Relativistic Fluid Dynamics

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Why Do We Need It?

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  - Rotating stars, black holes (general relativity)
  - And more!
Why Fluid Dynamics?

When Is the Fluid Dynamics Description Valid?

Definition

- **Fluid element**: basic element in the fluid with typical length $L$
- **Streamline**: trajectory of a particular fluid element

Two conditions:

(i) enough particles in a single fluid element:

$$nL^3 \gg 1$$
Why Fluid Dynamics?

When Is the Fluid Dynamics Description Valid?

Definition

- **Fluid element**: basic element in the fluid with typical length $L$
- **Streamline**: trajectory of a particular fluid element

Two conditions:

(i) enough particles in a single fluid element:

\[ nL^3 \gg 1 \]

(ii) moderate size of fluid element:

\[ l_{mfp} \ll L \ll l_{system} \]
Two Approaches of Fluid Dynamics

People use two different viewpoint to study fluid dynamics:

- **Eulerian approach**: the observer stands at a fixed position,

  \[ a = a(\vec{r}, t), \quad \vec{v} = \vec{v}(\vec{r}, t) \]
Two Approaches of Fluid Dynamics

People use two different viewpoint to study fluid dynamics:

- **Eulerian approach**: the observer stands at a fixed position,

  \[ a = a(\vec{r}, t), \quad \vec{v} = \vec{v}(\vec{r}, t) \]

- **Lagrangian approach**: the observer moves with a certain fluid element,

  \[ a = a(\vec{r}(t), t), \quad \vec{v} = \vec{v}(\vec{r}(t), t) \]
A perfect/ideal fluid has the following properties:

- reversible \( (dQ = TdS) \)
- adiabatic \( (dS = 0) \)
- isotropic
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Result
(a) isentropic flow
(b) local thermal equilibrium (LTE)
(c) no dissipative process: viscosity, heat conductance, radiation, etc
Constructing $T^{\mu \nu}$ of Perfect Fluid

In the local rest frame, the energy-stress tensor is

$$T^{\mu \nu}_{(0)} = \begin{pmatrix}
\varepsilon & 0 & 0 & 0 \\
0 & -\sigma_{11} & -\sigma_{12} & -\sigma_{13} \\
0 & -\sigma_{21} & -\sigma_{22} & -\sigma_{23} \\
0 & -\sigma_{31} & -\sigma_{32} & -\sigma_{33}
\end{pmatrix}$$

stress tensor
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\end{pmatrix}$$

stress tensor

Let’s perform a Lorentz transformation to the lab frame:

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix}
\gamma & \gamma \vec{\beta} \\
\gamma \vec{\beta} & 1 \left(1 + (\gamma - 1) \frac{\vec{\beta} \cdot \vec{\beta}}{\beta^2}\right)
\end{pmatrix},$$
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Let's perform a Lorentz transformation to the lab frame:

$$\therefore \Lambda^{\mu \nu} = \begin{pmatrix} \gamma & \gamma \vec{\beta} \\ \gamma \vec{\beta} & 1 + (\gamma - 1) \frac{\vec{\beta} \cdot \vec{\beta}}{\beta^2} \end{pmatrix},$$

$$T^{\mu \nu} = \Lambda^{\mu \alpha} \Lambda^{\nu \beta} T^{\alpha \beta}(0) = (\varepsilon + P) u^\mu u^\nu - P g^{\mu \nu}$$
Conservation Laws in the Local Rest Frame

\[ \therefore \partial_{\mu} T_{(0)}^{\mu \nu} = 0 \]

\[ \Rightarrow \frac{\partial \varepsilon}{\partial t} = \frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = \frac{\partial P}{\partial z} = 0, \quad \therefore \varepsilon = \varepsilon(\vec{r}), \quad P = P(t) \]
Conservation Laws

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Besides, we also have particle number conservation:

\[ \therefore j^{\mu} = n_0 u^{\mu}, \ \partial_{\mu} j^{\mu} = 0 \quad \Rightarrow \ n_0 = n_0(\vec{r}), \]

where \( n_0 \) is the proper particle density.
Conservation Laws from $\partial_\mu T^{\mu\nu} = 0$

Energy Conservation:

$$\frac{\partial}{\partial t} \left( \gamma^2 (\varepsilon + P) - P \right) + \nabla \cdot \left( \gamma^2 (\varepsilon + P) \vec{v} \right) = 0$$
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Energy Conservation:

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Momentum Conservation:

$$\frac{\partial}{\partial t} \left( \gamma^2 (\varepsilon + P) \vec{v} \right) + \nabla \cdot \left( \gamma^2 (\varepsilon + P) \vec{v} \vec{v} + P \mathbf{1} \right) = 0$$
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Energy Conservation:

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Particle number conservation:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \vec{v}) = 0,$$

where $n = \gamma n_0$ is the particle density seen in the lab.
Conservation Laws in the Non-Relativistic limit

\[
\therefore c \to \infty, \quad \beta \to 0, \quad \gamma \to 1 + \frac{v^2}{2c^2}, \quad \varepsilon \to n_0(mc^2 + U)
\]
Conservation Laws in the Non-Relativistic limit

\[ \therefore c \to \infty, \quad \beta \to 0, \quad \gamma \to 1 + \frac{v^2}{2c^2}, \quad \epsilon \to n_0(mc^2 + U) \]

\[ \Rightarrow \begin{cases} \\ 
\frac{\partial n}{\partial t} + \nabla \cdot (n \vec{v}) = 0 \\
\frac{\partial}{\partial t} \left( \frac{1}{2} nmv^2 + nU \right) + \nabla \cdot \left( \left( \frac{1}{2} nmv^2 + nU + P \right) \vec{v} \right) = 0 \\
\frac{\partial}{\partial t} \left( nm\vec{v} \right) + \nabla \cdot \left( nm\vec{v} \vec{v} \right) = -\nabla P \\
\end{cases} \]

Reduced to classical fluid dynamics!
Hello Navier-Stokes Equation

We can further simplify the equation of the momentum conservation:

\[ \frac{\partial}{\partial t} \left( nm\vec{v} \right) + \nabla \cdot \left( nm\vec{v} \vec{v} \right) = -\nabla P, \]

\[ \frac{\partial n}{\partial t} + \nabla \cdot \left( n\vec{v} \right) = 0 \]

\[ \Rightarrow \frac{\partial n}{\partial t} \vec{v} + n \frac{\partial \vec{v}}{\partial t} + \vec{v} \nabla \cdot \left( n\vec{v} \right) + n \left( \vec{v} \cdot \nabla \right) \vec{v} = -\frac{\nabla P}{m} \]
We can further simplify the equation of the momentum conservation:

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\frac{\partial n}{\partial t} + \nabla \cdot (n \vec{v}) = 0
\]

\[
\Rightarrow \frac{\partial n}{\partial t} \vec{v} + n \frac{\partial \vec{v}}{\partial t} + \vec{v} \nabla \cdot (n \vec{v}) + n (\vec{v} \cdot \nabla) \vec{v} = -\frac{\nabla P}{m}
\]

\[
\Rightarrow \begin{cases} 
\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{\nabla P}{nm} & \text{(Eulerian approach)} \\
\frac{d\vec{v}}{dt} = -\frac{\nabla P}{nm} & \text{(Lagrangian approach)}
\end{cases}
\]

which is the Navior-Stokes equation (without external force and viscosity), also known as Euler equation!
Let Equation of State Join the Game!

It is quite natural to introduce *equation of state (EoS)* to connect thermodynamic quantities.

\[ PV^\gamma = f(S_2) \]

\[ PV^\gamma = f(S_1) \]

\((S_2 > S_1)\)

Isentropic flow

• For isentropic flows, \( P = a(S) \).

• Assume \( S = \) constant throughout the fluid, then \( P \approx an^\gamma \).

Let’s verify it:

\( \because P \uparrow \Rightarrow V \downarrow \Rightarrow n \uparrow \)
It is quite natural to introduce equation of state (EoS) to connect thermodynamic quantities.

\[ PV^\gamma = f(S_2) \]
\[ PV^\gamma = f(S_1) \]

\((S_2 > S_1)\)

Isentropic flow

- For isentropic flows, \( P = a(S)n^\gamma \) along a streamline.
- Assume \( S = \) constant throughout the fluid, then \( P \approx a n^\gamma \).

Let’s verify it: \( P \uparrow \Rightarrow V \downarrow \Rightarrow n \uparrow \)
Five Variables But Six Equations? Be Careful!

Revisit the energy-stress tensor:

\[ T^{\mu\nu} = (\varepsilon + P)u^{\mu}u^{\nu} - Pg^{\mu\nu} \Rightarrow 5 \text{ variables!} \]
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Particle/Energy/Momentum conservations plus EoS

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In fact, it can be shown that the energy conservation is automatically satisfied and hence redundant.

\[ \therefore \text{EoS is necessary to completely solve the fluid problem.} \]
Entropy is “Frozen-in”!

\[ dS = 0 \implies \frac{dS}{dt} = 0 \implies \left( \frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) S = 0 \]
Entropy is “Frozen-in”!

\[ dS = 0 \quad \Rightarrow \quad \frac{dS}{dt} = 0 \quad \Rightarrow \quad \left( \frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) S = 0 \]

We say that \( S \), as a scalar field, is frozen-in. But \( S \) is not alone; there are more frozen-in objects.
Entropy is “Frozen-in”!

\[ dS = 0 \Rightarrow \frac{dS}{dt} = 0 \Rightarrow \left( \frac{\partial}{\partial t} + \bar{v} \cdot \nabla \right) S = 0 \]

We say that \( S \), as a scalar field, is frozen-in. But \( S \) is not alone; there are more frozen-in objects.

Note

- In the covariant form, the frozen-in condition of entropy \( S \) per particle is written as
  \[ u^\nu \partial_\nu S = 0 \]

- It is trivial if \( S = \text{constant} \).

- One can rewrite it into the conservation form:
  \[ \Rightarrow \frac{\partial}{\partial t} (nS) + \nabla \cdot (nS \bar{v}) = 0 \quad (\text{Entropy conservation}) \]
“Vorticity” is Also Frozen-in!

\[
\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{\gamma}{\gamma - 1} \nabla \left( \frac{P}{nm} \right) \quad \text{(Euler equation)}
\]

\[
(\vec{v} \cdot \nabla) \vec{v} = \nabla \left( \frac{v^2}{2} \right) - \vec{v} \times (\nabla \times \vec{v})
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\[
\begin{align*}
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\end{align*}
\]

(Euler equation)

\[
\nabla \times \Rightarrow \quad \frac{\partial}{\partial t} \left( \nabla \times \vec{v} \right) - \nabla \times \left( \vec{v} \times \left( \nabla \times \vec{v} \right) \right) = 0
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\[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{\gamma}{\gamma - 1} \nabla \left( \frac{P}{nm} \right) \]  
(Euler equation)

\[ (\vec{v} \cdot \nabla) \vec{v} = \nabla \left( \frac{v^2}{2} \right) - \vec{v} \times (\nabla \times \vec{v}) \]

According to Helmholtz theorem,

\[ \vec{v} = -\nabla \phi + \nabla \times \vec{\psi}. \]

\[ \Rightarrow \text{define } \vec{w} \equiv \nabla \times \vec{v} \text{ (vorticity) such that } \frac{\partial \vec{w}}{\partial t} - \nabla \times (\vec{v} \times \vec{w}) = 0 \]

\[ \Rightarrow \frac{\partial \vec{w}}{\partial t} + (\vec{v} \cdot \nabla) \vec{w} - (\vec{w} \cdot \nabla) \vec{v} + \vec{w} \left( \nabla \cdot \vec{v} \right) = 0 \]
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\[ \Rightarrow \frac{\partial \vec{w}}{\partial t} + (\vec{v} \cdot \nabla) \vec{w} - (\vec{w} \cdot \nabla) \vec{v} + \vec{w} \left( \nabla \cdot \vec{v} \right) = 0 \]

For incompressible flows, \( \nabla \cdot \vec{v} = 0 \), so \( \vec{w} \) is frozen-in!
Moreover, not only is $\vec{w}$ frozen-in, but also its flux is:

$$\frac{d}{dt} \int \vec{w} \cdot d\vec{a} = \frac{d}{dt} \oint \vec{v} \cdot d\vec{l} = 0,$$

which is related to angular momentum conservation.
Another Helmholtz Theorem

Moreover, not only is \( \vec{w} \) frozen-in, but also its flux is:

\[
\frac{d}{dt} \int \vec{w} \cdot d\vec{a} = \frac{d}{dt} \oint \vec{v} \cdot d\vec{l} = 0,
\]

which is related to angular momentum conservation. It’s also called Helmholtz theorem (or Alfven’s theorem when describing \( \vec{B} \) in the perfect conduction fluid).

Note

In general,

\[
\frac{\partial}{\partial t} \left( \frac{\vec{w}}{n} \right) + \vec{v} \cdot \nabla \left( \frac{\vec{w}}{n} \right) - \left( \frac{\vec{w}}{n} \cdot \nabla \right) \vec{v} = 0
\]
Classical Mechanics is Nothing but a Potential Flow!

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V = 0$$  \hspace{1cm} \text{(Hamilton-Jacobi equation)}$$

Define $\vec{v} \equiv \nabla S/m$ and note that $\nabla \left( \frac{(\nabla S)^2}{2} \right) = (\nabla S \cdot \nabla) \nabla S$
Classical Mechanics is Nothing but a Potential Flow!

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Define \( \vec{v} \equiv \nabla S / m \) and note that \( \nabla \left( \frac{(\nabla S)^2}{2} \right) = (\nabla S \cdot \nabla) \nabla S \)

\[ \Rightarrow \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{\nabla V}{m} \]

Therefore, classical mechanics corresponds to a "potential flow".

Comparison

\[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{\nabla P}{nm} \]  \hspace{1cm} \text{(Euler equation)}
• Relativistic fluid dynamics can be constructed through $T^\mu{}^\nu$
Conclusion

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- Navier-Stokes equation can be derived via the $c \to \infty$ limit
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- Relativistic fluid dynamics can be constructed through $T^\mu_\nu$
- Navier-Stokes equation can be derived via the $c \to \infty$ limit
- Equation of state as well as thermodynamics are required
- Perfect fluid is too perfect to capture all real physics
- It can be generalized to describe fluid in curved spacetime
Reference

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Thanks for your attention!