Reading Seminar on "The Theory of Partitions" by George E. Andrews

Resfricted Partitions and Permutations

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Sept. 10th, 2021

The Generating Function for Restricted Partitions

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So the generating function

$$G(N, M; q) = \sum_{n \geq 0} p(N, M, n) q^n$$

is a polynomial of degree NM.

Generating Function

Theorem

For $M, N \geq 0$,

$$G(N, M; q) = \frac{(1-q^{N+M})(1-q^{N+M-1})\cdots(1-q^{M+1})}{(1-q^{N})(1-q^{N-1})\cdots(1-q)}$$

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The Gaussian polynomial $\begin{bmatrix} n \\ m \end{bmatrix}_{(q)}$ is defined by

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- $\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m \end{bmatrix} + q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}$
- $\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + q^{m} \begin{bmatrix} n-1 \\ m \end{bmatrix}$
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Remark

$G(N, M; q) - G(N, M-1; q) = q^M G(N-1, M; q)$

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Proof.

$$\lim_{q\to 1} \begin{bmatrix} n \\ m \end{bmatrix} = \lim_{q\to 1} \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-m+1})}{(1-q^m)(1-q^{m-1})\cdots(1-q)}$$

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&= \frac{n}{m} \cdot \frac{n-1}{m-1} \cdots \frac{n-m+1}{1} \\
&= \frac{n!}{m!(n-m)!}.
\end{aligned}$$

$\square$

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Remark

$$\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m}.$$

Generating Functions

Theorem

1. $(z)_N = \sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix} (-1)^j z^j q^{\frac{j(j-1)}{2}}$
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$$\begin{aligned} (z)_N &= \frac{(z)_\infty}{(zq^N)_\infty} = \sum_{j=0}^{\infty} \frac{(q^{-N})_j z^j q^{Nj}}{(q)_j} \\ &= \sum_{j=0}^{\infty} \frac{(1-q^{-N})(1-q^{-N+1})\cdots(1-q^{-N+j-1})\, z^j q^{Nj}}{(1-q)\,(1-q^2)\cdots(1-q^j)} \end{aligned}$$

Generating Functions

Theorem

1. $(z)_N = \sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix} (-1)^j z^j q^{\frac{j(j-1)}{2}}$
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Proof.

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(z)_N &= \frac{(z)_\infty}{(zq^N)_\infty} = \sum_{j=0}^{\infty} \frac{(q^{-N})_j z^j q^{Nj}}{(q)_j} \\
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&= \sum_{j=0}^{N} \frac{(-1)^j q^{-Nj+\frac{j(j-1)}{2}}(1-q^N)\cdots(1-q^{N-j+1}) z^j q^{Nj}}{(q)_j}
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Recall

Theorem

If $|q| < 1$ and $|t| < 1$, then

$$1 + \sum_{n=1}^{\infty} \frac{(1-a)(1-aq)\cdots\left(1-aq^{n-1}\right)t^n}{(1-q)\left(1-q^2\right)\cdots\left(1-q^n\right)} = \prod_{n=0}^{\infty} \frac{(1-atq^n)}{(1-tq^n)}.$$

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} t^n = \frac{(atq)_\infty}{(tq)_\infty}.$$

Theorem

$$\sum_{j=0}^{m}(-1)^j \begin{bmatrix} m \\ j \end{bmatrix} = \begin{cases} (q;q^2)_n, & \textit{if } m = 2n; \\ 0, & \textit{if } m \textit{ is odd}; \end{cases}$$

$$\begin{bmatrix} n+m+1 \\ m+1 \end{bmatrix} = \sum_{j=0}^{n} q^j \begin{bmatrix} m+j \\ m \end{bmatrix}, \quad m, n \geq 0;$$

$$\sum_{k=0}^{h} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ n-k \end{bmatrix} q^{(n-k)(h-k)} = \begin{bmatrix} m+n \\ h \end{bmatrix};$$

$$\sum_{r \geq 0} \begin{bmatrix} M-m \\ r \end{bmatrix} \begin{bmatrix} N+m \\ m+r \end{bmatrix} \begin{bmatrix} m+n+r \\ M+N \end{bmatrix} q^{(N-r)(M-r-m)} = \begin{bmatrix} m+n \\ M \end{bmatrix} \begin{bmatrix} n \\ N \end{bmatrix}$$

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Remark

$$\binom{m+n}{h} = \sum_{k=0}^{h} \binom{m}{k} \binom{n}{r-k}$$

Definition

For $m_1, \ldots, m_r \geq 0$, we define the *Gaussian multinomial coefficient* (or *q-multinomial coefficient*) by

$$\begin{bmatrix} m_1 + \cdots + m_r \\ m_1, \ldots, m_r \end{bmatrix} = \frac{(q)_{m_1 + \cdots + m_r}}{(q)_{m_1} \cdots (q)_{m_r}}.$$

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A *multiset* is a set with possibly repeated elements. A multiset can be expressed as an ordered pair (M, f) where M is a set and

$$\begin{aligned} f : M &\to \mathbb{N} \cup \{0\} \\ m &\mapsto f(m) = \text{multiplicity of } m \end{aligned}$$

$M = \{1, 2, 2, 2, 3, 4, 2, 1, 4\}$

$$M = \{1^2 2^4 3 4^2\}$$

Namely, $f(1) = f(4) = 2$, $f(3) = 1$, and $f(2) = 4$.

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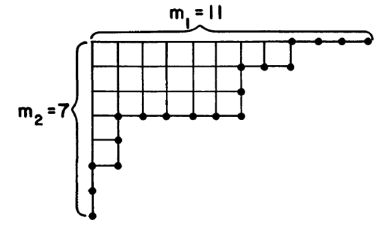
$m_1 = 11,\ m_2 = 7,\ \lambda = (8, 6, 6, 1, 1)$



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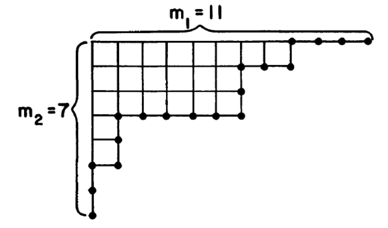

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Theorem (P. A. MacMahon)

For $r \geq 1$, $\sum_{n \geq 0} inv(m_1, \ldots, m_r; n) q^n = \begin{bmatrix} m_1 + \cdots + m_r \\ m_1, \ldots, m_r \end{bmatrix}$.

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For inductive step, it suffices to show

$$inv(m_1, \ldots, m_r; n) = \sum_{j=0}^{n} inv(m_1 + \cdots + m_{r-1}, m_r; j) inv(m_1, \ldots, m_{r-1}; n-j).$$

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- Take a permutation of $\{1^{m_1+\cdots+m_{r-1}} 2^{m_r}\}$ with j inversions.
- Replace each 2 by an r and replace the 1 by a permutation of $\{1^{m_1} \cdots (r-1)^{m_{r-1}}\}$ with $n-j$ inversions.

Greater index

Definition

Let $ind(m_1, \ldots, m_r; n)$ be the number of permuations $\xi_1 \cdots \xi_{m_1+\cdots+m_r}$ of $\{1^{m_1} \cdots r^{m_r}\}$ for which

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$$\sum_{i=1}^{m_1+\cdots+m_r-1} \chi(\xi_i) = n,$$

where

$$\chi(\xi_i) = \begin{cases} i & \xi_i > \xi_{i+1}; \\ 0, & \text{otherwise.} \end{cases}$$

Here this sum is called the *greater index* of the permutation.

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For $r \geq 1$, $\sum_{n \geq 0} ind(m_1, \ldots, m_r; n) q^n = \begin{bmatrix} m_1 + \cdots + m_r \\ m_1, \ldots, m_r \end{bmatrix}$.

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Claim

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Corollary

$$ind(m_1, \ldots, m_r; n) = inv(m_1, \ldots, m_r; n)$$

Unimodality

Definition

1. A polynomial $p(q) = a_0 + a_1 q + \cdots + a_n q^n$ is called *reciprocal* if for each i, $a_i = a_{n-i}$, equivalently $q^n p(1/q) = p(q)$.

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2. A polynomial $p(q) = a_0 + a_1 q + \cdots + a_n q^n$ is called *unimodal* if there exists m such that

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Theorem

Let $p(q)$ and $r(q)$ be reciprocal, unimodal polynomials with nonnegative coefficients; then $p(q)r(q)$ is also a reciprocal, unimodal polynomial with nonnegative coefficients.

Theorem

For all $N, M, n \geq 0$
$$p(N, M, n) = p(M, N, n);$$
$$P(N, M, n) = p(N, M, NM - n);$$
$$p(N, M, n) - p(N, M, n-1) \geq 0, \quad \textit{for} \quad 0 < n \leq NM/2.$$

Theorem

For all $m_1, m_2, \ldots, m_r, n \geq 0$

- $$ind(m_1, \ldots, m_r; n) = ind(m_{i_1}, \ldots, m_{i_r}; n) = inv(m_1, \ldots, m_r; n) = inv(m_{i_1}, \ldots, m_{i_r}; n)$$
  where $\{i_1, \ldots, i_r\}$ *is a permutation of* $\{1, 2, \ldots r\}$;
- $$ind(m_1, \ldots, m_r; n) = ind(m_1, \ldots, m_r; S - n) = inv(m_1, \ldots, m_r; n) = inv(m_1, \ldots, m_r; S - n)$$
  where $S = \sum_{1 \leq i < j \leq r} m_i m_j$ *is the secondary symmetric function of* m_i;
- $0 < n \leq \frac{S}{2}$: $ind(m_1, \ldots, m_r; n) - ind(m_{i_1}, \ldots, m_{i_r}; n-1) = inv(m_1, \ldots, m_r; n) - inv(m_{i_1}, \ldots, m_{i_r}; n-1) \geq 0.$

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