Rational Disagreement and the Fragility of Social Learning: Identification Failures with Noisy Communication

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Abstract

We examine how agents learn when information from original sources only reaches them after noisy relay. In the presence of random mutation of message content and transmission failures, there is a sharp threshold such that a receiver fully learns if and only if they have access to more chains than the threshold number and they perfectly understand the noise process. However, even slight uncertainty over the relative rates of mutations makes learning from long chains impossible, no matter how many distinct sources information emerges from. The identification failure is that an agent cannot distinguish uncertainty about the state from uncertainty about the noise in communication. This result rationalizes long-run disagreement: even agents with a common prior and access to an arbitrary number of primary sources can end up with different beliefs if their network positions place them at different distances relative to primary sources.

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1 Introduction

People disagree on issues even after extensive communication and despite the presence of many primary sources of information in their social network. Yet standard models of social learning predict consensus and move on to study the correctness of limiting beliefs and the speed of convergence. This basic discrepancy between the foundational models and reality is a challenge to the social-learning literature. Can rational agents who repeatedly relay information in a network disagree in the long-run? In a simple model of social learning resembling the children's game of 'telephone,' we show that non-convergence and limited learning is generically the only outcome. The key difference from many standard models of social learning is the introduction of noisy communication.

A set of agents are connected in a network. Some of these agents, who we call primary sources, receive exogenous and independent binary signals about a binary state of the world (e.g., is eating avocados good or bad for one's cholesterol). They noisily relay this information to their neighbors, who relay it to their neighbors, and so on. We model noise as independent errors at each step of communication, whereby one type of message is sent or misheard as another. We consider the perspective of an agent (the 'learner') who has a slight uncertainty about the communication process; e.g., about the propensity of agents to overstate the health benefits of avocados or about the chance that someone misinterprets something they hear. The learner is at some distance from the primary sources in the network. We ask whether this agent can learn the true state.

We show that even under ideal conditions—when the learner is path connected to arbitrarily (even infinitely) many independent primary sources, knows the exact distance between themself and those sources, and carries out perfect Bayesian updating—they cannot learn the state. We also show that this implies that learners who have a common prior but are simply located at different distances from the source continue to have different beliefs regardless of the number of primary sources. These failures of learning and agreement must persist *a fortiori* in less ideal settings with more complex networks and noisy communication.

Noise and distance to primary sources diminish how much an agent can learn from any given number of primary sources. Still, if the learner knew the *exact* probabilities with which such "mutations" occur, they could learn the true state from sufficiently many sources, no matter how far away those sources may be.

However, we show that if the learner has even the slightest uncertainty about the *relative probability* that messages mutate in either direction, then they cannot perfectly learn the true state once the primary sources are sufficiently far away, no matter how many independent primary sources they hear from. If the learner hears, for example, a larger fraction of messages supporting the view that avocados are good for cholesterol, this may be because that is true or instead because it is false but people have more of a propensity to exaggerate the health benefits of avocados when they have no health impact, than to claim they have no effect when they do have a benefit. Indeed, *nothing* can be learned about the state in the limit as the distance to sources grows, even with infinitely-many sources and small uncertainty about mutation rates.

As we show, this limitation on learning stems from a basic identification failure: the learner cannot disentangle uncertainty about the state from uncertainty about the noise in the communication process. Either could account for the patterns of communication that an agent observes.

Related Literature

Our paper belongs to a broad literature on social learning.

Golub and Sadler (2017) conclude a review of social-learning models by noting that: "Long-run consensus is a central finding throughout this literature... The consistency of this finding may cause some discomfort because we often observe disagreement empirically, even about matters of fact... A theory explaining long-run disagreement, especially one with rational foundations and appropriate sensitivity to network structure, would constitute a valuable contribution." Rationalizing long-run disagreement in networks is indeed the objective of our model. Our theory suggests that agents with varying proximity to primary sources can entertain different beliefs, with disagreement persisting even as the number of such primary sources grows large.

There are models of learning that can rationalize people holding different beliefs in response to similar information. These include differences on the basis of biased updating (e.g., Fryer et al. (2019)), having optimal models that diverge based on slight sample differences (e.g., Haghtalab et al. (2021)), believing that others are using a misspecified model (Acemoglu, Chernozhukov and Yildiz (2016)), or having misperceptions of others' news access (Bowen et al. (2021)). Our results complement these other explanations, showing that even agents who are fully Bayesian and have a common prior can end up with different posterior beliefs if they face uncertainty about the nature of the noise in the communication process and have different network positions relative to primary sources. Understanding each of these forces is necessary to help design policies that correct learning failures. In a companion paper (Jackson, Malladi and McAdams (2022)), we examine optimal policies to alleviate the learning failures due to noisy communication and the identification problem that we document here.

Agents in our model communicate with noise. We do not microfound the source of this noise because our analysis only depends on the probabilities with which mutations occur at each step of the communication process. Thus, our results hold regardless of how this noise is generated; e.g., whether mutations arise from unintentional mistakes in interpretation, intentional miscommunication, or some technological imperfection. For other perspectives on the behavior of biased agents in the spread of incorrect information, see Acemoglu, Ozdaglar and ParandehGheibi (2010) and Bloch, Demange and Kranton (2018).

2 The Base Model of Noisy Information Transmission

We begin by studying how noise builds up along a chain that can travel a path of length T from an original source to a Bayesian "learner."

Information passes by "word of mouth." This can be oral, written, via social media, etc.

There are two possible states of the world, $\omega \in \{0, 1\}$. Let $\theta \in (0, 1)$ be the prior probability that the state is 1.

A sequence of agents $\{1, 2, \ldots, T\}$, referred to as a "chain," successively relays a signal of the state via word of mouth, terminating with the learner at $T \geq 1$.

We do not model what the learner does with this information, but one can think of the learner preferring to match their action with the state. For instance, the learner may hear from friends about whether a certain diet is good for cardiovascular health and decide whether to adopt it.[1]

A first agent in a chain, interpreted as "a primary source," observes a noisy signal of the state, $s_1 \in \{0, 1, \emptyset\}$.[2] That signal is transmitted with noise becoming $s_2 \in \{0, 1, \emptyset\}$, and so on until signal s_T reaches the learner.

The "null signal," $s_t = \emptyset$, indicates that no signal was received, in which case no signal is transmitted. Another possibility is that something was received, but that the information was sent along in some incoherent manner: one person hears from another but cannot understand what was said and so has no information to pass along. In particular, if agent $t \geq 1$ receives the null signal $s_t = \emptyset$, then all subsequent agents (including the learner) also receive the null signal.

If agent $t \geq 1$ receives a signal $s_t \in \{0, 1\}$, then that agent passes a signal along ($s_{t+1} \neq \emptyset$) with probability p_1 if $s_t = 1$, and with probability p_0 if $s_t = 0$. Thus, for instance, if $p_1 > p_0$ then agents are more likely to transmit a signal if they heard a 1, and vice versa if $p_1 < p_0$. With the remaining probabilities of $1 - p_1$ and $1 - p_0$, respectively, the signal is dropped and $s_{t+1} = \emptyset$.

Each time a non-null signal is transmitted, that signal mutates from 0 to 1 with probability $\mu_{01} \in [0, 1/2)$, or from 1 to 0 with probability $\mu_{10} \in [0, 1/2)$; we let $M \equiv 1 - \mu_{01} - \mu_{10}$. We focus on the case where mutation rates are less than 1/2: signals are more likely to be transmitted faithfully than flip at each step. Again, these mutations could be from a person deliberately changing a message to suit their personal preference, or could be due to some misunderstanding or other noise in communication.

Our reduced-form model of communication suffices for our study of the capacity for receivers to learn. We emphasize that for any microfoundation of senders' (potentially hetero-

[1] The learner may have information from sources outside of its network. If these sources are not direct, then they can be modeled as part of the network. Otherwise, we can think of this external information as being reflected in the prior. We are interested in studying what the learner is able to learn from messages conveyed within their network.

[2] We focus on a binary world to crystallize the main ideas. Extensions to richer state spaces and signal structures are left for future research.

geneous) incentives, only the resulting average probability of mutation and message dropping at each step matter for our analysis.

In summary, if $s_{t-1} = 1$, the next agent (including $t = 1$) hears: $s_t = 1$ with probability $p_1(1 - \mu_{10})$, $s_t = 0$ with probability $p_1\mu_{10}$, and $s_t = \emptyset$ with probability $1 - p_1$. Similarly, conditional on $s_{t-1} = 0$: $s_t = 1$ with probability $p_0\mu_{01}$, $s_t = 0$ with probability $p_0(1 - \mu_{01})$, and $s_t = \emptyset$ with probability $1 - p_0$. If $s_t = \emptyset$ for some t, then $s_{t+1} = \emptyset$. This defines a 3×3 Markov chain in which $\emptyset$ is an absorbing state.[3]

Our analysis presumes that the learner has access to some number $n \geq 1$ of length T chains of messages, relayed through an *information network*. This network is a depth T directed tree, with nodes representing agents, n leaves representing the primary sources, the root representing the learner, and edges representing the direction of relay. Each path from the leaves to the root is a chain along which messages are forwarded. We let $R(n, T)$ be the set of such trees, with three examples pictured in Figure 1.

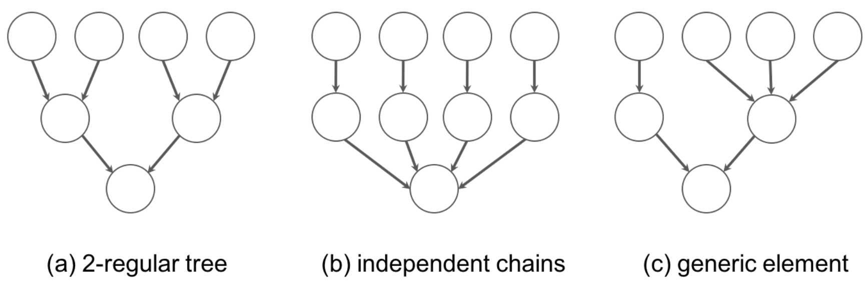

Figure 1: Three trees in $R(4, 2)$, the set of depth-2 directed trees with 4 leaves.

Conditionally independent signals of the state are independently relayed along each of these chains of length T via the same noisy process to the same learner. An example of the communication process over a 2-regular, depth 4 tree is pictured in Figure 2. The learner's ability to learn depends only on the number of primary sources, n, they are connected to and their distance, T. We therefore leave the precise structure of the information network within $R(n, T)$ unspecified in the statement of our results, and refer to the number of sources at distance T, $n(T)$.

3 Learning from Message Content

We first explore whether (and how much) the learner can learn about the true state in a special case of our model in which all messages are equally likely to be transmitted; i.e., $p_1 = p_0 = p$. In Section 4, we allow for the possibility that message content impacts the likelihood of transmission.

[3]Note that the setting is stationary in that the initial signal s_1 is derived from the original state in the same way as any other s_t depends on s_{t-1}, as if nature were "agent 0" in the chain with signal s_0 equal to the state. This assumption simplifies the expressions, but our analysis easily extends to allow first-signal accuracy and transmission failure rates to differ from subsequent ones.

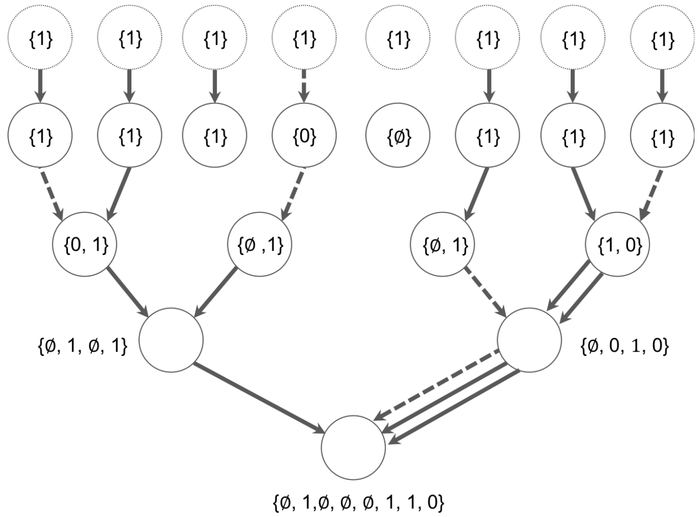

Figure 2: The root node ("learner") receives messages passed through eight paths, each starting from a different source. The absence of an arrow from one node to the one below it indicates that no message was sent, a dashed arrow indicates the message was delivered but mutated, and a solid arrow indicates that the message was delivered un-mutated. In this example, the true state is 1 and paths 1-3 and 6-8 begin with a correct initial signal, while path 4 begins with an incorrect initial signal and path 5 begins with no signal received. Initial messages are delivered on paths 1,2,4, and 6-8, mutating from 1 to 0 on path 1 and from 0 to 1 on path 4, and undelivered on paths 3 and 5. Messages are then relayed on path 2,4, and 6-8, mutating from 1 to 0 on path 6, but dropped on path 1. Finally, messages are re-relayed on paths 2 and 6-8, mutating from 0 to 1 on path 6, but dropped on path 4. Overall, the learner hears four messages, of which two never mutated, one mutated once, and one mutated twice.

Considering these cases in turn is essentially without loss of generality in our model. In Appendix 5 we show that limit learning is equivalent to the better of either using a threshold of signal content (e.g., whether more than a given fraction of signals are 1s vs 0s) or using a survival threshold (e.g., whether more than a given number of signals are observed). Thus, for understanding whether an observer can learn the state, it is enough to focus on each of these cases in turn.

3.1 Learning From Content Along a Single Chain

Lemma 1 characterizes learning along a single chain of messages when the message originates from a node at distance t from the observer.

LEMMA **1** *Suppose that $p = p_0 = p_1 > 0$ and consider any mutation rates $\mu_{01}, \mu_{10} \in (0, 1/2)$. If the state is 0 and agent $t \geq 1$ receives a non-null message, then the message is 0 (matching* *the true state) with probability*

$$X_0(t) = \frac{\mu_{10} + \mu_{01} M^t}{\mu_{10} + \mu_{01}}.$$

where $M = 1 - \mu_{10} - \mu_{01}$*. If the state is 1 and agent* $t \geq 1$ *receives a non-null message, the message is 1 (matching the true state) with probability*

$$X_1(t) = \frac{\mu_{01} + \mu_{10} M^t}{\mu_{01} + \mu_{10}}.$$

It follows that the difference between the probability of getting a given signal in its corresponding state compared to the other state is

$$X_0(t) - (1 - X_1(t)) = X_1(t) - (1 - X_0(t)) = M^t.$$

As t *grows, regardless of the starting state, the limit probabilities that a surviving message is a 0 or a 1, respectively, are*

$$\pi_0 = \frac{\mu_{10}}{\mu_{10} + \mu_{01}} \quad \text{and} \quad \pi_1 = \frac{\mu_{01}}{\mu_{10} + \mu_{01}}.$$

Finally, if $\mu_{01} = \mu_{10} = \mu$*, then the message matches the true state with probability*

$$X(t) = \frac{1 + M^t}{2}. \tag{1}$$

Note that $X(t) > 1/2$ for all t, $\lim_{t \to \infty} X(t) = 1/2$, and $X(t) - 1/2$ decreases exponentially at rate $1 - 2\mu$. Intuitively, the rate of decay, $1 - 2\mu = (1 - \mu) - \mu$, is how much more likely one is to get an unmutated signal than a mutated one from one period to the next.

3.2 Learning from Content Along Many Chains

We now characterize the threshold number of independent word-of-mouth chains that a Bayesian learner needs to access in order to have an accurate view of the true state.

Suppose that the learner has access to $n(T)$ independent chains of length T. We index n by T, because we seek to characterize how many chains are needed as a function of their length. Longer chains are more likely to be null or to have an incorrect message and so more are needed to deliver an equivalent amount of information. Let $I_{n(T)}$ be the vector of (potentially null) random messages that the learner receives from the chains, and let the random variable $b(n(T), T) = \Pr_T(\omega = 1 | I_{n(T)})$ be the posterior probability that the state equals 1 conditional on the information from $n(T)$ originating signals that have each independently traveled T steps.

Definition 1 **(Threshold for learning)** *We say that* $\tau(T)$ *is a threshold for learning if (i) Plim* $b(n(T), T) = 1$ *or* 0 *whenever* $n(T)/\tau(T) \to \infty$ *and (ii) Plim* $b(n(T), T) = \theta$ *whenever* $n(T)/\tau(T) \to 0$*.*

Note that if $Plim\ b(n(T), T) = 1$ or 0, then Bayesian-updated beliefs are correct with a probability going to 1. Thus, a threshold for learning is sharp in that if the number of chains of messages is of higher order, then the receiver learns the true state with a probability going to one, while if it is of lower order, the receiver learns *nothing*.

LEMMA **2** *Consider a learner who is connected to $n(T)$ primary sources at distance T and who knows p, μ_{01}, μ_{10}. Then $\frac{1}{p^T M^{2T}}$ is a threshold for learning.*

The threshold in Lemma 2 is sharp, and translates directly into a threshold for the average degree in the tree. For instance, suppose that the learner receives word-of-mouth messages through a random tree generated by a Galton-Watson branching process in which average degree distribution F places weight 0 on degree 0 and has finite variance.[4] Then, one can easily show that $Plim\ b(t) = 1$ or 0 whenever $\mathbb{E}_{d\sim F}[d] > \frac{1}{pM^2}$ and (ii) $Plim\ b(t) = \theta$ whenever $\mathbb{E}_{d\sim F}[d] < \frac{1}{pM^2}$.

3.3 The Impossibility of Learning with Uncertain Mutation Rates

Lemma 2 shows that learning is possible with sufficiently many primary sources, no matter how far away these sources may be, so long as the learner knows the mutation rates μ_{10} and μ_{01}. In practice, however, agents are at least somewhat uncertain about these mutation rates. As we show next, slight uncertainty about the ratio of these rates can dramatically limit what can be learned even from infinitely-many sources. Moreover, how much an agent with infinitely-many sources can learn itself converges to nothing as the distance to those sources goes to infinity, meaning that learning is *completely precluded* in the limit as the distance to sources grows.

PROPOSITION **1** *Suppose the learner does not know μ_{10}/μ_{01} but has an atomless prior over this ratio with convex support. Consider a learner who is connected to $n(T)$ primary sources at distance T.*

1. *There exists $\epsilon, \delta > 0$ and $\underline{T}$ such that for any $T > \underline{T}$ and any $n(T)$,*

$$Pr(|b(n(T), T) - \theta| > \epsilon) > \delta.$$

2. *Moreover, for any $n : \mathbb{N} \to \mathbb{N}$, $b(n(T), T)$ converges in probability to θ as T goes to infinity.*

The first part of Proposition 1 says that once sources are sufficiently distant, the probability that the learner learns is bounded away from 1 no matter how many independent sources she indirectly can access. The second part of Proposition 1 says that as the distance to sources grows, the learner learns nothing about the state no matter how quickly the

[4]This condition ensures that the tree does not die out and so has at least some paths of depth t with probability one. The analysis can be adapted to allow for extinction, but no new insight emerges.

number of sources grows. Note that Proposition 1 makes very weak assumptions about the nature of uncertainty over μ_{10}/μ_{01}. In particular, even if the learner is nearly certain about μ_{10}/μ_{01}; e.g., has a prior with a very narrow support, she cannot learn if T is sufficiently large.

When networks are sufficiently shallow, small enough uncertainties over mutation rates do not preclude learning. The receiver can still learn by hearing from sufficiently many sources. But the broad intuition for the first part of Proposition 1 is that depth compounds the effects of even small uncertainties and induces an identification problem for the receiver: once chains are sufficiently long, full learning is no longer possible. Observing more 1 messages than some threshold can either indicate that the state is 1 or that people are slightly more biased towards mutating 0's to 1's than the receiver anticipated. The next subsection discusses this more explicitly.

The intuition for the second part of Proposition 1 is that, as T grows, the fraction of 1 versus 0 messages converges to $\frac{\mu_{01}}{\mu_{10}}$. Learning comes from the fact that there is a bias away from $\frac{\mu_{01}}{\mu_{10}}$ in favor of the starting state, but that bias vanishes as T grows. Lemma 2 says that, if there are enough signals as a function of T (so $n(T)$ grows fast enough), then that slight bias can be discerned. However, any uncertainty about $\frac{\mu_{01}}{\mu_{10}}$ completely swamps the vanishing difference in the relative frequency of signals that reflects the starting state.

3.4 The Lack of Identification and the Failure of Learning

To hone intuition for Proposition 1 and understand the identification failure, consider a hypothetical extreme case in which there are *infinitely-many* independent chains. In this case, the learner at distance T gets an infinite number of signals that come in ratios proportional to their expected values. We *heuristically* reason at the limit to show that the results are due to identification, and not an order of limits argument.

Recall from Lemma 1 that, when the state is 1 or 0, each surviving (non-null) message's likelihood of being 1 equals $X_1(T)$ or $1 - X_0(T)$, respectively. As the number of primary sources goes to infinity, fraction $X_1(T)$ of surviving messages are 1 if the state is 1, while fraction $1 - X_0(T)$ are 1 if the state is 0. Recall also that

$$X_1(T) - (1 - X_0(T)) = M^T = (1 - \mu_{01} - \mu_{10})^T,$$

which implies $1 - X_0(T) = X_1(T) - M^T$. Note that the fraction of 1's heard by the learner is slightly smaller when the state is 0 than when the state is 1, but only by the vanishing amount M^T.

Suppose that a learner at distance T from primary sources sees a fraction f of messages that are 1. What are the possibilities about the state and mutation rates that can rationalize what the learner has observed? *Either* the state is 1 and μ_{01}, μ_{10} are such that they solve $f = X_1(T)$ *or* the state is 0 and μ_{01}, μ_{10} are such that they solve $f = X_1(T) - M^T$.

Note that as T becomes large, both $X_1(T)$ and $X_1(T) - M^T$ converge to $\frac{\mu_{01}}{\mu_{01}+\mu_{10}}$. Therefore, the fraction of 1's heard, f, is almost entirely driven by the mutation rates and nearly the same μ_{01}, μ_{10} can solve both equations. Since the learner can easily rationalize either equation with similar mutation rates, there is no way for the learner to distinguish which state it must be—unless the learner knows μ_{01}, μ_{10} sufficiently precisely to rule out very similar combinations of these mutation rates. As T grows, the required precision converges to requiring the learner know the ratio of mutation rates exactly.

In contrast, for small T, so that the learner is close to the original sources, M^T is nontrivial and the mutation rates that could solve the two equations are substantially different and some might be ruled out by the learner's prior. Thus, at least partial learning can be possible in the face of uncertain mutation rates if the learner is sufficiently close to the sources. However, given that M^T decays exponentially and that information is often relayed nontrivial distances, this can be demanding.

We note that being connected to primary sources at different distances, and being able to track those distances, can improve learning. If the learner could observe many messages at distance t and then again at distance $t + 1$, *and can distinguish how far a message has traveled*, then she could see how messages change with additional distance. This would help her further identify μ_{01}, μ_{10}. Thus, at least partial learning in the face of uncertainty is possible if the learner is *either* close to all sources *or* can precisely identify the distance that messages have traveled, both of which are demanding conditions.

3.5 Rational Disagreement

We have so far taken the perspective of a single learner who is some distance away from the primary sources. However, we may very well consider the updating problem of other nodes in the learner's information network. Suppose all nodes start with a common prior about the state. As the distance between these nodes and primary sources varies, so too does their ability to learn.

Consider a situation where network is so dense that, absent any uncertainty over mutations rates, all nodes would be able to learn the state with high certainty. Introducing a small uncertainty over the mutation rates makes it so that nodes close to the source continue to learn perfectly while sufficiently distant nodes update very little. Nodes at intermediate distances learn a bounded amount with some probability.

Therefore, even though all agents are Bayesian and started with a common prior, they entertain different beliefs based on their network position. Uncertainty over noise can therefore rationalize long-run disagreement between path-connected agents even after extensive communication.

4 Learning from Message Survival

Proposition 1 shows how even slight uncertainty about relative mutation rates can preclude learning, in the special case of our model in which all messages are equally likely to transmitted at each step. But what if the content of a message affects its likelihood of being retransmitted; i.e., what if $p_1 \neq p_0$? In that context, receivers can learn not only from the content of received messages but also from how often they have heard about a given issue. For example, it is often observed that statistically significant results are more likely to be shared and published than insignificant ones. If researchers in an academic field know that many people are working on a topic, but do not hear of many results, then they may infer that most results were insignificant, even without paying attention to the content of the studies that were published and shared.

We show in this section that, although agents can learn from message survival in this context (Sections 4.1-4.2), learning from messages received over sufficiently long chains remains impossible so long as agents have any uncertainty about the relative mutation rates (Section 4.3). Finally, we consider how well boundedly rational learners who pay attention to either the number of messages they have heard (ignoring the content) or instead simply to the relative fraction of 1s to 0s compare in efficiency to a Bayesian learner who pays attention to both dimensions (Section 4.4). Without loss of generality, we focus on the case in which $p_1 > p_0$, meaning that people are more likely to pass along signal 1 than 0.

4.1 Learning from Survival Along a Single Chain

We first characterize learning along a single chain of messages when learning purely from survival – where $\mu_{01} = \mu_{10} = \mu$ – to build intuition as to how this compares to learning purely from content (the $p = p_0 = p_1$ case of Lemma 1).

In the case where only message content is informative ($p_0 = p_1$), the content of a single message becomes nearly meaningless as chains grow long (due to mutation). In contrast, when $p_0 \neq p_1$, message survival continues to be informative about the true state of the world even as the chain of messages grows long – although survival becomes decreasingly likely.

Let

$$z \equiv \frac{p_1}{p_0}\left(1 + (1-2\mu)\frac{(p_1 - p_0)}{p_0 + \mu(p_1 - p_0)}\right).$$

Note that if $p_1 > p_0$ and $0 \geq \mu \leq 1/2$, then $z \geq \frac{p_1}{p_0} > 1$.

LEMMA **3** *Suppose that* $1 \geq p_1 > p_0 > 0$ *and* $\mu_{01} = \mu_{10} = \mu \in (0, 1/2]$.[5]

1. *The relative probability of message survival over a chain of length* t *conditional on state 1 versus state 0 is uniformly bounded away from* $\frac{p_1}{p_0}$*:*

$$\frac{Pr(s_t \neq \emptyset|\omega = 1)}{Pr(s_t \neq \emptyset|\omega = 0)} \geq z \geq \frac{p_1}{p_0} \text{ for all } t \geq 1, \tag{2}$$

with strict inequalities when $\mu < 1/2$.

[5]If $\mu = 0$ then $\frac{Pr(s_t \neq \emptyset|\omega=1)}{Pr(s_t \neq \emptyset|\omega=0)} = (p_1/p_0)^t$, which diverges, and the problem becomes trivial. Similarly, if $p_0 = 0$ then $Pr(s_t \neq \emptyset|\omega = 0) = 0$ and the problem becomes trivial. Note that here we do not require that $\mu < 1/2$ since survival at the first step contains information, even if subsequent steps are completely random. This contrasts with the case in which $p_1 = p_0$, in which learning is precluded when $\mu = 1/2$.

2. *The ratio in (2) converges as chain-length grows:* $y \equiv \lim_{t\to\infty} \frac{Pr(s_t \neq \emptyset|\omega=1)}{Pr(s_t \neq \emptyset|\omega=0)}$ *exists.*

3. *Upon seeing a surviving message, the learner's updated belief* $Pr(\omega = 1|s_t \neq \emptyset)$ *is uniformly bounded below by* $\frac{\theta}{\theta+(1-\theta)/z} > \theta$ *and bounded above in the limit by* $\frac{\theta}{\theta+(1-\theta)/y} < 1$.

4. *In the limit, updating is entirely due to signal survival and not content:* $\lim_{t\to\infty} Pr(\omega = 1|s_t = 1) = \lim_{t\to\infty} Pr(\omega = 1|s_t = 0) = \lim_{t\to\infty} Pr(\omega = 1|s_t \neq \emptyset)$.

Unlike the content of a single message, which becomes nearly meaningless as chains grow long (due to mutation), the information conveyed by a single message's survival does not vanish in the long-chain limit. For intuition, suppose for a moment that only the first agent in each chain was biased in favor of message 1, and other agents transmit with probability $\widehat{p}$ regardless of signal content. The likelihood of survival to t is $p_1 (\widehat{p})^{t-1}$ if the first agent saw signal 1 or $p_0 (\widehat{p})^{t-1}$ if the first agent saw signal 0. Thus, the relative likelihood of survival equals $p_1/p_0 > 1$ (favoring signal 1) no matter how long the chain. Moreover, biasing all agents in favor of transmitting message 1 further increases the relative likelihood of survival from state 1, since signal 1 is more likely to be received at each step along the chain when the true state is 1 rather than 0.

4.2 Learning from Survival Along Many Chains

We next consider the challenge of learning for a receiver who only counts messages without checking what they say. In parallel to the case of learning from signal content only ($p_0 = p_1$), the learner can discern the state from just signal frequency as long as transmission is more likely after one signal than the other ($p_0 \neq p_1$), there are sufficiently many starting sources of information, and the learner knows the transmission differences perfectly.

LEMMA 4 *Consider a learner who is connected to* $n(T)$ *primary sources at distance* T. *Suppose that* $\mu_{01} = \mu_{10} = \mu \in (0, 1/2]$ *and* $1 > p_1 > p_0 > 0$.[6] *There exists* $\lambda(T) = c + o(1)$ *for some* $c \in (0, 1)$, *such that a threshold for learning when conditioning only upon signal survival is*

$$\frac{1}{(p_1\lambda(T) + (1 - \lambda(T))p_0)^T}.$$

Given that messages mutate, the probability that any agent transmits a message lies somewhere between p_1 and p_0. Conditional on the initial message being 1, the overall probability that a message is transmitted all the way to the end of a length-T chain must therefore take the form $(p_1\lambda + (1 - \lambda)p_0)^T$ for some $\lambda \in (0, 1)$. Only if the number of sequences $n(T)$ grows faster than this would a growing number of signals survive, conditional on the state being 1. The learner can then discern the state (perfectly in the limit) based on the actual

[6] If $\mu = 0$, then it is easy to check that the threshold is an expected degree of $1/p$, which is then the threshold for messages to survive conditional upon state $\omega = 1$, which are the more likely to survive.

number of signals that survive. As with the case of learning only from content, the threshold $n(T)$ grows exponentially in chain length.

4.3 Impossibility of Learning from Message Survival with Uncertain and Asymmetric Transmission Rates

The identification failure and learning-impossibility finding of Proposition 1 persists when agents are able to learn from message survival. As before, even slight uncertainty about relative mutation rates completely precludes learning from distant sources. To see why, note that as T grows, only a vanishing fraction of chains survive. When $p_1 \neq p_0$, slightly changing μ_{01}/μ_{10} changes that fraction by orders of magnitude even though it will still be vanishing. This crowds out the information about the original state that can be gleaned from survival, which dies out over the sequence.[7]

We remark that other forms of uncertainty can also hamper learning. For example, agents may have uncertainty about the average degree in the network. Even if $p_0 \neq p_1$, as distance to sources grows, survival rates of messages converge across states as messages are increasingly likely to have mutated. Thus learning becomes increasingly dependent on knowing how many chains of messages originated from original sources. In that case, uncertainty over the network can preclude learning from survival. Our broader message is that when message transmission is noisy, uncertainty about the communication environment can lead to an identification problem and make learning impossible.

4.4 Alternatives to Full Bayesian Learning

Our analysis dispels the mystery of non-consensus by considering the impact of noisy communication alone in an otherwise idealized model (i.e., with Bayesian learners, simple and known networks, and a preponderance of communication). Our results suggest all the more that one should expect failure of consensus in settings with bounded learners and imperfectly known and complex networks. On the other hand, this leaves open the question of what the most important bottleneck to learning might be in such settings.

One obvious possibility is that people may simply not communicate much on certain issues, so beliefs never have a chance to converge. But what is the true bottleneck to learning in the many important situations where communication is abundant? An explanation suggested by our model is that the messages agents receive over social communication channels

[7] In particular, to see this (wlog) consider a case in which $1 > p_1 > p_0 > 0$. Let $Z_{\mu_{01}/\mu_{10}}(T, s)$ be the probability that a signal survives T periods conditional on starting out as a signal s, given μ_{01}/μ_{10}. The key observation is that $Z_{\pi}(t, 0)/Z_{\mu_{01}/\mu_{10}-\varepsilon}(T, 1)$ grows without bound as T grows, for any ε. Both probabilities are tending to 0, but eventually they mix. The probabilities are strings of products of combinations of p_1s and p_0s, and tilting that combination one way or the other eventually accumulates arbitrarily in terms of *relative* probabilities as things are exponentiated. Even a small shift in the fraction of mutations completely overturns the advantage of the starting state. Then tiny uncertainty about μ_{01}/μ_{10} introduces much larger swings in the survival rates than the starting states.

are sufficiently uninformative that agents correctly update their beliefs relatively little and by different amounts, depending on where they sit in the communication network. An alternative possibility is that agents fail to properly update beliefs despite the informativeness of social communication.

Comparing these possibilities theoretically requires positing a model of bounded learning. In Appendix B, we consider two forms of siimple bounded learning that are natural in the context of our model. First, for any parameters of the communication and noise process, consider a Bayesian learner who faces no uncertainty about those parameters. We show that, for this Bayesian learner, the number of primary sources needed for learning as distances grows large is the same as for the better of two types of bounded learners who only pay attention to either (i) the average content of the messages received or (ii) the number of messages received. This suggests that imperfect updating need not be a bottleneck for learning in settings like ours with abundant messages.

5 Concluding Remarks

We introduced a benchmark model of social learning via relayed signals in the presence of mutations and transmission failures. We showed that, even with a perfect understanding of the transmission process, learning is challenging in that it requires an exponentially growing number of original sources as the length of the chains over which information is relayed grows. Moreover, the slightest uncertainty over relative mutation rates renders learning from distant sources impossible regardless of the number of chains observed.

This finding rationalizes disagreement in networks and further suggests that forces which lengthen chains of communication (e.g., certain forms of online communication) can severely disrupt social learning, even if they increase the frequency of communication.

The difficulty of learning from distant sources naturally motivates learners to seek out information from closer, trusted contacts, and to down-weight or ignore more distantly-sourced information. We explore policies that can help agents learn from closer sources in another paper, Jackson et al. (2022).

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Appendix A: Proofs

Proof of Lemma 1: We derive the expressions of X_0, X_1, which can also be deduced from standard Markov chain results, but it may be useful for the reader to see the derivation. The proof is by induction. We give the proof for X_0, when the state is 0. The proof for X_1 is symmetric and the expression for X is a special case.

First, note that if $t = 1$ then this expression simplifies to $1 - \mu_{01}$, which is exactly the probability that the message has not mutated, and so this holds for $t = 1$.

Then for the induction step, supposing that the claimed expression is correct for $t - 1$, we show it is correct for t.

The probability of matching the true state at t is the probability of not matching at $t-1$ times μ_{10} plus the probability of matching at $t - 1$ times $1 - \mu_{01}$, which by the induction assumption can be written as

$$\begin{aligned}
&\left[1 - \frac{\mu_{10} + \mu_{01}M^{t-1}}{\mu_{10} + \mu_{01}}\right] \mu_{10} + \left[\frac{\mu_{10} + \mu_{01}M^{t-1}}{\mu_{10} + \mu_{01}}\right] (1 - \mu_{01}) \\
&= \mu_{10} + \left[\frac{-\mu_{10}^2 - \mu_{01}\mu_{10}M^{t-1} + \mu_{10} - \mu_{10}\mu_{01} + \mu_{01}M^{t-1} - \mu_{01}^2 M^{t-1}}{\mu_{10} + \mu_{01}}\right] \\
&= \mu_{10} + \left[\frac{-\mu_{10}^2 + \mu_{10} - \mu_{10}\mu_{01} + \mu_{01}M^{t-1}(1 - \mu_{10} - \mu_{01})}{\mu_{10} + \mu_{01}}\right] \\
&= \frac{\mu_{10} + \mu_{01}M^t}{\mu_{10} + \mu_{01}}.
\end{aligned}$$

as claimed. ▮

Proof of Lemma 2:

Consider the case in which the true state is 0, as the other case is analogous.

The expected number of 0 messages is $n(t)p^t X_0(t)$ while the expected number of 0 messages when the state is 1 is $n(t)p^t(1 - X_1(t))$. The difference in the expected number of 0 messages across states is

$$D(t) \equiv n(t)p^t X_0(t) - n(t)p^t(1 - X_1(t)) = n(t)p^t M^t.$$

If the standard deviation of the number of 0 messages in both states divided by $D(t)$ goes to 0, then by Chebychev's inequality, the probability of seeing more than $n(t)p^t(1-X_1(t))+\frac{D(t)}{2}$ 0 messages when the state is 1 goes to zero. On the other hand, the probability of seeing fewer than this many 0 messages when the state is 0 goes to zero.

When the ratio of standard deviation to $D(t)$ goes to infinity, the likelihood ratio between both states is within $1-\epsilon(t)$ on a $1-\delta(t)$ measure of messages (say, according to the measure when the state is 0), where $\epsilon, \delta \to 0$. Therefore, there is no learning in the limit.

It is therefore enough to show that the ratio of standard deviation to $D(t)$ goes to either infinity or zero when $n(t)$ is above or below the threshold, respectively. Now, the standard deviation of the number of 0 messages in the 0 state divided by the amount above is

$$\frac{(X_0(1-X_0)n(t)p^t)^{1/2}}{n(t)p^t M^t} = \frac{(X_0(1-X_0))^{1/2}}{(n(t)p^t)^{1/2} M^t}.$$

Note that the numerator converges to a constant, and so this expression either goes to 0 or infinity depending on whether $(n(t)p^t)^{1/2}M^t$ goes to 0 or infinity, which depends on whether $(n(t)pM^2)^t$ goes to 0 or infinity.

The expressions for all the other standard deviations and differences are analogous. ▮

Proof of Proposition 1:

We give the proof for the case in which $p^t n(t) \to \infty$. (With fewer paths there are even fewer signals from which to learn.) The following lemma is straightforward (and so its proof is omitted) but it useful.

Lemma **5** *Consider a sequence of $k \leq m$ such that $m \to \infty$ and $\frac{k}{m} \to a$. The maximizer of $z^k(1-z)^{m-k}$ is $z(m,k) = \frac{k}{m}$, and*

$$\frac{z(m,k)^k(1-z(m,k))^{m-k}}{z^k(1-z)^{m-k}} \to \infty$$

for any $z \neq a$, the size of this ratio increases with the distance of z from a (as $z^a(1-z)^{1-a}$ is strictly concave). Moreover, for any atomless and continuous probability measure G on z that has connected support and includes a in its interior

$$\frac{\int_{a-\varepsilon}^{a+\varepsilon} z^k(1-z)^{m-k} dG(z)}{\int_0^1 z^k(1-z)^{m-k} dG(z)} \to 1,$$

for any $\varepsilon > 0$.

Via a standard calculation for the limiting distribution for a two-state Markov chain, the steady state limit probability that the message is 1 is

$$\frac{\mu_{01}}{\mu_{01}+\mu_{10}} \equiv \rho.$$

The probability that some sequence ends in a 1 conditional on survival, ρ and starting in state $\omega = 1$ is

$$\rho + (1-\rho)M^t.$$

The probability that some sequence ends in a 1 conditional on survival, ρ and starting in state $\omega = 0$ is

$$\rho(1-M^t).$$

Similar calculations provide probabilities of ending in a 0.

The chance of observing k 1s, conditional on m sequences reaching the receiver, on μ_{01} and on the starting state being $\omega = 1$ is then

$$P_{k,m,t,\rho}(1) = \binom{m}{k} \left[\rho + (1-\rho)M^t\right]^k \left[(1-\rho)(1-M^t)\right]^{m-k}.$$

Then the chance of observing k 1s out of m sequences that reach the receiver conditional on the starting state being $\omega = 0$ is then

$$P_{k,m,t,\rho}(0) = \binom{m}{k} \left[\rho(1-M^t)\right]^k \left[(1-\rho) + \rho M^t\right]^{m-k}.$$

First consider the case where ρ is known, and suppose the state is 1 (the argument for the case where the state is 0 is analogous). As the number of signals grows large (keeping t fixed), $\frac{k}{m-k} \to \frac{\mu_{01}+\mu_{10}M^t}{\mu_{10}-\mu_{10}M^t} \equiv a_{t,1}$ in probability. Suppose without loss of generality that $\mu_{01} \geq \mu_{10}$ so $a_{t,1} > 1$ for all t. Now, the Bayesian's posterior that the state is $\omega = 1$ conditional upon seeing k 1's out of m sequences that reached the receiver

$$\frac{\theta P_{k,m,t,\rho}(1)}{\theta P_{k,m,t,\rho}(1) + (1-\theta)P_{k,m,t,\rho}(0)},$$

Let k_n and m_n be the random number of 1's and messages received respectively with n length t chains. By Lemma 5, $\frac{P_{k_n,m_n,t,\rho}(0)}{P_{k_n,m_n,t,\rho}(1)} \to 0$ in probability as the number of signals n grow large, so it follows that

$$\frac{\theta P_{k_n,m_n,t,\rho}(1)}{\theta P_{k_n,m_n,t,\rho}(1) + (1-\theta)P_{k_n,m_n,t,\rho}(0)} \to 1.$$

Therefore, since the agent can learn the true state with sufficiently many paths for any given t, it follows that the agent can learn the true state as $t \to \infty$ if $n(t)$ grows quickly enough.

Now we consider the case when ρ is unknown but follows an atomless distribution F with connected support. A Bayesian's posterior that the state is $\omega = 1$ conditional upon seeing k 1's out of m sequences that reached the receiver is

$$\frac{\theta \int_\rho P_{k,m,t,\rho}(1)dF(\rho)}{\theta \int_\rho P_{k,m,t,\rho}(1)dF(\rho) + (1-\theta)\int_\rho P_{k,m,t,\rho}(0)dF(\rho)},$$

and so if we can show that $\int_\rho P_{k,m,t,\rho}(1)dF(\rho)/\int_\rho P_{k,m,t,\rho}(0)dF(\rho)$ converges to one in probability, then we conclude the proof.

Given a true $\mu_{01}{}^*, \mu^*_{10}$ such that $\frac{\mu^*_{10}}{\mu^*_{01}}$ in the interior of the support of F, the realized k, m will be such that $\frac{k}{m-k} - \frac{\mu^*_{01}+\mu^*_{10}M^t}{\mu^*_{10}-\mu^*_{10}M^t} = \frac{k}{m-k} - \frac{\rho^*+(1-\rho^*)M^t}{(1-\rho^*)-(1-\rho^*)M^t}$ converges to 0 in probability, and $\frac{\rho^*+(1-\rho^*)M^t}{(1-\rho^*)-(1-\rho^*)M^t} \to \frac{\rho^*}{1-\rho^*} = a$.

By the first part of Lemma 5, for any k, m, $P_{k,m,t,\rho}(1)$ is maximized when ρ equals $\rho(t,k,m,1)$ such that

$$\rho(t,k,m,1) + (1-\rho(t,k,m,1))M^t = \frac{k}{m},$$

and $P_{k,m,t,\rho}(0)$ is maximized when ρ equals $\rho(t,k,m,0)$ such that

$$\rho(t,k,m,0)(1-M^t) = \frac{k}{m}.$$

$\rho(t,k,m,1)$ and $\rho(t,k,m,0)$ converge to each other, and to ρ^*, as well in probability. It therefore follows from Lemma 5 that

$$plim\ \frac{\int_\rho P_{k,m,t,\rho}(1)dF(\rho)}{\int_\rho P_{k,m,t,\rho}(0)dF(\rho)} = plim\ \frac{\int_{\rho(t,k,m,1)-\varepsilon}^{\rho(t,k,m,1)+\varepsilon} P_{k,m,t,\rho}(1)dF(\rho)}{\int_{\rho(t,k,m,0)-\varepsilon}^{\rho(t,k,m,0)+\varepsilon} P_{k,m,t,\rho}(0)dF(\rho)}$$

for any $\varepsilon > 0$.

Letting $[l,h]$ be the support of ρ, note that (a) $P_{k,m,t,\rho(t,k,m,1)}(1) = P_{k,m,t,\rho(t,k,m,0)}(0)$, and (b) that $\frac{d}{d\rho}P_{k,m,t,\rho}(1)$ evaluated at ρ' and $\frac{d}{d\rho}P_{k,m,t,\rho}(0)$ evaluated at ρ'' are the same when $P_{k,m,t,\rho'}(1) = P_{k,m,t,\rho''}(0)$. It follows that

$$P_{k,m,t,\rho(t,k,m,1)+\delta}(1) = P_{k,m,t,\rho(t,k,m,0)+\delta}(0)$$

for any $\delta \in \mathbb{R}$ such that both $\rho(t,k,m,1)+\delta$ and $\rho(t,k,m,0)+\delta$ fall in (l,h). In particular, if we let $\varepsilon_t = \frac{1}{2}\min\{\rho(t,k,m,1)-l, h-\rho(t,k,m,0)\}$ and if $\varepsilon_t > 0$, the intervals $[\rho(t,k,m,1)-\varepsilon_t, \rho(t,k,m,1)+\varepsilon_t]$ and $[\rho(t,k,m,0)-\varepsilon_t, \rho(t,k,m,0)+\varepsilon_t]$ strictly lie in (l,h). So by the earlier observation,

$$\int_{\rho(t,k,m,1)-\varepsilon_t}^{\rho(t,k,m,1)+\varepsilon_t} P_{k,m,t,\rho}(1)dF(\rho) = \int_{\rho(t,k,m,1)-\varepsilon_t}^{\rho(t,k,m,1)+\varepsilon_t} P_{k,m,t,\rho}(0)dF(\rho)$$

Moreover $plim\ \varepsilon_t = \frac{1}{2}\min\{\rho^* - l, h - \rho^*\} > 0$. Therefore, by the continuous mapping theorem,

$$plim\ \frac{\int_\rho P_{k,m,t,\rho}(1)dF(\rho)}{\int_\rho P_{k,m,t,\rho}(0)dF(\rho)} = plim\ \frac{\int_{\rho(t,k,m,1)-\varepsilon_t}^{\rho(t,k,m,1)+\varepsilon_t} P_{k,m,t,\rho}(1)dF(\rho)}{\int_{\mu_{01}(t,k,m,0)-\varepsilon_t}^{\rho(t,k,m,0)+\varepsilon_t} P_{k,m,t,\rho}(0)dF(\rho)} = 1,$$

which concludes the proof. ▮

Proof of Lemma 3, Part 1:

For ease of notation, let $P_{1S}^t \equiv Pr(s_t \neq \emptyset | \omega = 1)$ and $P_{0S}^t \equiv Pr(s_t \neq \emptyset | \omega = 0)$. These are the probabilities of signal survival to time t conditional on the first signal.

First we prove that $\frac{P_{1S}^t}{P_{0S}^t} \geq \frac{p_1}{p_0}$, with strict inequality when $\mu < 1/2$ and $t > 1$.

This is proven by induction. First, $P_{1S}^1 = p_1 > p_0 = P_{0S}^1$. Next, let us show that $\frac{P_{1S}^t}{P_{0S}^t} \geq \frac{p_1}{p_0}$ given that $P_{1S}^{t-1} > P_{0S}^{t-1}$. Note that given the stationarity of the process, $P_{1S}^{t-1} = Pr(s_t \neq \emptyset | s_1 = 1)$ and $P_{0S}^{t-1} = Pr(s_t \neq \emptyset | s_1 = 0)$, and then we can write[8] The first part

$$P_{1S}^t = p_1 \left[(1-\mu)Pr(s_t \neq \emptyset | s_1 = 1) + \mu Pr(s_t \neq \emptyset | s_1 = 0)\right],$$

and so then it follows that

$$P_{1S}^t = p_1(1-\mu)P_{1S}^{t-1} + p_1\mu P_{0S}^{t-1}.$$

Then by the inductive step ($P_{1S}^{t-1} > P_{0S}^{t-1}$) and so it follows that

$$P_{1S}^t \geq p_1(1-\mu)P_{0S}^{t-1} + p_1\mu P_{1S}^{t-1},$$

with strict inequality when $\mu < 1/2$ and $t > 1$. Similarly,

$$P_{0S}^t = p_0(1-\mu)P_{0S}^{t-1} + p_0\mu P_{1S}^{t-1}.$$

Therefore

$$\frac{P_{1S}^t}{P_{0S}^t} \geq \left(\frac{p_1}{p_0}\right)\frac{(1-\mu)P_{0S}^{t-1} + \mu P_{1S}^{t-1}}{(1-\mu)P_{0S}^{t-1} + \mu P_{1S}^{t-1}} = \frac{p_1}{p_0},$$

with strict inequality when $\mu < 1/2$ and $t > 1$, as claimed.

Now we complete the proof of the first part of the lemma. Note that (from above)

$$\frac{P_{1S}^t}{P_{0S}^t} = \left(\frac{p_1}{p_0}\right)\frac{(1-\mu)P_{1S}^{t-1} + \mu P_{0S}^{t-1}}{(1-\mu)P_{0S}^{t-1} + \mu P_{1S}^{t-1}}.$$

Therefore,

$$\frac{P_{1S}^t}{P_{0S}^t} = \left(\frac{p_1}{p_0}\right)\left(\frac{(1-\mu)P_{0S}^{t-1} + \mu P_{1S}^{t-1} + (1-2\mu)(P_{1S}^{t-1} - P_{0S}^{t-1})}{(1-\mu)P_{0S}^{t-1} + \mu P_{1S}^{t-1}}\right),$$

[8] The starting state s_0 is 1 in this calculation and so then there is a probability p that the signal survives to the first period, and then the calculation inside the $[\cdot]$ handles the two possible values of the first period signal and then the probability the signal survives to t if it has made it to the first period in the two possible values it could have in the first period.

and then since $p_1 > p_0$ and $\frac{P_{1S}^{t-1}}{P_{0S}^{t-1}} \geq \frac{p_1}{p_0}$, with strict inequality when $\mu < 1/2$ and $t > 1$, it follows that

$$\frac{P_{1S}^t}{P_{0S}^t} = \left(\frac{p_1}{p_0}\right)\left(1 + (1-2\mu)\frac{P_{1S}^{t-1} - P_{0S}^{t-1}}{P_{0S}^{t-1} + \mu(P_{1S}^{t-1} - P_{0S}^{t-1}})\right) \geq \frac{p_1}{p_0}\left(1 + (1-2\mu)\frac{(p_1 - p_0)}{p_0 + \mu(p - p_0)}\right),$$

with strict inequality when $\mu < 1/2$ and $t > 1$ (and it directly follows that this expression (z) is strictly larger than p_1/p_0 when $\mu < 1/2$), as claimed. ∎

The following result is useful in the proofs of the remaining parts of Lemma 3.

LEMMA **6** *Fix* $\theta \in (0,1), \mu_{01} = \mu_{10} = \mu \in (0, 1/2], 0 < p_0 \leq p_1 \leq 1$. *For all* $t > 0$,

$$Pr(s_t = 1|\omega = 1) \geq Pr(s_t = 0|\omega = 1).$$

Moreover, either there exists T *large enough such that*

$$Pr(s_t = 1|\omega = 0) \geq Pr(s_t = 0|\omega = 0) \quad \textit{for all} \quad \textit{for all } t \geq T,$$

or

$$Pr(s_t = 1|\omega = 0) < Pr(s_t = 0|\omega = 0) \textit{ for all } t.$$

Finally, the sequence

$$\frac{Pr(s_t = 1|\omega = 1)}{\min\{Pr(s_t = 1|\omega = 0), Pr(s_t = 0|\omega = 0)\}}$$

is bounded above.

Proof of Lemma 6:

The first claim is proven by induction:

Since $\mu \leq 1/2$, $Pr(s_1 = 1|\omega = 1) \geq Pr(s_1 = 0|\omega = 1)$. Suppose $Pr(s_t = 1|\omega = 1) \geq Pr(s_t = 0|\omega = 1)$. Note that,

$$\begin{aligned} Pr(s_{t+1} = 1|\omega = 1) &= p_1(1-\mu)Pr(s_t = 1|\omega = 1) + p_0\mu Pr(s_t = 0|\omega = 1) \\ Pr(s_{t+1} = 0|\omega = 1) &= p_1\mu Pr(s_t = 1|\omega = 1) + p_0(1-\mu)Pr(s_t = 0|\omega = 1). \end{aligned}$$

The result then follows from the inductive hypothesis and the facts that $p_1 \geq p_0$ and $\mu \leq 1/2$.

Next, to show the second claim in the lemma, note that

$$\begin{aligned} Pr(s_{t+1} = 1|\omega = 0) &= p_1(1-\mu)Pr(s_t = 1|\omega = 0) + p_0\mu Pr(s_t = 0|\omega = 0) \\ Pr(s_{t+1} = 0|\omega = 0) &= p_1\mu Pr(s_t = 1|\omega = 0) + p_0(1-\mu)Pr(s_t = 0|\omega = 0). \end{aligned}$$

Then if $Pr(s_t = 1|\omega = 0) \geq Pr(s_t = 0|\omega = 0)$ for some $t = T$, the same will hold for all $t > T$ by a similar inductive proof. Otherwise $Pr(s_t = 1|\omega = 0) < Pr(s_t = 0|\omega = 0)$ for all t, and then the result holds directly.

Finally, we show the third part of the claim. By the second part of this lemma, there are two cases to consider. If $Pr(s_t = 1|\omega = 0) < Pr(s_t = 0|\omega = 0)$ for all t. Then

$$\frac{Pr(s_t = 1|\omega = 1)}{\min\{Pr(s_t = 1|\omega = 0), Pr(s_t = 0|\omega = 0)\}} = \frac{Pr(s_t = 1|\omega = 1)}{Pr(s_t = 1|\omega = 0)}$$

If instead there is a T such that for all $t \geq T$, $Pr(s_t = 1|\omega = 0) \geq Pr(s_t = 0|\omega = 0)$, then

$$\begin{aligned}
\frac{Pr(s_t = 1|\omega = 1)}{\min\{Pr(s_t = 1|\omega = 0), Pr(s_t = 0|\omega = 0)\}} &= \frac{Pr(s_t = 1|\omega = 1)}{Pr(s_t = 0|\omega = 0)} \\
&= \frac{p_1(1-\mu)Pr(s_{t-1} = 1|\omega = 1) + p_0\mu Pr(s_{t-1} = 0|\omega = 1)}{p_1\mu Pr(s_{t-1} = 1|\omega = 0) + p_0(1-\mu)Pr(s_{t-1} = 0|\omega = 0)} \\
&\leq \frac{(p_1(1-\mu) + p_0\mu)Pr(s_{t-1} = 1|\omega = 1)}{p_1\mu Pr(s_{t-1} = 1|\omega = 0) + p_0(1-\mu)Pr(s_{t-1} = 0|\omega = 0)} \\
&< \frac{p_1(1-\mu) + p_0\mu}{p_1\mu}\frac{Pr(s_{t-1} = 1|\omega = 1)}{Pr(s_{t-1} = 1|\omega = 0)},
\end{aligned}$$

where the second to last inequality uses the first part of this lemma. We can therefore handle both cases simultaneously by showing that the sequence $\frac{Pr(s_t=1|\omega=1)}{Pr(s_t=1|\omega=0)}$ is bounded above.

To that end, note that

$$\begin{aligned}
Pr(s_t = 1|\omega = 0) &\geq Pr(s_t = 1|\omega = 0, s_1 = 1)Pr(s_1 = 1|\omega = 0) \\
&= Pr(s_{t-1} = 1|\omega = 1)p_0\mu.
\end{aligned}$$

So,

$$\frac{Pr(s_t = 1|\omega = 1)}{Pr(s_t = 1|\omega = 0)} \leq \frac{Pr(s_t = 1|\omega = 1)}{Pr(s_{t-1} = 1|\omega = 1)}\frac{1}{p_0\mu}.$$

It then suffices to show that $Pr(s_t = 1|\omega = 1) \leq Pr(s_{t-1} = 1|\omega = 1)$, since then from above

$$\frac{Pr(s_t = 1|\omega = 1)}{Pr(s_t = 1|\omega = 0)} \leq \frac{1}{p_0\mu},$$

which is finite given that $p_0 > 0$ and $\mu > 0$. To see that $Pr(s_t = 1|\omega = 1) \leq Pr(s_{t-1} = 1|\omega = 1)$,

$$\begin{aligned}
Pr(s_t = 1|\omega = 1) &= p_1(1-\mu)Pr(s_t = 1|s_1 = 1) + p_1\mu Pr(s_t = 1|s_1 = 0) \\
&= p_1(1-\mu)Pr(s_{t-1} = 1|\omega = 1) + p_1\mu Pr(s_{t-1} = 1|\omega = 0) \\
&\leq p_1(1-\mu)Pr(s_{t-1} = 1|\omega = 1) + p_1\mu Pr(s_{t-1} = 1|\omega = 1) \\
&= p_1 Pr(s_{t-1} = 1|\omega = 1),
\end{aligned}$$

where the inequality follows from the first part of the lemma, establishing the claim. ∎

Proof of Lemma 3, Part 2:

We show that $\lim_{t\to\infty}\frac{P^t_{1S}}{P^t_{0S}} = \lim_{t\to\infty}\frac{p_1 Pr(s_{t-1}=1|\omega=1)+p_0 Pr(s_{t-1}=0|\omega=1)}{p_1 Pr(s_{t-1}=1|\omega=0)+p_0 Pr(s_{t-1}=0|\omega=0)}$ exists.

The sequence is bounded above by the first and last part of Lemma 6: it is bounded above by either $\frac{Pr(s_t=1|\omega=1)}{Pr(s_t=1|\omega=0)}$ or $\frac{Pr(s_t=1|\omega=1)}{Pr(s_t=0|\omega=0)}$, both of which are bounded above. Furthermore, the sequence is bounded below by the first part of Lemma 3.

To complete the proof that the limit exists, we show that the sequence is monotone. For this, we will start by writing, r_t, the t^{th} term in the sequence, as $\frac{Pr(s_{t-1}=1|\omega=1)+\ell_1 Pr(s_{t-1}=0|\omega=1)}{Pr(s_{t-1}=1|\omega=0)+\ell_1 Pr(s_{t-1}=0|\omega=0)}$, where $\ell_1 = p_0/p_1$. Now the $t+1^{st}$ is

$$\begin{aligned}
r_{t+1} &= \frac{Pr(s_t = 1|\omega = 1) + \ell_1 Pr(s_t = 0|\omega = 1)}{Pr(s_t = 1|\omega = 0) + \ell_1 Pr(s_t = 0|\omega = 0)} \\
&= \frac{(p_1(1-\mu) + \ell_1 p_1 \mu) Pr(s_t = 1|s_1 = 1) + (p_0\mu + \ell_1 p_0(1-\mu)) Pr(s_t = 0|s_1 = 1)}{(p_1(1-\mu) + \ell_1 p_1 \mu) Pr(s_t = 1|s_1 = 0) + (p_0\mu + \ell_1 p_0(1-\mu)) Pr(s_t = 0|s_1 = 0)} \\
&= \frac{Pr(s_{t-1} = 1|\omega = 1) + \ell_2 Pr(s_{t-1} = 0|\omega = 1)}{Pr(s_{t-1} = 1|\omega = 0) + \ell_2 Pr(s_{t-1} = 0|\omega = 0)},
\end{aligned}$$

where $\ell_2 = \frac{p_0}{p_1}\frac{\mu + l(1-\mu)}{(1-\mu)+l\mu}$. Consider the sequence ℓ_t, where $\ell_{t+1} = \frac{p_0}{p_1}\frac{\mu+\ell_t(1-\mu)}{(1-\mu)+\ell_t\mu}$ and $\ell_1 = \frac{p_0}{p_1}$. Note that ℓ_t is non-decreasing in t given that $\mu \le 1/2$ and it is strictly increasing when $\mu < 1/2$. Iterating on the above logic

$$r_t = \frac{Pr(s_1 = 1|\omega = 1) + \ell_{t-1} Pr(s_1 = 0|\omega = 1)}{Pr(s_1 = 1|\omega = 0) + \ell_{t-1} Pr(s_1 = 0|\omega = 0)}.$$

To see that r_t is monotone in t, note that the sign of the derivative of r_t with respect to ℓ_t only depends on the sign of $Pr(s_1 = 0|\omega = 1)Pr(s_1 = 1|\omega = 0) - Pr(s_1 = 1|\omega = 1)Pr(s_1 = 0|\omega = 0))$, and so it is monotone given the monotonicity of ℓ_t in t. ▮

Proof of Lemma 3, Part 3:

That $Pr(\omega = 1|s_t \neq \emptyset) \ge \frac{\theta z}{1+\theta(z-1)}$ for any $t > 1$, with strict inequality when $\mu < 1/2$, follows from Part 1 and Bayes rule (and it is evident from the proof that this lower bound is not tight). Therefore, it remains to show that $\lim_{t\to\infty} Pr(\omega = 1|s_t \neq \emptyset)$ exists, a step which is deferred to the proof of Part 4.

The fact that $\lim_{t\to\infty} Pr(\omega = 1|s_t \neq \emptyset) = \frac{\theta}{\theta+(1-\theta)/y} < 1$ follows from Part 2 and Bayes' Rule. ▮

Proof of Lemma 3, Part 4:

It suffices to show that $\lim_{t\to\infty} Pr(\omega = 1|s_t = 1) = \lim_{t\to\infty} Pr(\omega = 1|s_t = 0)$, as this implies that $\lim_{t\to\infty} Pr(\omega = 1|s_t \neq \emptyset)$ exists and has the same value. This limiting equality between posterior distributions can equivalently be expressed in terms of likelihood ratios:

$$\begin{aligned}
&\lim_{t\to\infty} \frac{Pr(s_t = 1|\omega = 0)}{Pr(s_t = 1|\omega = 1)} = \lim_{t\to\infty} \frac{Pr(s_t = 0|\omega = 0)}{Pr(s_t = 0|\omega = 1)} \\
\iff &\lim_{t\to\infty} \frac{Pr(s_t = 1|s_t \neq \emptyset, \omega = 0)}{Pr(s_t = 1|s_t \neq \emptyset, \omega = 1)} = \lim_{t\to\infty} \frac{Pr(s_t = 0|s_t \neq \emptyset, \omega = 0)}{Pr(s_t = 0|s_t \neq \emptyset, \omega = 1)}. \qquad (3)
\end{aligned}$$

We show that

$$\lim_{t\to\infty} Pr(s_t = 1|s_t \neq \emptyset, \omega = 0) = \lim_{t\to\infty} Pr(s_t = 1|s_t \neq \emptyset, \omega = 1), \qquad (4)$$

since this implies that both sides of equation 3 are equal to 1.[9]

[9] Subtract each side of equation 4 from 1 before taking ratios to see that the right side of equation 3 is also 1.

Denote by S a sequence of signals that evolve according to our process, starting with $S_0 = 1$ and S' another (independent) sequence of signals with $S'_0 = 0$. Let $\tau = \min\{t | S'_t = 1\}$, where $\tau = \infty$ if S' is dropped at some step before mutating to signal 1, or if $S'_t = 0$ for all t.

In this notation, equation 4 can equivalently be expressed as: $\lim_{t\to\infty} Pr(S_t = 1|S_t \neq \emptyset) = \lim_{t\to\infty} Pr(S'_t = 1|S'_t \neq \emptyset)$. Note the following relationship between the two independent paths:[10]

$$\begin{aligned} Pr(S'_t = 1|S'_t \neq \emptyset) &= \sum_{i=1}^{t} Pr(S'_t = 1|S'_t \neq \emptyset, \tau = i) Pr(\tau = i|S'_t \neq \emptyset) \\ &= \sum_{i=1}^{t} Pr(S_{t-i} = 1|S_{t-i} \neq \emptyset) Pr(\tau = i|S'_t \neq \emptyset) \\ &\equiv \left(\sum_{i=1}^{t} Pr(S_{t-i} = 1|S_{t-i} \neq \emptyset) w_i^t \right), \end{aligned} \tag{5}$$

where $w_i^t = Pr(\tau = i|S'_t \neq \emptyset)$.

The result then follows from the following three claims, to be proved:

1. For any $\varepsilon > 0$ and positive integer k, for all sufficiently large t, $\sum_{i=t-k}^{t} w_i^t < \varepsilon$.

2. $\lim_{t\to\infty} Pr(S_t = 1|S_t \neq \emptyset)$ exists.

3. $\sum_{i=1}^{t} w_i^t + w_\infty^t = 1$. Moreover, $w_\infty^t \to 0$ as $t \to \infty$, i.e., the probability that the signal never mutated conditional on survival to t goes to 0 as t grows.

To see that these claims imply the result, note that by claim 1, most of the weight falls on the first $t-k$ terms of the sum in equation 5 for large enough t. By claim 2, for a large enough k (growing slower than t), these first $t-k$ terms will be close to $\lim_{t\to\infty} Pr(S_t = 1|S_t \neq \emptyset)$, and therefore by claim 3 the limiting weighted sum of these terms converges to this value as well.

Claim 3 is clear, so we prove the other two.

First we prove claim 1. Note that $p_0^i(1-\mu)^{i-1}\mu$ is the probability of survival with no mutation through $i-1$ and then survival with mutation at $t = i$, i.e., $Pr(\tau = i) = p_0^i(1-\mu)^{i-1}\mu$. Second, let m_i be number of mutations through time i. Obviously, $Pr(S'_i \neq \emptyset) > Pr(S'_i \neq \emptyset$ and $m_i = 1)$. Third, if survival were always at rate p_0, then $Pr(S'_i \neq \emptyset$ and $m_i = 1) = ip_0^i(1-\mu)^{i-1}\mu$. However, since survival likelihood immediately after the first mutation, p, is strictly higher than p_0 and mutations sometimes occur (note, we assume $\mu > 0$), $Pr(S'_i \neq \emptyset$ and $m_i = 1) > ip_0^i(1-\mu)^{i-1}\mu$. Putting these observations together, we have

$$\lim_{i\to\infty} \frac{Pr(\tau = i)}{Pr(S'_i \neq \emptyset)} < \lim_{i\to\infty} \frac{p_0^i(1-\mu)^{i-1}\mu}{ip_0^i(1-\mu)^{i-1}\mu} = \lim_{i\to\infty} \frac{1}{i} = 0, \tag{6}$$

[10] Note that if $\tau > t$, then the probability that $S'_t = 1$ is 0.

where, as noted earlier, the inequality arises from replacing $Pr(S'_i \neq \emptyset)$ with a lower bound on the probability of exactly one mutation occurring over the course of the first i periods, and all the ways this could happen, and then noting that $p_0 < p_1$. Now

$$\begin{aligned} Pr(\tau = i | S'_t \neq \emptyset) &= \frac{Pr(S'_t \neq \emptyset | \tau = i) Pr(\tau = i)}{Pr(S'_t \neq \emptyset)} \\ &= \frac{Pr(S_{t-i} \neq \emptyset) Pr(\tau = i)}{Pr(S'_t \neq \emptyset)} \\ &= \frac{Pr(S_{t-i} \neq \emptyset) Pr(\tau = i)}{Pr(S'_{t-i} = 1) Pr(S_i \neq \emptyset) + Pr(S'_{t-i} = 0) Pr(S'_i \neq \emptyset)} \\ &< \frac{Pr(S_{t-i} \neq \emptyset)}{Pr(S'_{t-i} \neq \emptyset)} \frac{Pr(\tau = i)}{Pr(S'_i \neq \emptyset)}, \end{aligned}$$

where the inequality follows from the fact that $Pr(S_i \neq \emptyset) > Pr(S'_i \neq \emptyset)$, by Lemma 3, Part 1. $\frac{Pr(S_{t-i} \neq \emptyset)}{Pr(S'_{t-i} \neq \emptyset)}$ is bounded by Lemma 3 Part 2 (as it has a limit), and $\frac{Pr(\tau=i)}{Pr(S'_i \neq \emptyset)}$ can be made arbitrarily small for large enough i by equation 6. Thus, for any δ and k we can find large enough t for which $w_i^t < \delta$ for $i > t - k$. Choosing $\delta = \varepsilon / k$ establishes claim 1.

Finally, we prove claim 2. The probability distribution of S_t is given by $e'_1 A^t$, where

$$A = \begin{bmatrix} p_1(1-\mu) & p_1\mu & 0 \\ p_0\mu & p_0(1-\mu) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is the Markov transition matrix for S. Let B be the principal 2×2 submatrix of A. By the partitioned matrix multiplication formula, $Pr(S_t = 1 | S_t \neq \emptyset) = \frac{e'_1 B^t e_1}{e'_1 B^t \mathbf{1}}$. Since B is strictly positive, the Perron-Frobenius theorem implies that this expression converges to the first entry of eigenvector corresponding to the largest eigenvalue of B. ∎

Proof of Lemma 4:

$\lim_{t\to\infty} \frac{P_{0S}^t}{P_{1S}^t} = r$ for some $r < 1$, by Lemma 3. Let r_t be the t^{th} term in the sequence.

Let $m(t)$ be the number of surviving signals. By Chernoff bounds, it follows that

$$Pr(m(t) > n(t) P_{1s}^t (1 + r_t)/2 | \omega = 1) \to 1$$

and

$$Pr(m(t) < n(t) P_{1s}^t (1 + r_t)/2 | \omega = 0) \to 1$$

provided that $n(t)P_{1s}^t \to \infty$. Given this separation, it is easy to the check that if $n(t)P_{1s}^t \to \infty$, the beliefs will converge to 0 or 1 in probability.

Next, note that if $n(t)P_{1s}^t \to 0$, then the expected number of surviving signals in either state is 0, and that happens with the probability going to 1 by Chebychev, and so there is no learning. So, the threshold is $1/P_{1s}^t$.

Note that survival lies between $1/p_1^t$ and $1/p_0^t$ and so

$$1/P_{1s}^t = \frac{1}{(p - 1\lambda(t) + (1 - \lambda(t))p_0)^t}.$$

The fact that $\lambda(t)$ converges to some λ then follows since this is a Markov chain and the probability that it survives in any given period (the third state with $s_t = \emptyset$ is absorbing) converges to a steady state distribution, which in this case lies between p_1 and p_0. ∎

Appendix B: Full Bayesian Learning vs Learning Only from Survival or Only from Content

In this section, we provide a bound on how much more likely a Bayesian agent using both signal survival and message content is to guess the true state compared to agents who use rules of thumb that account only for signal survival or only for average signal content.

Without loss of generality, we focus on the case in which $p_1 \geq p_0$.

First, let us consider the case in which the learner has access to a single chain and needs to predict the state based on the signal $s_t \in \{0, 1, \emptyset\}$. We consider four different ways in which the learner might guess.

- A "Bayesian agent," B, guesses the most likely state conditional on both signal survival and signal content.
- A "survival rule-of-thumb agent," S, guesses 1 if a signal is received and guesses 0 if no signal is received.
- A "content rule-of-thumb agent," C, guesses 1 if signal 1 is received, 0 if signal 0 is received, and guesses in favor of the prior if no message is received (flipping a coin if $\theta = 1/2$).
- A "naive agent," N, always guesses in favor of the prior.

S, C, N are collectively referred to as "limited learners" since they make their guess based on less information than is available.

Proposition 2 *Suppose that* $1 \geq p_1 \geq p_0 \geq 0$ *and* $\mu_{01} = \mu_{10} = \mu \in [0, 1/2]$. *The probability that a Bayesian agent is correct in guessing the state is at most* $\frac{4}{3}$ *higher than the best limited learner when* $t = 1$, *and at most* $\frac{3}{2}$ *higher than the best limited learner for all* $t > 1$.[11] *Moreover, as* t *grows, this upper bound converges to 1.*

Proposition 2 implies that, when word-of-mouth chains are long, a belief-updating strategy that uses only message survival or only message content is approximately equivalent to one that uses all available information, no matter what the parameters and no matter what the realized state.[12]

[11] We conjecture that the bound is $\frac{4}{3}$ for any $t > 1$.

[12] This is obvious when $\mu = 1/2$, in which case message content contains no information, or when $p_0 = p_1$, in which case message survival contains no information.

Next, suppose that the learner observes multiple chains. In this context, define “C” to be an agent who guesses 1 whenever the fraction of 1 messages compared to 0 messages is above a threshold, and define “S” to be the an agent who guesses 1 whenever the number of messages that survive is above or below a threshold. These thresholds are the conditional Bayesian ones, but these agents only consider one aspect of the information available.

It is difficult to give tight bounds on the relative performance of the Bayesian agent and the best of the limited learners when there are many sequences. However, we establish limiting results. In particular, everywhere in the parameter space, the threshold for learning for agent B is the same as for one of the limited learners. Thus, there is no number of starting messages for which a Bayesian agent can learn but none of the naive agents can. Indeed, for large t full learning can be obtained from just one dimension, and we get the following result.

PROPOSITION **3** *For any $\theta, p_0, p_1 \in [0,1]$ and $\mu_{01} = \mu_{10} = \mu \in [0, \frac{1}{2}]$, the threshold for learning is the same for B as it is for the better of C or S.*

The next lemma is useful in the proof of Proposition 2.

Let P_1^t (P_0^t) denote the Bayesian posterior probability that the state is 1 conditional upon a signal being received at time t and being 1 (0). Similarly, let $P_\emptyset^t$ (P_S^t) denote the Bayesian posterior probability that the state is 1 conditional upon no signal (some signal) being received at time t.

LEMMA **7** *If $p_1 > p_0$, then $P_1^t \geq P_0^t$ and $P_1^t \geq P_\emptyset^t$.*

Proof of Lemma 7

Let s^t denote the state of the signal at period t. That $P_1^t \geq P_0^t$ holds when $t = 1$ is easy to check from Bayes rule, given that $p_1 > p_0$ and $\mu \leq 1/2$. Now suppose $P_1^t \geq P_0^t$ for some t. Then by the law of total probability, it follows that

$$\begin{aligned} P_1^{t+1} &= Pr(s^t = 0|s^{t+1} = 1)P_0^t + Pr(s^t = 1|s^{t+1} = 1)P_1^t \\ &= \frac{p_0\mu Pr(s^t = 0)}{p_0\mu Pr(s^t = 0) + p_1(1-\mu)Pr(s^t = 1)}P_0^t + \frac{p_1(1-\mu)Pr(s^t = 1)}{p_0\mu Pr(s^t = 0) + p_1(1-\mu)Pr(s^t = 1)}P_1^t \end{aligned}$$

Similarly,

$$P_0^{t+1} = \frac{p_0(1-\mu)Pr(s^t = 0)}{p_0(1-\mu)Pr(s^t = 0) + p_1\mu Pr(s^t = 1)}P_0^t + \frac{p_1\mu Pr(s^t = 1)}{p_0(1-\mu)Pr(s^t = 0) + p_1\mu Pr(s^t = 1)}P_1^t$$

Since $P_1^t \geq P_0^t$ by the inductive hypothesis, it suffices to show that

$$\frac{p_1(1-\mu)Pr(s^t = 1)}{p_0\mu Pr(s^t = 0) + p_1(1-\mu)Pr(s^t = 1)} \geq \frac{p_1\mu Pr(s^t = 1)}{p_0(1-\mu)Pr(s^t = 0) + p_1\mu Pr(s^t = 1)}$$

i.e., that

$$\frac{1}{1+\frac{p_0}{p_1}\frac{Pr(s^t=0)}{Pr(s^t=1)}\frac{\mu}{1-\mu}} \geq \frac{1}{1+\frac{p_0}{p_1}\frac{Pr(s^t=0)}{Pr(s^t=1)}\frac{1-\mu}{\mu}}$$

which follows, since $\mu \leq 1-\mu$.

To see that $P_1^t \geq P_\emptyset^t$, note that it suffices to prove that $P_S^t \geq P_\emptyset^t$, since P_S^t is a convex combination of P_1^t and P_0^t, and we just proved $P_1^t \geq P_0^t$. Now the statement follows directly from part 1 of Proposition 3. ▮

Proof of Proposition 2:

First, note that we can focus on the case in which $p_1 \neq p_0$ as otherwise there is nothing to be learned from signal survival, and agent C does as well as B. Without loss of generality we take $p_1 > p_0$. Similarly, if $\mu = 1/2$, then all learning is from survival and S does as well as B, and so we can take $\mu < 1/2$.

Note that by Lemma 7, $P_1^t \geq P_0^t$ and $P_1^t \geq P_\emptyset^t$. In order for B to do strictly better in expectation than the other agents, it must be that $P_1^t > 1/2$ and at least one of P_0^t and $P_\emptyset^t$ are less than 1/2. To see this note that if all three are on the same side of 1/2, then they must lie on the same side as the prior.[13] If $\theta \neq 1/2$ then N gets the same payoff as B. If $\theta = 1/2$, then for all three to lie on the same side of the prior it must be that $p_1 = p_0$, in which case there is nothing learned from survival and C does as well as B in expectation.

Thus, $P_1^t > 1/2$ and at least one of P_0^t and $P_\emptyset^t$ are less than 1/2. If it is just $P_\emptyset^t$ that is less than 1/2, then S guesses the same as B (or equivalently in expected payoff terms). Thus, we need $P_0^t < 1/2$ to have a difference.

If is just P_0^t that is less than 1/2, then C guesses the same as B except if $\theta \leq 1/2$. But for such a θ, it must be that $P_\emptyset^t \leq 1/2$ and so C guesses as well as B.

So, consider the case in which $P_1^t > 1/2$ and $P_0^t < 1/2$ and $P_\emptyset^t < 1/2$. For C to guess differently than B, it must be that $\theta \geq 1/2$.

We can compute the expected payoff's for the three most relevant agents for this remaining case (we ignore N now, since in these conditions it is dominated by one of the others) for a given (p_1, p_0, μ, θ) satisfying the above constraints.

Letting U_B, U_C, U_S be the expected payoffs of agents B, C[14] and S respectively, it follows that

$$\begin{aligned}
U_B &= Pr(s_t=1)P_1^t + Pr(s_t=0)(1-P_0^t) + (1-Pr(s_t=1)-Pr(s_t=0))(1-P_\emptyset^t)\\
U_C &= Pr(s_t=1)P_1^t + Pr(s_t=0)(1-P_0^t) + (1-Pr(s_t=1)-Pr(s_t=0))P_\emptyset^t\\
U_S &= Pr(s_t=1)P_1^t + Pr(s_t=0)P_0^t + (1-Pr(s_t=1)-Pr(s_t=0))(1-P_\emptyset^t)
\end{aligned}$$

[13] They cover three disjoint events whose union is all possibilities, and so the overall probability of a 1 is a convex combination of these conditionals, and so it is impossible to have them all weakly and some strictly greater (or all weakly and some strictly less) than the prior.

[14] C has expected payoff $U_C = Pr(s_t=1)P_1^t + Pr(s_t=0)(1-P_0^t) + (1-Pr(s_t=1)-Pr(s_t=0))\left(I_{\theta>1/2}P_\emptyset^t + I_{\theta=1/2}1/2\right)$. The expression in the main text is obtained by noting that the worst ratio for this compared to B will be in cases for which $\theta > 1/2$

First, note that if $p_0 < 1$ and $\mu > 0$, then as $t \to \infty$, then $Pr(s_t = \emptyset) \to 1$ and $P_\emptyset \to 1/2$, in which case the ratio of B to either of these goes to 1. If $p_0 < 1$ and $\mu = 0$, then B does as well as S for every t. If $p_1 = p_0 = 1$, then B does as well as C for every t. These facts together establish the last claim in the proposition that as $t \to \infty$, the ratio $\frac{U_B}{\max\{U_S, U_C\}} \to 1$.

That the ratio is bounded above by $3/2$ can be seen as follows. Since $\theta \geq 1/2$ and $p_1 > p_0$, it follows that

$$Pr(s_t = 1) \geq Pr(s_t = 0), \quad P_1^t \geq (1 - P_0^t), \text{ and so } Pr(s_t = 1)P_1^t \geq Pr(s_t = 0)(1 - P_0^t).$$

Then if $Pr(s_t = 0)(1 - P_0^t) \leq (1 - Pr(s_t = 1) - Pr(s_t = 0))(1 - P_\emptyset^t)$ it follows that $U_S \geq U_B 2/3$. If $Pr(s_t = 0)(1 - P_0^t) \geq (1 - Pr(s_t = 1) - Pr(s_t = 0))(1 - P_\emptyset^t)$ then it follows that $U_C \geq U_B 2/3$.

To complete the proof, we compute

$$\max_{p_1, p_0, \theta, \mu \in [0,1]} \frac{U_B}{\max\{U_S, U_C\}}.$$

for $t = 1$. We can rewrite the payoffs of agents B, S and C in the case $P_1^1 > 1/2$ and $P_0^1 < 1/2$ and $P_\emptyset^1 < 1/2$ as follows:

$$\begin{aligned} U_B &= \theta p_1(1 - \mu) + (1 - \theta)(1 - p_0\mu) \\ U_C &= \theta(1 - p_1\mu) + (1 - \theta)p_0(1 - \mu) \\ U_S &= \theta p_1 + (1 - \theta)(1 - p_0) \end{aligned}$$

where

$$\theta p_1 \mu \leq p_0(1 - \theta)(1 - \mu) \tag{7}$$
$$\theta(1 - p_1) \leq (1 - \theta)(1 - p_0) \tag{8}$$
$$\theta \geq 1/2 \tag{9}$$
$$\mu \leq 1/2 \tag{10}$$
$$p_1 \geq p_0 \tag{11}$$
$$p_1, p_0, \mu, \theta \in [0, 1]. \tag{12}$$

Case 1: $U_S \leq U_C$.

This condition can be rewritten as

$$\theta(p_1\mu + (p_1 - 1)) \leq (1 - \theta)(p_0(1 - \mu) + (p_0 - 1)) \tag{13}$$

The program with this additional constraint can be written as

$$\begin{aligned} \max_{p_1, p_0, \theta, \mu \text{ satisfy 7-13}} \frac{U_B}{U_C} &\equiv \max_{p_1, p_0, \theta, \mu \text{ satisfy 7-13}} \frac{\theta p_1(1 - \mu) + (1 - \theta)(1 - p_0\mu)}{\theta(1 - p_1\mu) + (1 - \theta)p_0(1 - \mu)} \\ &= \max_{p_1, p_0, \theta, \mu \text{ satisfy 7-13}} \frac{\theta p_1 + (1 - \theta) - \mu(\theta p_1 + (1 - \theta)p_0)}{\theta + (1 - \theta)p_0 - \mu(\theta p_1 + (1 - \theta)p_0)} \end{aligned}$$

$$\leq \max_{p_1,p_0,\theta,\mu \text{ satisfy 7-13}} \frac{\theta + (1-\theta)2p_0 - 2\mu(\theta p_1 + (1-\theta)p_0)}{\theta + (1-\theta)p_0 - \mu(\theta p_1 + (1-\theta)p_0)}$$

where the inequality is from rearranging constraint 13, as $\theta p_1 + (1-\theta) \leq \theta + (1-\theta)2p_0 - \mu(\theta p_1 + (1-\theta)p_0)$, and plugging this into the numerator. It is easily verified that the above ratio is decreasing in μ for any values of the remaining parameters [15]. Moreover, reducing μ to 0 only relaxes constraints 7, 10 and 13, and leaves the other constraints unaffected. Therefore,

$$\max_{p_1,p_0,\theta,\mu \text{ satisfy 7-13}} \frac{U_B}{U_C} \leq \max_{p_1,p_0,\theta \text{ satisfy 8-13}} \frac{\theta + (1-\theta)2p_0}{\theta + (1-\theta)p_0}$$

It is clear that smaller values of θ increase this ratio, and by constraint 9, the smallest value of θ is $\frac{1}{2}$. But while reducing θ down to $\frac{1}{2}$ for given p_1 and p_0 relaxes constraint 8, doing so may violate constraint 13. We therefore separately consider the cases where either 13 or 9 bind, since at least one of them must at the optimum.

Subcase 1: 13 is satisfied with equality, i.e., $\theta(1-p_1) = (1-\theta)(1-2p_0)$. Plugging this in, the objective then becomes $2\frac{1+\theta(p_1-1)}{1+\theta(p_1-1)+\theta}$, which is decreasing in θ, so it is optimal to set $\theta = \frac{1}{2}$. The objective is then $2\frac{1+p_1}{2+p_1} \leq 4/3$. Note that at $p_1 = 1, p_0 = \frac{1}{2}, \theta = \frac{1}{2}, \mu = 0$, $\frac{U_B}{U_C} = \frac{4}{3}$, so this upper bound is tight.

Subcase 2: $\theta = 1/2$. Then

$$\frac{U_B}{U_C} = \frac{p_1 + 1 - \mu(p_1 + p_0)}{p_0 + 1 - \mu(p_1 + p_0)},$$

which is weakly increasing in μ by constraint 11. Constraint 13 can be rearranged to be

$$\mu \leq \frac{2p_0 - p_1}{p_1 + p_0},$$

which, first, implies that

$$\frac{U_B}{U_C} \leq \frac{2(p_1 - p_0) + 1}{(p_1 - p_0) + 1},$$

and second, along with the condition that $\mu \geq 0$, implies that

$$p_0 \geq \frac{p_1}{2}.$$

Since $\frac{2(p_1-p_0)+1}{(p_1-p_0)+1}$ is decreasing in p_0, this expression is maximized under the given constraints when $p_0 = \frac{p_1}{2}$. Therefore, $\frac{U_B}{U_C} \leq \frac{p_1+1}{\frac{p_1}{2}+1}$, which is maximized when $p_1 = 1$ and equals $4/3$.

Case 2: $U_S \geq U_C$. The new constraint is

$$\theta(p_1\mu + (p_1 - 1)) \geq (1-\theta)(p_0(1-\mu) + (p_0 - 1)), \tag{14}$$

and the relevant maximization program is

[15] $\frac{d}{dx}\frac{A-2x}{B-x} \leq 0$ if $A \leq 2B$ and $A, B > 0$.

$$\max_{p_1,p_0,\theta,\mu \text{ satisfy 8-12, 14}} \frac{U_B}{U_S} \equiv \max_{p_1,p_0,\theta,\mu \text{ satisfy 8-12, 14}} \frac{\theta p_1(1-\mu)+(1-\theta)(1-p_0\mu)}{\theta p_1+(1-\theta)(1-p_0)}.$$

Notice that the ratio $\frac{\theta p_1(1-\mu)+(1-\theta)(1-p_0\mu)}{\theta p_1+(1-\theta)(1-p_0)}$ is linear and decreasing in μ, and the constraints are linear in μ as well. Constraint 7 only places an upper bound on μ, so it is not relevant in pinning down this value at the optimum. On the other hand, constraint 14, which can be rewritten as

$$2p_0(1-\theta)-p_1\theta-(1-2\theta) \leq (p_1\theta+p_0(1-\theta))\mu$$

and the constraint that $\mu \geq 0$ are relevant. There are two cases:

Subcase 1: $2p_0(1-\theta)-p_1\theta-(1-2\theta) \geq 0$, $\mu = \frac{2p_0(1-\theta)-p_1\theta-(1-2\theta)}{p_1\theta+p_0(1-\theta)}$.

Then

$$\begin{aligned}\frac{U_B}{U_S} &= \frac{\theta p_1+(1-\theta)-\mu(\theta p_1+(1-\theta)p_0)}{\theta p_1+(1-\theta)-(1-\theta)p_0}\\ &= \frac{\theta p_1+(1-\theta)-2(1-\theta)p_0+p_1\theta+(1-2\theta)}{\theta p_1+(1-\theta)-(1-\theta)p_0}\\ &= \frac{2\theta p_1+(2-3\theta)-2(1-\theta)p_0}{\theta p_1+(1-\theta)-(1-\theta)p_0}\\ &= 2\frac{\theta p_1+(1-\frac{3}{2}\theta)-(1-\theta)p_0}{\theta p_1+(1-\theta)-(1-\theta)p_0}\end{aligned}$$

Clearly, the ratio is decreasing in θ, and moreover, decreasing θ only relaxes constraints 7 and 8. Therefore, constraint 9 binds and $\theta = \frac{1}{2}$ at the optimum, so

$$\frac{U_B}{U_S} = 2\frac{p_1+\frac{1}{2}-p_0}{p_1+1-p_0}$$

Since $\mu = \frac{2p_0-p_1}{p_1+p_0}$ at $\theta = \frac{1}{2}$, constraints 7 and 8 reduce to just $p_1 \geq p_0$. Since the ratio is increasing in $p_1 - p_0$, the only binding constraint is that $\mu \geq 0$, i.e., $2p_0 \geq p_1$. Therefore at the optimum, $p_1 = 1$, $p_0 = \frac{1}{2}$, $\mu = 0$, $\theta = \frac{1}{2}$, and $\frac{U_B}{U_S} = \frac{4}{3}$.

Subcase 2: $2p_0(1-\theta)-p_1\theta-(1-2\theta) \leq 0$, $\mu = 0$. In this case, the problem reduces to

$$\max_{p_1,p_0,\theta, \text{ satisfy 8,9,11, 12, 14}} \frac{\theta p_1+(1-\theta)}{\theta p_1+(1-\theta)(1-p_0)},$$

which is decreasing in θ. Now

$$\begin{aligned}2p_0(1-\theta)-p_1\theta-(1-2\theta) &\leq 0\\ \iff \theta(2-2p_0-p_1) &\leq 1-2p_0\end{aligned}$$

Suppose $2-2p_0-p_1 < 0$. Since $1-2p_0 \leq 1-2p_0+(1-p_1) = 2-2p_0-p_1 < 0$, it follows that $\theta(2-2p_0-p_1) \geq \theta(1-2p_0) \geq 1-2p_0$. Therefore, the only way to satisfy the constraint is if $p_1 = 1$ and $\theta = 1$, in which case $\frac{U_B}{U_S} = 1$.

If $2-2p_0-p_1 \geq 0$, then $\theta = \frac{1}{2}$ at the optimum, and so the constraint in this sub-subcase becomes $2p_0 \leq p_1$, while the objective function is $\frac{p_1+1}{p_1+1-p_0}$. This constraint binds at the optimum and again the optimal value is $\frac{4}{3}$ at $p_1 = 1$ and $p_0 = \frac{1}{2}$. ∎

The next lemma is useful in the proof of the second part of Proposition 3.

LEMMA **8** *Let $p_0 = p_1$ and suppose a Bayesian agent with prior θ on the state being* 1 *observes m signals of which k are* 1*s. Then the agent's posterior that the state is 1 is*

$$\frac{\theta X_1(t)^k(1-X_1(t))^{m-k}}{\theta X_1(t)^k(1-X_1(t))^{m-k} + (1-\theta)(1-X_0(t))^k X_0(t)^{m-k}}.$$

The proof is direct and omitted.

Proof of Proposition 3 :

We proceed by cases for different values of the parameters. We concentrate on situations in which $\mu < 1/2$ since if $\mu = 1/2$ then content is completely uninformative and the result is direct.

Case 1: $\mu = 0$. Suppose without loss of generality that $p_0 \leq p_1$. Any signal that reaches the agent is perfectly informative of the state, so a threshold for learning for agents B and C is the threshold for at least one signal to survive, which (following the logic of the proofs above) is $\frac{1}{p_1^t}$.

Case 2: $p_1 = p_0$ and $\mu > 0$. By Proposition 2, the threshold for learning for agent B is $\frac{1}{p_1^t(1-2\mu)^{2t}}$. In this case there is no information from signal survival, and by Lemma 8, agent B's posterior is the same as agent C's posterior. Therefore, agent C has the same threshold for learning as B.

Case 3: $p_0 \neq p_1$ and $\mu > 0$. Without loss of generality let $p_1 > p_0$. Then $\tau(t) = \frac{1}{P_{1S}^t}$ is a threshold for learning for an agent conditioning only on signal survival, as shown in the proof of Lemma 4. Let $b(t)$ denote the beliefs of agent B after observing the outcome of $n(t)$ original sources of information sent along chains of depth t. Since agent B conditions on survival and signal content, *plim* $b(t) \to 1$ or 0 whenever $n(t)/\tau(t) \to \infty$. When $n(t)/\tau(t) \to 0$, then the probability of even a single signal surviving to reach the agent approaches 0. This holds regardless of the starting state by Lemma 3 part 2, so *plim* $b(t) \to \theta$. Therefore, agent B and S have the same thresholds for learning in this case.[16]. ∎

[16]Strictly speaking, we only showed that they share a common threshold, but it is easy to see that being a threshold for learning for B, for S or for neither partitions the space of functions on $\mathbb{N} \to \mathbb{N}$.