# Reading Seminar on "The Theory of Partitions" by George E. Andrews

# The Asymptotics of Infinite Product Generating Functions

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October 29, 2021

Recall that the generating function for integer partitions $p(n)$ is

$$F(q) = \sum_{n=0}^{+\infty} p(n) q^n = \prod_{n=1}^{+\infty} (1 - q^n)^{-1} .$$

In 1954 G. Meinardus considered the asymptotics of generating functions of the form

$$f(\tau) = \prod_{n=1}^{+\infty} (1 - q^n)^{-a_n} = 1 + \sum_{n=1}^{+\infty} r(n) q^n,$$

where $q = e^{-\tau}$ and $Re\, \tau > 0$, and the $a_n$ are nonnegative real numbers. He proved a general result, known as Meinardus' first theorem, which includes asymtpotic formulas for many partition functions.

# Definitions and assumptions

Consider the generating function

$$f(\tau) = \prod_{n=1}^{+\infty} (1 - q^n)^{-a_n} = 1 + \sum_{n=1}^{+\infty} r(n) q^n,$$

where $q = e^{-\tau}$ and $Re\,(\tau) > 0$, and the $a_n$ are nonnegative real numbers.

Define the auxiliary Dirichlet series:

$$D(s) = \sum_{n=1}^{+\infty} \frac{a_n}{n^s} \qquad (s = \sigma + it),$$

which it is assumed

(i) to be convergent for $\sigma > \alpha$, for some positive real number $\alpha$;

(ii) to posses an analytic continuation in the region $\sigma \geq -C_0$, where $0 < C_0 < 1$. In this region $D(s)$ is analytic except for a simple pole at $s = \alpha$, with residue $A$;

(iii) $D(s) = O(|t|^{C_1})$ uniformly in $\sigma \geq -C_0$ as $|t| \to +\infty$, where $C_1$ is a fixed positive real number.

# Meinardus' First Theorem

**Theorem 6.2** As $n \to +\infty$,

$$r(n) = C\, n^{\kappa}\, exp\Big[n^{\frac{\alpha}{1+\alpha}}\left(1+\frac{1}{\alpha}\right)(A\Gamma(\alpha+1)\,\zeta(\alpha+1))^{\frac{\alpha}{1+\alpha}}\Big](1+O(n^{-\kappa_1}))$$

where

$$C = e^{D'(0)}[2\pi(1+\alpha)]^{-1/2}[A\Gamma(\alpha+1)]\zeta(\alpha+1)]^{(1-2D(0))/(2+2\alpha)},$$

$$\kappa = \frac{D(0)-1-1/(2\alpha)}{1+\alpha} \text{ and } \kappa_1 = \frac{\alpha}{\alpha+1}\min\left(\frac{C_0}{\alpha}-\frac{\delta}{4}, \frac{1}{2}-\delta\right),$$

$\delta$ is an arbitrary real number.

# Additional assumptions

The proof will use the function

$$g(\tau) = \sum_{n=1}^{+\infty} a_n q^n, \qquad q = e^{-\tau}.$$

If $\tau = y + 2\pi i x$, we shall assume that for $|\arg \tau| > \pi/4$ and $|x| \leq 1/2$,

$$R(g(\tau)) - g(y) \leq -C_2\, y^{-\epsilon}$$

for sufficiently small $y$, where $\epsilon$ is an arbitrary but fixed positive number, and $C_2$ is a positive real number depending on $\epsilon$.

# Proof of Theorem 6.2

By the Cauchy integral formula

$$\begin{aligned} r(n) &= \int_{\tau_0}^{\tau_0+2\pi i} f(\tau)e^{n\tau}\, d\tau \\ &= \int_{-1/2}^{1/2} f(y+2\pi i x)e^{ny+2\pi i n x}\, dx \end{aligned}$$

The proof is based on applying the saddle point method. In order to apply this method one needs information of the behavior of $f(\tau)$ in the half plane $Re(\tau) > 0$, and near $\tau = 0$.

**Lemma 1.** Under our assumptions on $f(\tau)$, $D(s)$, and $g(\tau)$, with $\tau = y + 2\pi i x$,

$$f(\tau) = \exp\left[A\Gamma(\alpha)\zeta(\alpha+1)\tau^{-\alpha} - D(0)\log\tau + D'(0) + O(y^{C_0})\right]$$

uniformly in $x$ as $y \to 0$, provided $|\arg\tau| \leq \pi/4$, $|x| \leq 1/2$.

**Proof of Lemma 1.** Let $q = e^{-\tau}$. Then

$$f(\tau) = \prod_{n=1}^{+\infty} \left(1 - e^{-n\tau}\right)^{-a_n}$$

$$\log f(\tau) = -\sum_{n=1}^{\infty} a_n \log(1 - e^{-n\tau})$$

$$= \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} a_n e^{-nk\tau} \qquad (1)$$

Recall that

$$e^{-\tau} = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \tau^{-s}\, \Gamma(s)\, ds \qquad (Re(\tau) > 0,\ \sigma_0 > 0) \qquad (2)$$

Using (2) in (1) we get

$$\log f(\tau) = \frac{1}{2\pi i} \int_{1+\alpha-i\infty}^{1+\alpha+i\infty} \tau^{-s}\Gamma(s)\zeta(s+1)D(s)ds. \quad (3)$$

Now we make the shift of the line of integration from $Re(s) = 1 + \alpha$ to $Re(s) = -C_0$. The integrand has two poles:

(i) a pole of order one at $\alpha$, with residue $\tau^{-\alpha}\Gamma(\alpha)\zeta(\alpha+1)A$;

(ii) a pole of order two at $s = 0$.

The residue at $s = 0$ can be obtained by first expanding the integrand near $s = 0$:

$$\tau^{-s}\Gamma(s)\zeta(s+1)D(s)$$

$$= (1 - s\,\log\tau + \cdots)\,(\frac{1}{s} - \gamma + \cdots)(\frac{1}{s} + \gamma + \cdots)(D(0) + D'(0)s + \cdots)$$

$$= \frac{1}{s^2} + (D'(0) + D(0)\log\tau)\frac{1}{s} + \cdots$$

and hence the residue at $s = 0$ is $D'(0) - D(0)\log\tau$.

From this shift we obtain

$$\begin{aligned}\log f(\tau) &= A\tau^{-\alpha}\Gamma(\alpha)\zeta(\alpha+1) - D(0)\log\tau + D'(0) + \\ &+ \frac{1}{2\pi i}\int_{-C_0-i\infty}^{-C_0+i\infty} \tau^{-s}\Gamma(s)\zeta(s+1)D(s)ds. \end{aligned} \tag{4}$$

The shift of the line of integration is permissible since for $arg(\tau)| \leq \pi/4$,

$$|\tau^{-s}| = |\tau|^{-\sigma} exp(t \arg(\tau)) \leq |\tau|^{-\sigma} exp(\pi|t|/4)$$

and by assumption, for $\sigma \geq C_0$, $D(s) = O(|t|^{C_1})$, while classical results give

$$\zeta(s+1) = O(|t|^{C_4}) \qquad \text{and} \qquad \Gamma(s) = O(\exp(-\frac{\pi}{2}|t|^{C_5})$$

as $t \to +\infty$.

We next show that as $\tau \to 0$ the integral in (4) tends to 0.

$$\Big|\frac{1}{2\pi i}\int_{-C_0-i\infty}^{-C_0+i\infty}\tau^{-s}\Gamma(s)\zeta(s+1)D(s)ds\Big|$$

$$= O\Big(|\tau|^{C_0}\int_{-\infty}^{+\infty}e^{-\pi|t|/4}|t|^{C_1+C_4+C_5}dt\Big)$$

$$= O(|\tau|^{C_0}) = O(y^{C_o}).$$

This and (4) now prove Lemma 1.

Let's go back to the integral representation of $r(n)$,

$$
\begin{aligned}
r(n) &= \int_{\tau_0}^{\tau_0+2\pi i} f(\tau) e^{n\tau}\, d\tau \\
&= \int_{-1/2}^{1/2} f(y+2\pi i x) e^{ny+2\pi i n x}\, dx
\end{aligned}
$$

The maximum value of the integrand occurs at $x = 0$, and for such $x$ Lemma 6.1 implies $f$ is well approximated by

$$
\exp\Big[A\Gamma(\alpha)\zeta(\alpha+1)y^{-\alpha} + ny\Big].
$$

The saddle point method suggests we should choose $y$ that minimizes the expression above.

That is, choose $y$ such that

$$\frac{d}{dy}\left(\exp\left[A\Gamma(\alpha)\zeta(\alpha+1)y^{-\alpha}+ny\right]\right)=0$$

This gives

$$y=n^{-1/(\alpha+1)}\left[A\Gamma(\alpha+1)\zeta(\alpha+1)\right]^{1/(\alpha+1)}.$$

For convenience, we define

$$m=ny=n^{\alpha/(\alpha+1)}\left[A\Gamma(\alpha+1)\zeta(\alpha+1)\right]^{1/(\alpha+1)}.$$

With this value of $y$ we split the integral into three parts:

$$\begin{aligned} r(n) &= e^m\int_{-1/2}^{-y^\beta} f(y+2\pi ix)e^{2\pi inx}\,dx+ \\ &\quad +e^m\int_{-y^\beta}^{y^\beta} f(y+2\pi ix)e^{2\pi inx}\,dx+ \\ &\quad +e^m\int_{y^\beta}^{1/2} f(y+2\pi ix)e^{2\pi inx}\,dx, \end{aligned}$$

where

$$\beta = 1 + \frac{\alpha}{2}\left(1 - \frac{\delta}{2}\right) \qquad \text{with} \;\; 0 < \delta < \frac{2}{3}.$$

W are going to show that under our assumption the contribution of the sum of the first and last integrals are small. Define

$$R_1 = \int_{-1/2}^{-y^\beta} f(y + 2\pi i x) e^{2\pi i n x}\, dx + \int_{y^\beta}^{1/2} f(y + 2\pi i x) e^{2\pi i n x}\, dx.$$

**Lemma 2.** There exists a positive $\epsilon_1$ such that

$$f(y + 2\pi ix) = O\Big(\exp[A\Gamma(\alpha)\zeta(\alpha+1)y^{-\alpha} - C_3 y^{-\epsilon_1}]\Big)$$

uniformly on $x$ with $y^\beta \leq |x| \leq 1/2$, as $y \to 0$, where

$$\beta = 1 + \frac{\alpha}{2}\left(1 - \frac{\delta}{2}\right) \qquad \text{with} \;\; 0 < \delta < \frac{2}{3}$$

and $C_3$ is a fixed real number.

**Sketch of Proof of Lemma 2.** The proof consists of considering two cases

(i) $y^\beta \le |x| \le \dfrac{y}{2\pi}$

(ii) $\dfrac{y}{2\pi} \le |x| \le \dfrac{1}{2}$.

Case (i) is proven by proceeding as in the beginning of the proof of Lemma 1, and then by showing that under the assumptions of Lemma 2 in the region considered in case (i)

$$\log|f(y+2\pi i x)| \le A\Gamma(\alpha)\zeta(\alpha+1)y^{-\alpha} - C_3 y^{-\epsilon_1}.$$

Case (ii) relies on te condition $Re(g(\tau)) - g(y) \leq -C_2 y^{-\epsilon}$. Since

$$\log f(\tau) = -\sum_{n=1}^{\infty} a_n \log(1 - e^{-n\tau}) = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} a_n e^{-nk\tau},$$

one has

$$\log |f(y + 2\pi i x)| - Re(g(\tau)) = \sum_{k=2}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} a_n e^{-nky} \cos(2\pi k n x).$$

Since all the $a_n$'s are nonnegative,

$$\log |f(y + 2\pi i x)| - Re(g(\tau)) \leq \sum_{k=2}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} a_n e^{-nky} = \log(f(y)) - g(y).$$

Hence in case (ii),

$$\begin{aligned}\log|f(y+2\pi ix)| &\leq \log f(y)+Re(g(\tau))-g(y)\\ &\leq A\Gamma(\alpha)\zeta(\alpha+1)y^{-\alpha}-C_8y^{-\epsilon}\\ &\leq A\Gamma(\alpha)\zeta(\alpha+1)y^{-\alpha}-C_3y^{-\epsilon_1},\end{aligned}$$

thus proving the lemma.

We are now go back to considering $R_1$,

$$R_1 = \int_{-1/2}^{-y^\beta} f(y+2\pi i x) e^{2\pi i n x}\, dx + \int_{y^\beta}^{1/2} f(y+2\pi i x) e^{2\pi i n x}\, dx.$$

By Lemma 2,

$$R_1 = O\Big( \exp\Big[ (\frac{m}{n})^{-\alpha} A\Gamma(\alpha)\zeta(\alpha+1) - C_3 (\frac{m}{n})^{-\epsilon_1} \Big] \Big)$$

as $n \to +\infty$ (i.e., $y = m/n \to 0$).
Hence

$$\exp(m) R_1 = O\Big( \exp\Big[ (1+\frac{1}{\alpha}) m - C_9 m^{\epsilon_2} \Big] \Big)$$

as $n \to +\infty$.

Recall

$$r(n) = e^m \int_{-y^\beta}^{y^\beta} f(y + 2\pi i x) e^{2\pi i n x}\, dx + e^m R_1,$$

and we just found bounds for $e^m R_1$. Now applying Lemma 1 and making the change of variable $2\pi x = (m/n)\omega$ we obtain

$$r(n) = \exp\Big[\Big(1 + \frac{1}{\alpha}\Big) m - (D(0) - 1)\log\frac{m}{n} + D'(0) - \log 2\pi\Big] I + e^m R_1,$$

where

$$I = \int_{-C_{10} m^{(1-\beta)/\alpha}}^{C_{10} m^{(1-\beta)/\alpha}} \exp(\phi(\omega)) d\omega,$$

where

$$\phi(\omega) = m\Big[\frac{1}{\alpha(1 + i\omega)^\alpha} - \frac{1}{\alpha} + i\omega\Big] - D(0)\log(1 + i\omega) + O(m^{-C_0/\alpha})$$

as $m \to +\infty$.

The final step of the proof consists in showing that

$$I = \left[\frac{2\pi}{m(\alpha+1)}\right]^{1/2} (1 + O(m^{-\mu_3}))$$

where

$$\mu_3 = \min\left(\frac{C_0}{\alpha} - \frac{\delta}{4}, \frac{1}{2} - \delta\right).$$

Putting all together one now has that

$$r(n) = \exp\left[(1+\frac{1}{\alpha})m - (D(0)-1)\log\frac{m}{n} + D'(0)\right](2\pi m(\alpha+1))^{-1/2} \cdot (1 + O(m^{-\mu_3}))$$

as $m \to +\infty$. The proof is now completed by replacing $m$ by a function of $n$.

# Application of Theorem 2.

**Theorem 6.3**

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\left(\frac{2}{3}\right)^{1/2} n^{1/2}\right).$$

**Theorem 6.4**. Let $H_{k,a}$ denote all positive integers congruent to $a$ modulo $k$. Then for $1 \leq a \leq k$,

$$p(\text{"}H_{k,a}\text{"}, n) \sim C n^{\kappa} \exp\left(\pi\left(\frac{2n}{3k}\right)^{1/2}\right)$$

where

$$C = \Gamma(\frac{a}{k})\pi^{-1+a/k}\, 2^{-3/2-a/(2k)}\, 3^{-a/(2k)}\, k^{-1/2+a/(2k)}$$

and

$$\kappa = -\frac{1}{2}\left(1 + \frac{a}{k}\right).$$