$. An equilibrium exists with likelihood α of innovation goodness if and only if $\alpha(\mathbf{a}) = \alpha$ for some $\mathbf{a} \in \mathcal{A}(\alpha)$.

The producer's incentive to invest is maximized when consumers use the adoption rule $\mathbf{a} = (1,0)$; so, α cannot possibly exceed $F(2\rho - 1)$ in any equilibrium. There are three relevant cases, depending on how $F(2\rho - 1)$ compares to the belief-thresholds ρ and $1-\rho$.

(i) Suppose that $F(2\rho - 1) < 1 - \rho$. Since $\alpha \leq F(2\rho - 1) < 1 - \rho$, consumers must find it optimal never to adopt in any equilibrium and, anticipating this, the producer has zero incentive to invest. Thus, all innovations are bad in the unique equilibrium, i.e., $\hat{\alpha} = 0$.

(ii) Suppose that $F(2\rho - 1) \in [1-\rho, \rho]$. In this case, an equilibrium exists with $\alpha = F(2\rho - 1)$ and consumer adoption rule $(1,0)$. Since the equilibrium likelihood of innovation goodness cannot exceed $F(2\rho - 1)$, we conclude that $\hat{\alpha} = F(2\rho - 1)$.

(iii) Suppose that $F(2\rho - 1) > \rho$. In this case, an equilibrium exists with $\alpha = \rho$ and adoption rule $\mathbf{a}^* = \left(1, 1 - \frac{F^{-1}(\rho)}{2\rho - 1}\right)$. (Given this adoption rule, the producer finds it optimal to invest with probability $F(\Delta R(\mathbf{a}^*)) = \rho$. Moreover, no equilibrium can exist with $\alpha > \rho$. Why not? In such an equilibrium, consumers would find it optimal to use adoption rule $(1,1)$, causing good and bad innovations to be equally adopted and hence giving the producer zero incentive to invest, a contradiction. We conclude that $\hat{\alpha} = \rho$. □

Proof of Proposition 8. Let $\Delta R(T;\alpha)$ denote the extra revenue earned by good innovations when consumers have ex ante belief α and the viral campaign stops at time T. Let $T^*(\alpha)$ be the optimal stopping time when consumers have ex ante belief $\alpha \in (1-\rho,\rho)$. Finally, let $S(\alpha) \equiv F(\Delta R(T^*(\alpha);\alpha))$ be the supply of good innovations when consumers have ex ante belief α and the viral campaign stops optimally at time $T^*(\alpha)$. An equilibrium exists with ex ante belief α if and only if $S(\alpha)=\alpha$.

Suppose that $\hat{\alpha} \equiv F(2\rho-1) \in (1/2,\rho)$. For all $\alpha \in (1/2,\rho)$, $\Delta R(T^*(\alpha);\alpha) < 2\rho-1$ (Proposition 6(i)); thus, $S(\alpha) < \hat{\alpha}$ for all $\alpha \in (1/2,\rho)$ and, in particular, $S(\alpha) < \alpha$ for all $\alpha \in [\hat{\alpha},\rho)$. On the other hand, because $\Delta R(0;1/2) = \Delta R(T^*(1/2);1/2) = 2\rho-1$ (Proposition 6(iii)), $S(1/2) = \hat{\alpha} > 1/2$. By a continuity argument (straightforward details omitted), there exists $\alpha \in (1/2,\hat{\alpha})$ such that $S(\alpha)=\alpha$. Overall, we conclude that $\alpha^* \in (1/2,\hat{\alpha})$.

Suppose next that $\hat{\alpha} \in (1-\rho,1/2)$. First, we show that no equilibrium exists with $\alpha \geq 1/2$. Suppose otherwise. With $\alpha \geq 1/2$, $\Delta R(T^*(\alpha);\alpha) \leq 2\rho-1$ (Proposition 6(i,iii)); thus, $S(\alpha) \leq \hat{\alpha} < 1/2$, contradicting the presumption that $S(\alpha)=\alpha \geq 1/2$. Next, for all $\alpha \in (1-\rho,1/2)$, $\Delta R(T^*(\alpha);\alpha) > 2\rho-1$ by Proposition 6(ii); thus, $S(\alpha) > \hat{\alpha}$ for all $\alpha \in (1-\rho,1/2)$. On the other hand, when $\alpha=1/2$, $\Delta R(0;1/2) = \Delta R(T^*(1/2);1/2) = 2\rho-1$ by Proposition 6(iii); thus, $S(1/2)=\hat{\alpha} < 1/2$. By continuity, there exists $\alpha \in (1-\rho,1/2)$ such that $S(\alpha)=\alpha$ and any such fixed point is strictly greater than $\hat{\alpha}$. All together, we conclude that $\alpha^* \in (\hat{\alpha},1/2)$.

The last case when $\hat{\alpha}=1/2$ is similar. $\Delta R(T^*(1/2);1/2) = \Delta R(0,1/2) = 2\rho-1$ (Proposition 6(iii)); thus, $S(1/2)=1/2$. On the other hand, for all $\alpha \in (1/2,\rho)$, $\Delta R(T^*(\alpha);\alpha) < 2\rho-1$ and hence $S(\alpha) < 1/2 < \alpha$. Thus, $\alpha^*=1/2$. □