The Probabilistic Method Ch. 2 & 3

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Linearlity if Expectation: Basics

THM. Let $X_1, \cdots, X_n$ be random variables, $X = c_1X_1 + \cdots + c_nX_n$. Linearity of expectation states that $E(X) = c_1E(X_1) + \cdots + c_nE(X_n)$

Remark: The power of this principle comes from there being no restrictions on the dependence or independence of X_i. In many instances, $E(X)$ can easily be calculated by a judicious decomposition into simple (often indicator) random variables X_i.

Linearity of Expectation: A Simple Example

Let σ be a random permutation on $\{1, \cdots, n\}$, uniformly chosen. Let $X(\sigma)$ be the number of fixed points of σ. To find $E(X)$, we decompose $X = X_1 + \cdots + X_n$, where X_i is the indicator random variable of the event $\sigma(i) = i$. Then

$$E(X_i) = Pr(\sigma(i) = i) = \frac{1}{n}$$

, so that

$$E(X) = \frac{1}{n} + \cdots + \frac{1}{n} = \frac{1}{n}n = 1$$

Splitting Graphs

THM. Let $G(V, E)$ be a graph with n vertices and e edges. Then G contains a bipartite subgraph with at least $\frac{e}{2}$ edges.

Splitting Graphs

Proof.

Let $T \subseteq V$ be a random subset given by $Pr(X \in T) = \frac{1}{2}$, these choices being mutually independent. Set $B = V - T$, call an edge $\{x, y\}$ crossing if exactly one of x, y is in T. Let X be the number of crossing edges. We decompose

$$X = \sum_{\{x,y\} \in E} X_{xy}$$

, where X_{xy} is the indicator random variable for $\{x, y\}$ being crossing. Then $E(X_{xy}) = \frac{1}{2}$, as two fair coin flips have probability $\frac{1}{2}$ of being different. Then

$$E(X) = \sum_{\{x,y\} \in E} E(X_{xy}) = \frac{e}{2}$$

Thus $X \geq \frac{e}{2}$ for some choice of T, and the set of those crossing edges forms a bipartite graph.

□

Splitting Graphs

THM. If G has $2n$ vertices and e edges, then it contains a bipartite subgraph with at least $\frac{en}{2n-1}$ edges. If G has $2n+1$ vertices and e edges, then it contains a bipartite subgraph with at least $\frac{e(n+1)}{2n+1}$ edges.

Proof.

When G has $2n$ vertices, let T be chosen uniformly from among all $n-$ element subsets of V, Any edge $\{x, y\}$ now has probability $\frac{n}{2n-1}$ of being crossing, and the proof concludes as before. When G has $2n+1$ vertices, choose T uniformly from among all $n-$ element subsets of V, and the proof is similar.

□

Example: Balancing Vectors

THM. Let $v_1, \cdots, v_n \in \mathbb{R}^n$, all $|v_i| = 1$. Then there exist $\epsilon_1, \cdots, \epsilon_n = \pm 1$ so that

$$|\epsilon_1 v_1 + \cdots + \epsilon_n v_n| \leq \sqrt{n}$$

and there also exist $\epsilon_1, \cdots, \epsilon_n = \pm 1$ so that

$$|\epsilon_1 v_1 + \cdots + \epsilon_n v_n| \geq \sqrt{n}$$

.

Example: Balancing Vectors

Proof.

Let $\epsilon_1, \cdots, \epsilon_n$ be selected uniformly and independently from $\{-1. +1\}$, set

$$X = |\epsilon_1 v_1 + \cdots + \epsilon_n v_n|^2$$

. Then

$$X = \sum_{i=1}^{n} \sum_{j=1}^{n} \epsilon_i \epsilon_j v_i v_j$$

Thus

$$E(X) = \sum_{i=1}^{n} \sum_{j=1}^{n} v_i v_j E(\epsilon_i \epsilon_j)$$

When $i \neq j$, $E(\epsilon_i \epsilon_j) = 0$. When $i = j$, ${\epsilon_i}^2 = 1$ so $E({\epsilon_i}^2) = 1$. Thus $E(X) = \sum_{i=1}^{n} v_i \cdot v_i = n$. Hence there exists specific $\epsilon_1, \cdots, \epsilon_n = \pm 1$ with $X \geq n$ and $X \leq n$. Taking square root gives the theorem. □

Example: Balancing Vectors

The next result includes part of the above theorem as a linear translation of the $p_1 = \cdots = p_n = \frac{1}{2}$ case.

THM. Let $v_1, \cdots, v_n$ in $\mathbb{R}^n$, all $|v_i| \leq 1$. Let $p_1, \cdots, p_n \in [0,1]$ be arbitrary, and set $w = p_1 v_1 + \cdots + p_n v_n$. Then there exist $\epsilon_1, \cdots, \epsilon_n \in \{0,1\}$ so that, setting $v = \epsilon_1 v_1 + \cdots + \epsilon_n v_n$,

$$|w - v| \leq \frac{\sqrt{n}}{2}$$

Example: Unbalancing Lights

THM. Let $a_{ij} = \pm 1$ for $1 \leq i, j \leq n$. Then there exists $x_i, y_j = \pm 1, 1 \leq i, j \leq n$, so that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i y_j \geq (\sqrt{\frac{2}{\pi}} + o(1)) n^{\frac{3}{2}}$$

.

Remark: This result has an amusing interpretation. Let an $n \times n$ array of lights be given, each either on ($a_{ij} = +1$) or off ($a_{ij} = -1$). Suppose for each row and each column there is a switch so that if the switch is pulled ($x_i = -1$ for row i and $y_j = -1$ for column j) all of the lights in that line will be "switched" on to off or off to on. Then for any initial configuration it is possible to perform switchings so that the number of lights on minus the number of lights off is at least $(\sqrt{\frac{2}{\pi}} + o(1)) n^{\frac{3}{2}}$.