Viral Social Learning*

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Abstract

An innovation (e.g., new product or idea) spreads like a virus, transmitted by those who have previously adopted it. We characterize equilibrium adoption dynamics and the resulting lifecycle of virally-spread innovations. Herding on adoption can occur but only early in the innovation lifecycle, and all virally-spread innovations eventually become obsolescent. A producer capable of advertising directly to consumers finds it optimal to wait and allow awareness to grow virally initially after launch. When most innovations would otherwise be high (low) quality absent any viral social learning, running an optimal-length viral campaign decreases (increases) equilibrium investment in innovation quality.

Keywords: social learning, viral marketing, SIR model, economic epidemic

JEL Classification: C72, D62, D83

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1 Introduction

Each fall, PhD advisers inform faculty they know at other schools about their students on the job market that year. Those faculty then evaluate the work and, if impressed, talk about it with others in the months before interviews occur at the Annual Meeting of the American Economic Association. Since better work tends to generate more word of mouth (controlling for observables such as topic and adviser fame), faculty at the Meetings rightly update their beliefs about a paper's quality based on whether and when they previously heard about it. This is an example of "viral social learning."

When a novel virus enters a population, infected hosts expose others who, if successfully infected, will start spreading the virus as well. Virus strains that are more successful at causing infection will therefore spread more quickly through the population, allowing individuals to make meaningful inferences about virus infectivity from how long it took for them to be exposed. In the same way, when a new product is launched or a new idea espoused, those who have purchased the product or accepted the idea naturally spread awareness and cause others to consider it as well. Awareness of better "innovations" (such as new products, ideas, or job-market candidates) will therefore tend to spread more quickly, allowing people to make inferences about quality from how long it took before they were exposed themselves.

In this paper, we examine the equilibrium dynamics of social learning about an innovation when awareness spreads virally and each consumer receives a private signal about innovation quality. We find that virally-spread innovations progress through an endogenous lifecycle consisting of four phases, each with its own distinctive pattern of consumer adoption and belief evolution. Initially after launch, during the first two phases, innovations enjoy positive word of mouth as consumers who learn quickly about an innovation infer that it is more likely to be of high quality. As time goes on, however, newly-exposed consumers grow less and less confident about quality until, eventually, a time is reached after which no one adopts regardless of their private signal, an endogenous obsolescence.

Our model of viral social learning adapts the susceptible-infected-recovered (SIR) model of viral epidemiology[1] to an economic context, allowing consumers to *decide* whether to become infected by a new innovation based on their endogenous equilibrium beliefs about innovation quality. There is a unit-mass population of consumers and an innovation that is "good" (state $\omega = g$) with probability α and "bad" ($\omega = b$) with probability $1 - \alpha$. When first exposed to the innovation, each consumer i observes the time t_i since

[1]The SIR model was formulated in 1908 by Ronald Ross (who also famously discovered that malaria is transmitted by mosquito) and developed further by Kermack and McKendrick [1927]. It remains the workhorse model of the field; see Anderson [1991] and Blackwood and Childs [2018].

launch and a conditionally i.i.d. private signal s_i that matches the true state with probability $\rho > 1/2$. Consumer i then makes a once-and-for-all decision whether to adopt the innovation, preferring to adopt whenever she believes that the innovation is more likely to be good than bad. Those who adopt are "infected" and subsequently expose others via random encounters that occur over continuous time, while those who choose not to adopt are "recovered" and will never adopt nor expose anyone else to the innovation. So long as anyone adopts the innovation at launch, all consumers will eventually be exposed.

The fraction of consumers exposed at launch is denoted by Δ. We focus in the main text on two special cases. In a "traditional ad campaign," $\Delta = 1$ and all consumers simultaneously decide at launch whether to adopt. By contrast, in a "viral campaign," $\Delta \approx 0$ and almost all consumers encounter the innovation socially. In an extension, we allow the innovation producer to run a blended campaign, launching the innovation virally but then ending the viral campaign at time $T \geq 0$ with an ad that reaches all still-unexposed consumers; $T = 0$ corresponds to a traditional ad campaign while $T = \infty$ corresponds to a viral campaign. For a producer who seeks to maximize the (undiscounted[2]) mass of adopters, we find that the optimal viral campaign length $T^* > 0$, meaning that running a viral campaign of some length is always better than not running a viral campaign at all.

In a further extension, we model the producer as deciding whether to produce a good innovation at some cost. We show that if most innovations would be good when marketed with a traditional ad campaign, then running an optimal-length viral campaign *reduces* equilibrium investment in innovation quality. This is surprising, since one might think that having a viral campaign would give a producer even more incentive to invest in quality, to drive positive word of mouth. However, as we show, viral campaigns not only spread awareness but also cause some consumers to adopt even after a negative private signal, reducing the importance of quality in driving adoption.[3] On the other hand, if most innovations would be bad when marketed with a traditional ad campaign, then running an optimal-length viral campaign *increases* equilibrium investment in innovation quality.

Illustrative case: viral campaign when most innovations are good. To gain intuition, suppose that most innovations are good ($\alpha > 1/2$) but bad innovations are common

[2]If the producer prefers quicker adoption, such as when adoption corresponds to purchasing an innovation and the producer is a firm that discounts profits *or* when the innovation becomes obsolescent at some exogenous rate, then running a traditional ad campaign may be optimal.

[3]Other forces work in the opposite direction, e.g., the fact that more consumers are exposed to good innovations during the early phases of the innovation lifecycle (when consumers adopt even after a bad signal) gives producers *more* incentive to invest than otherwise. The overall effect of introducing viral social learning on investment is therefore far from obvious.

enough that consumers prefer not to adopt after a bad private signal ($\alpha < \rho$), and that the producer runs a viral campaign. The small mass of consumers exposed at launch will adopt if and only if they have gotten a good private signal. But what about those exposed immediately after launch? They hear about it from someone who adopted at launch and hence got a good signal. Such consumers will therefore act as if they have gotten two signals: their own private signal *plus* a good signal. Such consumers have an incentive to adopt even after a bad private signal, and so will herd on adoption.

As the "herding phase" that commences immediately after launch continues, the set of infected consumers consists of (i) the small mass of "original adopters" who all got a good private signal and (ii) an ever-growing mass of "herders" who adopted regardless of their private signal. As time goes on, consumers view exposure as less and less of a positive signal about quality, since more and more of those spreading awareness adopted regardless of their private signal; that is, consumers' interim belief upon exposure, denoted $p(t)$, is declining in t. Eventually, a time t_1 is reached at which $p(t_1) = \rho$, so that newly-exposed consumers are indifferent whether to adopt after a bad private signal, and the herding phase ends.

After that, the innovation lifecycle enters a "partially-herding phase" from time t_1 to time t_2 during which consumers randomize whether to adopt after a bad private signal (and consumers' interim belief remains constant at $p(t) = \rho$), followed by a "sensitive-to-signals phase" during which consumers only adopt after a good private signal and $p(t)$ is once again decreasing in t. Finally, once consumers' interim belief reaches $1 - \rho$ at finite time t_3, the innovation lifecycle enters its final "obsolescence phase," during which consumers never adopt and $p(t)$ declines to zero as $t \to \infty$.

What if the viral campaign does not last forever? Suppose in particular that viral spread of the innovation ends at time $T > 0$, when all remaining "susceptible" consumers become aware of the innovation via a non-social broadcast. We show that using an *optimal-length* viral campaign always increases the mass of adopting consumers, relative to a traditional ad campaign. In the case considered here when $\alpha \in (1/2, \rho)$, we show that the ex ante expected mass of adopting consumers is maximized at $T^* = t_2$, with all $T < t_2$ and all $T > t_3$ being strictly suboptimal, meaning that an optimal viral campaign must progress through the first two phases of the innovation lifecycle but end before viral obsolescence.

Relation to the literature. In the classic social learning model (Bikhchandani et al. [1992], Banerjee [1992], Smith and Sorensen [2000]), infinitely many agents are arrayed in a line and decide whether to adopt, one at a time and in order, based on their own private signal and all decisions made by those before them. A large subsequent literature has extended the classic model in several fascinating directions: with agents observing only some of the prior decision history (Lee [1993], Celen and Kariv [2004], Acemoglu et al. [2011], Lobel and Sadler [2015]); observing only the number of others taking each action but not their positions in the decision sequence (Banerjee and Fudenberg [2004], Gale and Kariv [2003], Callander and Horner [2009], Smith and Sorensen [2013]); not knowing their own position in the sequence (Guarino et al. [2011], Herrera and Horner [2013], Monzon and Rapp [2014]); being able to acquire an informative signal or sample an available option (Hendricks et al. [2012], Mueller-Frank and Pai [2016], Ali [2014], Board and Meyer-ter-Vehn [2018]); or selecting which part of the previous history to observe (Kultti and Miettinen [2006], Kultti and Miettinen [2007], Celen [2008], Song [2016]).

This paper takes a different tack, following Banerjee [1993]'s pioneering work on the economics of rumors—marrying economics to infectious-disease epidemiology and bringing together insights from both fields. Banerjee [1993] captures the idea that agents are more likely to hear about a rumor when others have acted on it, and therefore can learn about rumor quality based on when they hear about it. The main difference with our paper is that Banerjee [1993] assumes that those who hear a rumor socially (after time 0) do not get any private information, while we assume that all agents receive an informative private signal.[4]

Because consumers encounter an innovation from others who have already adopted it,[5] our model captures the notion that innovations can "go viral," the number of adopters growing exponentially as awareness of the innovation spreads socially. This idea that ideas can spread like a virus is widely appreciated[6] and well-studied, with some going even further to explore how ideas mutate as they circulate through a population; see e.g., Adamic et al. [2016], Simmons et al. [2011], and Jackson et al. [2019]. We abstract from the possibility of mutation, but push the literature forward by modeling *idea-infection as an economic choice*. In doing so, we endogenize the parameters of the diffusion process, showing moreover how equilibrium diffusion dynamics change over time, passing through "lifecycle phases" with distinctive patterns of adoption.

[4]Much as in our paper, agents in Banerjee [1993] may herd on adoption immediately after launch but such herding cannot last forever due to the diminishing informativeness of exposure, two insights that also apply in our context and that determine "Phase I" of the innovation lifecycle.

[5]Classic social learning reemerges if one assumes (i) all infected and recovered consumers meet others and spread awareness at the same rate and (ii) each consumer is able to observe the history of decisions made along the entire *chain* of consumers leading to their awareness. In that context, consumers along each chain behave exactly as in the classic model, all those sufficiently far along each chain ignoring their own private signal ("herding") and making the wrong decision with non-vanishing probability.

[6]See e.g., "The Age of the Viral Idea" by Bill Davidow, *The Atlantic*, Nov 17, 2011 and "The Internet Catches a Viral Epidemic" by Bill Wasik, *Wired*, April 16, 2013.

The main contribution vis-a-vis the epidemiology literature is to expand the scope of infection models to "innovations" broadly understood, and to endogenize their economic-epidemic spread.[7] From an epidemiological perspective, the most interesting specific finding is that innovation infectivity (the likelihood that any given newly-exposed consumer will adopt) goes to zero in finite time, what we refer to as "viral obsolescence." By contrast, during a viral epidemic, infectivity remains constant (unless the virus mutates or new control measures are put in place) and new hosts will continue to be infected as long as anyone remains infected.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 characterizes equilibrium dynamics of adoption and innovation diffusion. We then present two extensions, allowing the producer to choose when to end the viral campaign (Section 4) and whether to make a costly investment in innovation quality (Section 5). Concluding remarks are in Section 6. Some formal proofs are relegated to an Appendix.

2 Model and preliminary analysis

Summary. A new innovation is launched at time $t = 0$. Consumers in a unit-mass population become aware of the innovation over time and, when first encountering the innovation, decide whether to adopt it. More widespread adoption speeds the viral diffusion process by which innovation awareness spreads. The time at which a consumer first encounters the innovation therefore conveys information about its quality, endogenously determining a time-varying path of consumer adoption, innovation diffusion, and belief formation that we refer to as "viral social learning." See Figure 1.

Consumer incentive to adopt. Each consumer i encounters a new innovation at (random) time t_i, at which point she observes the time, receives a private signal s_i about the innovation's unobservable quality, and decides whether or not to adopt. The innovation may be "good" or "bad". Consumers get payoff $u_g > 0$ when adopting a good innovation, $-u_b < 0$ when adopting a bad innovation, or zero when not adopting, and seek to maximize their own expected payoff. To simplify equations, suppose that $u_g = u_b$ so that each consumer strictly prefers to adopt if and only if they believe that the innovation's

[7] The literature on controlling infectious disease routinely endogenizes diffusion dynamics, both in terms of optimal and equilibrium control of a disease; see e.g., Newman [2002] on disease spread over a social network, Lipsitch et al. [2007] on when best to use an antiviral agent during a flu outbreak, Bauch and Bhattacharyya [2012] on the dynamics of vaccine scares, Laxminarayan and Brown [2001] and McAdams [2017] on when to switch to a new antibiotic in the face of rising resistance, and McAdams et al. [2019] on how diagnostic-targeted control measures may slow or even reverse the rise of antibiotic resistance.

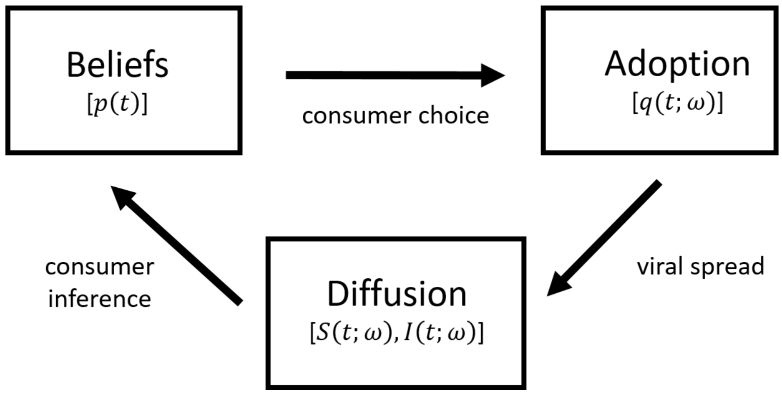


Figure 1: Illustration of viral social learning, whereby the dynamics of innovation adoption drive the epidemiological dynamics of innovation awareness, which in turn determine the dynamics of consumer beliefs about innovation quality.

likelihood of being good exceeds 1/2.[8]

Consumer belief formation. Let $\alpha \in (0,1)$ be the ex ante likelihood that the innovation is good, let $p(t_i)$ be the probability that the innovation is good conditional on encountering it at time t_i, and let $p(t_i, s_i)$ denote its likelihood of being good conditional on t_i and private signal s_i. We refer to $\alpha = p(0)$ as consumers' "ex ante belief," $p(t_i)$ as consumer i's "interim belief," and $p(t_i, s_i)$ as her "ex post belief." Interim beliefs are determined according to Bayes' Rule, with each consumer i updating her belief to $p(t_i) = \frac{\alpha f(t_i|\omega=g)}{\alpha f(t_i|\omega=g)+(1-\alpha)f(t_i|\omega=b)}$ or, equivalently,

$$\frac{p(t_i)}{1-p(t_i)} = \frac{\alpha}{1-\alpha} \times \frac{f(t_i|\omega = g)}{f(t_i|\omega = b)}$$

where $f(t_i|\omega)$ denotes the endogenous[9] p.d.f. of t_i conditional on the true state $\omega \in \{g, b\}$. Consumers' private signals are i.i.d. conditional on the state, with precision $\rho = \Pr(s_i = G|\omega = g) = \Pr(s_i = B|\omega = b) \in (1/2, 1)$. Given private signal s_i, consumer i

[8] This normalization is without loss of generality. If $u_g \neq u_b$, then consumers will adopt based on belief threshold $u_b/(u_b + u_g)$ rather than 1/2 but, otherwise, our analysis carries through directly. For example, suppose that good innovations generate value V, bad innovations generate value 0, and all innovations are sold at price P. Then $u_g = V - P$, $u_b = -P$, and our analysis can be used to construct the innovation's *demand curve* when marketed through a viral campaign.

[9] We will characterize the equilibrium distribution of $t_i|\omega$, showing that $f(t_i|\omega)$ exists and is continuous in t_i at all but finitely-many points.

updates her belief further to $p(t_i, s_i)$, defined by

$$\frac{p(t_i, s_i)}{1 - p(t_i, s_i)} = \frac{p(t_i)}{1 - p(t_i)} \times \frac{\Pr(s_i|\omega = g)}{\Pr(s_i|\omega = b)}$$

Among consumers who encounter the innovation at time t ("time-t consumers"), those who get a good signal have updated belief $p(t, G) = \frac{p(t)\rho}{p(t_i)\rho + (1-p(t_i))(1-\rho)}$ while those who get a bad signal have updated belief $p(t, B) = \frac{p(t)(1-\rho)}{p(t_i)(1-\rho)+(1-p(t_i))\rho}$.

Lemma 1 (Adoption patterns)**.** *(i)* Herding on adoption: *If $p(t) > \rho$, then all time-t consumers strictly prefer to adopt. (ii)* Herding on non-adoption: *If $p(t) < 1 - \rho$, then all time-t consumers strictly prefer not to adopt. (iii)* Sensitive to signals: *If $p(t) \in (1 - \rho, \rho)$, then all time-t consumers strictly prefer to adopt after a good private signal but not after a bad private signal.*

Proof. Time-t consumers strictly prefer to adopt if and only if their ex post belief $p(t, s_i) > 1/2$. The desired results follow immediately from the fact that $p(t, B) > 1/2$ if and only $p(t) > \rho$ and $p(t, G) > 1/2$ if and only $p(t) > 1 - \rho$. □

Viral diffusion. Innovation awareness spreads through the consumer population like a virus, according to a Susceptible-Infected-Recovered (SIR) model (Kermack and McKendrick [1927]). At each point in time, each consumer is in one of three epidemiological states: *susceptible*, if she has not yet been exposed to the innovation; *infected*, if she previously chose to adopt the innovation; or *recovered*, if she previously chose not to adopt. We assume that mass $\Delta > 0$ of consumers are exposed to the innovation at time $t = 0$ regardless of innovation quality. Those who adopt then become infected and spread innovation awareness virally, meeting another randomly-selected consumer at rate $\beta > 0$, and exposing that other consumer to the innovation. If susceptible, that other consumer receives a private signal and decides whether or not to adopt, then transitions to either the infected state (if adopting) or the recovered state (if not adopting).

Let $S_\omega(t)$, $I_\omega(t)$, and $R_\omega(t)$ denote the mass of susceptible, infected, and recovered consumers at time t, conditional on the unobserved innovation-quality state $\omega \in \{g, b\}$. Since the population has unit mass, $R_\omega(t) = 1 - S_\omega(t) - I_\omega(t)$ and the overall epidemiological process is described by $(S_\omega(t), I_\omega(t) : t \geq 0, \omega = g, b)$. Let $q_\omega(t)$ denote time-t consumers' likelihood of adopting conditional on ω. Epidemiological dynamics are characterized by the system of differential equations

$$S'_\omega(t) = -\beta I_\omega(t) S_\omega(t) \tag{1}$$

$$I'_\omega(t) = q_\omega(t)\beta I_\omega(t) S_\omega(t) \tag{2}$$

Equation (1) follows from the fact that each infected consumer meets another consumer at rate $\beta > 0$ and fraction $S_\omega(t)$ of others remain suspectible, generating a state-dependent flow $\beta I_\omega(t) S_\omega(t)$ of newly-exposed consumers who are then no longer susceptible. Equation (2) follows from the fact that fraction $q_\omega(t)$ of these newly-exposed consumers choose to adopt. Note that epidemiological dynamics are completely determined by the adoption process $(q_\omega(t) : t \geq 0, \omega = g, b)$.

We will henceforth normalize $\beta = 1$. This is without loss of generality since, with any $\hat{\beta} \neq 1$, epidemiological dynamics are exactly the same but happen $\hat{\beta}$ times faster than when $\beta = 1$. The model has three remaining parameters: $\Delta \in (0, 1]$, the fraction of consumers exposed at launch; $\alpha \in [0, 1]$, the ex ante likelihood of a good innovation; and $\rho \in (1/2, 1)$, the precision of consumers' private signals.

Viral social learning. Since the consumer population has unit mass, the flow of newly-exposed consumers can be interpreted as the density of the time-until-exposure t, i.e., $f(t|\omega) = |S'_\omega(t)| = S_\omega(t) I_\omega(t)$, where $|S'_\omega(t)|$ is the flow of consumers being exposed at time t when the innovation is good ($\omega = g$) or bad ($\omega = b$). Thus, time-t consumers' interim belief is given by

$$\frac{p(t)}{1 - p(t)} = \frac{\alpha}{1 - \alpha} \times \frac{S_g(t) I_g(t)}{S_b(t) I_b(t)} = \frac{\alpha}{1 - \alpha} \times \frac{|S'_g(t)|}{|S'_b(t)|}. \tag{3}$$

Equilibrium. Our solution concept is perfect Bayesian equilibrium (PBE or simply "equilibrium"). We will show by construction that a PBE exists and that this equilibrium is essentially unique, in the sense that all PBE generate the same population-wide epidemiological dynamics $(S_\omega(t), I_\omega(t) : t \geq 0; \omega \in \{g, b\})$.

Discussion: observability of time since launch. We assume that, when consumers encounter the innovation, they are able to observe how much time has elapsed since launch. However, our analysis can be easily extended to a setting in which fraction $\eta \in [0, 1]$ of consumers are unable to observe the time. Since all consumers will eventually be exposed to the innovation (so long as anyone adopts initially at launch), a consumer who is unable to observe the time will not make any inference about innovation quality and so will decide whether to adopt *as if* encountering the innovation at launch. The overall likelihood that a consumer exposed at time $t > 0$ will adopt in innovation-quality state $\omega \in \{g, b\}$ is therefore $\tilde{q}_\omega(t) = \eta q_\omega(0) + (1-\eta) q_\omega(t)$, where $q_\omega(0)$ and $q_\omega(t)$ are the likelihoods that consumers who *can* observe the time will adopt, respectively, at time 0 and time t. The rest of our analysis then carries over, with more complex formulas but little additional insight.

3 Equilibrium Innovation Lifecycle

This section characterizes consumers' equilibrium adoption behavior and the resulting epidemiological dynamics. In so doing, we characterize the endogenous lifecycle of a new innovation subject to viral social learning. To keep the presentation as simple as possible, we focus on the case in which the fraction of consumers who encounter the innovation at launch is small, i.e., $\Delta \approx 0$.[10]

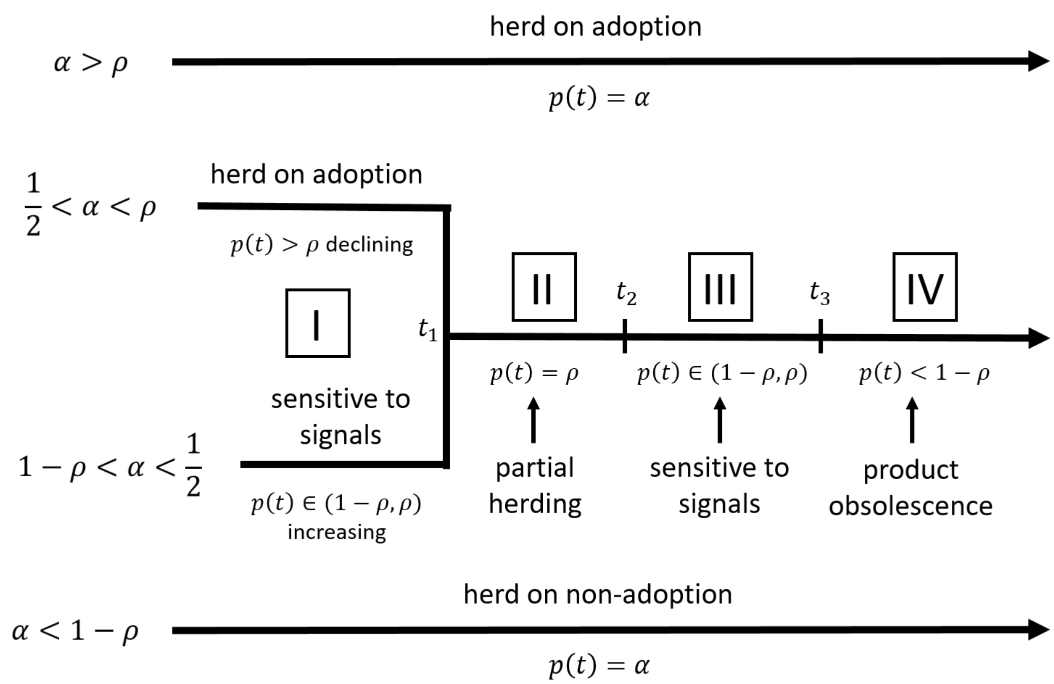


Figure 2: Visual summary of equilibrium adoption behavior and interim beliefs over the innovation lifecycle, depending on consumers' ex ante belief $\alpha = p(0)$.

[10]Our analysis and key conclusions (including uniqueness of equilibrium dynamics) extend easily to any $\Delta \in (0,1]$, but some qualitative features of equilibrium consumer behavior change when Δ is sufficiently large. For instance, in the case when $\alpha \in (1-\rho, 1/2)$, consumers' interim beliefs increase immediately after launch if $\Delta \approx 0$ but decrease immediately after launch if $\Delta > 1/2$.

Figure 2 illustrates the innovation's equilibrium lifecycle in three main cases: when the innovation is sufficiently likely to be good that $\alpha > \rho$ (top); when the innovation is sufficiently likely to be bad that $\alpha < 1-\rho$ (bottom); and when the innovation has an intermediate likelihood of being good so that $\alpha \in (1-\rho, \rho)$ (middle). Proposition 1 characterizes consumers' equilibrium behavior in the first two cases.

Proposition 1. *(i) When $\alpha > \rho$, all consumers adopt the innovation in equilibrium. (ii) When $\alpha < 1-\rho$, no consumers adopt the innovation in equilibrium.*

Proof. When $\alpha > \rho$, consumers herd on adoption at time $t = 0$ and, with an equal mass of "infected" consumers spreading awareness no matter whether the innovation is good or bad, later-exposed consumers infer nothing about innovation quality from their time of exposure. These later-exposed consumers must therefore also herd on adoption. Since all consumers eventually encounter the innovation, all consumers end up adopting in the unique equilibrium. On the other hand, when $\alpha < 1-\rho$, consumers herd on non-adoption at time $t = 0$ and, with no one "infected" to spread awareness virally, no one else is ever exposed in the unique equilibrium. □

The rest of our analysis focuses on the remaining case in which $\alpha \in (1-\rho, \rho)$, so that consumers who encounter the innovation at launch are sensitive to signals.[11] Theorem 1 summarizes our main findings, that equilibrium epidemiological dynamics are uniquely determined and that consumer behavior transitions through (up to) four distinct phases during the innovation's lifecycle. Behavior in Phase I immediately after launch depends on whether α is greater or less than $1/2$ but, no matter what, there is always a period of partial herding (Phase II), a period in which consumers are sensitive to signals (Phase III), and a final period with zero adoption (Phase IV).

Theorem 1. *Suppose that $\alpha \in (1-\rho, \rho)$ and $\Delta \approx 0$. All equilibria generate the same epidemiological dynamics $(S_\omega(t), I_\omega(t) : t \geq 0; \omega \in \{g, b\})$ and time-path of interim beliefs $p(t)$. Consumers' post-launch equilibrium behavior transitions through four distinct phases.*

Phase I: *(i) If $\alpha \in (1/2, \rho)$, then consumers herd on adoption and interim belief $p(t) > \rho$ decreases until time $t_1 > 0$ is reached at which $p(t_1) = \rho$. (ii) If $\alpha \in (1-\rho, 1/2)$, then*

[11] We ignore the boundary cases when $\alpha = \rho$ and $\alpha = 1-\rho$. These cases are more complex because consumers have multiple best responses at launch, but this extra complexity does not lead to any additional insight. For example, if $\alpha = 1-\rho$, consumers exposed at launch will adopt with some probability $a_G \in [0,1]$ after a good signal but not adopt after a bad signal, resulting in initial infected mass $I_g(0+) = a_G\rho\Delta$ when the innovation is good and $I_b(0+) = a_G(1-\rho)\Delta$ when it is bad and initial belief $p(0+) = 1/2$. For each $a_G \in (0,1]$, subsequent equilibrium dynamics are then uniquely determined by similar arguments as used here for the case when $\alpha \in (1-\rho, \rho)$.

consumers are sensitive to signals and $p(t) \in (1/2, \rho)$ *increases until time* $t_1 > 0$ *at which* $p(t_1) = \rho$*. (iii) If* $\alpha = 1/2$*, then* $p(0+) \equiv \lim_{\epsilon \to 0} p(\epsilon) = \rho$ *and Phase I does not occur, i.e.,* $t_1 = 0$*.*

Phase II: *Consumers partially herd on adoption, adopting always after a good signal and with probability* $a_B(t) \in (0, 1)$ *after a bad signal, where* $a_B(t)$ *is decreasing in* t*, until time* $t_2 > t_1$ *is reached at which* $a_B(t_2) = 0$*. Consumers' interim belief* $p(t) = \rho$ *for all* $t \in [t_1, t_2]$*.*

Phase III: *Consumers are sensitive to signals and interim belief* $p(t) \in (1 - \rho, \rho)$ *is decreasing in* t *until time* $t_3 > t_2$ *is reached at which* $p(t_3) = 1 - \rho$*.*

Phase IV: *Consumers forevermore herd on non-adoption, what we refer to as "viral obsolescence," and consumers' interim belief* $p(t) < 1 - \rho$ *continues to decrease with* $\lim_{t \to \infty} p(t) = 0$*.*

The rest of this section provides the proof of Theorem 1 through a series of lemmas and propositions, characterizing consumers' equilibrium behavior throughout the innovation lifecycle, from launch to obsolescence.

3.1 Beginning of innovation lifecycle (Launch and Phase I)

This section characterizes consumers' equilibrium behavior at launch (Prop 2) and after launch until time t_1 is reached at which consumers' interim belief equals ρ (Props 3-4). We refer to the period of time from launch until t_1 as "Phase I".

Innovation launch. Because $\alpha = p(0) \in (1 - \rho, \rho)$, consumers exposed at launch are sensitive to signals (Lem 1(iii)). Thus, fraction ρ of those exposed at launch adopt when the innovation is good, or fraction $1 - \rho$ when it is bad. In particular, at any time $t \approx 0$, there are more "infected" consumers when the innovation is good: $I_g(t) \approx \rho\Delta$ vs. $I_b(t) \approx (1 - \rho)\Delta$, where Δ is the mass of consumers exposed at launch. On the other hand, there is approximately the same number of still-unexposed "susceptible" consumers: $S_g(t) \approx S_b(t) \approx 1 - \Delta$.

Since more consumers adopt good innovations at launch, others are more likely to encounter an innovation shortly after launch when it is good. Those quickly exposed therefore interpret their quick awareness as good news about innovation quality and update upward their beliefs about innovation quality. In particular, consumers exposed at time $t \approx 0$ hold interim belief $p(t) = \frac{\alpha S_g(t) I_g(t)}{\alpha S_g(t) I_g(t) + (1-\alpha) S_b(t) I_b(t)} \approx p(0+) = \frac{\alpha \rho}{\alpha \rho + (1-\alpha)(1-\rho)}$. Proposition 2 summarizes these findings.

Proposition 2 (Launch). *Suppose that $\alpha \in (1-\rho, \rho)$.*[12] *In any equilibrium, consumers are sensitive to signals at time $t = 0$ and those exposed immediately after launch have interim belief $p(0+) = \frac{\alpha\rho}{\alpha\rho + (1-\alpha)(1-\rho)} > \alpha$.*

Whether the "good news" of quick exposure is good enough to prompt quickly-exposed consumers to herd on adoption depends on their ex ante belief α. If $\alpha > 1/2$, then $p(0+) > \rho$ and consumers will herd on adoption immediately after adoption. On the other hand, if $\alpha < 1/2$, then $p(0+) \in (1/2, \rho)$ and consumers will continue to be sensitive to signals immediately after adoption. Finally, if $\alpha = 1/2$, then $p(0+) = \rho$ and Phase I does not occur, i.e., $t_1 = 0$.

Phase I: herding on adoption case. Suppose first that $\alpha \in (1/2, \rho)$, so that consumers herd on adoption immediately after launch. We will show that consumers' interim belief $p(t)$ declines until time $t_1 > 0$ is reached at which $p(t_1) = \rho$.

By equation (3), the likelihood ratio $\frac{p(t)}{1-p(t)} = \frac{\alpha}{1-\alpha} \times \frac{S_g(t)I_g(t)}{S_b(t)I_b(t)}$; so,

$$\begin{aligned}\frac{\mathrm{d}\log\left(\frac{p(t)}{1-p(t)}\right)}{\mathrm{d}t} &= \frac{S_g'(t)}{S_g(t)} - \frac{S_b'(t)}{S_b(t)} + \frac{I_g'(t)}{I_g(t)} - \frac{I_b'(t)}{I_b(t)} \\ &= -I_g(t) + I_b(t) + q_g(t)S_g(t) - q_b(t)S_b(t) \\ &\equiv X(t) \qquad (4)\end{aligned}$$

where $\frac{S_\omega'(t)}{S_\omega(t)} = -I_\omega(t)$ and $\frac{I_\omega'(t)}{I_\omega(t)} = q_\omega(t)S_\omega(t)$ by equations (1-2). So, $\frac{p(t)}{1-p(t)}$ grows exponentially at rate $X(t)$; in particular, $p'(t) \gtrless 0$ iff $X(t) \gtrless 0$.

Lemma 2 summarizes these findings, emphasizing the implications of equation (4) when consumers herd on adoption, are sensitive to signals, or herd on non-adoption.

Lemma 2. $\frac{p(t)}{1-p(t)}$ *increases exponentially at rate $X(t)$, where*

$$X(t) = \left(q_g(t)S_g(t) - q_b(t)S_b(t)\right) - \left(I_g(t) - I_b(t)\right). \qquad (5)$$

In particular: (i) Suppose that consumers herd on adoption at time t. $p'(t) < 0$ if $I_g(t) > I_b(t)$ and $S_g(t) < S_b(t)$. (ii) Suppose that consumers are sensitive to signal at time t. $p'(t) > 0$ if and only if the following inequality holds:

$$\rho S_g(t) - (1-\rho)S_b(t) > I_g(t) - I_b(t). \qquad \text{(SS)}$$

[12]Results that do not explicitly state "$\Delta \approx 0$" hold no matter how many consumers are exposed at launch.

(We refer to this as "Condition SS," mnemonic for "sensitive to signal.") (iii) Suppose that consumers herd on non-adoption at time t. $p'(t) < 0$ if $I_g(t) > I_b(t)$.

Proof. The proof is immediate from equation (4), given that (i) $q_g(t) = q_b(t) = 1$ when consumers herd on adoption; (ii) $q_g(t) = \rho$ and $q_b(t) = 1 - \rho$ when consumers are sensitive to signals; and (iii) $q_g(t) = q_b(t) = 0$ when consumers herd on non-adoption. □

Since consumers herd on adoption, $X(t) < 0$ during Phase I so long as good innovations have greater cumulative adoption ($I_g(t) > I_b(t)$) and greater cumulative exposure ($S_g(t) < S_b(t)$) than bad innovations (Lemma 2(i)). To establish these results, we use Lemma 3 (proven in the Appendix).

Lemma 3. *Suppose that $p(t) > \alpha$ for all $t \in (0, \hat{t})$ for some $\hat{t}$. Then $I_g(t) > I_b(t)$, $I_g'(t) > I_b'(t)$, $S_g(t) < S_b(t)$, and $S_g'(t) < S_b'(t)$ for all $t \in (0, \hat{t})$.*

Because $p(0+) > \rho > \alpha$, Lemma 3 implies that $I_g(t) - I_b(t)$ and $S_b(t) - S_g(t)$ are positive and increasing so long as consumers' interim belief $p(t)$ remains above ρ. We conclude that $X(t) < 0$ and $X'(t) < 0$, and hence that $\frac{p(t)}{1-p(t)}$ declines exponentially at an increasing rate during Phase I. Consumers' interim belief must therefore reach ρ in finite time; let t_1 be the first time at which $p(t_1) = \rho$.

Proposition 3 summarizes these findings on Phase I when $\alpha \in (1/2, \rho)$.

Proposition 3 (Phase I: herding on adoption)**.** *Suppose that $\alpha \in (1/2, \rho)$. There exists $t_1 > 0$ such that, in any equilibrium at all times $t \in (0, t_1)$, consumers herd on adoption and consumers' interim belief $p(t) > \rho$ is decreasing over time with $p(t_1) = \rho$.*

Phase I: sensitive to signals case. Suppose next that $\alpha \in (1 - \rho, 1/2)$, so that consumers are sensitive to signals after launch. We will show that consumers' interim belief $p(t)$ increases until time $t_1 > 0$ is reached at which $p(t_1) = \rho$.

Since consumers are sensitive to signals, interim belief is increasing during Phase I so long as Condition SS is satisfied. Recall that, at times $t \approx 0$, $S_g(t) \approx S_b(t) \approx 1 - \Delta$, $I_g(t) \approx \rho\Delta$, and $I_b(t) \approx (1 - \rho)\Delta$. Thus, $X(t) \approx (2\rho - 1)(1 - 2\Delta) \approx 2\rho - 1 > 0$ and interim beliefs are initially increasing. However, the rate at which $\frac{p(t)}{1-p(t)}$ increases itself decreases over time. Why? By equations (1-2), $S_g'(t) = -I_g(t)S_g(t)$, $S_b'(t) = -I_b(t)S_b(t)$, $I_g'(t) = \rho I_g(t)S_g(t)$, and $I_b'(t) = (1 - \rho)I_b(t)S_b(t)$; thus,

$$X'(t) = -2\left(\rho I_g(t)S_g(t) - (1 - \rho)I_b(t)S_b(t)\right). \tag{6}$$

Note that $X'(t) \geq 0$ iff $\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)} \leq \frac{1-\rho}{\rho}$. By equation (3), that is only possible at times when $\frac{p(t)}{1-p(t)} \leq \frac{\alpha(1-\rho)}{(1-\alpha)\rho}$ which, since $\alpha < 1/2$, implies that $p(t) < 1-\rho$. We conclude that, so long as $p(t) > 1-\rho$ and consumers are sensitive to signal, $X'(t) < 0$.

Because $X'(t) < 0$, consumers' interim beliefs may begin to decline if consumers are sensitive to signal for long enough. However, because our assumption that Δ is small,[13] this does not happen for a long time. To see why, note that the total mass of consumers exposed by any given time $\tilde{t}$ can be made arbitrarily small by beginning with a sufficiently small initial mass Δ of consumers exposed at launch. In particular, for any time $\tilde{t}$ and any small $\epsilon > 0$, we can find Δ sufficiently small so that (i) $S_g(t), S_b(t) \in (1-\epsilon, 1)$ for all $t < \tilde{t}$ and (ii) $I_g(t), I_b(t) \in (0, \epsilon)$ for all $t < \tilde{t}$, implying that $X(t) > (\rho(1-\epsilon) - \epsilon) - (1 - \rho - 0) = 2\rho - 1 - \epsilon(1+\rho) > 0$ for all $t \in (0, \tilde{t})$. Recall by Lemma 2 that the likelihood ratio $\frac{p(t)}{1-p(t)}$ rises exponentially at rate $X(t)$; so, for small Δ, $\frac{p(t)}{1-p(t)}$ rises exponentially at approximate rate $2\rho - 1$ until a time t_1 is reached at which consumers' interim belief equals ρ.

We conclude that, in any equilibrium, consumers must be sensitive to signal at all times $t \in (0, t_1)$ (since interim belief is in $(1/2, \rho)$ at such times) and must not continue to be sensitive to signal immediately after t_1 (since then interim belief would rise above ρ, a contradiction). This completes the proof and, in addition, uniquely characterizes t_1 as the first time at which $p(t_1) = \rho$.

Proposition 4 summarizes these findings on Phase I when $\alpha \in (1-\rho, 1/2)$.

Proposition 4 (Phase I: sensitive to signals)**.** *Suppose that $\alpha \in (1-\rho, 1/2)$ and $\Delta \approx 0$. There exists $t_1 > 0$ such that, in any equilibrium at all times $t \in (0, t_1)$, consumers are sensitive to signals and consumers' interim belief $p(t) \in (1/2, \rho)$ is increasing over time with $p(t_1) = \rho$.*

3.2 Middle of innovation lifecycle (Phase II)

This section characterizes equilibrium behavior after time t_1. We find that, for a non-empty interval of time, consumers randomize whether to adopt after a bad signal, what we call "partial herding." Over that period of time, consumers' interim belief remains equal to ρ and the likelihood $a_B(t)$ that consumers adopt after a bad signal declines continuously until, at some time t_2, $a_B(t) = 0$ and consumers become sensitive to signals. We refer to the period from t_1 until t_2 as "Phase II".

Proposition 5 (Phase II)**.** *Suppose that $\alpha \in (1-\rho, \rho)$ and $\Delta \approx 0$. There exists $t_2 > t_1$ such that, in any equilibrium at all times $t \in (t_1, t_2)$, consumers partially herd with probability*

[13] When Δ is sufficiently large, $p(t)$ never reaches ρ and Phase I proceeds directly to Phase III.

$a_B(t) \in (0,1)$ of adoption after a bad signal and consumers' interim belief $p(t) = \rho$ is constant. Moreover, $a_B(t)$ is decreasing over time with $a_B(t_2) = 0$.

Proof. We begin by showing that consumers' interim belief must remain constant immediately after time t_1. First, suppose that $p(t)$ were to rise after time t_1, causing consumers to herd on adoption. Since $p(t) > \alpha$ for all $t \in (0, t_1)$ (shown previously[14]), Lemma 3 implies that, at time t_1, good innovations must be more widely seen ($S_g(t_1) < S_b(t_1)$) and more widely adopted ($I_g(t_1) > I_b(t_1)$). Thus, at all times shortly after t_1, $S_g(t) - I_g(t) < S_b(t) - I_b(t)$ and consumers' interim belief must decline over time by Lemma 2(i), a contradiction.

Next, suppose that $p(t)$ were to fall after time t_1, causing consumers to be sensitive to signals. As discussed in the proof of Prop 4, the assumption here of a small launch ($\Delta \approx 0$) implies that only a small mass of consumers are exposed to the innovation prior to Phase II; in particular, $S_g(t_1), S_b(t_1) \in (1-\epsilon, 1)$ and $I_g(t_1), I_b(t_1) \in (0, \epsilon)$ for some small ϵ. Consequently, for all times t shortly after t_1, Condition SS holds. Were consumers to be sensitive to signals immediately after time t_1, consumers' interim belief would therefore increase over time by Lemma 2(ii), a contradiction. We conclude that, in any equilibrium, consumers' interim belief must remain $p(t) = \rho$ for some period of time after t_1.

By equation (3), interim belief $p(t) = \rho$ requires that $\frac{\rho}{1-\rho} = \frac{\alpha I_g(t) S_g(t)}{(1-\alpha) I_b(t) S_b(t)}$ or, equivalently, $\frac{I_g(t) S_g(t)}{I_b(t) S_b(t)} = \frac{(1-\alpha)\rho}{\alpha(1-\rho)}$. In order for this ratio not to change over time, the ratio of derivatives $\frac{(I_g(t) S_g(t))'}{(I_b(t) S_b(t))'}$ must also equal $\frac{(1-\alpha)\rho}{\alpha(1-\rho)}$. Taking derivatives, using equations (1-2), and re-arranging yields

$$\begin{aligned}\frac{(1-\alpha)\rho}{\alpha(1-\rho)} &= \frac{I_g'(t) S_g(t) + I_g(t) S_g'(t)}{I_b'(t) S_b(t) + I_b(t) S_b'(t)} = \frac{I_g(t) S_g^2(t) q_g(t) - I_g^2(t) S_g(t)}{I_b(t) S_b^2(t) q_b(t) - I_b^2(t) S_b(t)} \\ &= \frac{I_g(t) S_g(t) (S_g(t) q_g(t) - I_g(t))}{I_b(t) S_b(t) (S_b(t) q_b(t) - I_b(t))}\end{aligned}$$

and so it must be that

$$S_g(t) q_g(t) - I_g(t) = S_b(t) q_b(t) - I_b(t). \tag{7}$$

Let $a_B(t)$ denote the likelihood that consumers exposed at time t adopt the innovation after getting a bad signal. The overall likelihood that a good innovation is adopted equals $q_g(t) = \rho + (1-\rho) a_B(t)$; similarly, the likelihood that a bad innovation is adopted equals

[14] When $\alpha = 1/2$, $t_1 = 0$. When $\alpha \in (1/2, \rho)$, $p(t) > \rho > \alpha$ for all $t \in (0, t_1)$ by Prop 3. Finally, when $\alpha \in (1-\rho, 1/2)$, $p(t) > 1/2 > \alpha$ for all $t \in (0, t_1)$ by Prop 4.

$q_b(t) = 1 - \rho + \rho a_B(t)$. Equation (7) can therefore be re-written as

$$\left(\rho S_g(t) - (1-\rho)S_b(t)\right) - \left(I_g(t) - I_b(t)\right) + a_B(t)\left((1-\rho)S_g(t) - \rho S_b(t)\right) = 0 \quad (8)$$

or, equivalently,

$$a_B(t) = \frac{\rho S_g(t) - (1-\rho)S_b(t) - \left(I_g(t) - I_b(t)\right)}{\rho S_b(t) - (1-\rho)S_g(t)} \quad (9)$$

In Section 3.1, we characterized the time t_1 at which Phase II begins and the initial conditions $\left(I_g(t_1), I_b(t_1), S_g(t_1), S_b(t_1)\right)$. Now, equation (9) uniquely determines $a_B(t_1+)$, consumers' equilibrium likelihood of adopting after a bad signal immediately after time t_1. Note that, since $I_g(t_1) > I_b(t_1)$ and $S_b(t_1) > S_g(t_1)$ (by Lemma 3), $a_B(t_1+) < 1$. Moreover, because Condition SS holds at time t_1 (discussed earlier), the numerator in (9) is positive; so, $a_B(t_1+) > 0$.

Equations (1,2,9) uniquely determine the path of $\left(a_B(t), S_g(t), S_b(t), I_g(t), I_b(t)\right)$, starting at time t_1 and so long as $a_B(t) \in (0,1)$. Let t_2 be the first time after t_1 at which $a_B(t) \in \{0,1\}$, or $t_2 = \infty$ if $a_B(t)$ remains forever between zero and one. In the Appendix, we complete the proof by showing that $a_B(t)$ is strictly decreasing and eventually reaches zero; so, $t_2 < \infty$ and $a_B(t_2) = 0$, as desired. □

Discussion: how Phase II ends. The time t_2 at which Phase II ends is the first time at which $\rho S_g(t) - (1-\rho)S_b(t) = I_g(t) - I_b(t)$, i.e., the first time that Condition SS is satisfied with equality (details in the Appendix). The reason for this, intuitively, is that sensitivity to signals spreads awareness of good innovations more rapidly than bad innovations and hence puts *upward pressure on beliefs* early in the innovation's lifecycle, when few people have been exposed to the innovation.[15] Partial herding is needed during Phase II to temper this upward pressure, to keep beliefs from rising above ρ. However, the upward pressure on beliefs due to signal sensitivity weakens over time until, eventually, beliefs begin to fall even if consumers are sensitive to signals. Time t_2 is the transition point after which there is *downward pressure on beliefs* even if consumers are sensitive to signals.

[15] Given our small-launch assumption that $\Delta \approx 0$, the mass of consumers who encounter the innovation by time t_1 is approximately zero and the upward pressure associated with signal sensitivity remains strong. If instead there were a large launch, our analytical method still applies but with different qualitative findings. In particular, in the case when $\alpha \in (1-\rho, 1/2)$ and $\Delta > 1/2$, consumer beliefs begin falling immediately after launch and the innovation's lifecycle begins in Phase III.

3.3 End of innovation lifecycle (Phase III and obsolescence)

This section characterizes equilibrium behavior after time t_2. We have two main findings. First, consumers remain sensitive to signals for a period of time but, even though newly-exposed consumers are only adopting after a good private signal, consumers' interim belief falls until a time t_3 is reached at which $p(t_3) = 1 - \rho$ (Prop 6). Second, consumers herd on non-adoption after time t_3, what we refer to as "viral obsolescence," and their interim beliefs continue to decline toward zero (Prop 7). We refer to the sensitive-to-signal period from t_2 to t_3 as "Phase III" and the obsolescent period after t_3 as "Phase IV".

Proposition 6 (Phase III). *Suppose that $\alpha \in (1-\rho, \rho)$ and $\Delta \approx 0$. There exists $t_3 > t_2$ such that, in any equilibrium at all times $t \in (t_2, t_3)$, consumers are sensitive to signal and consumers' interim belief $p(t)$ declines over time from $p(t_2) = \rho$ to $p(t_3) = 1 - \rho$. Moreover, at all times $t \leq t_3$, the cumulative mass of consumers who have been exposed and who have adopted is higher when the innovation is good than when it is bad, i.e., $S_g(t) < S_b(t)$ and $I_g(t) > I_b(t)$ for all $t \in [0, t_3]$.*

Proposition 7 (Phase IV). *Suppose that $\alpha \in (1-\rho, \rho)$ and $\Delta \approx 0$. After time t_3 in any equilibrium, consumers herd on non-adoption and $p(t)$ declines with* $\lim_{t \to \infty} p(t) = 0$.

Proof. We prove Props 6-7 together, relegating the most complex techincal details in the Online Appendix. For ease of exposition, we divide the proof into four main steps.

Step 1: After time t_2, $\frac{p(t)}{1-p(t)}$ declines exponentially at an increasing rate until some time $\tilde{t}$ at which $p(\tilde{t}) = \max\{1-\rho, \underline{\alpha}\}$, where $\underline{\alpha} \equiv \frac{(1-\rho)\alpha}{(1-\rho)\alpha + \rho(1-\alpha)} \in \left(\frac{(1-\rho)^2}{(1-\rho)^2+\rho^2}, \frac{1}{2}\right)$.

By Lemma 2, $\frac{p(t)}{1-p(t)}$ declines exponentially at rate $X(t)$. So, it suffices to show that $X(t) < 0$ and $X'(t) < 0$ at all times after t_2 until a time $\tilde{t}$ is reached at which $p(\tilde{t}) = \max\{1-\rho, \underline{\alpha}\}$. By the proof of Prop 5: $p(t_2) = \rho$; consumers are sensitive to signal at time t_2 (because $a_B(t_2) = 0$); and $X(t_2) = \left(\rho S_g(t_2) - I_g(t_2)\right) - \left((1-\rho)S_b(t_2) - I_b(t_2)\right) = 0$. It suffices to show that $X'(t) < 0$ at all times $t \in [t_2, \tilde{t})$, since then it must also be that $X(t) < 0$ at all times $t \in (t_2, \tilde{t})$.

By equation (6), $X'(t) = -2\left(\rho S_g(t) I_g(t) - (1-\rho) S_b(t) I_b(t)\right)$ while consumers are sensitive to signals. Thus, $X'(t) < 0$ so long as $\frac{S_g(t_2)I_g(t_2)}{S_b(t_2)I_b(t_2)} > \frac{1-\rho}{\rho}$. By equation (3), $\frac{p(t)}{1-p(t)} = \frac{\alpha S_g(t_2) I_g(t_2)}{(1-\alpha) S_b(t_2) I_b(t_2)}$; so, $\frac{S_g(t_2)I_g(t_2)}{S_b(t_2)I_b(t_2)} > \frac{1-\rho}{\rho}$ if and only if $p(t) > \underline{\alpha}$ or, equivalently, $\frac{p(t)}{1-p(t)} > \frac{\alpha(1-\rho)}{(1-\alpha)\rho} = \frac{\underline{\alpha}}{1-\underline{\alpha}}$. In other words:

$$\text{when consumers are sensitive to signal, } X'(t) \gtrless 0 \text{ iff } p(t) \gtrless \underline{\alpha} \tag{10}$$

At time t_2, consumers are sensitive to signal and $p(t_2) = \rho > \underline{\alpha}$; so, $X'(t_2) < 0$. Moreover, $X'(t) < 0$ at times $t \in (t_2, \tilde{t})$ since (i) consumers remain sensitive to signal (because $p(t) \in (1-\rho, \rho)$) and (ii) $p(t) > \underline{\alpha}$. We conclude that $\frac{p(t)}{1-p(t)}$ decreases exponentially at an increasing rate from time t_2 until time $\tilde{t}$.

What about after time $\tilde{t}$? There are two relevant cases. First, suppose that $\alpha \in (1-\rho, 1/2]$, so that $\underline{\alpha} \leq 1-\rho$. In this case, $p(\tilde{t}) = 1-\rho$ and Phase III ends at time $\tilde{t}$, i.e., $t_3 = \tilde{t}$. Second, suppose that $\alpha \in (1/2, \rho)$. In this more challenging case, $\underline{\alpha} \in (1-\rho, 1/2)$ and the argument so far shows that $\frac{p(t)}{1-p(t)}$ declines at an increasing rate until time $\tilde{t}$, when consumers' interim belief hits $\underline{\alpha}$. However, we still need to show that consumers' interim belief *continues* falling long enough after time $\tilde{t}$ to reach $1-\rho$.

Step 2: In the case when $\alpha \in (1/2, \rho)$, $\frac{p(t)}{1-p(t)}$ declines exponentially at a decreasing rate from time $\tilde{t}$ until time t_3 at which $p(t_3) = 1-\rho$.

The argument in Step 1 established that $p(\tilde{t}) = \underline{\alpha} \in (1-\rho, 1/2)$ and $X(\tilde{t}) < 0$; thus, consumers' interim belief continues to fall below $\underline{\alpha}$ right after time $\tilde{t}$. By condition (10), we conclude that $X'(t) > 0$ right after $\tilde{t}$ and at all times $t > \tilde{t}$ so long as consumers' interim belief remains between $1-\rho$ and $\underline{\alpha}$.

This leaves three possibilities for what happens after time $\tilde{t}$: (i) $p(t)$ stops decreasing (and starts increasing) at some time $\hat{t}$ before reaching $1-\rho$; (ii) $p(t)$ continues decreasing forever but never reaches $1-\rho$; or (iii) $p(t)$ continues decreasing until a time t_3 at which point $p(t_3) = 1-\rho$ and Phase III ends. We need to prove that possibility (iii) always occurs. Here in the main text, we show that possibility (ii) cannot occur; the proof that possibility (i) cannot occur is more technically challenging and relegated to the Appendix.

As shorthand, define $X(\infty) = \lim_{t\to\infty} X(t)$, $I_g(\infty) = \lim_{t\to\infty} I_g(t)$, and so on.

Suppose for the sake of contradiction that possibility (ii) occurs, i.e., consumers' interim belief continues falling forever after time t_2 but never reaches $1-\rho$. This is only possible if $X(\infty) = 0$, which in turn requires that $I_g(\infty) - \rho S_g(\infty) = I_b(\infty) - (1-\rho)S_b(\infty)$. Since all consumers eventually encounter the innovation, $S_g(\infty) = S_b(\infty) = 0$. Thus, it must be that $I_g(\infty) = I_b(\infty)$. We will reach a contradiction by showing that $I_g(\infty) > I_b(\infty)$.

Recall that we are focusing here on the case in which $\alpha \in (1/2, \rho)$. We have shown: consumers are sensitive to signals at launch ($t = 0$), adopting good innovations with probability ρ and bad ones with probability $1-\rho$; consumers herd on adoption in Phase I ($t \in (0, t_1)$), adopting all innovations with probability one; and consumers partially herd on adoption in Phase II ($t \in (t_1, t_2)$), adopting good innovations with probability $\rho + a_B(t)(1-\rho)$ and bad ones with probability $1-\rho + a_B(t)\rho$. Moreover, given the presumption that possibility (ii) is occurring, consumers are again sensitive to signals at all times $t > t_2$. Overall, the mass of consumers who adopt a good innovation therefore takes the form:

$$I_g(\infty) = \rho\Delta + \int_0^{t_1} |S_g'(t)|dt + \int_{t_1}^{t_2} (\rho + (1-\rho)a_B(t))|S_g'(t)|dt + \int_{t_2}^{\infty} \rho|S_g'(t)|dt$$
$$= \rho + \int_0^{t_1} (1-\rho)|S_g'(t)|dt + \int_{t_1}^{t_2} (1-\rho)a_B(t)|S_g'(t)|dt \quad (11)$$

where $|S_g'(t)|$ is the flow of consumers being exposed at time t and $\Delta + \int_0^\infty |S_g'(t)|dt = 1$ because the consumer population has unit mass. Similarly, the overall share of consumers who adopt a bad innovation takes the form:

$$I_b(\infty) = (1-\rho)\Delta + \int_0^{t_1} |S_b'(t)|dt + \int_{t_1}^{t_2} (1-\rho+\rho a_B(t))|S_b'(t)|dt + \int_{t_2}^{\infty} (1-\rho)|S_b'(t)|dt$$
$$= (1-\rho) + \int_0^{t_1} \rho|S_b'(t)|dt + \int_{t_1}^{t_2} \rho a_B(t)|S_b'(t)|dt \quad (12)$$

Since consumers' interim belief exceeds ρ throughout Phase I and equals ρ throughout Phase II, $|S_g'(t)| > |S_b'(t)|$ for all $t \in (0, t_2)$ by Lemma 3. Thus,

$$I_b(\infty) < (1-\rho) + \int_0^{t_1} \rho|S_g'(t)|dt + \int_{t_1}^{t_2} \rho a_B(t)|S_g'(t)|dt \quad (13)$$

(11, 13) together imply

$$I_g(\infty) - I_b(\infty) > (2\rho - 1)\left(1 - \int_0^{t_1} |S_g'(t)|dt - \int_{t_1}^{t_2} a_B(t)|S_g'(t)|dt\right). \quad (14)$$

Finally, note that $\int_0^{t_1} |S_g'(t)|dt = (1-\Delta) - S(t_1)$ and, since $a_B(t) < 1$ for all $t \in (t_1, t_2)$, $\int_{t_1}^{t_2} a_B(t)|S_g'(t)|dt < S(t_1) - S(t_2)$. We conclude that $I_g(\infty) - I_b(\infty) > (2\rho - 1)(\Delta + S(t_2)) > 0$; so, $I_g(\infty) > I_b(\infty)$, completing the desired contradiction.

Step 3: At all times $t \leq t_3$, $S_g(t) < S_b(t)$ and $I_g(t) > I_b(t)$.

Let $\Delta S(t) \equiv S_b(t) - S_g(t)$ denote the "exposure gap," the extra share of consumers who have been exposed to good innovations by time t, and let $\Delta I(t) \equiv I_g(t) - I_b(t)$ denote the "adoption gap," the extra share who have adopted. At launch, $S_g(0) = S_b(0) = \Delta$, $I_g(t) = \rho\Delta$, and $I_b(t) = (1-\rho)\Delta$; so, $\Delta S(0) = 0$ and $\Delta I(0) = (2\rho - 1)\Delta > 0$. Here we will show that $\Delta S(t) > 0$ and $\Delta I(t) > 0$ at all times $t \in (0, t_3)$.

By Steps 1-2, consumers' interim belief $p(t)$ declines throughout Phase III, from ρ at time t_2 to $1-\rho$ at time t_3; so, there is a unique time $\hat{t} \in (t_2, t_3)$ at which $p(\hat{t}) = \alpha$. Note that $p(t)$ exceeds consumers' ex ante belief α at all times $t \in (0, t_2]$ by Props 3-

5 and that $p(t) > \alpha$ for all $t \in (t_2, \hat{t})$ by definition of $\hat{t}$. Lemma 3 therefore implies that $\Delta S'(t) = S_b'(t) - S_g'(t) > 0$ and $\Delta I'(t) = I_g'(t) - I_b'(t) > 0$ for all $t \in (0, \hat{t})$. Since $\Delta S(0) = 0$ and $\Delta I(0) > 0$, we conclude that $\Delta S(t) > 0$ and $\Delta I(t) > 0$ for all $t \in (0, \hat{t})$.

What about at times $t \in (\hat{t}, t_3)$ when consumers' interim belief $p(t)$ lies between $1 - \rho$ and α? Proof details for this case are relegated to the Appendix, but it is helpful to highlight some key observations. First, because consumers' interim belief is less than their ex ante belief after time $\hat{t}$, the flow of newly-exposed consumers $S_b(t)I_b(t) = -S_b'(t)$ when the innovation is bad must be greater than the flow $S_g(t)I_g(t) = -S_g'(t)$ when the innovation is good. Consequently, the exposure gap $\Delta S(t)$ must shrink after time $\hat{t}$. Moreover, the adoption gap $\Delta I(t)$ may (or may not) also begin to shrink before the end of Phase III. However, neither the exposure gap nor the adoption gap ever completely closes—good innovations are always more encountered and more widely adopted than bad innovations.

Step 4: After time t_3, $\frac{p(t)}{1-p(t)}$ declines exponentially at a constant rate, $\Delta I(t)$ is constant, and $\Delta S(t)$ is decreasing but positive.

Consumers' interim belief at time t_3 equals $1 - \rho$, making them indifferent whether to adopt after a good private signal. Let $a_G(t_3) \in [0, 1]$ be the probability with which consumers exposed to the innovation at time t_3 adopt after a good signal. Note that

$$X(t_3) = a_G(t_3) \left(\rho S_g(t_3) - (1 - \rho) S_b(t_3) \right) - \left(I_g(t_3) - I_b(t_3) \right).$$

To establish that consumers' interim belief continues declining below $1 - \rho$, it suffices to show that $X(t_3) < 0$. However, this follows immediately from the facts that $X(t_3-) < 0$ (proven in Step 2), $I_g(t_3) > I_b(t_3)$ (proven in Step 3), and $a_B(t_3) \in [0, 1]$.

Once consumers' interim belief falls below $1 - \rho$, immediately after time t_3, consumers herd on non-adoption; so, $X(t_3+) = -\left(I_g(t_3) - I_b(t_3) \right) < 0$ by Step 3 and beliefs continue to fall. Consumers therefore still herd on non-adoption, meaning that $I_g(t) = I_g(t_3)$, $I_b(t) = I_b(t_3)$, and hence $X(t) = X(t_3)$ and $\Delta I(t) = \Delta I(t_3)$ for all $t > t_3$. We conclude that all adoption ceases after time t_3 and that $\frac{p(t)}{1-p(t)}$ forevermore declines exponentially at the constant rate $|X(t_3)|$. In particular, $\lim_{t \to \infty} p(t) = 0$.

Finally, as discussed in Step 3, the fact that $p(t) < \alpha$ implies that $S_b(t)I_b(t) > S_g(t)I_g(t)$; hence, the exposure gap must shrink during obsolescence, i.e., $\Delta S'(t) < 0$ for all $t > t_3$. At the same time, because $I_g(t) > I_b(t)$, the condition $S_b(t)I_b(t) > S_g(t)I_g(t)$ is only possible if $S_b(t) > S_g(t)$; thus, $\Delta S(t) > 0$ for all $t > t_3$. □

4 Stopping the Viral Campaign

Here we extend the analysis to allow the producer to decide *how long* to continue the viral campaign. Suppose that, at any time $T \geq 0$, the producer can stop the viral campaign by running a "broadcast advertisement" (or simply "broadcast") that reaches all still-unexposed consumers. In this section, we characterize the optimal time at which to stop the viral campaign. We find that immediately stopping the viral campaign ($T = 0$) and allowing it to continue forever ($T = \infty$) are each suboptimal.

To keep the analysis as simple as possible, we assume that the producer must choose the broadcast time $T \in [0, \infty]$ before launch and before knowing whether its innovation will be good or bad; running the broadcast is costless; and the producer's objective is to maximize the expected mass of consumers who adopt the innovation.[16] As in Section 3, we focus on the case when $\alpha \in (1-\rho, \rho)$,[17] so that consumers are sensitive to signals at launch, and assume a small initial launch ($\Delta \approx 0$).

Broadcast-updated beliefs. Consumers who see the broadcast at time T update their belief about innovation quality based on the fact that they did not encounter the innovation during the preceding viral campaign. Let $p_{BR}(T)$ denote consumers' updated belief after seeing the broadcast. Conditional on the innovation being good or bad, each consumer will encounter the innovation via broadcast with ex ante probability $S_g(T)$ or $S_b(T)$, respectively. By Bayes' Rule:

$$\frac{p_{BR}(T)}{1-p_{BR}(T)} = \frac{\alpha}{1-\alpha} \times \frac{S_g(T)}{S_b(T)} \tag{15}$$

Lemma 4 (proven in the Appendix) gathers key facts about broadcast-updated beliefs.

Lemma 4. *Suppose that $\alpha \in (1-\rho, \rho)$ and $\Delta \approx 0$. (i) $p_{BR}(0+) = \alpha \approx p_{BR}(t_1)$. (ii) $\frac{p_{BR}(T)}{1-p_{BR}(T)}$ falls exponentially at rate $I_g(T) - I_b(T) > 0$ for all T. (iii) $p_{BR}(T) < p(T)$ for all T. (iv) There is a threshold time $\overline{T} \in (t_1, t_3)$ such that $p_{BR}(T) > 1-\rho$ for all $T < \overline{T}$ and $p_{BR}(T) < 1-\rho$ for all $T > \overline{T}$. (v) If $\alpha \in (1/2, \rho)$, then $\overline{T} \in (t_2, t_3)$.*

Discussion of Lemma 4. During Phase I of the innovation lifecycle when few consumers have encountered the innovation, someone seeing the broadcast will infer little from the

[16] For simplicity, we assume that the producer does not care about the timing of adoption. Introducing discounting complicates the analysis but does not generate any additional insight.

[17] The other cases are trivial. In the high-quality case when $\alpha > \rho$, $I_g(T) = I_b(T) > 0$ for all T since consumers herd on adoption; thus, consumers do not update their beliefs and all consumers adopt no matter when (or whether) the producer decides to run the broadcast. Similarly, in the low-quality case when $\alpha < 1-\rho$, $I_g(T) = I_b(T) = 0$ for all T and no consumers ever adopt.

fact that they did not encounter the innovation virally (Lemma 4(ii)). However, as the innovation begins to "mature" during Phase II—with viral awareness growing faster when the innovation is good than when it is bad—consumers who see the broadcast will interpret that as "bad news" about innovation quality. Less obviously, consumers' negative inference when seeing the broadcast gets worse as time goes on (Lemma 4(ii)) and is worse than the inference they would make if encountering the innovation virally at the same time (Lemma 4(iii)).

If the viral campaign continues long enough, a threshold time $\overline{T} > 0$ is eventually reached at which $p_{BR}(\overline{T}) = 1 - \rho$. Those seeing the broadcast after $\overline{T}$ herd on non-adoption, while those seeing it before $\overline{T}$ are sensitive to signals. We refer to $\overline{T}$ as the moment of "broadcast obsolescence," which always occurs prior to viral obsolescence. Moreover, $\overline{T} \in (t_1, t_3)$ for all $\alpha \in (1-\rho, \rho)$ (Lemma 4(iv)) and, if $\alpha \in (1/2, \rho)$, $\overline{T} \in (t_2, t_3)$ (Lemma 4(v)).

Optimal stopping time. Let T^* denote the earliest optimal stopping time.

Theorem 2. *Suppose that $\alpha \in (1-\rho, \rho)$ and $\Delta \approx 0$. (i) If $\overline{T} \geq t_2$, then $T^* = t_2$ (the end of Phase II). (ii) If $\overline{T} < t_2$, then* either *$T^* = \overline{T}$ (broadcast obsolescence)* or *$T^* = t_3$ (viral obsolescence). Moreover, $T^* = t_3$ if and only if $\overline{T} < t_2$ and*

$$\alpha \left(\int_{\underline{T}}^{t_2} a_B(t)(1-\rho)|S_g'(t)|\mathrm{d}t - S_g(t_3) \right) + (1-\alpha) \left(\int_{\underline{T}}^{t_2} a_B(t)\rho|S_b'(t)|\mathrm{d}t - S_b(t_3) \right) \geq 0 \tag{16}$$

where $\left(S_g(t), S_b(t), a_B(t) : t \geq 0\right)$ were derived in Section 3.

Discussion: Always optimal to run a limited-time viral campaign. An implication of our analysis is that a producer seeking to maximize adoption will always choose to run a viral campaign. In particular, the producer is strictly better off running a viral campaign *either* of length $t_2 > 0$ (if $\alpha \in [1/2, \rho)$) *or* of length $\overline{T}$ (if $\alpha \in (1-\rho, 1/2)$) than not running any viral campaign at all.[18]

When is it optimal to stop the viral campaign? Theorem 2 lays out three possibilities:

(a) if $\overline{T} \geq t_2$, then stopping at time t_2 is optimal;

(b) if $\overline{T} < t_2$ and inequality (16) is satisfied, then never stopping is optimal; or

(c) if $\overline{T} < t_2$ and inequality (16) is not satisfied, then stopping at time $\overline{T}$ is optimal.

[18] In the trivial cases when $\alpha \geq \rho$ or $\alpha \leq 1-\rho$, consumers herd on adoption or on non-adoption, respectively, regardless of how long the viral campaign lasts.

Interestingly, only possibilities (a) and (c) arise in practice. Why? When $\alpha \in (1/2, \rho)$, Lemma 4(v) implies $\overline{T} \geq t_2$, meaning that only possibility (a) arises in this case. When $\alpha \in (1-\rho, 1/2]$, the theory is unclear but numerical simulation reveals that stopping time $\overline{T}$ is always better than never stopping. Over the relevant parameter space $\{(\alpha, \rho) : \rho \in (1/2, 1), \alpha \in (1-\rho, 1/2]\}$, we solved the system of differential equations (1,2) to determine the equilibrium epidemiological dynamics and computed the mass of consumers who adopt good and bad innovations when the producer uses stopping time $\overline{T}$ (which depends on α and ρ) versus never stopping the viral campaign. As illustrated in Figure 4, stopping the campaign at time $\overline{T}$ is strictly better across the entire parameter space, increasing adoption by as much as 72% for some parameter values.

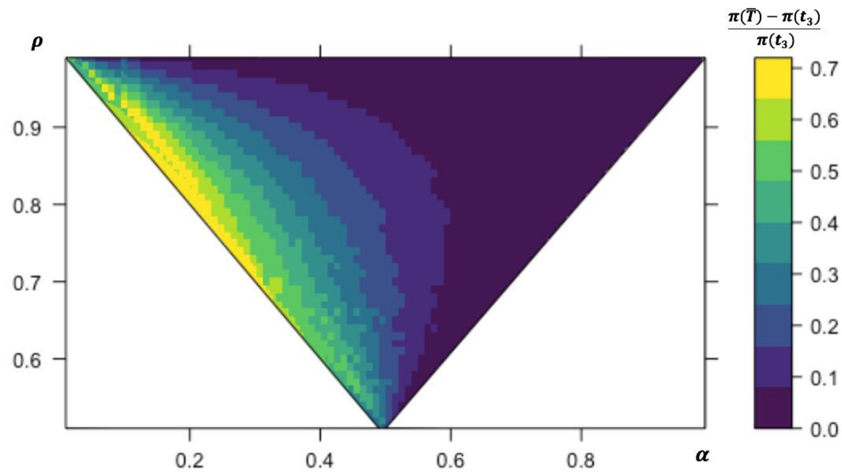


Figure 3: Numerically-simulated percentage increase in the producer's expected profit $\pi(T)$ when stopping the viral campaign at broadcast obsolescence ($T = \overline{T}$) versus viral obsolescence ($T = t_3$). For all combinations of (α, ρ), stopping at $T = \overline{T}$ is more profitable.

The rest of this section proves Theorem 2.

Go/no-go threshold for broadcast. Should the viral campaign continue until time $\overline{T}$, the producer must decide whether to run the broadcast right at that moment, so that still-unexposed consumers are willing to adopt after a good signal ("go"), or never run the broadcast at all, allowing the campaign to continue until viral obsolescence ("no-go").

Case #1: when $\overline{T} \geq t_2$, always "go". Suppose first that $\overline{T} \geq t_2$ so that the go/no-go threshold is in Phase III; this occurs if and only if $p_{BR}(t_2) \geq 1-\rho$. In this case, the producer unambiguously prefers to "go". To see why, consider what would happen if the viral campaign continued after time $\overline{T}$. The consumers who remain unexposed at time $\overline{T}$ will encounter the innovation later either (i) during the remainder of Phase III and be sensitive to signals or (ii) during Phase IV and herd on non-adoption. By comparison, should the producer run the broadcast at (or infinitesimally before) time $\overline{T}$, all of these consumers will be sensitive to signals—leading to strictly more adoption, whether the innovation is good or bad.

Case #2: when $\overline{T} \geq t_2$, "no go" if and only if inequality (16) holds. Suppose next that $\overline{T} < t_2$ so that the go/no-go threshold is in Phase II; this occurs if and only if $p_{BR}(t_2) < 1 - \rho$. As before, running the broadcast at time $\overline{T}$ ensures that all still-unexposed consumers will be sensitive to signals, avoiding the downside that consumers exposed in Phase IV never adopt. However, there is also a benefit associated with continuing to run the viral campaign, that consumers exposed in the remainder of Phase II (at times $t \in (\overline{T}, t_2)$) will sometimes adopt after getting a negative private signal as well as after a positive signal.[19] Whether the producer prefers to continue the viral campaign past time $\overline{T}$ depends on the magnitudes of these countervailing effects.

The downside of continuing the viral campaign is that all consumers who get a positive signal and would have been exposed during Phase IV choose to adopt under the time-$\overline{T}$ broadcast but not under the continued viral campaign. These consumers have mass $\rho S_g(t_3)$ when the innovation is good and mass $(1-\rho)S_b(t_3)$ when it is bad. Overall, then, the "viral downside" equals $\alpha\rho S_g(t_3) + (1-\alpha)(1-\rho)S_b(t_3)$.

The upside of continuing the viral campaign is that some consumers who get a negative signal and would have been exposed during the remainder of Phase II choose to adopt under the continued viral campaign but not under the time-$\overline{T}$ broadcast. These consumers have mass $\int_{\overline{T}}^{t} a_B(t)(1-\rho)|S'_g(t)|\mathrm{d}t$ when the innovation is good and mass $\int_{\overline{T}}^{t} a_B(t)\rho|S'_b(t)|\mathrm{d}t$ when it is bad, where $a_B(t)$ is consumers' equilibrium likelihood of adopting after a bad signal during Phase II. Overall, then, the "viral upside" equals $\int_{\overline{T}}^{t} a_B(t)(1-\rho)|S'_g(t)|\mathrm{d}t + (1-\alpha)\int_{\overline{T}}^{t} a_B(t)\rho|S'_b(t)|\mathrm{d}t$, and the upside exceeds the downside if and only if inequality (16) holds.

Summarizing our progress thus far: (i) If $p_{BR}(t_2) \geq 1-\rho$, then $\overline{T} \geq t_2$ and the optimal stopping time is during Phase III prior to broadcast obsolescence. (ii) If $p_{BR}(t_2) < 1-\rho$, then $\overline{T} < t_2$ and it is optimal *either* to allow the viral campaign to continue until viral obsolescence (if (16) holds) *or* to stop at time $\overline{T}$ during Phase II (if (16) does not hold).

Stopping the viral campaign prior to time $\min\{t_2, \overline{T}\}$ is suboptimal. The producer is strictly better off extending the viral campaign until time $\min\{t_2, \overline{T}\}$. Proof details are in the Appendix, but the idea is simple and intuitive. Suppose that some time $T' <$

[19] We can ignore the consumers exposed in Phase III, since they are sensitive to signals and hence adopt exactly as they would have under a time-$\overline{T}$ broadcast.

$\min\{t_2, \overline{T}\}$ has been reached and the producer is considering whether to run the broadcast at that moment or wait until (just before) time $\min\{t_2, \overline{T}\}$. Either way, all consumers who see the broadcast and get a positive private signal will adopt. However, under the longer viral campaign, consumers who encounter the innovation between $\max\{T', t_1\}$ and $\min\{t_2, \overline{T}\}$ (the portion of Phase II that is after T' and before $\overline{T}$) also sometimes adopt after a negative private signal, due to partial herding. Thus, lengthening the viral campaign until time $\min\{t_2, \overline{T}\}$ unambiguously increases overall adoption.

Putting these pieces together, we can now complete the proof.

First, consider the case when $p_{BR}(t_2) \geq 1 - \rho$ so that $\overline{T} \geq t_2$. We have shown that it is suboptimal to stop the viral campaign prior to time t_2 *and* suboptimal to allow it to continue beyond time $\overline{T}$. What about the remaining time interval from t_2 to $\overline{T}$? Consumers who encounter the innovation during the interval $[t_2, \overline{T}]$ are sensitive to signals, regardless of whether they encounter the innovation virally or through the broadcast. Thus, all broadcast times within this time interval (including time t_2) generate exactly the same pattern of consumer adoption and hence must all be optimal for the producer. This completes the proof of Thm 2(i).

Second, consider the case when $p_{BR}(t_2) < 1 - \rho$ so that $\overline{T} < t_2$. As argued earlier, any stopping time prior to $\overline{T}$ is suboptimal and waiting until viral obsolescence (any stopping time $T \geq t_3$ is better than $\overline{T}$ if and only if inequality (16) holds. Finally, because no one adopts after seeing a broadcast later than $\overline{T}$, stopping at any time between $\overline{T}$ and t_3 is strictly worse than stopping at $\overline{T}$. We conclude that waiting until viral obsolescence is optimal if inequality (16) holds and that stopping at time $\overline{T}$ is optimal if inequality (16) does not hold. This completes the proof of Thm 2(ii). □

5 Investment in Innovation Quality

This section considers an extension in which the producer decides whether to invest in innovation quality. We find that, when most innovations would otherwise be good (or bad) absent viral social learning, a producer who runs an optimal-length viral campaign will invest less (or more) than one who must reach consumers through advertising only.

Suppose that the producer earns revenue equal to the mass of consumers who adopt the innovation; bad innovations can be produced at zero cost; and producing a good innovation requires paying c, a private cost drawn from continuous c.d.f. $F(\cdot)$ with full support on the interval $[0, \bar{c}]$.

Suppose further that consumers' ex ante belief $\alpha \in (1 - \rho, \rho)$, so that those who encounter the innovation at launch are sensitive to signals. (For the moment, view α as a fixed parameter; later, we will endogenize α.) Let $R_g(T)$ and $R_b(T)$ denote the revenue earned by good and bad innovations, respectively, for any given viral-campaign stopping time $T \geq 0$. As shown in Section 4: if $T > \overline{T}$ so that those exposed at the time-T broadcast do not adopt, then $R_g(T) = I_g(T)$ and $R_b(T) = I_b(T)$; otherwise, if $T \leq \overline{T}$ so that those exposed at time T are sensitive to signals, then

$$R_g(T) = I_g(T) + \rho S_g(T) \text{ and } R_b(T) = I_b(T) + (1-\rho)S_b(T) \tag{17}$$

Let $\Delta R(T) = R_g(T) - R_b(T)$ denote the extra revenue earned by good innovations; we refer to $\Delta R(T)$ as "the incentive to invest."

Incentive to invest prior to a traditional ad campaign. Since consumers are sensitive to signals at launch, fraction ρ (or $1-\rho$) of consumers will adopt a good (or bad) innovation; so, $R_g(0) = \rho$ and $R_b(0) = 1-\rho$. The producer therefore finds it optimal to invest in quality whenever $c < \Delta R(0) = 2\rho - 1$.

Incentive to invest prior to a viral campaign. Suppose next that the producer runs a viral campaign that stops at time $T > 0$. Anticipating how the viral social learning process will unfold, the producer finds it optimal to invest whenever $c < \Delta R(T)$. For all $T \leq \overline{T}$, equations (1-2,17) imply

$$\begin{aligned} R_g'(T) &= I_g'(T) + \rho S_g'(T) = S_g(T) I_g(T) \left(\rho a_G(T) + (1-\rho) a_B(T) - \rho\right) \\ R_b'(T) &= I_b'(T) + (1-\rho) S_b'(T) = S_b(T) I_b(T) \left((1-\rho) a_G(T) + \rho a_B(T) - (1-\rho)\right) \end{aligned}$$

where $a_G(T)$ and $a_B(T)$ are the likelihoods that virally-exposed consumers adopt at time T after a good or bad signal, respectively. Since virally-exposed consumers with good signals adopt until viral obsolescence at time t_3 and since $\overline{T} < t_3$ (Lem 4), it must be that $a_G(T) = 1$ for all $T \leq \overline{T}$. We conclude that

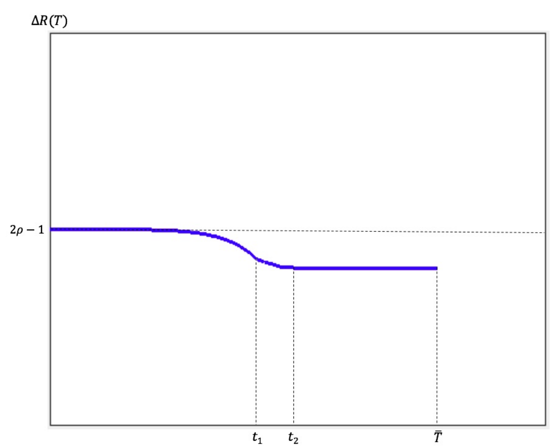
$$\Delta R'(T) = a_B(T) \left((1-\rho) S_g(T) I_g(T) - \rho S_b(T) I_b(T)\right) \tag{18}$$

for all $T \leq \overline{T}$. In particular, over this range, (i) $R'(T) > 0$ if and only if $a_B(T) > 0$ and $\frac{S_g(T)I_g(T)}{S_b(T)I_b(T)} > \frac{\rho}{1-\rho}$ and (ii) $R'(T) < 0$ if and only if $a_B(T) > 0$ and $\frac{S_g(T)I_g(T)}{S_b(T)I_b(T)} < \frac{\rho}{1-\rho}$.

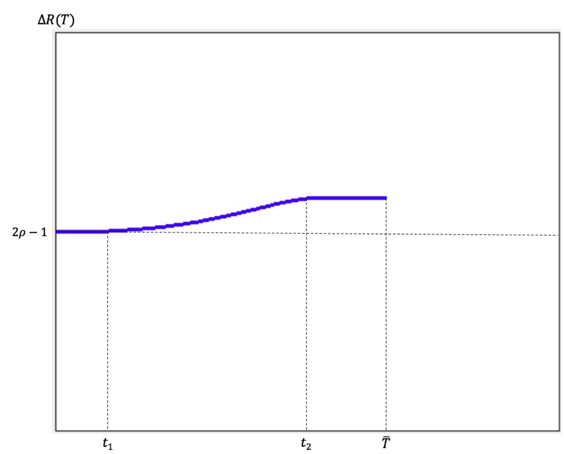
Recall that, by Bayes' Rule, consumers' interim belief $p(t)$ at any time $t > 0$ during the viral campaign satisfies $\frac{p(t)}{1-p(t)} = \frac{\alpha}{1-\alpha} \times \frac{S_g(t)I_g(t)}{S_b(t)I_b(t)}$ and, in particular, $\frac{p(0+)}{1-p(0+)} = \frac{\alpha}{1-\alpha} \times \frac{\rho}{1-\rho}$. Thus, the condition $\frac{S_g(T)I_g(T)}{S_b(T)I_b(T)} \gtrless \frac{\rho}{1-\rho}$ is equivalent to $p(T) \gtrless p(0+)$ or, in words, "whether consumers interim belief at stopping time T is higher or lower than it would be after observing a good private signal and nothing else."

Case #1: Most innovations are good. Suppose that $\alpha \in (1/2, \rho)$. In this case: $p(0+) > \rho$ (Prop 2) and $\overline{T}$ is in Phase III (Lem 4); during Phase I, $a_B(t) = 1$ and $p(t) \in (\rho, p(0+))$ (Prop 3); during Phase II, $a_B(t) \in (0,1)$ and $p(t) = \rho$ (Prop 5); and during Phase III, $a_B(t) = 0$ (Prop 6). Thus, $R'(T) < 0$ for all T in Phase I and Phase II and $R'(T) = 0$ for all T in Phase III. We conclude that $\Delta R(T) < \Delta R(0)$ for all $0 < T \leq \overline{T}$, i.e., the producer has *less* incentive to invest that in a traditional ad campaign.

Case #2: Most innovations are bad. Suppose that $\alpha \in (1-\rho, 1/2)$. In this case: $p(0+) \in (1/2, \rho)$ (Prop 2) and $\overline{T}$ is in Phase II or Phase III (Lem 4); during Phase I, $a_B(t) = 0$ and $p(t) \in (p(0+), \rho)$ (Prop 4); during Phase II, $a_B(t) \in (0,1)$ and $p(t) = \rho$ (Prop 5); and during Phase III, $a_B(t) = 0$ (Prop 6). Thus, $R'(T) = 0$ for all T in Phase I and $R'(T) > 0$ for all T in Phase II and Phase III. We conclude that $\Delta R(T) \geq \Delta R(0)$ for all $0 < T \leq \overline{T}$ with strict inequality for all $T \in (t_1, \overline{T})$, i.e., the producer has *more* incentive to invest than in a traditional ad campaign.



(a) $\alpha \in (\frac{1}{2}, \rho)$



(b) $\alpha \in (1-\rho, \frac{1}{2})$

Figure 4: Incentive to invest $\Delta R(T)$, depending on the viral campaign's stopping time T, up until broadcast obsolescence at time $\overline{T}$. (At time $\overline{T}$, there is a discontinuity in $\Delta R(T)$, not shown to keep the figure as simple as possible.) Panel (a) illustrates the case with parameters $\alpha = 0.55$ and $\rho = 0.65$, and panel (b) the case when $\alpha = 0.45$ and $\rho = 0.65$.

Figure 4 illustrates how the producer's incentive to invest varies with the length of the viral campaign, depending on whether $\alpha \in (1/2, \rho)$ or $\alpha \in (1-\rho, 1/2)$. Proposition 8 summarizes our findings on the producer's incentive to invest, in light of our previous results on adoption dynamics during the viral campaign (Theorem 1) and optimal stopping time T^* (Theorem 2).

Proposition 8. *Suppose that consumers believe that fraction $\alpha \in (1-\rho, \rho)$ of innovations are good and behave as characterized in Theorem 1. (i) If $\alpha \in (1/2, \rho)$, then $\Delta R(T^*) < \Delta R(0)$. (ii) If $\alpha \in (1-\rho, 1/2)$, then $\Delta R(T^*) > \Delta R(0)$. (iii) If $\alpha = 1/2$, then $\Delta R(T^*) = \Delta R(0)$.*

Proof. (i) When $\alpha \in (1/2, \rho)$, the earliest optimal stopping time $T^* = t_2 < \overline{T}$ (Lem 4, Thm 2) and, as shown earlier, $\Delta R(T) < \Delta R(0)$ for all $T \leq \overline{T}$; so, $\Delta R(T^*) < \Delta R(0)$. (ii) When $\alpha \in (1-\rho, 1/2)$, $T^* = \overline{T}$ (Fig 3), $\overline{T} \in (t_1, t_3)$ and, as shown earlier, $\Delta R(T) > \Delta R(0)$ for all $T \in (t_1, t_3)$; so, $\Delta R(T^*) > \Delta R(0)$. (iii) When $\alpha = 1/2$, $t_1 = 0$ (Thm 1), $p(0+) = \rho$, and $T^* \in (0, t_3)$. In this case: Phase I does not occur; $\frac{S_g(t)I_g(t)}{S_b(t)I_b(t)} = \frac{\rho}{1-\rho}$ for all $t \in (0, t_2]$ (Phase II); and $a_B(t) = 0$ for all $t \in [t_2, t_3]$ (Phase III). By equation (18), we conclude that $\Delta R'(T) = 0$ for all $T \in (0, t_3)$; so, $\Delta R(T^*) = \Delta R(0)$. □

Equilibrium investment with a traditional ad campaign. The following "traditional marketing game" serves as a useful benchmark. First, the producer observes its private cost c and decides whether to produce a good innovation ("invest"). Then, the innovation is advertised to all consumers at time $t = 0$, who decide whether to adopt based only on their own private signals.

Proposition 9. *Let $\hat{\alpha}$ denote the maximal equilibrium likelihood of a good innovation in the traditional marketing game. (i) If $F(2\rho - 1) < 1 - \rho$, then $\hat{\alpha} = 0$. (ii) If $F(2\rho - 1) \in [1-\rho, \rho]$, then $\hat{\alpha} = F(2\rho - 1)$. (iii) If $F(2\rho - 1) > \rho$, then $\hat{\alpha} = \rho$.*

Proof. The proof is in the Appendix. □

To gain intuition, consider the case when $F(2\rho - 1) \in [1-\rho, \rho]$. In the equilibrium with ex ante belief $\hat{\alpha} = F(2\rho - 1)$, all consumers adopt after a good signal but not after a bad one, causing mass ρ of consumers to adopt good innovations while mass $1-\rho$ adopt bad ones. Anticipating this, the producer invests whenever its cost $c < 2\rho - 1$, which occurs with probability $\hat{\alpha}$.

Equilibrium investment in "viral marketing game." Consider the following game. First, the producer commits to its marketing strategy, choosing the stopping time T for the viral campaign.[20] Second, the producer observes its cost c and privately decides whether to produce a good innovation. Finally, the viral campaign unfolds as in Section 4, with consumers adopting optimally and innovation awareness spreading virally until time T, when all still-susceptible consumers are exposed non-socially.

[20]The alternative game in which the producer chooses the stopping time after observing c is more complex, since consumers could potentially make meaningful inferences about quality based on the stopping time, but has equilibria with similar qualitative features.

Any equilibrium of this game must satisfy three conditions. First, the producer must invest optimally, whenever the cost c is less than the extra revenue earned by good innovations. Second, consumers' ex ante belief α must be correct, equal to the true share of good innovations. Finally, the stopping time T must be optimal for the producer given α, as characterized in Theorem 2.

Optimal-length viral marketing may increase or decrease equilibrium investment, depending on how much equilibrium investment can be supported in the traditional marketing game. In particular, viral marketing leads to less investment if $\hat{\alpha} > 1/2$, more investment if $\hat{\alpha} < 1/2$, and equal investment if $\hat{\alpha} = 1/2$.

Proposition 10. *Let α^* denote the maximal equilibrium likelihood of a good innovation in the viral marketing game. (i) If $\hat{\alpha} \in (1/2, \rho)$, then $\alpha^* \in (1/2, \hat{\alpha})$. (ii) If $\hat{\alpha} \in (1-\rho, 1/2)$, then $\alpha^* \in (\hat{\alpha}, 1/2)$. (iii) If $\hat{\alpha} = 1/2$, then $\alpha^* = 1/2$.*

Proof. Let $\Delta R(T; \alpha)$ denote the extra revenue earned by good innovations when consumers have ex ante belief α and the viral campaign stops at time T. Let $T^*(\alpha)$ be the optimal stopping time when consumers have ex ante belief $\alpha \in (1-\rho, \rho)$. Finally, let $S(\alpha) \equiv F(\Delta R(T^*(\alpha); \alpha))$ be the supply of good innovations when consumers have ex ante belief α and the viral campaign stops optimally at time $T^*(\alpha)$. An equilibrium exists with ex ante belief α if and only if $S(\alpha) = \alpha$.

Suppose that $\hat{\alpha} \equiv F(2\rho - 1) \in (1/2, \rho)$. For all $\alpha \in (1/2, \rho)$, $\Delta R(T^*(\alpha); \alpha) < 2\rho - 1$ (Prop 8(i)); thus, $S(\alpha) < \hat{\alpha}$ for all $\alpha \in (1/2, \rho)$ and, in particular, $S(\alpha) < \alpha$ for all $\alpha \in [\hat{\alpha}, \rho)$. On the other hand, because $\Delta R(0; 1/2) = \Delta R(T^*(1/2); 1/2) = 2\rho - 1$ (Prop 8(iii)), $S(1/2) = \hat{\alpha} > 1/2$. By a continuity argument (straightforward details omitted), there exists $\alpha \in (1/2, \hat{\alpha})$ such that $S(\alpha) = \alpha$. Overall, we conclude that $\alpha^* \in (1/2, \hat{\alpha})$.

Suppose next that $\hat{\alpha} \in (1-\rho, 1/2)$. First, we show that no equilibrium exists with $\alpha \geq 1/2$. Suppose otherwise. With $\alpha \geq 1/2$, $\Delta R(T^*(\alpha); \alpha) \leq 2\rho - 1$ (Prop 8(i,iii)); thus, $S(\alpha) \leq \hat{\alpha} < 1/2$, contradicting the presumption that $S(\alpha) = \alpha \geq 1/2$. Next, for all $\alpha \in (1-\rho, 1/2)$, $\Delta R(T^*(\alpha); \alpha) > 2\rho - 1$ by Prop 8(ii); thus, $S(\alpha) > \hat{\alpha}$ for all $\alpha \in (1-\rho, 1/2)$. On the other hand, when $\alpha = 1/2$, $\Delta R(0; 1/2) = \Delta R(T^*(1/2); 1/2) = 2\rho - 1$ by Prop 8(iii); thus, $S(1/2) = \hat{\alpha} < 1/2$. By continuity, there exists $\alpha \in (1-\rho, 1/2)$ such that $S(\alpha) = \alpha$ and any such fixed point is strictly greater than $\hat{\alpha}$. All together, we conclude that $\alpha^* \in (\hat{\alpha}, 1/2)$.

The last case when $\hat{\alpha} = 1/2$ is similar.$\Delta R(T^*(1/2); 1/2) = \Delta R(0, 1/2) = 2\rho - 1$ (Prop 8(iii)); thus, $S(1/2) = 1/2$. On the other hand, for all $\alpha \in (1/2, \rho)$, $\Delta R(T^*(\alpha); \alpha) < 2\rho - 1$ and hence $S(\alpha) < 1/2 < \alpha$. Thus, $\alpha^* = 1/2$. □

6 Concluding remarks

This paper introduces and analyzes an economic-epidemiological model of viral social learning, in which awareness of an innovation (e.g., new product or practice, scientific finding, rumor, etc.) spreads by word of mouth from those who have already "adopted" it. In this context, consumers' beliefs about the quality of an innovation depend on when they first encounter it. We characterize the equilibrium epidemiological dynamics of innovation diffusion and adoption, determining the lifecycle of innovations subject to such "viral social learning." Moreover, we show that using a viral campaign to launch an innovation increases total adoption relative to a traditional advertisement campaign, but that using an optimal-length viral campaign may increase or decrease producers' incentive to invest in innovation quality.

The model captures a rich interplay between the epidemiological dynamics of diffusion and the economic dynamics of consumer beliefs. As such, the paper serves to bridge the economic literature on social learning and the epidemiological literature on social transmission, combining ideas and methods from both fields.

The paper's economic contribution builds on the pioneering work of Banerjee [1993], the first to bring infectious-disease insights to social learning. The main difference is that all agents in our model receive a private signal about innovation quality whereas, in Banerjee [1993], only those who encounter the innovation at "launch" are privately informed. This distinction generates new qualitative findings, including an endogenous obsolescence. (In Banerjee [1993], the spread of a rumor slows down but never stops; in our model, all innovations eventually stop being adopted.)

From an epidemiological point of view, the paper's most important methodological contribution is to expand the scope of the classic Susceptible-Infected-Recovered (SIR) model to *economic epidemics*, in which the parameters of the diffusion model depend on equilibrium economic incentives. In particular, we endogenize infectivity, the likelihood that a newly-exposed host will become infected, and show how infectivity changes throughout an economic epidemic. In future work, our methodology could be extended in several interesting directions to endogenize other key parameters, most notably, the transmission rate and the informativeness of agents' private signals. Such work could also explore the consumer-welfare implications of viral social learning.[21]

Several other natural directions for future research could build on our analysis, a few of which we highlight here.

[21] Being exposed virally to an innovation provides a valuable "social signal" about its quality. Holding innovation quality fixed, consumers are therefore clearly better off (ex ante) when an innovation is marketed virally. As we have shown, however, viral marketing can in some cases lead to less investment in quality.

Pricing. Suppose that a new product generates value v_g or v_b when of good or bad quality, respectively, that the product is launched virally, and that "adoption" corresponds to buying the product at price $p \in (v_b, v_g)$. This fits our model with gain $u_g = v_g - p$ when buying a good product and loss $u_b = p - v_b$ from a bad product. Thus, for any given price p,[22] our analysis characterizes the proportion of consumers who will buy good products, $Q_g(p) \equiv \lim_{t\to\infty} I_g(t; p)$, and bad products, $Q_b(p) \equiv \lim_{t\to\infty} I_b(t; p)$. Computing the fixed price p^* that maximizes ex ante expected revenue, $R(p) = p\left(\alpha Q_g(p) + (1-\alpha)Q_b(p)\right)$, is then a straightforward numerical exercise. But other important pricing issues remain for future research, including (i) how prices change over time throughout a product's lifecycle, if the producer is able to set prices dynamically, and (ii) how much consumers are able to learn from prices, if the producer has information about quality when setting prices.

Temporary infectiousness. Suppose that consumers who adopt only remain infectious for a limited period of time. For example, people who choose to play a new game may eventually get bored and stop introducing it to others. In this context, how widely an innovation spreads through the consumer population depends on innovation quality. For instance, suppose that those who adopt encounter an average of two others before "recovering," that consumers are sensitive to signals, and that their private signals have precision $\rho = 2/3$. Early on during the diffusion process, each infected consumer will on average infect 4/3 others when the innovation is good but only 2/3 others when it is bad—inducing an epidemic spread of good innovations but an endogenous extinguishment of bad innovations.

The option to wait. An important simplifying feature of this paper's model is that consumers decide whether or not to adopt the innovation when they first encounter it. This assumption could be appropriate in some contexts but, in many situations, consumers have the option to wait and learn more. For instance, in our job-market example, a faculty member might wait until she has heard others promoting a paper before starting to promote it herself. Similarly, a consumer with a bad personal impression of a new product might still wind up buying it at a later date if, over time, she encounters enough others who have also done so.

Adding the option to wait complicates the analysis, but our methodology can be generalized to this more complex setting. A consumer i who has not yet adopted by time t is in a susceptible state with history h_{it}, where h_{it} is a vector encoding the previous times

[22] In the trivial cases when price $p \geq u_g$ (or $p \leq u_b$), the unique equilibrium has no one (or everyone) buying the product.

at which consumer i encountered someone else who was infected. For each susceptible history h_{it}, the likelihood ratio of that state occurring when the innovation is good versus bad determines consumer i's belief about innovation quality, and the dynamics of belief evolution determine the option value of waiting. At each point of time, then, there will be a subset of susceptible states from which consumers will choose to adopt. Overall, as in this paper, the dynamics of adoption determine the dynamics of consumer beliefs, which in turn determine the dynamics of adoption. However, there are important differences and complications. For instance, once consumers have the option to wait, there might in some cases be multiple equilibria with different time-paths of adoption.

Reversible adoption decisions. Another important simplifying assumption is that consumers' adoption decisions are irreversible. This assumption also can be relaxed within our basic analytical framework. Suppose that, rather than "buying" the innovation irreversibly, each consumer decides at each instant whether or not to "rent" it.[23] Each consumer will choose to rent whenever their epidemiological state is such that their belief about innovation quality exceeds the threshold for adoption (50% in our main analysis). Because consumers' decisions at each point in time depend only on their current states, equilibria in this context can be derived relatively simply, much as in this paper, with current incentives only depending on patterns of past behavior. An interesting open question in this context is whether consumers will over time successfully aggregate their initially-dispersed information. That is, as time $t \to \infty$, will the fraction of consumers renting a good innovation converge to 100% while the fraction renting a bad innovation converges to 0%?

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[23]This alternative model corresponds to the Susceptible-Infected-Susceptible (SIS) model, used in infectious-disease epidemiology to model infection dynamics when recovery from infection does *not* provide immunity from re-infection. Here, the decision to stop renting corresponds to "recovery" while the decision to (re-)start renting corresponds to (re-)infection.

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A Appendix

A.1 Proof of Prop 1

Suppose that $\alpha = p(0) > \rho$. Consumers exposed at launch herd on adoption (Lemma 1(i)). By equation (3), $\frac{p(0+)}{1-p(0+)} = \frac{\alpha}{1-\alpha}\frac{S_g(0+)I_g(0+)}{S_b(0+)I_b(0+)}$; so, $p(0+) = \alpha$. Consumers exposed right after launch therefore also herd on adoption and, so long as consumers have herded on adoption up to time t, $I_g(t) = I_b(t)$ and $S_g(t) = S_b(t)$, implying that $p(t) = \alpha$ and that consumers must herd on adoption at time t as well. We conclude that consumers must herd on adoption at all times $t > 0$ and that doing so constitutes an equilibrium.

Suppose next that $\alpha = p(0) < 1 - \rho$. Consumers exposed to the innovation at launch herd on non-adoption (Lemma 1(ii)) and, with no one infected, no one is subsequently exposed to the innovation. Thus, $I_g(t) = I_b(t) = 0$ and $S_g(t) = S_b(t) = 1 - \Delta$ for all $t \geq 0$. Consumers' off-equilibrium beliefs (should they encounter someone who has previously adopted) are indeterminate, and many equilibria exist with different off-equilibrium beliefs. However, all such equilibria are outcome equivalent and exhibit zero adoption. □

A.2 Proof of Lemma 3

At launch, mass Δ of consumers are exposed to the innovation, of whom fraction $q_g(t)$ or $q_b(t)$ adopt when the innovation is good or bad, respectively; so, $S_g(0) = S_b(0) = 1 - \Delta$, $I_g(0) = q_g(0)\Delta$, and $I_b(0) = q_b(0)\Delta$. By Lemma 1, each exposed consumer is always at least as likely to adopt when the innovation is good than when it is bad. In particular, it must be that $q_g(0) \geq q_b(0)$, implying that there are at least as many consumers "infected" at launch when the innovation is good than when it is bad.

Suppose that $p(t) > \alpha$. Since $\frac{p(t)}{1-p(t)} = \frac{\alpha}{1-\alpha} \times \frac{I_g(t)S_g(t)}{I_b(t)S_b(t)}$, it must be that $\frac{I_g(t)S_g(t)}{I_b(t)S_b(t)} > 1$ and hence $I_g(t)S_g(t) = -S'_g(t) > -S'_b(t) = I_b(t)S_b(t)$ by equation (1). $I'_g(t) = -q_g(t)S'_g(t)$ and $I'_b(t) = -q_b(t)S'_b(t)$ by equation (2), and $q_g(0) \geq q_b(0)$ since exposed consumers are always at least as likely to adopt good innovations than bad ones (Lemma 1; thus, $I'_g(t) > I'_b(t)$. We conclude that, if $p(t) > \alpha$ for all $t \in (0, \hat{t})$, then $S'_g(t) < S'_b(t)$, $I'_g(t) > I'_b(t)$, $S_g(\hat{t} < S_b(t)$, and $I_g(t) > I_b(t)$ for all $t \in (0, \hat{t})$. □

A.3 Proof of Lemma 4

$\frac{\mathrm{d}\log\left(S_g(T)/S_b(T)\right)}{\mathrm{d}T} = \frac{S'_g(T)}{S_g(T)} - \frac{S'_b(T)}{S_b(T)} = -(I_g(T) - I_b(T))$ by equation (1); thus, $\frac{p_{BR}(T)}{1-p_{BR}(T)}$ falls exponentially at rate $I_g(T) - I_b(T)$. Since $I_g(T) > I_b(T)$ for all T (Prop 6), $p_{BR}(T)$ is strictly decreasing in T; so, $\overline{T}$ is well-defined. Because $S_g(t) \approx S_b(t) \approx 1$ throughout

Phase I (Section 3.1), $p_{BR}(T) \approx \alpha$ for all $t \leq t_1$; so, $\overline{T} > t_1$. Moreover, $I_g(T) > I_b(T)$ implies $p_{BR}(T) < p(T)$ by equations (3,15). Since $p(T)$ is non-increasing after Phase I and reaches $1-\rho$ at time t_3, we conclude that $\overline{T} < t_3$; so, $\overline{T} \in (t_1, t_3)$.

So far, we have shown that $\overline{T}$ must occur during Phase II or Phase III. As discussed in Section 3.2, condition (SS) holds throughout Phase II (corresponding to the intuition that there is "upward pressure" on beliefs when consumers are sensitive to signals) but fails to hold throughout Phase III. In the case when $\alpha \in (\frac{1}{2}, \rho)$, the fact that $\frac{\alpha S_g(\overline{T})}{(1-\alpha)S_b(\overline{T})} = \frac{(1-\rho)}{\rho}$ (by definition of $\overline{T}$) implies $\frac{S_g(\overline{T})}{S_b(\overline{T})} < \frac{(1-\rho)}{\rho}$ and hence $\rho S_g(\overline{T}) - (1-\rho)S_b(\overline{T}) < 0$. Because $I_g(\overline{T}) - I_b(\overline{T}) > 0$, condition (SS) fails at time $\overline{T}$; thus, $\overline{T} \in (t_2, t_3)$ whenever $\alpha \in (1/2, \rho)$.

A.4 Proof of Prop 5

In the main text, we showed that $p(t)$ must remain equal to ρ for some period of time after t_1 and derived equation (9) characterizing how consumers' likelihood of adopting after a bad signal $a_B(t)$ evolves over time. Here we show that $a_B(t)$ is strictly decreasing after t_1 and eventually reaches zero. For ease of reference, we reproduce equation (9):

$$a_B(t) = \frac{\rho S_g(t) - (1-\rho)S_b(t) - \big(I_g(t) - I_b(t)\big)}{\rho S_b(t) - (1-\rho)S_g(t)}$$

By Lemma 3, $S_b(t) > S_g(t)$ so long as consumers' interim belief continues to exceed the initial belief α.

Let t_2 denote the first time after t_1 at which consumers no longer partially herd, i.e., $a_B(t_2) = 0$ and $a_B(t) > 0$ for all $t \in (t_1, t_2)$, or $t_2 = \infty$ if consumers partially herd forever. Since $p(t) > \alpha$ throughout Phase I and $\rho > \alpha$, Lemma 3 implies that $S_b(t) > S_g(t)$, ensuring that the denominator of (9) remains positive. Moreover, t_2 is the first time as which the numerator of (9) equals zero, i.e., when Condition SS holds with equality.

Next, note that

$$a_B'(t) = \frac{\begin{array}{c}(\rho S_g'(t) - (1-\rho)S_b'(t) - (I_g'(t) - I_b'(t))(\rho S_b(t) - (1-\rho)S_g(t)) \\ -(\rho S_g(t) - (1-\rho)S_b(t) - (I_g(t) - I_b(t)))(\rho S_b'(t) - (1-\rho)S_g'(t))\end{array}}{(\rho S_b(t) - (1-\rho)S_g(t))^2}.$$

Rearranging and simplifying the numerator, we have

$$\begin{aligned}\text{numerator} =&(\rho^2 - (1-\rho)^2)(S_g'(t)S_b(t) - S_b'(t)S_g(t)) \\ &- (I_g'(t) - I_b'(t))(\rho S_b(t) - (1-\rho)S_g(t)) \\ &+ (I_g(t) - I_b(t))(\rho S_b'(t) - (1-\rho)S_g'(t)).\end{aligned}$$

By (1-2), the second term above can be re-written as

$$\begin{aligned}&- (I_g'(t) - I_b'(t))(\rho S_b(t) - (1-\rho)S_g(t)) \\ =&- (I_g(t)S_g(t)(\rho + (1-\rho)a_B(t)) - I_b(t)S_b(t)(1-\rho+\rho a_B(t)))(\rho S_b(t) - (1-\rho)S_g(t)) \\ =&- I_b(t)(I_g(t) - I_b(t))(\rho S_b(t) - (1-\rho)S_g(t)) \\ &- (I_g(t) - I_b(t))S_g(t)(\rho + (1-\rho)a_B(t))(\rho S_b(t) - (1-\rho)S_g(t))\end{aligned} \tag{19}$$

Similarly, the third term above can be re-written as

$$\begin{aligned}&(I_g(t) - I_b(t))(\rho S_b'(t) - (1-\rho)S_g'(t)) \\ =&- (I_g(t) - I_b(t))(\rho I_b(t)S_b(t) - (1-\rho)I_g(t)S_g(t)) \\ =&- I_b(t)(I_g(t) - I_b(t))(\rho S_b(t) - (1-\rho)S_g(t)) \\ &+ (I_g(t) - I_b(t))S_g(t)(1-\rho)(I_g(t) - I_b(t))\end{aligned} \tag{20}$$

To establish that the entire numerator is negative, we will show that the first term is negative and that the sum of the second term (19) and third term (20) is negative. To that end, recall that $I_g(t) > I_b(t)$, $I_g'(t) > I_b'(t)$, $S_g(t) < S_b(t)$, and $S_g'(t) < S_b'(t)$ at all times $t < t_2$ (Lem 3). The fact that the first term is negative now follows immediately from (1-2), since $S_g'(t)S_b(t) - S_b'(t)S_g(t) = -S_g(t)S_b(t)(I_g(t) - I_b(t)) < 0$. Moreover, $\rho S_b(t) > (1-\rho)S_g(t)$ because $S_b(t) > S_g(t)$ and $\rho > 1/2$; so, the first part of (19) and the first part of (20) are negative. To show that the sum of (19) and (20) is negative, it therefore suffices to show that $(\rho + (1-\rho)a_B(t))(\rho S_b(t) - (1-\rho)S_g(t)) > (1-\rho)(I_g(t) - I_b(t))$. But this follows immediately from the fact that $\rho S_b(t) - (1-\rho)S_g(t) > I_g(t) - I_b(t)$ (since Condition SS remains satisfied) and $\rho + (1-\rho)a_B(t) > 1-\rho$ (since $\rho > 1/2$ and $a_B(t) \geq 0$).

Overall, we conclude that $a_B(t) > 0$ but that $a_B'(t) < 0$ so long as the numerator of equation (9) continues to be positive, i.e., so long as Condition SS continues to be satisfied. Moreover, there is a finite time t_2 at which partial herding ceases. To see why, suppose for the sake of contradiction that consumers were to partially herd forever. Because all consumers are eventually exposed to the innovation, $\lim_{t\to\infty} S_g(t) = \lim_{t\to\infty} S_b(t) = 0$. On the other hand, because $I_g'(t) > I_b'(t)$ so long as $a_B(t) > 0$, $\lim_{t\to\infty}(I_g(t) - I_b(t)) > I_g(t_1) - I_g(t_1) > 0$. All together, then, the numerator of (9) must eventually become negative, a contradiction. □

A.5 Proof of Prop 9

Proof. Let $\mathbf{a} = (a_G, a_B)$ denote consumers' "adoption rule" in the traditional marketing game, with a_G and a_B being, respectively, their likelihood of adopting after a good or bad private signal. Let $\mathcal{A}(\alpha)$ denote the set of optimal adoption rules, depending on the likelihood α of innovation goodness: if $\alpha > \rho$, then $\mathcal{A}(\alpha) = (1,1)$; if $\alpha = \rho$, then $\mathcal{A}(\alpha) = \{(1, a_B) : a_B \in [0,1]\}$; if $\alpha \in (1-\rho, \rho)$, then $\mathcal{A}(\alpha) = (1,0)$; if $\alpha = 1-\rho$, then $\mathcal{A}(\alpha) = \{(a_G, 0) : a_G \in [0,1]\}$; and if $\alpha < 1-\rho$, then $\mathcal{A}(\alpha) = (0,0)$. Good innovations earn revenue $R_g(\mathbf{a}) = \rho a_G + (1-\rho)a_B$, bad innovations earn revenue $R_b(\mathbf{a}) = (1-\rho)a_G + \rho a_B$, and good innovations earn extra revenue $\Delta R(\mathbf{a}) = (2\rho - 1)(a_G - a_B)$.

Let $\alpha(\mathbf{a}) = F(\Delta R(\mathbf{a}))$ denote the likelihood that the producer finds it optimal to invest when consumers use adoption rule $\mathbf{a}$. An equilibrium exists with likelihood α of innovation goodness if and only if $\alpha(\mathbf{a}) = \alpha$ for some $\mathbf{a} \in \mathcal{A}(\alpha)$.

The producer's incentive to invest is maximized when consumers use the adoption rule $\mathbf{a} = (1,0)$; so, α cannot possibly exceed $F(2\rho - 1)$ in any equilibrium. There are three relevant cases, depending on how $F(2\rho-1)$ compares to the belief-thresholds ρ and $1-\rho$.

(i) Suppose that $F(2\rho - 1) < 1-\rho$. Since $\alpha \le F(2\rho-1) < 1-\rho$, consumers must find it optimal never to adopt in any equilibrium and, anticipating this, the producer has zero incentive to invest. Thus, all innovations are bad in the unique equilibrium, i.e., $\hat{\alpha} = 0$.

(ii) Suppose that $F(2\rho-1) \in [1-\rho, \rho]$. In this case, an equilibrium exists with $\alpha = F(2\rho - 1)$ and consumer adoption rule $(1,0)$. Since the equilibrium likelihood of innovation goodness cannot exceed $F(2\rho-1)$, we conclude that $\hat{\alpha} = F(2\rho-1)$.

(iii) Suppose that $F(2\rho - 1) > \rho$. In this case, an equilibrium exists with $\alpha = \rho$ and adoption rule $\mathbf{a}^* = \left(1, 1 - \frac{F^{-1}(\rho)}{2\rho-1}\right)$. (Given this adoption rule, the producer finds it optimal to invest with probability $F\left(\Delta R(\mathbf{a}^*)\right) = \rho$. Moreover, no equilibrium can exist with $\alpha > \rho$. Why not? In such an equilibrium, consumers would find it optimal to use adoption rule $(1,1)$, causing good and bad innovations to be equally adopted and hence giving the producer zero incentive to invest, a contradiction. We conclude that $\hat{\alpha} = \rho$. □

B Online appendix

B.1 Omitted steps in proof of Props 6-7

The proof of Props 6-7 is divided into four steps. Step 1 and Step 4 were completely proven in the main text. Here we provide omitted details for Step 2 and Step 3.

Step 2: "possibility (i)" cannot occur. This part of the proof establishes that, in the case when $\alpha \in (1/2, \rho)$ during Phase III (when consumers are once again sensitive to signals), consumers' interim belief must fall until a time $t_3 > t_2$ is reached at which $p(t_3) = 1 - \rho$. In the main text, we ruled out the the possibility that $p(t)$ might continue to fall over time but never reach $1 - \rho$ ("possibility (ii)"), the lowest belief given which consumers are willing to adopt after a good private signal. Here, we need to rule out the possibility that $p(t)$ stops declining at some time t' before reaching $1 - \rho$ ("possibility (i)").

Suppose for the sake of contradiction that there exists $t' > t_2$ such that $X(t) < 0$ for all $t \in (t_2, t')$, $X(t') = 0$, and $p(t') > 1 - \rho$. For future reference, note that $X(t') = 0$ requires that $\rho S_g(t') - I_g(t') = (1-\rho)S_b(t') - I_b(t')$. Also recall that, since $X(t_1) = 0$ and $p(t_1) = \rho > \underline{\alpha}$, condition (10) implies that $X'(t_1) < 0$ and that $X(t)$ grows more negative until time $\tilde{t}$ at which $p(\tilde{t}) = \underline{\alpha}$. Thus, it must be that $t' > \tilde{t}$ and that $p(t') \in (1-\rho, \underline{\alpha})$ or equivalently, given equation (3), $\frac{(1-\rho)(1-\alpha)}{\rho\alpha} < \frac{I_g(t')S_g(t')}{I_b(t')S_b(t')} < \frac{1-\rho}{\rho}$.

Several equations that follow are quite complex, so we introduce the following shorthand: $a = S_g(t_2)$; $b = S_b(t_2)$; $c = \rho S_g(t_2) - I_g(t_2) = (1-\rho)S_b(t_2) - I_b(t_2)$; and $d = -(\rho S_g(t') - I_g(t')) = -((1-\rho)S_b(t') - I_b(t'))$.

We know that

$$\begin{aligned} c + d &= (\rho S_g(t_2) - I_g(t_2)) - (\rho S_g(t') - I_g(t')) \\ &= \int_{t_2}^{t'} 2\rho I_g(t) S_g(t) \mathrm{d}t = 2(I_g(t') - I_g(t_2)) = -2\rho(S_g(t') - S_g(t_2)) \\ &= \int_{t_2}^{t'} 2(1-\rho) I_b(t) S_b(t) \mathrm{d}t = 2(I_b(t') - I_b(t_2)) = -2(1-\rho)(S_b(t') - S_b(t_2)), \end{aligned} \tag{21}$$

which implies that

$$\begin{aligned} I_g(t') - I_g(t_2) &= I_b(t') - I_b(t_2) = \frac{c+d}{2} \\ S_g(t') - S_g(t_2) &= -\frac{c+d}{2\rho} \\ S_b(t') - S_b(t_2) &= -\frac{c+d}{2(1-\rho)}. \end{aligned}$$

Therefore,

$$\begin{aligned}\frac{I_g(t')S_g(t')}{I_b(t')S_b(t')} &= \frac{(I_g(t_2)+I_g(t')-I_g(t_2))(S_g(t_2)+S_g(t')-S_g(t_2))}{(I_b(t_2)+I_b(t')-I_b(t_2))(S_b(t_2)+S_b(t')-S_b(t_2))} \\ &= \frac{(a-\frac{c+d}{2\rho})((a\rho-c)+\frac{c+d}{2})}{(b-\frac{c+d}{2(1-\rho)}))((b(1-\rho)-c)+\frac{c+d}{2})} \\ &= \frac{a(a\rho-c)+\frac{c^2-d^2}{4\rho}}{b(b(1-\rho)-c)+\frac{c^2-d^2}{4(1-\rho)}}\end{aligned}$$

We already know that $\frac{a(a\rho-c)}{b(b(1-\rho)-c)} = \frac{(1-\alpha)\rho}{\alpha(1-\rho)} > 1$. Hence, no matter whether $c^2-d^2 \geq 0$ or $c^2-d^2 < 0$, $\frac{I_g(t')S_g(t')}{I_b(t')S_b(t')} > \frac{1-\rho}{\rho}$, a contradiction. This proves that interim beliefs will keep decreasing until t_3 such that $p(t_3) = 1-\rho$.

Step 3: $\Delta I(t) > 0$ and $\Delta S(t) > 0$ for all $t \in [t_2, t_3]$. In the main text, we showed that $\Delta I(t) = I_g(t) - I_b(t)$ and $\Delta S(t) = I_b(t) - I_g(t)$ are positive and increasing over the range $t \in (0, t_2]$. It remains to show that $\Delta I(t') > 0$ and $\Delta S(t') > 0$ for all $t' \in (t_2, t_3]$.

We begin by showing that the "adoption gap" $\Delta I(t)$ exceeds $\Delta I(t_2)$ during all of Phase III. Fix any $t' \in (t_2, t_3)$. Recall that $X(t_2) = 0$ (shown in the proof of Prop 5), $X(t') < 0$ (proven in Step Two), and $X(t) = (\rho S_g(t) - I_g(t)) - ((1-\rho)S_b(t) - I_b(t))$ for all $t \in [t_2, t_3)$ (by Lemma 2, because consumers are sensitive to signals). Thus,

$$(\rho S_g(t') - I_g(t')) - ((1-\rho)S_b(t') - I_b(t')) < (\rho S_g(t_2) - I_g(t_2)) - ((1-\rho)S_b(t_2) - I_b(t_2)).$$

Rearranging and reformulating terms as in equation (21) yields

$$\int_{t_2}^{t'} -2\rho I_g(t)S_g(t)\mathrm{d}t < \int_{t_2}^{t'} -2(1-\rho)I_b(t)S_b(t)\mathrm{d}t. \tag{22}$$

Since $I_g'(t) = \rho I_g(t)S_g(t)$ and $I_b'(t) = (1-\rho)I_b(t)S_b(t)$, inequality (22) implies that $I_g(t') - I_g(t_2) > I_b(t') - I_b(t_2)$, which in turn implies that $\Delta I(t') > \Delta I(t_2)$. Since $\Delta I(t_2) > 0$, we conclude that $\Delta I(t') > 0$ for all $t' \in (t_2, t_3]$, as desired.

The "exposure gap" $\Delta S(t) = S_b(t) - S_g(t)$ is non-monotone during Phase III, for reasons discussed in the main text, but we can show that $\Delta S(t) > 0$ for all $t \in (t_2, t_3]$. Recall that $p(t) > \alpha$ for all $t \in [t_2, \hat{t})$ and $p(t) < \alpha$ for all $t \in (\hat{t}, t_3]$, where $\hat{t} \in (t_2, t_3)$ is the unique time during Phase III at which consumers' interim belief $p(t)$ equals their ex ante belief α. Also recall that, by equation (3),

$$p(t) \gtrless \alpha \text{ iff } S_g(t)I_g(t) \gtrless S_b(t)I_b(t) \text{ iff } -S_g'(t) \gtrless -S_b'(t). \tag{23}$$

Prior to time $\hat{t}$, $p(t) > \alpha$ and condition (23) implies that $\Delta S'(t) > 0$, i.e., the exposure gap is increasing and hence obviously still positive. After time $\hat{t}$, $p(t) < \alpha$ and condition (23) implies that $S_g(t)I_g(t) < S_b(t)I_b(t)$; since $I_g(t) > I_b(t)$, this is only possible if $S_b(t) > S_g(t)$. Thus, even though the exposure gap tightens after time $\hat{t}$, it must remain positive throughout Phase III. □