1. Find the eigenvalues and eigenvectors of matrices

\[
\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
\]

2. Recall that \( I = \sigma_0, X = \sigma_x = \sigma_1, Y = \sigma_y = \sigma_2 \) and \( Z = \sigma_z = \sigma_3 \) denote the Pauli matrices.
   
   (a) Diagonalize \( X, Y, Z, X \otimes X, Y \otimes Y \) and \( Z \otimes Z \).
   
   (b) Show that \( X \otimes X \) commute with \( Z \otimes Z \), and therefore they are simultaneously diagonalizable. What is the basis in which \( X \otimes X \) and \( Z \otimes Z \) are simultaneously diagonalizable?
   
   (c) Show that \( \{I, X, Y, Z\} \) is a basis for the space of operators acting on \( \mathbb{C}^2 \).
   
   (d) For an arbitrary operator \( A \) acting on \( \mathbb{C}^2 \), find the coefficients \( a_0, a_1, a_2, a_3 \) in the expansion \( A = \sum_{i=0}^{3} a_i \sigma_i \) as a function of \( A \) (Hint: You can use the Hilbert-Schmidt inner product introduced in Exercise 2.39 of Nielsen&Chuang).

3. Find the eigenvalues of \( n_x \sigma_x + n_y \sigma_y + n_z \sigma_z \), where \( n_{x,y,z} \in \mathbb{R} \) are real numbers. Find the set of \( n_x, n_y, n_z \) for which \( I + n_x \sigma_x + n_y \sigma_y + n_z \sigma_z \) is a positive operator.

4. (Exercise 2.35 of Nielsen&Chuang) Let \( \hat{n} \) be any real three-dimensional unit vector and \( \theta \) a real number. Prove that

\[
e^{i \theta \hat{n} \cdot \vec{\sigma}} = \cos \theta I + i \sin \theta \hat{n} \cdot \vec{\sigma},
\]

where \( \hat{n} \cdot \vec{\sigma} = n_x \sigma_x + n_y \sigma_y + n_z \sigma_z \).

5. Suppose \( A \) and \( B \) are commuting Hermitian operators, i.e. \( [A, B] = 0 \). Prove that

\[
e^{\alpha (A+B)} = e^{\alpha A} e^{\alpha B},
\]

for any \( \alpha \in \mathbb{C} \). Note that this equality does not hold, in general, if \( [A, B] \neq 0 \).

6. Suppose \( A \) and \( B \) are two non-degenerate Hermitian operators defined on the same vector space. Prove that there exists a unitary \( U \) such that \( U A U^\dagger = B \). What is the unitary \( U \) that satisfies \( U Y U^\dagger = Z \), where \( Y \) and \( Z \) are Pauli matrices?

7. Consider the tensor product of two \( d \)-dimensional vector spaces, denoted by \( \mathbb{C}^d \otimes \mathbb{C}^d \). The Swap operator on this tensor product space is defined by equation

\[
\text{Swap}(|\psi\rangle \otimes |\phi\rangle) = |\phi\rangle \otimes |\psi\rangle,
\]

where \( |\psi\rangle \) and \( |\phi\rangle \) are arbitrary vectors in \( \mathbb{C}^d \). In words, this operator swaps the vectors in the two vector spaces.
(a) Let \( \{|i\rangle, i = 1, \ldots, d\} \) be an orthonormal basis for \( \mathbb{C}^d \). The set of outer products \( \{|i\rangle \langle j| : i, j = 1, \ldots, d\} \) forms a basis for the space of linear operators acting on \( \mathbb{C}^d \). Similarly, the set of outer products \( \{|i\rangle \langle j| \otimes |k\rangle \langle l| : i, j, k, l = 1, \ldots, d\} \) forms a basis for the space of operators acting on \( \mathbb{C}^d \otimes \mathbb{C}^d \). Write the Swap operator as the linear combination of these outer products.

(b) Show that the Swap operator is both unitary and Hermitian.

(c) Show that the eigenvalues of Swap are \( \pm 1 \).

(d) In the case of \( d = 2 \), Swap acts on the 4-dimensional vector space corresponding to a pair of qubits. Find an orthonormal basis in which Swap is diagonal.

(e) Let \( |0\rangle \) and \( |1\rangle \) be the normalized eigenvectors of \( Z \), corresponding to the eigenvalues \( +1 \) and \( -1 \), respectively. Write down the matrix representation of \( X \otimes X, Y \otimes Y, \) and \( Z \otimes Z \) in the computational basis, i.e. \( \{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\} \). In the case of \( d = 2 \), show that the Swap operator can be written as a linear combination of \( X \otimes X, Y \otimes Y, \) and \( Z \otimes Z \) and \( I \otimes I \).

(f) For general \( d \in \mathbb{N} \), find the dimension of the eigen-subspace of Swap with eigenvalue \( +1 \) (Hint: Find \( \text{Tr}(\text{Swap}) \) as a function of \( d \), and use the fact that all eigenvalues are \( \pm 1 \)).

(g) For \( \theta \in \mathbb{R} \) show that

\[
e^{i\theta} \text{Swap} = \cos \theta \ 1_d \otimes 1_d + i \sin \theta \ \text{Swap},
\]

where \( 1_d \) is the identity operator on \( \mathbb{C}^d \).

(h) For any pair of operators \( A \) and \( B \) acting on \( \mathbb{C}^d \), show that

\[
\text{Swap}(A \otimes B)\text{Swap} = B \otimes A.
\]

For any arbitrary unitary \( U \) acting on \( \mathbb{C}^d \) show that

\[
(U \otimes U)\text{Swap}(U^\dagger \otimes U^\dagger) = \text{Swap}.
\]

(i) For any pair of operators \( A \) and \( B \), show that

\[
\text{Tr}(\text{Swap}(A \otimes B)) = \text{Tr}(AB).
\]

8. Let \( |\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \otimes |i\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d \) and \( 1_d \) be the identity operator on \( \mathbb{C}^d \).

(a) Prove that

\[
(A \otimes 1_d)|\Psi\rangle = (B \otimes 1_d)|\Psi\rangle,
\]

if and only if \( A = B \).

(b) Show that

\[
(A \otimes 1_d)|\Psi\rangle = (1_d \otimes A^T)|\Psi\rangle,
\]

where \( A^T \) is the transpose of \( A \) in the \( \{|i\rangle : i = 1, \ldots, d\} \) basis.

(c) For any unitary \( U \) acting on \( \mathbb{C}^d \) show that

\[
(U \otimes U^*)|\Psi\rangle = |\Psi\rangle.
\]