Econ205 – Intermediate Microeconomics with Calculus
Chapter 1*

Margaux Luflade

May 1st, 2016

Contents

I Basic consumer theory 3

1 Overview 3
1.1 What? 3
1.1.1 Self-interest 3
1.1.2 Economic equilibrium 3
1.2 How? 4
1.2.1 Models 4
1.2.2 Analysis 4

2 Preferences & utility 6
2.1 The concept of preferences 6
2.1.1 Commodity bundles 6
2.1.2 Preferences 6
2.1.3 Goods, bads and neuters 6
2.2 The concept of utility 7
2.2.1 Utility representation and utility functions 7
2.2.2 Marginal rate of substitution 7
2.2.3 Examples: common utility functions 8

3 Consumer optimality 9
3.1 Objective & constraint: utility & budget 9
3.2 Solution 9
3.2.1 Solution method: Lagrange 9
3.2.2 Solution function: Marshallian demand 10
3.3 Value function: indirect utility 10
3.4 Lagrange multipliers: shadow prices 11
3.5 Dual problem 11
3.5.1 Objective & constraint: costs & utility level 11
3.5.2 Solution method & solution function 11
3.5.3 Hicksian demand 11
3.5.4 Value function: expenditure function 11

*The structure of these notes is largely borrowed from Pr. Curt Taylor. All errors and typos remain mine. Comments are welcome and should be sent to margaux.luflade@duke.edu.
4 Demand: properties and positive analysis

4.1 Properties of demand functions ........................................... 12
  4.1.1 Duality properties (1) .............................................. 12
  4.1.2 Mathematical concepts ............................................ 13
  4.1.3 Duality properties (2) .............................................. 14

4.2 Substitution and income effects .......................................... 15
  4.2.1 Monotonicity of Hicksian demand .................................. 15
  4.2.2 Normal and inferior goods .......................................... 15
  4.2.3 Slutsky equation ................................................... 15

4.3 Effects of distortionary taxes and subsidies .............................. 15
Part I
Basic consumer theory

1 Overview

1.1 What?

As every individual, therefore, endeavors as much as he can both to employ his capital in the support of domestic industry, and so to direct that industry that its produce may be of the greatest value; every individual necessarily labors to render the annual revenue of the society as great as he can. He generally, indeed, neither intends to promote the public interest, nor knows how much he is promoting it. He intends only his own gain, and he is in this, as in many other cases, led by an invisible hand to promote an end which was no part of his intention.

In the above famous passage from his 1776 treatise, *An Inquiry into the Nature and causes of the Wealth of Nations*, Adam Smith identifies the two principles that still form the core of modern economic analysis. In brief, these are the principles of *self interest* (i.e., rational or optimizing behavior) and *economic equilibrium*. In large measure, we will be concerned throughout the semester with formalizing Smith’s notions of self interest and economic equilibrium. In addition, we will see that the equilibrium of an economic system often cannot be improved upon, and that even when it can, care must be taken in order not to create a worse situation instead of a better one.

1.1.1 Self-interest

**Individual optimization, utility & preferences.** Economists formalize the notion of self interest through the mathematics of optimization. Specifically, the assumption underlying nearly all modern economic analysis is simply that each individual in society endeavors to make himself as happy as possible. In particular, we say that each agent *maximizes his utility* subject to the constraints he faces. This does not mean that economists believe everyone is or should be “selfish”. People derive happiness from such selfless endeavors as teaching kindergarten or volunteering in a soup kitchen. Economists are seldom concerned with the formation of an individual’s tastes. Some people spend money on cocaine, and others spend money on season tickets to the opera. Economists are agnostic about what gives an individual pleasure (i.e., utility).

**Constraints.** Individuals are not unconstrained in their pursuit of self-interest. The constraints facing an economic agent depend on the system of property rights and the laws and customs of the society in which he lives. They also depend importantly on the state of technology and the initial endowment of resources. We will primarily be concerned with studying the optimization problems facing agents (consumers and firm owners) who operate in a private-ownership market economy where property rights are well-defined, perfectly enforced, and may be legally exchanged for money. (This is—to some extent—an abstraction. Even in our own market economy, there are many goods which are illegal to own or sell; e.g., drugs, vital organs, or babies.)

1.1.2 Economic equilibrium

**Equilibrium of the economic system.** Economic agents do not operate in a vacuum. Rather, they are typically part of an economic system. The actions taken by each agent in the system often impact either the utility or the constraints of other agents (although this impact is sometimes very small). We have several different notions of ‘equilibrium’ in economics, but generally, we say that an economic system attains equilibrium when each agent is simultaneously maximizing his utility subject to the constraints he faces. This is a powerful concept and forms the basis for nearly all economic analysis.

**Solving for and analyzing equilibrium solutions.** Throughout this semester, we will be concerned with formalizing the principles of self interest and economic equilibrium. Specifically, we will solve economic
optimization problems (e.g., maximizing utility or minimizing cost) and analyze the solutions we obtain. We will also combine the optimization problems of all the agents in a system in order to find an equilibrium.

1.2 How?

1.2.1 Models

There are, therefore, two kinds of problems in economics, optimization problems and equilibrium problems. In either case we may analyze the solution to the problem at hand in order to make predictions about economic behavior (positive analysis) or evaluate the solution according to some criterion (normative analysis).

Definition. The engine for performing economic analysis is a mathematical model which is a simplified representation of a real-world situation. A good economic model captures the salient features of the environment without including extraneous details that unnecessarily complicate the problem. A model should be as simple as possible, but no simpler!

Exogenous vs. endogenous variables. There are two kinds of variables in any model, exogenous variables and endogenous variables. In the natural sciences these are often called independent and dependent variables respectively. The exogenous or independent variables are the parameters of the model that define the environment. The endogenous or dependent variables are the object of the analysis.

In an optimization problem the endogenous variables are the choice variables that the agent in question selects in order to optimize his objective function. For example, in a standard consumer problem, the individual regards the prices of goods as exogenous variables or parameters, and she regards the amount she purchases of each good as an endogenous choice variable. Hence, she maximizes her utility (her objective function) by choice of consumption levels of each good taking prices as constants.

In an equilibrium problem, the endogenous variables depend on the model under study. For instance, in a supply and demand model the endogenous variables are the aggregate quantity demanded, the aggregate quantity supplied, and the market price. Equilibrium consists of a market price at which the quantity demanded and the quantity supplied are equal. Exogenous variables include, the prices of other goods, the tastes of consumers, and the technology of firms.

1.2.2 Analysis

Positive analysis. The most common form of positive analysis performed by microeconomics is called comparative static analysis. It is a prediction about how the endogenous variables in a model will respond to changes in one (or more) of the exogenous variables.

Example 1.1 (Individual Firm Supply Slopes Up). As a simple example of a comparative static result, consider a firm that produces output \( q \geq 0 \) and has a cost function \( C(q) \). (As we will see, the cost function also depends on input prices and technology, but we suppress this notationally for now.) We suppose that the firm is a price taker and thus regards the price at which it can sell output \( p \geq 0 \) as an exogenous parameter. Its objective is to choose the endogenous variable, \( q \), so as to maximize its profit

\[
\max_{q} \pi \equiv pq - C(q).
\]

Consider two exogenous values for the price, \( p_1 \) and \( p_0 \) and assume \( p_1 > p_0 \). Suppose that \( q_1 \) maximizes profit at \( p_1 \) and \( q_0 \) maximizes profit at \( p_0 \). We wish to establish the following comparative static result

**Proposition 1.2.** the firm’s supply function slopes (weakly) up, \( q_1 \geq q_0 \).

**Proof.** The proof will be covered in discussion session. Prepare it as an exercise before the first discussion session.
Normative analysis. Normative analysis is typically more controversial since policy makers may not agree on the social objective; i.e., the criterion used to evaluate outcomes. For example, if the production of some good creates carbon emissions that are associated with climate change, then it may be desirable to tax emissions in order to induce a reduction in the equilibrium level of pollution. Just how large the tax should be depends on the objective of the policy maker. Economists might, for example, argue that the tax should be set so as to maximize social surplus (i.e., allocative efficiency), but even this prescription is likely to be controversial because of the imprecise measurement of costs and benefits. Just how much is a polar bear worth?
2 Preferences & utility

2.1 The concept of preferences

2.1.1 Commodity bundles.

Definition 2.1. A commodity bundle (or market basket) is a vector of quantities of a set of goods.

Let \( A \) denote a set of market baskets.

2.1.2 Preferences.

Ordering. Let the preferences of a given individual be represented by the symbol \( \succeq \). This binary relation describes the economically relevant aspects of a consumer’s psychology, his preferences. Specifically, \( a \succeq b \) means that the individual in question weakly prefers the collection of commodities in market basket \( a \) to that in \( b \). In other words, bundle \( a \) leaves the individual in question at least as well off as bundle \( b \). We represent strict preference (i.e., \( a \) is strictly preferred to \( b \)) notationally by writing \( a \succ b \). The preferences of a consumer are usually assumed to obey three properties that define an ordering:

Assumption 2.2 (Ordering). The preference relation \( \succeq \) is an ordering or ranking over the elements of the set of commodity bundles \( A \). That is, it satisfies the following three conditions.

- Reflexivity: For any commodity bundle \( a \) in \( A \), \( a \succeq a \).
- Completeness: For any two commodity bundles \( a \) and \( b \) in \( A \), either \( a \succeq b \) or \( b \succeq a \).
- Transitivity: For any three commodity bundles \( a \), \( b \), and \( c \) in \( A \), \( a \succeq b \) and \( b \succeq c \) implies \( a \succeq c \).

Definition 2.3. Any binary relation that is reflexive, complete, and transitive on a set of items is called an order or ranking of the items.

Can you explain these axioms in words? Do they sound reasonable to you?

Indifference.

Definition 2.4. A consumer is said to be indifferent between commodity bundles \( a \) and \( b \) (denoted \( a \sim b \)) if \( a \succeq b \) and \( b \succeq a \).

Definition 2.5. An indifference set for a consumer is the subset of commodity bundles in \( A \) among which he is indifferent; i.e., commodity bundles \( a \) and \( b \) are in the same indifference set if and only if \( a \sim b \).

Consider the following assumption:

Assumption 2.6 (Continuity). For any two bundles \( a \) and \( b \) in \( A \) such that \( a \succ b \), there exists a bundle \( c \) sufficiently close to \( a \) such that \( c \succ b \).

Can you explain this property in words?

How do indifference sets look like if commodities are infinitely divisible and preferences are continuous?

2.1.3 Goods, bads and neuters

Definition. A commodity is called a good if the consumer in question prefers more of it to less. It is called a bad if he prefers less of it to more. And, it is called a neuter if he is indifferent about having more or less of it.
**Graphical characterization.** For ease of exposition, we will often restrict attention to commodity bundles containing only two commodities. Let \((x, y)\) be a market basket with \(x\) units of commodity \(X\) and \(y\) units of commodity \(Y\).

On a graph with quantities of \(X\) on the \(x\)-axis, and quantities of \(Y\) on the \(y\)-axis, can you draw indifference curves (i) if \(X\) and \(Y\) are goods? (ii) if \(Y\) is a bad? (iii) if \(Y\) is a neuter?

Consider the following property:

**Assumption 2.7 (Non-Satiation).**

- **Weak**: If \(a\) and \(b\) are any two commodity bundles such that \(a\) contains at least as much of every commodity as \(b\), then \(a \succeq b\), and if \(a\) contains strictly more of every good than \(b\), then \(a \succ b\).

- **Strong**: If \(a\) and \(b\) are any two commodity bundles such that \(a\) contains at least as much of every commodity as \(b\), then \(a \succeq b\), and if \(a\) contains strictly more of at least one good than \(b\), then \(a \succ b\).

Can you explain this property in words? Does it sound reasonable to you?

### 2.2 The concept of utility

#### 2.2.1 Utility representation and utility functions

**Existence of a representation** When a consumer’s preferences are reflexive, complete, and transitive (i.e., when they order the set of all commodity bundles), it is possible to represent \(\succeq\) with a utility function or utility index. A utility function \(U\) mapping the set of bundles \(\mathcal{A}\) to the real numbers \(\mathbb{R}\) is said to represent the preference relation \(\succeq\) if, for any two bundles \(a\) and \(b\) in \(\mathcal{A}\),

\[
a \succeq b \iff U(a) \geq U(b).
\]

A utility function assigns an index number to each commodity bundle. Bundles with higher utility indices are on higher indifference curves. Bundles which deliver the same level of utility are on the same indifference curve.

**Ordinal vs. cardinal representation.** It is important to remember that a utility function is only an ordinal ranking (i.e., an index) not a cardinal measure of happiness. For instance, a commodity bundle that gives a consumer utility of 20 makes him better off than one that gives him utility of 10, but not necessarily twice as well off.

What does this last remark tells you about uniqueness of a utility representation?

#### 2.2.2 Marginal rate of substitution

**Definition: mathematical characterization and intuition.** The slope of an indifference curve at a given point is called the individual’s marginal rate of substitution (MRS) of \(Y\) for \(X\) at that point.

Can you provide a formal (mathematical) expression of the MRS? Can you provide intuition about the meaning of the concept (like: if the MRS of the individual at \((x_0, y_0)\) is 3, what does that mean, in terms of goods \(X\) and \(Y\)?

Can you discuss the uniqueness of utility representation in terms of the MRS?

**Convexity of preferences.** A consumer’s preferences depend importantly on the shape of his indifference curves. The following assumption (which does not always hold) implies that indifference curves are bowed toward the origin.
Assumption 2.8 (Convexity).

**Weak:** If \((x_1, y_1) \sim (x_2, y_2)\), then for any \(\lambda\) between 0 and 1
\[
\lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2) \succeq (x_1, y_1).
\]

**Strict:** If \((x_1, y_1) \sim (x_2, y_2)\), then for any \(\lambda\) between 0 and 1
\[
\lambda(x_1, y_1) + (1 - \lambda)(x_2, y_2) \succ (x_1, y_1).
\]

Can you explain this property in words?

### 2.2.3 Examples: common utility functions

Can you draw indifference curves for the preferences represented by the following utility functions?

**Perfect substitutes.** These preferences are represented by (any increasing transformation of) the utility function
\[
U(x, y) = \alpha x + \beta y,
\]
where \(\alpha\) and \(\beta\) are positive constants (exogenous parameters). An indifference curve is given by the equation
\[
y = -\frac{\alpha}{\beta}x + \frac{u}{\beta}.
\]
Can you provide intuition for why this function represents *perfect substitutes*?

**Perfect complements (or Leontief preferences).** These preferences are represented by (any increasing transformation of) the utility function
\[
U(x, y) = \min\{\alpha x, \beta y\},
\]
where (as before) \(\alpha\) and \(\beta\) are positive constants.
Can you provide intuition for why this function represents *perfect complements*?

**Cobb-Douglas utility.** The most widely used utility index is the Cobb-Douglas function given by (any increasing transformation of)
\[
U(x, y) = x^\alpha y^\beta,
\]
where \(\alpha\) and \(\beta\) are positive constants. This utility function is intermediate between perfect substitutes and perfect complements. In particular, the indifference curves exhibit smoothly diminishing MRS.

**Globally satiated preferences.** As an example, suppose that an individual has preferences represented by the utility function
\[
U(x, y) = b - (x - x_0)^2 - (y - y_0)^2,
\]
where \(b\), \(x_0\), and \(y_0\) are positive constants.
Why is this function representing *globally satiated* preferences?
3 Consumer optimality

3.1 Objective & constraint: utility & budget

Budget constraint. The main factor constraining choice for most individuals in a market economy is that they have limited wealth to spend on goods. Suppose that a consumer’s income is $I$ and that the prices of goods X and Y (the only two things he consumes) are $p_X$ and $p_Y$. Then, the consumer can afford to purchase any bundle $(x, y)$ such that

$$p_X x + p_Y y \leq I.$$ 

This is called the individual’s budget constraint. The frontier of this constraint (sometimes called the budget line) is the set of bundles that completely exhausts the individual’s income.

Can you draw the individual’s feasible set?

How does your graph change if (keeping all other things constant) the individual’s income increases? What if it decreases?

How does your graph change if (keeping all other things constant) the price of X increases? What if it decreases?

How does your graph change if (keeping all other things constant) the price of Y increases? What if it decreases?

What is the interpretation of the intercept in your graph?

In your graph, what does the price ratio $\frac{p_X}{p_Y}$ correspond to? Can you give an intuition about the meaning of the price ratio (like: if the price ratio is 3, what does that mean, in terms of goods X and Y?) How does this interpretation differ from the one you gave for the MRS?

Utility maximization. Economists typically assume that the objective of a consumer is to maximize his utility. A consumer cannot, however, consume as much of every good as he wants because scarce goods generally have positive prices and consumers have limited wealth. Hence, a consumer must pick the best bundle of commodities he can afford. That is, he maximizes his utility subject to his budget constraint. Formally this maximization program is written:

$$\max_{(x,y)} U(x, y) \text{ subject to } p_X x + p_Y y \leq I. \quad (1)$$

In this program the consumer’s utility function $U(\cdot, \cdot)$ is the objective function, the amounts of the goods X and Y are the choice variables and the consumer’s income $I$ is the resource constraint. Technically we should also include non-negativity constraints on consumption: $x \geq 0$ and $y \geq 0$, but these are usually only implicit.

3.2 Solution

3.2.1 Solution method: Lagrange

Provided the utility function is strictly quasi-concave and differentiable, we can solve (1) using the method of Lagrange. Specifically, we write the unconstrained program

$$\max_{(x, y, \lambda)} L = U(x, y) + \lambda(I - p_X x - p_Y y). \quad (2)$$

The variable, $\lambda$, is called the Lagrange multiplier. It can be shown that $\lambda = 0$ if the budget constraint does not bind and $\lambda > 0$ only if it does.\footnote{Formally the Karush-Kuhn-Tucker (KKT) conditions necessary for a maximum are: (Primal feasibility) $I - p_X x - p_Y y \geq 0$, (Dual feasibility) $\lambda \geq 0$, and (Complementary slackness) $\lambda(I - p_X x - p_Y y) = 0.$}

What can you say about $\lambda$ if preferences are continuous and non-satiated?

There are two possible types of solution to (2), an interior solution in which the consumer buys positive amounts of both goods or a corner solution in which he buys zero units of one of the goods.
**Interior solutions**  If the utility function is differentiable, then the first-order necessary conditions for an interior solution are:

\[
\frac{\partial L}{\partial x} = 0, \quad (3) \\
\frac{\partial L}{\partial y} = 0, \quad (4) \\
\text{and} \\
\frac{\partial L}{\partial \lambda} = 0. \quad (5)
\]

Use this system of three equations to solve for the optimal values of \(x, y\) and \(\lambda\) given \(p_X, p_Y\) and \(I\).

**Corner solutions.** A corner solution exists at \(x^* = 0\) and \(y^* = I/p_Y\) if

\[
\frac{\partial U}{\partial x} \frac{\partial U}{\partial y} < \frac{p_X}{p_Y}
\]

at every point \((x, y)\) on the budget frontier for which \(x \geq 0\) and \(y \geq 0\).

Can you provide an interpretation of this condition? Why does optimality at the corner point make sense in that case?

Give a condition for \(x^* = I/p_X\) and \(y^* = 0\) to be a corner solution.

### 3.2.2 Solution function: Marshallian demand

The solution \((x^*, y^*)\) gives the consumer’s *Marshallian* (also called *uncompensated* or *ordinary*) demand for goods \(X\) and \(Y\) given prices and income \((p_X, p_Y, I)\). In fact, if we let \(p_X\) vary holding fixed \(p_Y, I,\) and preferences, then \(x^*\) will vary yielding the consumer’s demand function for \(X\).

**Example 3.1** (Deriving Ordinary Demand). Suppose an individual has symmetric Cobb-Douglas preferences

\[U(x, y) = xy.\]

His budget constraint is

\[p_X x + p_Y y = I.\]

1. Consider \((p_X, p_Y, I)\) as given, and find the quantities of goods \(X\) and \(Y\) demanded by the consumer.
2. On a graph, represent the quantities of \(X\) and \(Y\) demanded when \((p_X, p_Y, I) = (1, 1, 50)\). On the same graph, represent the quantities of \(X\) and \(Y\) demanded when \((p_X, p_Y, I) = (2.5, 1, 50)\).
3. On the same graph, represent the Marshallian demand function of the consumer.
4. What is the effect on quantities demanded of a 1% increase in the price of \(X\)?
5. What is the effect on quantities demanded of a 1% increase in the consumer’s income?

### 3.3 Value function: indirect utility

When the general solution to an optimization problem is substituted back into the objective function, the result is called a *value function*. It specifies how the optimal value of the objective depends on the underlying parameters. In the current context, the solution to a consumer’s optimization problem yields his ordinary demand functions \(x^*(p_X, p_Y, I)\) and \(y^*(p_X, p_Y, I)\). Specifically, these functions specify the optimal levels for the endogenous choice variables \(x\) and \(y\) as a function of the exogenous parameters \(p_X, p_Y,\) and \(I\). If we substitute the demand functions into the consumer’s utility function we obtain the value function known as the consumer’s *indirect utility function*:

\[V(p_X, p_Y, I) \equiv U(x^*(p_X, p_Y, I), y^*(p_X, p_Y, I)) \quad (6)\]

The indirect utility function tells us how an individual’s welfare is related to the environmental variables (parameters) he faces, namely prices and income.
Example 3.2 (Deriving Indirect Utility). Find the indirect utility function from the previous example. What utility the optimizing consumer gets when \((p_X, p_Y, I) = (1, 1, 50)\)? And when \((p_X, p_Y, I) = (2.5, 1, 50)\)?

3.4 Lagrange multipliers: shadow prices

At an interior optimum, it is possible to solve equations (3), (4), and (5) above not only for the optimal consumption levels \(x^*\) and \(y^*\) but also for the “optimal” value for the Lagrange multiplier \(\lambda^*\). In general, the optimal value for the multiplier in a constrained optimization program gives the marginal change in the value function from relaxing the constraint. This is sometimes called the shadow price of the constraint. In a utility maximization problem, \(\lambda^*\) is, thus, the increase in the optimal level of utility from a marginal increase in income; i.e., it represents the marginal (indirect) utility of income:

\[
\lambda^* = \frac{\partial V}{\partial I}.
\]  

(7)

In the context of utility maximization, this result is admittedly of very limited use since marginal utility has no meaning beyond its sign (positive or negative) because utility is only ordinal. The interpretation of the Lagrange multiplier in problems with cardinal objectives is, however, very useful!

Example 3.3 (The Shadow Price of the Constraint). What is the shadow price of the budget constraint when \((p_X, p_Y, I) = (1, 1, 50)\)? And when \((p_X, p_Y, I) = (2.5, 1, 50)\)?

3.5 Dual problem

Associated with every constrained maximization problem is a conjugate constrained minimization problem and vice versa. We call the original program the primal and the conjugate program the dual.

3.5.1 Objective & constraint: costs & utility level

For instance, if the primal program is to maximize utility subject to a budget (or expenditure) constraint, then the dual program is to minimize expenditure subject to a utility constraint. That is, rather than thinking about finding the highest indifference curve on a given budget line, the dual is to find the lowest budget (isoexpenditure) line on a given indifference curve. Can you write the dual optimization problem?

3.5.2 Solution method & solution function

1. Can you write the Lagrangian function of the dual problem?
2. What are necessary conditions for interior solutions?
3. When do corner solutions arise?

3.5.3 Hicksian demand

4. What are the solution functions of the dual problem? These are called Hicksian demand functions.
5. What is the relationship between Hicksian and Marshallian demand functions?

3.5.4 Value function: expenditure function

6. What is the value function of the dual problem? What is its interpretation?
4 Demand: properties and positive analysis

4.1 Properties of demand functions

4.1.1 Duality properties (1)

Inverse functions. The indirect utility function $V(p_X, p_Y, I)$ specifies the highest level of utility that can be attained given expenditure level $I$ when prices are $p_X$ and $p_Y$. Conversely, the expenditure function $e(p_X, p_Y, u)$ gives the lowest level of expenditure necessary to attain utility level $u$ when prices are $p_X$ and $p_Y$. Given this, it should come as no surprise that the indirect utility function and the expenditure function are inverses of each other. Mathematically

$$V(p_X, p_Y, I) = e^{-1}(p_X, p_Y, I) \Rightarrow e(p_X, p_Y, V(p_X, p_Y, I)) = I \quad (8)$$

or equivalently

$$e(p_X, p_Y, u) = V^{-1}(p_X, p_Y, u) \Rightarrow V(p_X, p_Y, e(p_X, p_Y, u)) = u. \quad (9)$$

Inverse Lagrange multipliers. Moreover, it follows from the Inverse Function Theorem that the Lagrange multipliers for the utility-max and expenditure-min problems are reciprocals. Can you recall the Inverse Function Theorem?

$$\lambda^* = \frac{\partial V}{\partial I} = \frac{\partial (e^{-1})}{\partial I} = \frac{1}{\partial e} = \frac{1}{\mu^c} \quad (10)$$

or equivalently

$$\mu^c = \frac{\partial e}{\partial u} = \frac{\partial (V^{-1})}{\partial u} = \frac{1}{\partial V} = \frac{1}{\lambda^*}. \quad (11)$$

What if $\lambda^* = 0$? Can you recall in which case this happens?

Consistency between Marshallian and Hicksian demands. Expressions (8) and (9) make our lives much easier since they say that we can derive the expenditure function simply by inverting the indirect utility function and vice versa. Hence, to find the value function for the dual problem, one can just invert the value function for the primal problem. This along with two results known as Shephard’s Lemma and Roy’s Identity (Propositions 4.7 and 4.8 discussed below) imply that we can obtain the solution to the dual problem directly from the solution to the primal problem (and vice versa). Recall also the claim made at the end of the last section that the solutions to the primal and dual are the same when the value of one problem equals the constraint in the other. Mathematically this is written

$$x^*(p_X, p_Y, I) = x^c(p_X, p_Y, V(p_X, p_Y, I)) \quad (12)$$

and

$$x^c(p_X, p_Y, u) = x^*(p_X, p_Y, e(p_X, p_Y, u)). \quad (13)$$

Can you explain these two equations in words, and write analogous equations that hold for $Y$?

Example 4.1 (Inverting the Expenditure Function). Can you write the dual problem to the utility maximization problem in the very first example of Section 3? Can you give the associated expenditure function without solving the dual problem?
4.1.2 Mathematical concepts

Homogeneity.

Definition 4.2 (Homogeneous Functions). A real-valued function $F(x_1, x_2, \ldots, x_n)$ is homogenous of degree $K$ (HOD-K) if for any constant $T > 0$,

$$F(T \times x_1, T \times x_2, \ldots, T \times x_n) = T^K \times F(x_1, x_2, \ldots, x_n).$$

To get a better grasp on the concept, show the following:

1. the function $f_1 : \mathbb{R}_+^2 \to \mathbb{R}, (x_1, x_2) \mapsto \frac{x_1}{x_2}$ is HOD-1;
2. the function $f_3 : \mathbb{R}_+^2 \to \mathbb{R}, (x_1, x_2) \mapsto x_1 + x_2^2$ is not homogeneous of any degree.

Two cases are of particular interest for economists, $K = 0$ and $K = 1$. Since $T^0 = 1$, a function that is HOD-0 does not change at all when its arguments are scaled up or down (by the same factor). A function which is HOD-1 changes by the same factor as its arguments. We sometimes say that a function that is HOD-1 exhibits constant returns to scale.

Can you give an intuition for why we say that a function that is HOD-1 exhibits constant returns to scale?

Proposition 4.3 (Homogeneity of Ordinary Demand). Ordinary (Marshallian) demands are HOD-0 in prices and income

$$x^*(T p_X, T p_Y, T I) = x^*(p_X, p_Y, I)$$

and


Proof. Do you know how to prove this result?

Proposition 4.4 (Homogeneity of Compensated Demand). Compensated demand functions are HOD-0 in prices.

Can you write that mathematically? We will prove the result in class.

The Envelope Theorem.

Proposition 4.5 (The Envelope Theorem). When evaluating the change in a value function from a change in a parameter, one need consider only the direct effect (i.e., only the derivative of the value function with respect to the parameter in question).

Proof. We prove this only for the simplest case. Suppose we wish to choose $x$ so as to maximize the function $f(x, a) : \mathbb{R}^2 \to \mathbb{R}$, where $a$ is a parameter. The first-order condition is

$$\frac{\partial f}{\partial x} = 0.$$

Denote the solution by $x^*(a)$. Then the value function is

$$F(a) \equiv f(x^*(a), a).$$

Now suppose that we are interested in knowing how much the value function changes due to a small change in the parameter $a$. In general we need to calculate

$$F'(a) = \frac{\partial f(x^*(a), a)}{\partial x} \frac{dx^*}{da} + \frac{\partial f(x^*(a), a)}{\partial a}. \tag{14}$$

1. Can you justify that the second term on the right of (14) is the direct effect on the value function from a change in $a$?
2. Can you justify that the first term on the right of (14) gives the indirect effect on the value function from a change in \(a\)?

3. To show the desired result, we need to show that the indirect effect is zero. Can you explain why the indirect effect is zero?

Example 4.6 (The Envelope Theorem). Consider

\[ f(x; a) = ax - 0.5x^2. \]

Here, we look at \(a\) as a parameter, and \(x\) as the variable per se. Can you illustrate the Envelope Theorem with this function? That is, show that if \(F\) denotes the value function of the maximization problem with \(f\) as objective function, then

\[ F'(a) = \frac{\partial f(x^*, a)}{\partial a}. \]

The Envelope Theorem greatly simplifies matters in many cases because it is not necessary to consider changes in the value function arising from the changes in the optimal values of the choice variables when a parameter changes. In particular, we need consider only the direct impact on the value function from changing the parameter itself. One implication of the Envelope Theorem is Shephard’s Lemma.

4.1.3 Duality properties (2)

Proposition 4.7 (Shephard’s Lemma). The compensated demand for a good is equal to the derivative of the expenditure function with respect to the price of that good: Can you write this as an equation? Does this make sense to you?

Proof. In class, we will prove this result using the Envelope Theorem.

Another result deriving from the Envelope Theorem is known as Roy’s Identity.

Proposition 4.8 (Roy’s Identity). Ordinary demand can be recovered from the indirect utility function as follows:

\[ x^*(p_X, p_Y, I) = -\frac{\partial V/\partial p_X}{\partial V/\partial I}, \]

and

\[ y^*(p_X, p_Y, I) = -\frac{\partial V/\partial p_Y}{\partial V/\partial I}. \]

Proof. In class, we will prove this result using the Envelope Theorem.

Together, Shephard’s Lemma, Roy’s Identity, and the duality relationships (12) and (13) allow us to recover Hicksian and Marshallian demands from the respective value functions.

Example 4.9 (Recovering Demands). Consider the expenditure function

\[ e(p_X, p_Y, u) = 2\sqrt{p_X p_Y u}. \]

1. Can you derive the associated Hickshian demand using Shephard’s Lemma?

2. Can you get the associated Marshallian demand using Roy’s identity? Can you think of another way to do it?
4.2 Substitution and income effects

4.2.1 Monotonicity of Hicksian demand

Proposition 4.10 (Substitution). *Compensated demand functions are weakly decreasing in their own prices:* Can you write this as equations?

Proof. We will prove the result using a method very similar to that used to show that individual firm supply is weakly increasing in the very first section. Can you try it? (or at least review the example from the first section?)

4.2.2 Normal and inferior goods

Definition 4.11 (Normal and Inferior Goods). An individual regards good $X$ as normal if he consumes more of it when his income rises:

$$\frac{\partial x^*}{\partial I} > 0.$$  

He regards $X$ as inferior if he consumes less of it as his income rises

$$\frac{\partial x^*}{\partial I} < 0.$$  

Can you find example or a normal good? and of an inferior good?

4.2.3 Slutsky equation

We are now in a position to investigate how an ordinary (Marshallian) demand reacts when its own price changes. To accomplish this, first recall that the solution to the primal and the dual problem coincide when the value function of one problem appears as the constraint of the other.

Specifically, recall the relationship between the Hicksian and Marshallian demands for good $X$ given in (13).

Differentiate both sides of the equation with respect to $p_X$.

Show that what you get implies:

Slutsky Equation:

$$\frac{\partial x^*}{\partial p_X} = \frac{\partial x^c}{\partial p_X} - \frac{\partial x^*}{\partial I} x^*$$

1. Provide an interpretation to the first term on the RHS. What is its sign?
2. Provide an interpretation to the second term on the RHS. What is its sign?
3. Justify the names of substitution effect and income effect given to these terms.
4. On a graph with quantities of good $X$ on the $x$-axis and quantities of good $Y$ on the $y$-axis: (i) draw a budget line corresponding to some prices $(p_X^0, p_Y)$; (ii) draw a budget line corresponding to some prices $(p_X^1, p_Y)$, with $p_X^1 > p_X^0$; (iii) draw indifference curves and represent optimal points under the two price schemes; (iv) illustrate substitution and income effects involved in the change of consumption bundle.
5. What would be a good for which the substitution effect is dominated by the income effect? What would happen then after a price increase?

4.3 Effects of distortionary taxes and subsidies

Consider an individual who consumes health care (commodity $X$) and a composite of all other goods (commodity $Y$). Suppose the individual has preferences that can be represented by the Cobb-Douglas utility function:

$$U(x, y) = x^{1/25} y^{24/25}. \quad (15)$$
Write the MRS. Are the individual’s preferences convex?

Suppose that $p_X = p_Y = 1$ thousand dollars per unit and that the individual has an income of 100 thousand dollars per year. Write the individual’s budget constraint. Find his optimal consumption bundle.

Now, suppose that the government imposes an income tax with a rate of $t = 1/4$, but exempts income spent on health care from taxation.

1. How does that affect the individual’s budget constraint? How does that affect his optimal consumption bundle?
2. How much is the IRS revenue from this tax (for that individual)?
3. Think about the alternative policy to reach the same revenue: a lump-sum tax on income. How would that look like? What would be the individual’s problem and optimal consumption bundle under this alternative policy?
4. From a welfare point of view, which policy is better?
5. The first tax scheme considered here is said to be distortionary, while the second is non-distortionary. Justify these names.