Reading Seminar on "The Theory of Partitions" by George E. Andrews

The Hardy-Ramanujan-Rademacher Expansion of $p(n)$

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Oct. 22nd, 2021

Recall

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Theorem (G.H.Hardy and S.Ramanujan)

$p(n)$ is the integer nearest

$$\frac{1}{2\sqrt{2}} \sum_{q=1}^{\nu} \sqrt{q} A_q(n) \psi_q(n).$$

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- Five terms of the series (1) predict the correct value of p(200).

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$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k} \left[\frac{d}{dx} \frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(x - \frac{1}{24}\right)}\right)}{\sqrt{x - \frac{1}{24}}} \right]_{x=n},$$

where

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$$\omega_{h,k} = \begin{cases} \left(\frac{-k}{h}\right) \exp\left\{-\pi i \left(\frac{2-hk-h}{4} + \frac{(k-k^{-1})(2h-h'+h^2h')}{12}\right)\right\}, & h \text{ odd} \\ \left(\frac{-h}{k}\right) \exp\left\{-\pi i \left(\frac{k-1}{4} + \frac{(k-k^{-1})(2h-h'+h^2h')}{12}\right)\right\}, & k \text{ odd} \end{cases}$$

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$$hh'\equiv -1 \pmod{k}.$$

Theorem (Cont'd)

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The Legendre symbol $\left(\frac{a}{p}\right)$ is defined for an odd prime p and $a \in \mathbb{Z}$, as

$$\left(\frac{a}{p}\right) = \begin{cases} 1, & \text{if } a \text{ is a quadratic residue modulo } p; \\ -1, & \text{if } a \text{ is not a quadratic residue modulo } p. \end{cases}$$

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Let $(P, Q) = 1$, $Q = q_1 q_2 \cdots q_t$, where the q_i's are odd primes , not necessarily distinct . Then the Jacobi symbo is defined by

$$\left(\frac{P}{Q}\right) = \prod_{j=1}^{t} \left(\frac{P}{q_j}\right).$$

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Let P and Q be odd, with $(P, Q) = 1$. Then,

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Also,

$$\frac{2-hk-h}{4} - \frac{k-1}{4} = \frac{2-hk-k-h+1}{4} = \frac{2-k(h+1)}{4}$$

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$$\frac{h + h'}{k + k'},$$

which has the smallest denominator.

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\theta'_{0,1} &= \frac{1}{N+1}; \\
\theta'_{h,k} &= \frac{h}{k} - \frac{h_0+h}{k_0+h}, \qquad h > 0; \\
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$$\theta''_{h,k} = \frac{h_1+h}{k_1+h} - \frac{h}{k}$$

$$\begin{aligned}
p(n) &= \frac{1}{2\pi i}\oint_C \frac{P(z)}{z^{n+1}}dz \\
\left[z=\rho e^{2\pi i t}\right] &= \frac{1}{2\pi i}\int_0^1 \frac{P(\rho e^{2\pi i t})}{(\rho e^{2\pi i t})^{n+1}} d(\rho e^{2\pi i t}) = \frac{1}{\rho^n}\int_0^1 \frac{P(\rho e^{2\pi i t})}{e^{2\pi i n t}}dt \\
&= \frac{1}{\rho^n} \sum_{\substack{k=1 \\ (h,k)=1 \\ 0\le h<k}}^{N} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} \frac{P\left(\rho \exp\left\{\frac{2\pi i h}{k} + 2\pi i \phi\right\}\right)}{\exp\left\{\frac{2\pi n i h}{k} + 2\pi n i \phi\right\}} d\phi
\end{aligned}$$

- $\rho = e^{-\frac{2\pi}{N^2}}$

- $\rho = e^{-\frac{2\pi}{N^2}}$ $z = k(N^{-2} - i\phi)$

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-

$$P(q) = \frac{q^{\frac{1}{24}}}{\eta(\tau)}$$

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-

$$P(q) = \frac{q^{\frac{1}{24}}}{\eta(\tau)} \Rightarrow P(e^{2\pi i \frac{h+iz}{k}}) = \omega_{h,k} z^{\frac{1}{2}} e^{\pi \frac{z^{-1}-z}{12k}} P\left(e^{\frac{2\pi i}{k}(h'+iz^{-1})}\right).$$

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- Then,

$$p(n) \quad = \quad e^{\frac{2\pi n}{N^2}} \sum_{k,h}^{N} e^{-\frac{2\pi ihn}{k}} \omega_{h,k} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} z^{\frac{1}{2}} e^{\pi \frac{z^{-1}-z}{12k}} \frac{P\left(e^{\frac{2\pi i}{k}(h'+iz^{-1})}\right)}{e^{2\pi in\phi}} d\phi$$

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$$P(q) = \frac{q^{\frac{1}{24}}}{\eta(\tau)} \Rightarrow P(e^{2\pi i \frac{h+iz}{k}}) = \omega_{h,k} z^{\frac{1}{2}} e^{\pi \frac{z^{-1}-z}{12k}} P\left(e^{\frac{2\pi i}{k}(h'+iz^{-1})}\right).$$

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- $\rho = e^{-\frac{2\pi}{N^2}}$ $z = k(N^{-2} - i\phi)$
- $$P(q) = \frac{q^{\frac{1}{24}}}{\eta(\tau)} \Rightarrow P(e^{2\pi i \frac{h+iz}{k}}) = \omega_{h,k} z^{\frac{1}{2}} e^{\pi \frac{z^{-1}-z}{12k}} P\left(e^{\frac{2\pi i}{k}(h'+iz^{-1})}\right).$$
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&= e^{\frac{2\pi n}{N^2}} \sum_{k,h}^{N} e^{-\frac{2\pi i h n}{k}} \omega_{h,k} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} z^{\frac{1}{2}} e^{\pi \frac{z^{-1}-z}{12k}} \frac{1 + \left[P\left(e^{\frac{2\pi i}{k}(h'+iz^{-1})}\right) - 1\right]}{e^{2\pi i n\phi}} d\phi \\
&= \Sigma_1 + \Sigma_2.
\end{aligned}$$

- $\rho = e^{-\frac{2\pi}{N^2}}$ $z = k(N^{-2} - i\phi)$
- $$P(q) = \frac{q^{\frac{1}{24}}}{\eta(\tau)} \Rightarrow P(e^{2\pi i \frac{h+iz}{k}}) = \omega_{h,k} z^{\frac{1}{2}} e^{\pi \frac{z^{-1}-z}{12k}} P\left(e^{\frac{2\pi i}{k}(h'+iz^{-1})}\right).$$
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As $z \to 0$, with $\Re z > 0$,

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As $z \to 0$, with $\Re z > 0$, $e^{\frac{2\pi i}{k}(h'+iz^{-1})} \to 0$

- $\rho = e^{-\frac{2\pi}{N^2}}$ $z = k(N^{-2} - i\phi)$
- $$P(q) = \frac{q^{\frac{1}{24}}}{\eta(\tau)} \Rightarrow P(e^{2\pi i \frac{h+iz}{k}}) = \omega_{h,k} z^{\frac{1}{2}} e^{\pi \frac{z^{-1}-z}{12k}} P\left(e^{\frac{2\pi i}{k}(h'+iz^{-1})}\right).$$
- Then,

$$\begin{aligned} p(n) &= e^{\frac{2\pi n}{N^2}} \sum_{k,h}^{N} e^{-\frac{2\pi i h n}{k}} \omega_{h,k} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} z^{\frac{1}{2}} e^{\pi \frac{z^{-1}-z}{12k}} \frac{P\left(e^{\frac{2\pi i}{k}(h'+iz^{-1})}\right)}{e^{2\pi i n\phi}} d\phi \\ &= e^{\frac{2\pi n}{N^2}} \sum_{k,h}^{N} e^{-\frac{2\pi i h n}{k}} \omega_{h,k} \int_{-\theta'_{h,k}}^{\theta''_{h,k}} z^{\frac{1}{2}} e^{\pi \frac{z^{-1}-z}{12k}} \frac{1 + \left[P\left(e^{\frac{2\pi i}{k}(h'+iz^{-1})}\right) - 1\right]}{e^{2\pi i n\phi}} d\phi \\ &= \Sigma_1 + \Sigma_2. \end{aligned}$$

As $z \to 0$, with $\Re z > 0$, $e^{\frac{2\pi i}{k}(h'+iz^{-1})} \to 0$ rapidly.

- $|\Sigma_2| \leq \cdots \leq CN^{-\frac{1}{2}} \exp\left(\frac{2\pi n}{N^2}\right)$

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- For $\Sigma_1 = e^{\frac{2\pi n}{N^2}} \sum_{k,h}^{N} e^{-\frac{2\pi i h n}{k}} \omega_{h,k} \underline{\int_{-\theta'_{h,k}}^{\theta''_{h,k}} z^{\frac{1}{2}} e^{\pi \frac{z^{-1}-z}{12k}} \frac{1}{e^{2\pi i n \phi}} d\phi}$,

- $|\Sigma_2| \leq \cdots \leq CN^{-\frac{1}{2}} \exp\left(\frac{2\pi n}{N^2}\right) \to 0$ as $N \to \infty$, for fixed n.
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$$I_{h,k} := \int_{N^{-2}+i\theta'_{h,k}}^{N^{-2}-i\theta''_{h,k}} (k\omega)^{\frac{1}{2}} \exp\left\{ \frac{\pi}{12k}\left(\frac{1}{k\omega} - k\omega\right) + 2\pi n(\omega - N^{-2}) \right\} i$$

$$= e^{-2\pi n N^2} k^{\frac{1}{2}} i^{-1} \int_{N^{-2}+i\theta'_{h,k}}^{N^{-2}-i\theta''_{h,k}} \underbrace{\omega^{\frac{1}{2}} \exp\left\{ 2\pi\left(n - \frac{1}{24}\right)\omega + \frac{\pi}{12k^2\omega} \right\}}_{g(\omega)}$$

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$$= e^{-2\pi n N^2} k^{\frac{1}{2}} i^{-1} \int_{N^{-2}+i\theta'_{h,k}}^{N^{-2}-i\theta''_{h,k}} \underbrace{\omega^{\frac{1}{2}} \exp\left\{2\pi\left(n - \frac{1}{24}\right)\omega + \frac{\pi}{12k^2\omega}\right\}}_{g(\omega)}$$

$$e^{2\pi n N^2} I_{h,k} = k^{\frac{1}{2}} i^{-1} \left(\int_{-\infty}^{0} - \int_{-\infty}^{-\varepsilon} - \int_{-\varepsilon}^{-\varepsilon-i\theta''_{h,k}} - \int_{-\varepsilon-i\theta''_{h,k}}^{N^{-2}-i\theta''_{h,k}} - \int_{N^{-2}+i\theta'_{h,k}}^{-\varepsilon+i\theta'_{h,k}} - \int_{-\varepsilon+i\theta'_{h,k}}^{-\varepsilon} - \int_{-\varepsilon}^{-\infty}\right) g(\omega) d\omega$$

$$= k^{\frac{1}{2}} i^{-1} (L_k - I_1 - I_2 - I_3 - I_4 - I_5 - I_6).$$

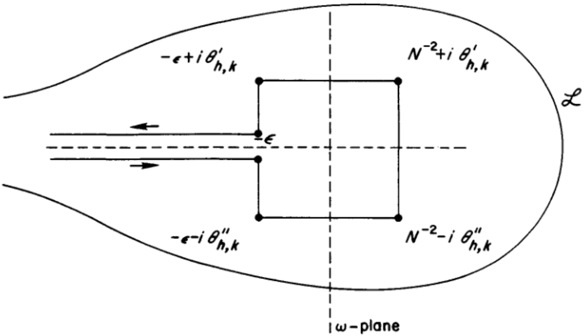
$-\epsilon+i\theta'_{h,k}$
$N^{-2}+i\theta'_{h,k}$
$\mathcal{L}$
$-\epsilon$
$-\epsilon-i\theta''_{h,k}$
$N^{-2}-i\theta''_{h,k}$
ω-plane

Figure 5.1

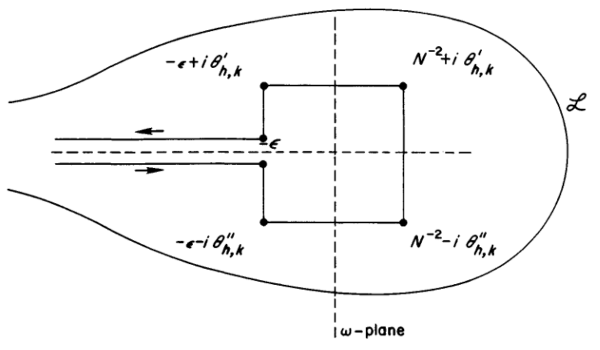

Figure 5.1

- I_2, I_3, I_4, and I_5 are ignorable.

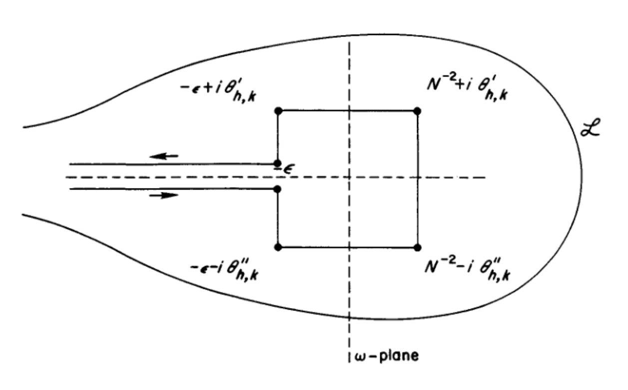

Figure 5.1

- I_2, I_3, I_4, and I_5 are ignorable.
- $$I_1 + I_6 = -2i \underbrace{\int_{\varepsilon}^{\infty} t^{\frac{1}{2}} \exp\left\{-2\pi\left(n - \frac{1}{24}\right)t - \frac{\pi}{12k^2 t}\right\} dt}_{H_k}.$$

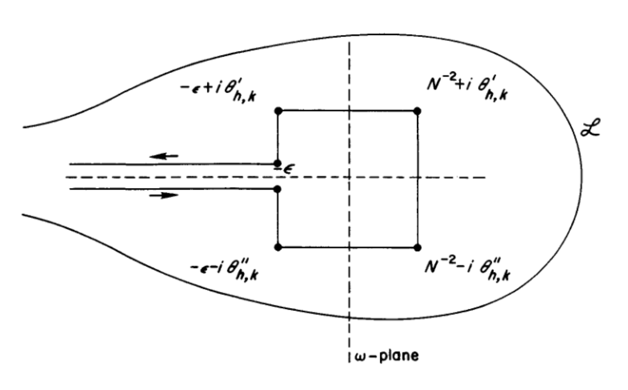

Figure 5.1

- I_2, I_3, I_4, and I_5 are ignorable.
- $$I_1 + I_6 = \underbrace{-2i\int_{\varepsilon}^{\infty} t^{\frac{1}{2}} \exp\left\{-2\pi\left(n-\frac{1}{24}\right)t - \frac{\pi}{12k^2 t}\right\} dt}_{H_k}.$$
- $$\psi_k(n) = \frac{k^{\frac{1}{2}}}{i} L_k + 2k^{\frac{1}{2}} H_k = \frac{k^{\frac{1}{2}}}{\pi\sqrt{2}} \frac{d}{dx}\bigg|_{x=n} \frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(x-\frac{1}{24}\right)}\right)}{\sqrt{x-\frac{1}{24}}}.$$

L_k

$$\frac{1}{i} L_k = \frac{1}{i} \int_{-\infty}^{0} \omega^{\frac{1}{2}} \exp\left\{ 2\pi \left(n - \frac{1}{24} \right) \omega + \frac{\pi}{12k^2\omega} \right\} d\omega$$

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L_k

$$
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\frac{1}{i} L_k &= \frac{1}{i} \int_{-\infty}^{0} \omega^{\frac{1}{2}} \exp\left\{ 2\pi \left(n - \frac{1}{24} \right) \omega + \frac{\pi}{12k^2\omega} \right\} d\omega \\
&= \frac{1}{i} \int_{-\infty}^{0} \omega^{\frac{1}{2}} \exp\left\{ 2\pi \left(n - \frac{1}{24} \right) \omega \right\} \sum_{s=0}^{\infty} \frac{\left(\frac{\pi}{12k^2\omega} \right)^s}{s!} d\omega
\end{aligned}
$$

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\frac{1}{i}L_k &= \frac{1}{i}\int_{-\infty}^{0}\omega^{\frac{1}{2}}\exp\left\{2\pi\left(n-\frac{1}{24}\right)\omega+\frac{\pi}{12k^2\omega}\right\}d\omega \\
&= \frac{1}{i}\int_{-\infty}^{0}\omega^{\frac{1}{2}}\exp\left\{2\pi\left(n-\frac{1}{24}\right)\omega\right\}\sum_{s=0}^{\infty}\frac{\left(\frac{\pi}{12k^2\omega}\right)^s}{s!}d\omega \\
\left[z=2\pi\left(n-\frac{1}{24}\right)\omega\right] &= 2\pi\sum_{s=0}^{\infty}\frac{\left(\frac{\pi}{12k^2}\right)^s}{s!}\left[2\pi\left(n-\frac{1}{24}\right)\right]^{s-\frac{3}{2}}\frac{1}{2\pi i}\int_{-\infty}^{0}e^z z^{-s+\frac{1}{2}}dz
\end{aligned}$$

L_k

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\frac{1}{i} L_k &= \frac{1}{i} \int_{-\infty}^{0} \omega^{\frac{1}{2}} \exp\left\{ 2\pi \left(n - \frac{1}{24} \right) \omega + \frac{\pi}{12k^2\omega} \right\} d\omega \\
&= \frac{1}{i} \int_{-\infty}^{0} \omega^{\frac{1}{2}} \exp\left\{ 2\pi \left(n - \frac{1}{24} \right) \omega \right\} \sum_{s=0}^{\infty} \frac{\left(\frac{\pi}{12k^2\omega} \right)^s}{s!} d\omega \\
\left[z = 2\pi \left(n - \frac{1}{24} \right) \omega \right] &= 2\pi \sum_{s=0}^{\infty} \frac{\left(\frac{\pi}{12k^2} \right)^s}{s!} \left[2\pi \left(n - \frac{1}{24} \right) \right]^{s - \frac{3}{2}} \frac{1}{2\pi i} \int_{-\infty}^{0} e^z z^{-s + \frac{1}{2}} dz \\
&= (2\pi)^{-\frac{1}{2}} \left(n - \frac{1}{24} \right)^{-\frac{3}{2}} \sum_{s=0}^{\infty} \frac{\left[\frac{\pi^2 \left(n - \frac{1}{24} \right)}{6k^2} \right]^s}{s! \Gamma(s - 1/2)}
\end{aligned}$$

L_k

$$
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&= \frac{1}{i}\int_{-\infty}^{0} \omega^{\frac{1}{2}} \exp\left\{2\pi\left(n-\frac{1}{24}\right)\omega\right\} \sum_{s=0}^{\infty} \frac{\left(\frac{\pi}{12k^2\omega}\right)^s}{s!} d\omega \\
\left[z = 2\pi\left(n-\frac{1}{24}\right)\omega\right] &= 2\pi \sum_{s=0}^{\infty} \frac{\left(\frac{\pi}{12k^2}\right)^s}{s!} \left[2\pi\left(n-\frac{1}{24}\right)\right]^{s-\frac{3}{2}} \frac{1}{2\pi i}\int_{-\infty}^{0} e^z z^{-s+\frac{1}{2}} dz \\
&= (2\pi)^{-\frac{1}{2}} \left(n-\frac{1}{24}\right)^{-\frac{3}{2}} \sum_{s=0}^{\infty} \frac{\left[\frac{\pi^2\left(n-\frac{1}{24}\right)}{6k^2}\right]^s}{s!\Gamma(s-1/2)}
\end{aligned}
$$

$$
\sum_{s=0}^{\infty} \frac{\left[\frac{Y^2}{4}\right]^s}{s!\Gamma(s-1/2)} = \frac{1}{2\sqrt{\pi}}\left(-1 + Y^2 \frac{d}{dY}\left(\frac{\cosh Y - 1}{Y}\right)\right)
$$

for $Y = (\pi/k)\sqrt{2/3(x-1/24)}$

H_k

$$H_k = -\frac{1}{2\pi\sqrt{2}} \cdot \frac{d}{dx}\bigg|_{x=n} \frac{\exp\left(-\frac{\pi}{k}\sqrt{\frac{2}{3}}\sqrt{x-\frac{1}{24}}\right)}{\sqrt{x-\frac{1}{24}}}$$

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- $$I_\nu(z) := \frac{\left(\frac{z}{2}\right)^\nu}{2\pi i}\int_{c-i\infty}^{c+i\infty} t^{-\nu-1}e^{t+\frac{z^2}{4t}}\,dt.$$

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$$H_k = -\frac{1}{2\pi\sqrt{2}} \cdot \frac{d}{dx}\bigg|_{x=n} \frac{\exp\left(-\frac{\pi}{k}\sqrt{\frac{2}{3}}\sqrt{x-\frac{1}{24}}\right)}{\sqrt{x-\frac{1}{24}}}$$

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- $$I_{\frac{3}{2}}(z) = \sqrt{\frac{2z}{\pi}} \cdot \frac{d}{dz}\left(\frac{\sinh z}{z}\right)$$

$$\begin{aligned}
\psi_k(n) \quad &:= \quad \frac{k^{\frac{1}{2}}}{i} L_k + 2k^{\frac{1}{2}} H_k \\
&= \quad \frac{k^{\frac{1}{2}}}{i}\int_{-\infty}^{0} \omega^{\frac{1}{2}} \exp\left\{2\pi\left(n-\frac{1}{24}\right)\omega + \frac{\pi}{12k^2\omega}\right\} d\omega \\
&\quad +2k^{\frac{1}{2}}\int_0^\infty t^{\frac{1}{2}} \exp\left\{-2\pi\left(n-\frac{1}{24}\right)t - \frac{\pi}{12k^2 t}\right\} dt
\end{aligned}$$