Jayce R. Getz, Heekyoung Hahn

An Introduction to Automorphic Representations

with a view toward Trace Formulae

April 22, 2022
To Angela and Adsila, with love
Preface

The instructor of an introductory course on automorphic representations, the student beginning their foray into the subject, and the experienced mathematical researcher in another discipline trying to obtain conversational knowledge in the field are all faced with a formidable task. The prerequisites are vast, as the theory of automorphic representations draws on, and makes substantial contributions to, much of algebraic geometry, harmonic analysis, number theory, and representation theory. Moreover, it is a subject in which one can spend a career moving in any direction one chooses.

The student should not be overly discouraged, however, because there is no person currently alive who has the necessary prerequisites to really understand the entire subject. This is meant as no disrespect to the luminaries in the field that, either through their work or personal contact, have made a profound impact on the mathematical community and the authors of this book. It is simply that broad and deep a subject of research. Extremely few important results that were proved by one individual (or even by a group working on a single project). Most of the important results in the field have been obtained by individuals or small teams with very different specialties working on various pieces of a larger whole, sometimes over the course of decades. We say this to encourage the reader; one need not know everything in order to get started and make important contributions.

In the authors’ experience, after scrounging around the literature one learns enough to begin discovering mathematics connected to their particular point of view. The goal of this book is to make this process easier. The more particular aim is to develop enough of the language and machinery of the subject that after reading this book one can start reading original research papers. We have also tried to be concise enough that the reader does not get mired in so many details that they are unable to see the overarching themes and goals of the subject. This means that we have not made any systematic attempt to prove every result that we use in the book. Our preference has been to omit proofs that require substantial digressions that interrupt the flow of ideas. However, we have included such proofs if they are
either not available in the literature or very difficult to extract. When proofs are not provided we have made an effort to provide precise references where the reader can find more details.

This book is written so that it can be used as a text for a semester or year-long course, a resource for self-study of more advanced topics, and as a reference for researchers. We hope that our efforts to provide definitions, proofs or precise references for proofs will be helpful to the last group. We treat general reductive groups over arbitrary global fields.

Prerequisites for the book include introductory graduate level courses in algebra and analysis together with undergraduate topology. Graduate level classes in algebraic number theory, algebraic geometry, representation theory, and algebraic topology are helpful at various places. However the book is written so that the reader who does not have this knowledge can still proceed, supplementing the material in the book from other resources when necessary.

We now describe several outlines for possible courses. The sections we have not included have valuable information that may be substituted or added as desired. Here is an outline for the sections one might cover for a semester long introduction to automorphic representation theory:

Affine Algebraic Groups §1.2, §1.3, §1.5, §1.6, §1.7, §1.8, §1.9
Adeles §2.1, §2.2, §2.3, §2.4, §2.5, §2.6, §2.7
Automorphic Representations §3.1, §3.2, §3.3, §3.4, §3.5, §3.6, §3.7
Representations of Totally Disconnected Groups §5.1, §5.2, §5.3, §5.4, §5.5, §5.6, §5.7
Unramified Representations §7.1, §7.2, §7.3, §7.5, §7.6, §7.7

It is useful to say in words what these chapters cover. Given a global field $F$ and a reductive algebraic group $G$ over $F$, an automorphic representation of $G$ in the $L^2$-sense is an irreducible unitary representation $G(A_F)^1$ which is isomorphic to a subquotient of $L^2(G(F)\backslash G(A_F)^1)$. The first goal of the course outline above is to make sense of this definition. The second goal, realized in Chapter 6, is to give a refinement of it that is technically very useful and make the connection with classical automorphic forms transparent. The precise relationship between the two definitions is explained in §6.6. Finally in Chapter 7 one finds a rough form of the Langlands functoriality conjecture.

For a second semester or an advanced class one has more flexibility. Chapter 8 covers deeper topics in nonarchimedean representation theory, including the crucial related notions of Jacquet modules and supercuspidal representations. It also includes a statement of the Bernstein-Zelevinsky classification (see §8.4). These topics, though important, come up only in passing later in the book, and could be omitted. Chapter 9 on the cuspidal spectrum gives a complete proof of the fact that the cuspidal spectrum decomposes discretely. It is a basic fact that is hard to extract from the literature, but it could easily be treated as a black box if desired. Chapter 10 outlines the theory of Eisen-
stein series. This foundational theory lies at the heart of many constructions (and analytic difficulties) in the theory of automorphic representations. However, the book is written so that Chapter 10 can be skipped without much loss of continuity.

With these comments in mind, a second course that does not cover simple relative trace formulae might have an outline as follows:

- The Cohomology of Locally Symmetric Spaces §15.2, §15.3, §15.4, §15.5, §15.6, §15.8

The last three chapters of the book together give a proof of a simple version of the (relative) trace formula. This implies the trace formula, twisted trace formula, etc. in simple cases. In a course these three chapters should be covered together, and they rely in particular on §14.3. If covering the simple relative trace formula is a priority for the instructor, they may consider dropping Chapter 15 or postponing it to the end of the semester.

The spectral side of the trace formula is considered in §16.1 and §16.2. The geometric side is the focus of Chapter 17. Chapter 9 on the discreteness of the cuspidal spectrum and Chapter 17 are easily the most technical chapters of the book. The first requires some analysis and the second some algebraic geometry. In some sense they are the also the most important, because they give complete proofs of results that are very difficult to extract from the literature, if they exist at all. That said, just as much of Chapter 9 can be taken as a black box, the same is true of Chapter 17.

In a course it might make sense to treat §17.1 and §17.3 only superficially and focus on the definition of the local relative orbital integrals in §17.4 and the definition of global relative orbital integrals in §17.7. Moving forward the key result is in Theorem 17.7.4. After this preparatory work the general simple relative trace formula in §18.2 follows easily. The sections §18.3, §18.4 §18.5 provide specializations of the simple relative trace formula that are important for illustrating what the formula means. Readers interested in analytic number theory should peruse §18.7, §18.8, and §18.9 to obtain a complete derivation of the famous Kuznetsov formula in a form suitable for applications to analytic number theory. After all the hard work required to derive (even simple versions) of relative trace formulae, it would be remiss not to discuss some applications. These are contained in Chapter 19. The most important sections are §19.2 and §19.3, but to really understand the theory in §19.3 the examples and clarifications in the remaining sections of that chapter are invaluable. Some suggestions for further work are contained in §19.6.
Most of the material in this book has appeared in one form or another in the literature. We have made an honest attempt to give accurate attributions for both results and expository treatments. We would appreciate being informed of any serious errors we have made either in mathematics or in attributing results. As much as possible we treat arbitrary reductive groups over arbitrary global fields, including the case of positive characteristic. We have also refrained from assuming that the affine algebraic groups under consideration are reductive until necessary. This both emphasizes the results that really depend on the fact that $G$ is reductive and develops the preliminaries necessary to understand, for example, unitary representations of nonreductive groups. This is not vacuous, as the theory of Jacobi forms already plays a role in automorphic forms [BS98].

The results on orbital integrals in Chapter 17 and the simple relative trace formula in Chapter 18 are based partially on work of the second author [Hah09] and both authors [GH15]. However they have not appeared previously in the generality in which we prove them here. In particular we treat arbitrary reductive groups over arbitrary global fields. Thus we allow positive characteristic.

In the interest of full disclosure, we should say a word about foundations. We define affine schemes (and in particular algebraic groups) as certain functors on $k$-algebras, and in §17.5 we make use of some more serious algebraic geometry involving sheaves of sets on a site. This puts us up against the usual set-theoretic difficulties. To handle these we assume the universe axiom, fix a universe containing an infinite cardinal, and assume (implicitly) that the categories of $k$-algebras, sets, etc., with which we are working are small with respect to that universe. This is a standard convention in algebraic geometry, though it is seldom mentioned. For more details we refer to [SGA72, §0]. It is likely that the reliance on the universe axiom could be removed at the cost of complicating our definitions.

Durham, NC, June, 2021

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Acknowledgements

The authors thank Francesc Castella, Andrew Fiori and Cameron Franc for typsetting the first draft of the lecture notes that formed the germ of this book, and thank Brian Conrad, Pam Gu, Chun-Hsien Hsu, Tasho Kaletha, Minhyong Kim, Chung-Ru Lee, Spencer Leslie, Siyan Daniel Li-Huerta, Jason Polak, Leslie Saper, Timo Richarz, and Chad Schoen for many useful corrections and comments. Many people have been generous in answering questions as this book was written, including James Arthur, Avner Ash, David Ginzburg, Nadya Gurevich, Michael Harris, Rahul Krishna, Erez Lapid, Freydoon Shahidi, Michal Zydor, and any others who we may have forgotten. The authors give special thanks to Solomon Friedberg, who taught a course out of a draft of this book, Alex Youcis for his very careful reading of the first twelve chapters, and to the anonymous referees. Their comments led to many clarifications and corrections. The authors also thank Ken Ono for suggesting this book project. The first author wishes to thank the National Science Foundation for support (DMS-1901883 and DMS-1405708) at various times during the preparation of this book. Part of this book was written while both authors were members at the Institute for Advanced Study in the spring of 2018 supported in part by C. Simonyi Endowment, and the final draft was completed while both authors were on sabbatical at Postech in South Korea in the 2020–2021 academic year and jointly supported in part by Postech Mathematical Institute (NRF-2017R1A2B2001807) and by YoungJu Choie (NRF-2018R1A4A1023590). We thank these institutions for their hospitality and providing us with excellent working conditions. We also thank the students of Hahn’s course at Postech for their questions which let to various improvements. Finally, we truly appreciate the encouragement of Mahdi Asgari, Peter Sarnak and many graduate students.
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Chapter 1
Affine Algebraic Groups

Before Genesis, God gave the devil free reign to construct much of creation, but dealt with certain matters himself, including semisimple groups.

Attributed to Chevalley by Harish-Chandra

Abstract We briefly recall some of the basic notions related to affine algebraic groups. If the reader is comfortable with reductive groups then this chapter can probably be skipped and then referred to later as needed.

1.1 Introduction

Someone beginning to learn the theory of automorphic representations does not need to internalize the content of this chapter. For the most part, one can get by with much less. For example, one can think of $GL_n$, which is really an affine group scheme over the integers, as a formal notational device already familiar from an elementary course in algebra: for commutative rings $R$, the notation $GL_n(R)$ refers to the invertible matrices with coefficients in $R$. Likewise, one knows that, for example, $Sp_{2n}(R)$ is the group of matrices with coefficients in $R$ that fix the standard symplectic form on $R^{2n}$. One can think of a reductive group $G$ as some gadget of this type that assigns a group to every ring of some type (say $\mathbb{Q}$-algebras) and behaves like $GL_n$. In fact, for a first reading of this book, whenever one sees the assumption that $G$ is a reductive group, one could assume that $G = GL_n$ for some integer $n$ and not lose a lot of the flavor.

That said, at some point, it is not a bad idea for any student of automorphic representation theory (and really, any mathematician) to come to
grips with the “modern” (post 1960s) terminology of algebraic geometry and algebraic groups. It is useful, beautiful, and despite its reputation, not really that counterintuitive or difficult. Thus we will briefly introduce it in this chapter, mostly as a means of fixing notation and conventions, and make use of it in the remainder of the book. We have chosen to focus on affine schemes because (for the most part) this is all we require and they are simpler.

Much of the material in this chapter can be found in one form or another in the three standard references for algebraic groups [Bor91, Hum75, Spr09]. Unfortunately they were all written in the archaic language of Weil used before Grothendieck’s profound reimagining of the foundations of algebraic geometry. The most complete reference is [DG74], but it is hard to penetrate if one does not have solid preparation in algebraic geometry. The reference [Wat79] is more accessible, but covers less. Fortunately, Milne has reworked the three standard references in modern language. His book [Mil17] was instrumental in the preparation of this chapter.

1.2 Affine schemes

Throughout this chapter, we let $k$ be a commutative Noetherian ring with identity. Let $\text{Alg}_k$ denote the category of commutative $k$-algebras with identity. For a $k$-algebra $A$, one obtains a functor

$$\text{Spec}(A) : \text{Alg}_k \rightarrow \text{Set}$$

$$R \mapsto \text{Hom}_k(A, R),$$

where the subscript $k$ denotes $k$-algebra homomorphisms. If $\phi : A \rightarrow B$ is a morphism of $k$-algebras, then we obtain, for each $k$-algebra $R$, a map

$$\text{Spec}(B)(R) \rightarrow \text{Spec}(A)(R)$$

$$\varphi \mapsto \varphi \circ \phi.$$ 

This collection of maps is an example of what is known as a natural transformation of functors. More precisely, to give a natural transformation $X \rightarrow Y$ of set-valued functors on $\text{Alg}_k$ is the same as giving for each $k$-algebra $R$ a map

$$X(R) \rightarrow Y(R)$$

such that, for all morphisms of $k$-algebras $R \rightarrow R'$, the following diagram commutes:
One can check that (1.2) is a natural transformation of functors.

**Definition 1.1.** An affine scheme over $k$ or an affine $k$-scheme is a functor on the category of $k$-algebras of the form $\text{Spec}(A)$. A morphism of affine $k$-schemes is a natural transformation of functors.

This definition gives us a category $\text{AffSch}_k$ of affine schemes over $k$, equipped with a contravariant equivalence of categories

$$\text{Spec} : \text{Alg}_k \to \text{AffSch}_k$$

(see Exercise 1.3). In other words, the category of affine schemes over $k$ is the category of $k$-algebras “with the arrows reversed.” If $k$ is understood, we often omit explicit mention of it. By way of terminology, if $X$ is an affine scheme then

$$X(R)$$

is called its $R$-valued points.

**Definition 1.2.** A functor $S : \text{Alg}_k \to \text{Set}$ is representable by a ring $A$ if $S = \text{Spec}(A)$. In this case, we write

$$\mathcal{O}(S) := A$$

and refer to it as the coordinate ring of $S$.

To ease comparison with other references, we note that we are defining schemes using their functors of points, whereas the usual approach is to define schemes as a topological space with a sheaf of rings on it satisfying certain desiderata and then associate a functor of points to the scheme. The two approaches are equivalent. The usual approach is desirable for many purposes, but the approach via the functor of points is more suitable for the study of algebraic groups. For more details on the usual approach and on the scheme theoretic concepts mentioned below, we point the reader to [Har77, EH00, Mum99, GW10]. The functor of points approach is used in [DG74].

In order to work with affine schemes in practice, one must usually impose additional restrictions on the representing ring $A$.

**Definition 1.3.** An affine $k$-scheme $\text{Spec}(A)$ is of finite type if $A$ is a finitely generated $k$-algebra.

Therefore $\text{Spec}(A)$ is of finite type over $k$ if and only if

$$A \cong k[t_1, \ldots, t_n]/(f_1, \ldots, f_m)$$
for some finite set of indeterminates \( t_1, \ldots, t_n \) and finite set of polynomials \( f_1, \ldots, f_m \). An important example of an affine scheme of finite type over \( k \) is affine \( n \)-space:

\[
\mathbb{A}^n_k := \text{Spec}(k[t_1, \ldots, t_n]).
\]

This is also denoted by \( \mathbb{A}^n \), but we avoid this notation because we will use the symbol \( \mathbb{A} \) for the adeles (which will be defined in Definition 2.5).

Here are two other nice properties that the schemes of interest to us will often enjoy:

**Definition 1.4.** An affine scheme \( X \) is **reduced** if \( \mathcal{O}(X) \) has no nilpotent elements and **irreducible** if the nilradical of \( \mathcal{O}(X) \) is prime. It is **integral** if \( \mathcal{O}(X) \) is an integral domain.

It is important to note that an affine scheme can be both reduced and irreducible (see Exercise 1.4).

Assume for the moment that \( k \) is a field. An affine scheme \( \text{Spec}(A) \) of finite type over \( k \) is **smooth** if the coordinate ring \( A \) is formally smooth. Here a \( k \)-algebra \( A \) is said to be **formally smooth** if for every \( k \)-algebra \( B \) with ideal \( I \leq B \) of square zero and any \( k \)-algebra homomorphism \( A \to B/I \), one has a morphism \( A \to B \) such that the diagram

\[
\begin{array}{ccc}
A & \longrightarrow & B/I \\
\uparrow & & \uparrow \\
k & \longrightarrow & B
\end{array}
\]

commutes.

For example, an affine scheme is smooth if it is isomorphic to an affine scheme

\[
\text{Spec}(k[t_1, \ldots, t_n]/(f_1, \ldots, f_{n-d}))
\]

where the ideal in \( k[t_1, \ldots, t_n] \) generated by the \( f_i \) and all the \((n-d) \times (n-d)\) minors of the matrix of derivatives \((\partial f_i/\partial t_j)\) is the whole ring \( k[t_1, \ldots, t_n] \). We have not and will not define open subschemes and Zariski covers of schemes, but for those familiar with this language, we remark that one can always cover a smooth affine scheme by open affine subschemes of the form (1.5).

We now discuss closed subschemes.

**Definition 1.5.** A morphism of affine \( k \)-schemes

\[
X \to Y
\]

is a **closed immersion** if the associated map \( \mathcal{O}(Y) \to \mathcal{O}(X) \) is surjective.

If \( X \to Y \) is a closed immersion then \( X(R) \to Y(R) \) is injective for all \( k \)-algebras \( R \) (see Exercise 1.5).
1.2 Affine schemes

If $X \to Y$ is a closed immersion then $\mathcal{O}(X) \cong \mathcal{O}(Y)/I$ for some ideal $I$ of $\mathcal{O}(Y)$. Motivated by this observation, one defines a **closed subscheme** of the affine scheme $Y$ to be an affine scheme of the form $\text{Spec}(\mathcal{O}(Y)/I)$ for some ideal $I \subseteq \mathcal{O}(Y)$. Thus affine schemes of finite type over $k$ are all closed subschemes of $\mathbb{G}^n_a$ for some $n$. When $k$ is an algebraically closed field, one thinks of the closed subscheme $\text{Spec}(k[t_1, \ldots, t_n]/(f_1, \ldots, f_m))$ as being the zero locus of $f_1, \ldots, f_m$.

The definition of a closed subscheme can also be given from a topological perspective as we now explain. There is a topological space attached to $\text{Spec}(A)$ which is again denoted $\text{Spec}(A)$. It is the collection of all primes $p \subseteq A$ equipped with the **Zariski topology**. To define this topology, for every subset $S \subseteq A$ we let

$$V(S) = \{ p \in \text{Spec}(A) : p \supseteq S \}. \quad (1.6)$$

One checks that the sets $V(S)$ are closed under infinite intersections and finite unions. Thus we can define the Zariski topology on $\text{Spec}(A)$ to be the topology with $V(S)$ as its closed sets. A morphism of schemes induces a morphism of the associated topological spaces, and the image of the injective morphism

$$\text{Spec}(A/I) \to \text{Spec}(A)$$

is $V(I)$ for an ideal $I$ of $A$ (see Exercise 1.1).

Since we have mentioned the associated topological space of an affine scheme, we state two definitions that use it.

**Definition 1.6.** A morphism of affine $k$-schemes $X \to Y$ is **dominant** if the induced map of topological spaces has dense image, and **surjective** if the induced map of topological spaces is surjective.

If $X$ and $Y$ are integral, then $X \to Y$ is dominant if and only if the map $\mathcal{O}(Y) \to \mathcal{O}(X)$ is injective [Sta16, Tag 0CC1].

**Lemma 1.2.1** Assume $k$ is a field and $X$ and $Y$ are affine schemes of finite type over $k$. Then $X \to Y$ is surjective if and only if $X(\overline{k}) \to Y(\overline{k})$ is surjective.

**Proof.** By Chevalley’s theorem, the image of the topological space of $X$ in $Y$ is constructible [GW10, Theorem 10.20]. But $Y(\overline{k})$, which may be identified with the set of closed points of $Y$, is very dense in $Y$ (see [GW10, §3.12] for the definition of very dense). The lemma follows (see [GW10, Remark 10.15]).

We close this section with the notion of fiber products. This is a simple and useful method to create new schemes from existing ones. Let

$$X = \text{Spec}(A), \quad Y = \text{Spec}(B), \quad Z = \text{Spec}(C)$$
be affine $k$-schemes equipped with morphisms $f : X \to Y$ and $g : Z \to Y$. We can then form the **fiber product**

$$X \times_Y Z := \text{Spec}(A \otimes_B C)$$

where $A$ and $C$ are regarded as $B$-algebras via the maps $B \to A$ and $B \to C$ induced by $f$ and $g$, respectively. This scheme has the property that for $k$-algebras $R$, one has that

$$(X \times_Y Z)(R) = X(R) \times_{Y(R)} Z(R),$$

where the fiber product on the right is that in the category of sets,

$$X(R) \times_{Y(R)} Z(R) := \{(x, z) \in X(R) \times Z(R) : f(x) = g(z)\}.$$

By far the case that will be used the most frequently in the sequel is the so-called **absolute product** in the category of affine $k$-schemes. To define it, note that every affine $k$-scheme $X = \text{Spec}(A)$ comes equipped with a morphism $X \to \text{Spec}(k)$, namely the morphism induced by the ring homomorphism $k \to A$ that gives $A$ the structure of a $k$-algebra. This morphism $X \to \text{Spec}(k)$ is called the **structure morphism**. Thus for any pair of affine $k$-schemes $X$ and $Z$, we can form the absolute product

$$X \times Z := X \times_k Z := X \times_{\text{Spec}(k)} Z.$$  

It follows from (1.7) that for $k$-algebras $R$,

$$(X \times Z)(R) = X(R) \times Z(R).$$

### 1.3 Affine group schemes

**Definition 1.7.** An **affine group scheme** over $k$ (or affine $k$-group scheme) is a functor

$$\text{Alg}_k \to \text{Group}$$

representable by a $k$-algebra. A **morphism** of affine $k$-group schemes $H \to G$ is a natural transformation of functors from $H$ to $G$.

Here a natural transformation of functors is defined as in the previous section, though this time we require that for each $k$-algebra $R$ the associated map

$$H(R) \to G(R)$$

is a group homomorphism, not just a map of sets. It is clear that an affine group scheme over $k$ is in particular an affine scheme over $k$. In this book, we
will only be interested in affine group schemes (as opposed to elliptic curves for instance).

The most basic examples of affine group schemes are the additive and multiplicative groups:

**Example 1.1.** The **additive group** $\mathbb{G}_a$ is the functor assigning to each $k$-algebra $R$ its additive group,

$$\mathbb{G}_a(R) := (R, +).$$

It is represented by the polynomial algebra $k[x]$:

$$\text{Hom}_k(k[x], R) = R.$$

**Example 1.2.** The scheme of $n \times n$ matrices $M_n$ is the functor assigning to each $k$-algebra $R$ the set

$$M_n(R)$$

of $n \times n$ matrices, viewed as a group under addition. It is represented by

$$k[[\{x_{ij} : 1 \leq i, j \leq n\}]].$$

It is isomorphic as an affine $k$-group scheme to $\mathbb{G}_a^n$.

**Example 1.3.** The **multiplicative group** $\mathbb{G}_m$ is the functor assigning to each $k$-algebra $R$ its multiplicative group,

$$\mathbb{G}_m(R) = R^\times.$$

It is represented by $k[x, y]/(xy - 1)$.

These affine group schemes are all abelian in the sense that their $R$-valued points are always abelian. The most basic example of a nonabelian group scheme is the general linear group $\text{GL}_n$. It is the functor taking a $k$-algebra $R$ to the group of $n \times n$ invertible matrices $(x_{ij})$ with coefficients $x_{ij}$ in $R$. The representing ring is

$$k[[\{x_{ij} : 1 \leq i, j \leq n\}]]/[y]/(y \det(x_{ij}) - 1).$$

Note that $\text{GL}_1 = \mathbb{G}_m$. If one wishes to be coordinate free, then for any finite rank free $k$-module $V$, one can define

$$\text{GL}_V(R) := \{R\text{-module automorphisms } V \otimes_k R \rightarrow V \otimes_k R\}.$$ 

A choice of isomorphism $V \cong k^n$ induces an isomorphism $\text{GL}_V \cong \text{GL}_n$.

We isolate a particularly important class of morphisms with the following definition:
Definition 1.8. A representation of an affine group scheme $G$ is a morphism of affine group schemes $r : G \to \text{GL}_V$. It is faithful if it is a closed immersion.

Assume that $k$ is a field and $r$ is injective in the sense that $r : G(R) \to \text{GL}_V(R)$ is injective for all $k$-algebras $R$. Then $r$ is faithful [Mil17, Corollary 3.35].

Definition 1.9. An affine group scheme $G$ is said to be linear if it admits a faithful representation $G \to \text{GL}_V$ for some $V$.

We will usually be concerned with linear group schemes. We shall see below in Theorem 1.5.1 that this is not much loss of generality if $k$ is a field.

Definition 1.10. Let $G$ be an affine group scheme over $k$. An affine subgroup scheme (or simply a subgroup) $H$ of $G$ is an affine subscheme $H$ of $G$ such that $H(R)$ is a subgroup of $G(R)$ for all $k$-algebras $R$.

We write $H \leq G$ to indicate that $H$ is a subgroup scheme of $G$.

We have never defined a subscheme; one defines first an open subscheme, and then defines a subscheme to be an closed subscheme of an open subscheme. We refer to any standard text on algebraic geometry for the definition of an open subscheme. One should be aware that an open subscheme of an affine scheme need not be affine, and this is why we have avoided defining the notion of an open subscheme. Happily, when $k$ is a field and $G$ and $H$ are affine of finite type over $k$, a subgroup is automatically closed (see the comments after Definition 1.11 below).

Much of the basic theory of abstract groups goes through for affine algebraic groups over a field without essential change, but the proofs are far more complicated. We will not develop the theory; we refer the reader to [Mil17] for details. However, it is useful to define monomorphisms and quotients in this category:

Definition 1.11. Let $k$ be a field. A morphism $H \to G$ of affine group schemes of finite type over $k$ is a monomorphism (resp. quotient map) if the corresponding map $\mathcal{O}(G) \to \mathcal{O}(H)$ is surjective (resp. faithfully flat).

For this paragraph, we continue to assume that $k$ is a field and that $H \to G$ is a morphism of affine group schemes of finite type over $k$. By definition, $H \to G$ is a monomorphism if and only if it is a morphism of affine group schemes and is a closed immersion. If $H \to G$ is a monomorphism then it is easy to verify that $H(R) \to G(R)$ is injective for all $k$-algebras $R$ (see Exercise 1.5), and the converse is also true [Mil17, Proposition 5.31]. In particular over a field, subgroups are closed. However, if $H \to G$ is a quotient map, then $H(R) \to G(R)$ need not be surjective for a given $k$-algebra $R$. A trivial example is the map $\mathbb{G}_m \to \mathbb{G}_m$ of affine group schemes over $\mathbb{Q}$ given on points by $x \mapsto x^n$ for an integer $n > 1$ (or $n < -1$). It is true, however, that
1.4 Extension and restriction of scalars

if $H \to G$ is a quotient map then $H(\mathbb{K}) \to G(\mathbb{K})$ is surjective [Mil17, §5.b], and the converse holds whenever $H$ is reduced (see Lemma 1.2.1 and [Mil17, Proposition 1.70]).

Example 1.4. The homomorphism $\mathbb{G}_a \to \text{GL}_2$ given on points in a $k$-algebra $R$ by

$$R \to \text{GL}_2(R)$$

$$x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

is a monomorphism. The homomorphism $\text{GL}_n \to \mathbb{G}_m$ given on points in a $k$-algebra by the determinant is a quotient map.

1.4 Extension and restriction of scalars

Let $k \to k'$ be a homomorphism of rings. Given a $k$-algebra $R$, one obtains a $k'$-algebra $R \otimes_k k'$. Moreover, given a $k'$-algebra $R'$, one can view it as a $k$-algebra in the tautological manner. This gives rise to a pair of functors

$$\text{Alg}_k \to \text{Alg}_{k'} \quad \text{and} \quad \text{Alg}_{k'} \to \text{Alg}_k$$

known as base change and restriction of scalars, respectively. These functors are useful in that they allow us to change the base ring $k$.

For affine schemes, we similarly have a base change functor

$$k' : \text{AffSch}_k \to \text{AffSch}_{k'}$$

given by $X_{k'}(R') = X(R')$; the ring representing $X_{k'}$ is simply $\mathcal{O}(X) \otimes_k k'$.

In fact, this is a special case of the fiber product construction of §1.2: in the notation of that section, one takes $Y = \text{Spec}(k)$ and $Z = \text{Spec}(k')$ and let us equipped with the map $Z \to Y$ induced by $k \to k'$.

The functor in the opposite direction is a little more subtle. For any affine $k'$-scheme $X'$ one can always define a set valued functor

$$\text{Res}_{k'/k}X'(R) := X'(R \otimes_k k')$$

called the Weil restriction of scalars. A priori this is just a set valued functor on $\text{Alg}_k$, but if $k'/k$ satisfies certain conditions then it is representable, and hence we obtain a functor

$$\text{Res}_{k'/k} : \text{AffSch}_{k'} \to \text{AffSch}_k.$$ 

For example, it is enough to assume that $k'/k$ is a field extension of finite degree, or more generally that $k'/k$ is finite and locally free [BLR90, Theorem 4, §7.6].
Example 1.5. The Deligne torus is

\[ S := \text{Res}_{\mathbb{C}/\mathbb{R}} \text{GL}_1. \]

One then has that \( S(\mathbb{R}) = \mathbb{C}^\times \) and \( S(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times \). If \( V \) is a real vector space, then to give a representation \( S \to \text{GL}_V \) is equivalent to giving a Hodge structure on \( V \) (see [Mil05] for instance).

Example 1.6. Let \( d \) be a square free integer and \( L = \mathbb{Q}(\sqrt{d}) \). If we view \( L \) as a 2-dimensional vector space over \( \mathbb{Q} \), one obtains an injection of \( L^\times \) into \( \text{GL}_2(\mathbb{Q}) \). This can be refined into a monomorphism

\[ \text{Res}_{L/\mathbb{Q}} \mathbb{G}_m \to \text{GL}_2 \]

of affine algebraic groups over \( \mathbb{Q} \). One can choose the monomorphism so the \( R \)-valued points of its image is

\[ \{ \left( \begin{array}{cc} a & db \\ b & a \end{array} \right) : a, b \in R, a^2 - db^2 \in R^\times \} \]

for \( \mathbb{Q} \)-algebras \( R \).

1.5 Reductive groups

We now assume that \( k \) is a field and let \( k^{\text{sep}} \leq \overline{k} \) be a separable (resp. algebraic) closure of \( k \). Much of the theory simplifies in this case.

One has the following definition:

Definition 1.12. An affine algebraic group over a field \( k \) is an affine group scheme of finite type over \( k \).

In other words, an affine group scheme \( G \) is algebraic if \( G \) is represented by a quotient of \( k[x_1, \ldots, x_n] \) for some \( n \) (namely \( \mathcal{O}(G) \)). There is a natural action of \( G \) on \( \mathcal{O}(G) \); if we think of \( \mathcal{O}(G) \) as functions on \( G \), then the action is that induced by right translation. For any \( k \)-subspace of \( \mathcal{O}(G) \) invariant under \( G \), we obtain a representation of \( G \), and this representation can be chosen to be faithful. If one fills in the details of this discussion, one obtains the following theorem [Mil17, §4.d]:

Theorem 1.5.1 An affine algebraic group over a field admits a faithful representation.

Thus affine algebraic groups are linear. Another nice feature of affine algebraic groups is that they are often smooth (see [Mil17, Proposition 1.26, Theorem 3.23]).

Theorem 1.5.2 Let \( G \) be an affine algebraic group over a field \( k \). If \( k \) has characteristic zero or if \( G_{\overline{k}} \) is reduced, then \( G \) is smooth.
We will also require the notion of connectedness. An affine group scheme \( G \) over a field \( k \) is **connected** if the only idempotents in \( \mathcal{O}(G) \) are 0 and 1. Though it is not true for general affine schemes, an affine algebraic group \( G \) over \( k \) is connected if and only if \( G \) is irreducible [Mil17, Summary 1.36]. Using this fact, one shows that if \( k \) is a subfield of \( \mathbb{C} \), then \( G \) is connected if and only if \( G(\mathbb{C}) \) is connected in the analytic topology. Every affine algebraic group has a normal subgroup \( G^0 \leq G \) that is the maximal connected subgroup of \( G \) containing the identity. It is called the **neutral component** of \( G \).

We now discuss the Jordan decomposition. Let \( M_n \) denote the scheme of \( n \times n \) matrices over \( k \). One reference is [Mil17, §9.b].

**Definition 1.13.** Let \( k \) be a perfect field. An element \( x \in M_n(k) \) is said to be **semisimple** if there exists \( g \in \text{GL}_n(\overline{k}) \) such that \( g^{-1}xg \) is diagonal, **nilpotent** if there exists a positive integer \( n \) such that \( x^n = 0 \), and **unipotent** if \( (x - I) \) is nilpotent. For an arbitrary linear group, we say that an element \( x \in G(k) \) is **semisimple** (resp. **unipotent**) if \( r(x) \) is so for some faithful representation \( r : G \to \text{GL}_V \).

It turns out that if \( x \in G(k) \) has the property that \( r(x) \) is semisimple (resp. unipotent) for some faithful representation \( r \) then it is semisimple (resp. unipotent) for any faithful representation \( r \).

**Theorem 1.5.3 (Jordan decomposition)** Let \( G \) be an affine algebraic group over a perfect field \( k \). Given \( x \in G(k) \), there exist unique \( x_s, x_u \in G(k) \) such that \( x = x_s x_u = x_u x_s \), where \( x_s \) is semisimple and \( x_u \) is unipotent.  

This leads us to the notion of a unipotent group. An affine algebraic group over \( k \) is **unipotent** if every representation of \( G \) has a fixed vector. If \( G \) is smooth then \( G(k) \) consists of unipotent elements if and only if \( G \) is unipotent [Mil17, Corollary 14.12]. We remark that unipotent groups are always upper-triangularizable. More precisely, an affine algebraic group \( G \) is unipotent if and only if there exists a faithful representation \( r : G \to \text{GL}_n \) such that the image of \( G \) is a subgroup of the group of upper triangular matrices in \( \text{GL}_n \) [Mil17, Theorem 14.5].

To define the weaker notion of a solvable group, we recall that the **derived subgroup** \( G^{\text{der}} := DG \) of an affine algebraic group \( G \) is the intersection of all normal subgroups \( N \leq G \) such that \( G/N \) is commutative. If \( G \) is connected then \( G^{\text{der}} \) is connected. One has that

\[
G^{\text{der}}(\overline{k}) = \langle xyx^{-1}y^{-1} : x, y \in G(\overline{k}) \rangle.
\]

In analogy with the case of abstract groups, we define

\[
D^nG := D(D^{n-1}G)
\]

inductively for \( n \geq 1 \) and say that \( G \) is **solvable** if \( D^nG \) is the trivial group for \( n \) sufficiently large.
Definition 1.14. Let \( G \) be a smooth affine algebraic group. The **unipotent radical** \( R_u(G) \) of \( G \) is the maximal connected normal unipotent subgroup of \( G \). The **(solvable) radical** \( R(G) \) of \( G \) is the maximal connected normal solvable subgroup of \( G \).

We remark that since a unipotent group is always solvable we have \( R_u(G) \leq R(G) \).

Definition 1.15. A smooth connected affine algebraic group \( G \) is said to be **reductive** if \( R_u(G_\mathbb{F}) = \{1\} \) and **semisimple** if \( R(G_\mathbb{F}) = \{1\} \).

If \( k \) is a perfect field, then \( R_u(G_\mathbb{F}) = R_u(G_\mathbb{F}) \). However, this is false in general for nonperfect fields. For more details see [CGP10]. We emphasize that, throughout this entire book, our convention is that reductive groups are connected.

For \( n > 1 \) the group of upper triangular matrices in \( \text{GL}_n \) is not reductive. The most basic example of a reductive group is \( \text{GL}_n \). It is not semisimple since \( Z_{\text{GL}_n} = R(\text{GL}_n). \) Here \( Z_G \) denotes the center of the affine algebraic group \( G \). The subgroup \( \text{SL}_n \leq \text{GL}_n \) is the derived group of \( \text{GL}_n \) and is semisimple (which implies reductive). In fact, for any reductive group \( G \), one has \( Z_G^0 = R(G) \) [Mil17, Corollary 17.62] and \( G^\text{der} \) is semisimple [Mil17, Proposition 19.21].

Assume that \( H_1 \) and \( H_2 \) are closed subgroup schemes of an affine algebraic group \( G \) over \( k \) such that \( H_1(R) \) normalizes \( H_2(R) \) for all \( k \)-algebras \( R \). We then have a product map \( H_1 \times H_2 \to G \) given on points in a \( k \)-algebra \( R \) by

\[
H_1(R) \times H_2(R) \to G(R)
(h_1, h_2) \mapsto h_1 h_2.
\]

We say \( H_1 H_2 = G \) if this is a quotient map.

Suppose that \( G \) is a reductive group. Then

\[
G = Z_G^0 G^\text{der}
\]  

[Mil17, §21.f]. We note that

\[
Z_G \cap G^\text{der}
\]

is the center of \( G^\text{der} \). It is a finite group scheme (see [Mil17, §11.a and 24.c]). For example, \( \text{GL}_n^\text{der} = \text{SL}_n \) and \( Z_{\text{GL}_n}(R) = R^e I_n \) for \( k \)-algebras \( R \).

We warn the reader that one has to be a bit careful with the statement that \( G = Z_G^0 G^\text{der} \), and with similar statements involving products of affine algebraic groups in a larger group below. In particular,

\[
Z_G^0(\mathbb{F}) \times G^\text{der}(\mathbb{F}) \to G(\mathbb{F})
\]

is surjective, but

\[
Z_G^0(k) \times G^\text{der}(k) \to G(k)
\]
need not be. This is false even for \( k = \mathbb{Q} \) and \( G = \text{GL}_2 \).

The following theorem of Mostow [Mos56] states that we can always break an affine algebraic group in characteristic zero into a reductive and unipotent part:

**Theorem 1.5.4** Let \( G \) be an affine algebraic group over a characteristic zero field \( k \). Then there is a subgroup \( M \leq G \) such that \( M^0 \) is reductive and

\[
G = MR_u(G).
\]

All such subgroups \( M \) are conjugate under \( R_u(G)(k) \).

This theorem motivates the following definition:

**Definition 1.16.** A subgroup \( M \leq G \) of a connected affine algebraic group \( G \) is a **Levi subgroup** if the restriction of the quotient map \( G_\bar{k} \to G_\bar{k}/R_u(G_\bar{k}) \) to \( M_\bar{k} \) induces an isomorphism

\[
M_\bar{k} \cong G_\bar{k}/R_u(G_\bar{k}).
\]

The reason that the definition is stated in terms of the algebraic closure \( \bar{k} \) is that, in positive characteristic, one can have \( R_u(G_\bar{k}) \leq N_\bar{k} \). This leads into the subject of pseudo-reductive groups (see [CGP10]). When there is a subgroup \( N \leq G \) such that \( N_\bar{k} = R_u(G_\bar{k}) \), if \( M \) is a Levi subgroup of \( G \) then \( G = MN \) and the product is semidirect. The decomposition \( G = MN \) is called a **Levi decomposition**. Levi decompositions do not exist in general in positive characteristic but they do exist for parabolic subgroups (see \$A.3\).

### 1.6 Lie algebras

Now that we have defined reductive groups, we could ask for a classification of them, or more generally for a classification of morphisms \( H \to G \) of reductive groups. The first step in this process is to linearize the problem using objects known as Lie algebras. We will return to the question of classification in \$1.7 and Theorem 1.8.3 below.

For simplicity we assume in this section that \( k \) is a field. For the general case we refer to [DG74, \$II].

**Definition 1.17.** A **Lie algebra** (over \( k \)) is a \( k \)-vector space \( \mathfrak{g} \) together with a bilinear pairing

\[
[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}
\]

satisfying the following assumptions:

(a) \([X, X] = 0\) for all \( X \in \mathfrak{g} \).
(b) For $X, Y, Z \in \mathfrak{g}$,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0. \quad (1.9)$$

Morphisms of Lie algebras are simply $k$-linear maps preserving $[\cdot, \cdot]$.

The pairing $[\cdot, \cdot]$ is known as the **Lie bracket** of the Lie algebra $\mathfrak{g}$ and the identity (1.9) is known as the **Jacobi identity**. We remark that if $k'/k$ is a field extension then $\mathfrak{g} \otimes_k k'$ inherits the structure of a Lie algebra over $k'$ in a natural manner.

Let $\text{LAG}_k$ denote the category of linear affine group schemes over $k$ and let $\text{LieAlg}_k$ denote the category of Lie algebras over $k$. There exists a functor

$$\text{Lie} : \text{LAG}_k \rightarrow \text{LieAlg}_k$$

defined by

$$\text{Lie} G = \ker(G(k[t]/t^2) \rightarrow G(k)).$$

Here the implicit map is evaluation at $t = 0$. From the definition just given, it is not clear that $\text{Lie} G$ is a Lie algebra, or even a $k$-vector space, for that matter. At the moment, it is only a set-valued functor. There is one important special case where it is easy to deduce a Lie algebra structure. Let $G = \text{GL}_V$ for a finite dimensional vector space $V$. In this case it is not hard to see that

$$\text{Lie} G = \{I_V + Xt : X \in \text{End}_{k, \text{linear}}(V)\} \xrightarrow{\sim} \text{End}_{k, \text{linear}}(V)$$

$$I_V + Xt \mapsto X.$$

This Lie algebra is traditionally denoted $\mathfrak{gl}_V$, or simply $\mathfrak{gl}_n$ when $V = k^n$. One defines

$$[X, Y] = X \circ Y - Y \circ X$$

and verifies that it satisfies the axioms for a Lie bracket.

In order to deduce that $\text{Lie} G$ is a Lie algebra in general, we essentially just embed $\text{Lie} G$ in $\mathfrak{gl}_V$, but in a functorial way. The first step is to give $\text{Lie} G$ a $k$-vector space structure. Let

$$\epsilon : \mathcal{O}(G) \rightarrow k \quad (1.10)$$

be the coidentity, i.e. the homomorphism corresponding to the identity element of $G(k)$.

**Lemma 1.6.1** There is a canonical bijection

$$\text{Lie} G \xrightarrow{\sim} \text{Hom}_{k, \text{linear}}(\ker(\epsilon)/\ker(\epsilon)^2, k).$$

In particular, $\text{Lie}$ is a functor from $\text{LAG}_k$ to the category of finite dimensional $k$-vector spaces.
Using the $k$-vector space structure of $\mathfrak{g}$ given above, we see that if $k'/k$ is a field extension then we obtain a map

$$(\operatorname{Lie} G) \otimes_k k' \to \operatorname{Lie} G_{k'}.$$ 

Since we have assumed $k$ is a field, it is bijective [Mil17, §10.19]. In other words, the functor $\operatorname{Lie}$ is compatible with base change.

**Proof.** An element of $\operatorname{Lie} G$ is a $k$-algebra homomorphism

$$\varphi : \mathcal{O}(G) \to k[t]/t^2$$

such that the composite with the natural map $k[t]/t^2 \to k$ is $\epsilon$. Thus $\varphi$ maps the ideal $\ker(\epsilon)$ into $t$ and hence factors through $\mathcal{O}(G)/\ker(\epsilon)^2$. Since $\mathcal{O}(G)/\ker(\epsilon)^2 = k \oplus \ker(\epsilon)/\ker(\epsilon)^2$ (see Exercise 1.9) we obtain a bijection

$$\operatorname{Lie} G \to \operatorname{Hom}_{k-\text{linear}}(\ker(\epsilon)/\ker(\epsilon)^2, k)$$

$$\varphi \mapsto \varphi|_{\ker(\epsilon)/\ker(\epsilon)^2}$$

as claimed. We leave it to the reader to check the functoriality assertion. $\square$

The group $G$ always admits a representation

$$\operatorname{Ad} : G \to \operatorname{GL}_{\operatorname{Lie} G}.$$

We define it briefly; more details are given in [Mil17, §10.d]. Let $R$ be a $k$-algebra; we must construct the homomorphism $\operatorname{Ad} : G(R) \to \operatorname{GL}_{\operatorname{Lie} G}(R)$. Note that the map $R \to R[t]/t^2$ giving $R[t]/t^2$ its $R$-algebra structure gives rise to an injection $G(R) \to G(R[t]/t^2)$. The conjugation action of $G(R[t]/t^2)$ on itself restricts to yield an action of $G(R)$ on $G(R[t]/t^2)$ that preserves $\ker(G(R[t]/t^2) \to G(R))$.

Arguing as in the proof of Lemma 1.6.1, the set $\ker(G(R[t]/t^2) \to G(R))$ can be given the structure of an $R$-module. Moreover the action of $G(R)$ on this module is $R$-linear. The natural map

$$(\operatorname{Lie} G) \otimes_k R \to \ker(G(R[t]/t^2) \to G(R))$$

is an $R$-module isomorphism, and we obtain the action denoted $\operatorname{Ad}$ above. It is known as the **adjoint representation**.

We finally give the reader a definition of the bracket on $\operatorname{Lie} G$. For this we note that, by functoriality of $\operatorname{Lie}$ as a $k$-vector space valued functor, the morphism $\operatorname{Ad}$ gives rise to a morphism of finite dimensional $k$-vector spaces

$$\operatorname{ad} : \operatorname{Lie} G \to \operatorname{Lie} GL_{\operatorname{Lie} G} = \mathfrak{gl}_{\operatorname{Lie} G}.$$ 

We then define
$[A, X] = \text{ad}(A)(X)$.

One checks that this is indeed a Lie bracket and that when $G = \text{GL}_V$ this recovers the earlier definition of the bracket.

In practice, to compute the Lie algebra of a linear affine group $G$, one just chooses an embedding $G \to \text{GL}_V$ and computes $\text{Lie} G$ in terms of the conditions that cut the group $G$ from $\text{GL}_V$. One then obtains the bracket for free; it is just the restriction of the bracket on $\text{GL}_V$.

**Example 1.7.** If $G = \text{SL}_n$, the special linear group of matrices in $\text{GL}_n$ of determinant 1, then

$$\det(I_n + Xt) = 1 + \text{tr}(X)t \pmod{t^2}$$

by Taylor expansion, so

$$\mathfrak{sl}_n := \text{Lie} \text{SL}_n = \{ X \in \mathfrak{gl}_n : \text{tr}(X) = 0 \}.$$

Notice how the use of the Taylor expansion amounted to differentiating some equation once and then taking the zero term; this method of determining Lie algebras works for at least all the classical groups, which are roughly subgroups of $\text{GL}_V$ defined as the subgroup fixing various perfect pairings on $V$ (see Exercise 1.9).

### 1.7 Tori

Throughout this section, we assume that $k$ is a field with separable closure $k^{\text{sep}}$ and algebraic closure $\overline{k}$. Set

$$\text{Gal}_k := \text{Gal}(k^{\text{sep}}/k).$$

**Definition 1.18.** An algebraic torus or simply torus is an affine algebraic group $T$ over $k$ such that $T_E \cong \mathbb{G}_m^n$ for some $n$. The integer $n$ is called the **rank of torus**.

We remark that if $T$ is a torus of rank $n$ over $k$ then there is a finite separable extension $L/k$ over which $T$ splits, in other words, $T_L \cong \mathbb{G}_m^n$. [Con14, Lemma B.1.5] (see Definition 1.20 below for the notion of a split torus).

**Definition 1.19.** A **character** of an affine algebraic group $G$ is an element of

$$X^*(G) = \text{Hom}(G, \mathbb{G}_m).$$

A **cocharacter** is an element of

$$X_*(G) = \text{Hom}(\mathbb{G}_m, G).$$
We observe that $X^*(G)$ is naturally an abelian group. If $G$ is abelian the set of cocharacters $X_*(G)$ is again a group, but this is false for general $G$.

We warn the reader that often the notation $X^*(G)_k$ is used for what we call $X^*(G)$ (with an analogous notation for cocharacter groups). The reason is that authors want to emphasize that the characters are defined over $k$ and not over some field extension. Writing $X^*(G)_k$ for $X^*(G)$ is redundant so we will not use this notation.

**Example 1.8.** The group $X^*(GL_n)$ is the free cyclic group generated by det. The homomorphism $\mathbb{G}_m \to GL_n$ given on points in a $k$-algebra $R$ by \[
R^\times \to GL_n(R) \\
t \mapsto \begin{pmatrix} t \\ I_{n-1} \end{pmatrix}
\]
is a cocharacter, as is any $GL_n(k)$-conjugate of it.

Characters determine tori in a sense we now make precise. Say a $\mathbb{Z}[\text{Gal}_k]$-module $M$ is continuous if the associated map $\text{Gal}_k \to GL_M(\mathbb{Z})$ is continuous when we give $\text{Gal}_k$ the usual profinite topology and $GL_M(\mathbb{Z})$ the discrete topology. In other words, $\text{Gal}_k \to GL_M(\mathbb{Z})$ is continuous if and only if it has finite image, or equivalently, factors through $\text{Gal}_{k'}$ for some finite separable extension $k' \leftarrow k$. For a free $\mathbb{Z}$-module $M$ let $k[M]$ denote the $k$-group algebra of $M$. If $M$ is a continuous $\mathbb{Z}[\text{Gal}_k]$-module, then the action of $\text{Gal}_k$ on $k^{sep}$ induces a $k^{sep}$-semilinear action of $\text{Gal}_k$ on $k^{sep}[M]$; we use this to define $(k^{sep}[M])^{\text{Gal}_k}$. For the proof of the following theorem we refer to [Mil17, §12.e, Theorem 12.23].

**Theorem 1.7.1** The association \[
T \mapsto X^*(T_{k^{sep}})
\]
defines a contravariant equivalence of categories between the category of algebraic tori defined over $k$ and finite rank $\mathbb{Z}$-torsion free continuous $\mathbb{Z}[\text{Gal}_k]$-modules. One has a canonical identification \[
\mathcal{O}(T) = (k^{sep}[X^*(T_{k^{sep}})])^{\text{Gal}_k}.
\]

We now record a few examples of tori.

**Example 1.9.** We define a special orthogonal group $SO_2 \leq GL_2$ by stipulating that for $\mathbb{Q}$-algebras $R$ one has that
\[ \text{SO}_2(R) = \{ \left( \begin{array}{cc} a & b \\ -b & a \end{array} \right) : a, b \in R \text{ and } a^2 + b^2 = 1 \} . \]

Over any field containing a square root of \(-1\), one has that
\[ \frac{1}{2} \left( \begin{array}{cc} 1 & i \\ i & 1 \end{array} \right) \left( \begin{array}{cc} a & b \\ -b & a \end{array} \right) \left( \begin{array}{cc} 1 & -i \\ -i & 1 \end{array} \right) = \left( \begin{array}{cc} a-bi & a+bi \\ -a+bi & a-bi \end{array} \right) . \tag{1.11} \]

It follows that \( (\text{SO}_2)_{\mathbb{Q}(i)} \cong \mathbb{G}_m \).

**Example 1.10.** Let \( L/k \) be a finite separable field extension and let
\[ N_{L/k} : \text{Res}_{L/k} \mathbb{G}_m \rightarrow \mathbb{G}_m \]
be the norm map; it is given on points by \( x \mapsto \prod_{\tau \in \text{Hom}_k(L,k)} \tau(x) \). Then the kernel of \( N_{L/k} \) is an algebraic torus. When \( k = \mathbb{Q} \) and \( L = \mathbb{Q}(i) \), this torus is isomorphic to the group \( \text{SO}_2 \) of the previous example.

**Example 1.11.** If \( L/k \) is any finite separable field extension, then \( \text{Res}_{L/k} \mathbb{G}_m \) is an algebraic torus. Moreover one can show that
\[ X^*((\text{Res}_{L/k} \mathbb{G}_m)_{L}) \cong \bigoplus_{\tau} \mathbb{Z}_\tau, \]
where the summation runs over the embeddings \( \tau : L \rightarrow k^{\text{sep}} \). In the special case where \( L/k \) is Galois, the left hand side admits a natural action of \( \text{Gal}(L/k) \) and, as a representation of \( \text{Gal}(L/k) \),
\[ X^*((\text{Res}_{L/k} \mathbb{G}_m)_{L}) \otimes \mathbb{C} \]
is isomorphic to the induced representation \( \text{Ind}_{1}^{\text{Gal}(L/k)}(1) \), where \( 1 \) denotes the trivial representation (of the trivial group).

The examples above illustrate that though an algebraic torus \( T \) satisfies \( T_{k^{\text{sep}}} \cong \mathbb{G}_m^n \), it may not be the case that \( T \cong \mathbb{G}_m^n \). This motivates the following definition:

**Definition 1.20.** An algebraic torus \( T \) over \( k \) is said to be **split** if \( T \cong \mathbb{G}_m^n \) over \( k \) or if equivalently \( X^*(T) \cong \mathbb{Z}^{\text{rank}(T)} \). An algebraic torus \( T \) is said to be **anisotropic** if \( X^*(T) = \{ \text{id} \} \).

Any torus \( T \) can be decomposed as \( T = T_s T_a \) where \( T_s \) is the maximal split subtorus, \( T_a \) is the maximal anisotropic subtorus, and \( T_a \cap T_s \) is finite.

**Definition 1.21.** A torus \( T \leq G \) is **maximal** if \( T_{\mathbb{C}} \) is a maximal element of the set of tori of \( G_{\mathbb{C}} \), partially ordered by inclusion. It is a **maximal split torus** if it is a maximal element of the set of split tori of \( G \), partially ordered by inclusion.

Thus a maximal split torus need not be a maximal torus. This terminology, though confusing, is standard.
1.7 Tori

**Theorem 1.7.2** Every connected smooth affine algebraic group $G$ over $k$ admits a maximal torus. All maximal split tori in $G$ are conjugate under $G(k)$.

*Proof.* The first result is due to Grothendieck (see [Con14, Appendix A] for instance). The second was announced by Borel and Tits; a reference is [CGP10, Theorem C.2.3].

In view of the second assertion of Theorem 1.7.2, the rank of a maximal split torus of $G$ is an invariant of $G$; it is known as the rank of $G$. For example, the rank of $\text{SL}_n$ is $n-1$, and the rank of $\text{GL}_n$ is $n$. The rank cannot decrease under base change, but it can increase: For example, the group $\text{SO}_2$ defined in Example 1.9 has rank 0, but $(\text{SO}_2)_{\mathbb{Q}(i)}$ has rank 1.

For the remainder of this section, $G$ is a reductive group and $T \hookrightarrow G$ is a torus. Let $N_G(T)$ be the normalizer of $T$ in $G$ and $C_G(T)$ is the centralizer of $T$ in $G$. The torus $T$ is maximal if and only if $C_G(T) = T$ [Mil17, Corollary 17.84].

**Definition 1.22.** The Weyl Group of $T$ in $G$ is

$$W(G \hookrightarrow T) = N_G(T) \rtimes C_G(T).$$

This brief definition hides some important subtleties as we now explain. A possible reference is [Con14, §3]. The basic reference for quotients by reductive groups in the affine case is [MFK94, §1.2]. This is explained in some detail in §17.1. The group schemes $N_G(T)$, $C_G(T)$ are algebraic subgroups of $G$ and $W(G, T)$ is the GIT quotient of $N_G(T)$ by $C_G(T)$.

It turns out that $W(G, T)$ is again a smooth (even étale) group scheme of finite type over $k$. In particular $W(G, T)(\overline{k})$ is finite. Moreover, one has that

$$W(G, T)(k) = W(G, T)(k^{\text{sep}})/C_G(T)(k^{\text{sep}}).$$

To describe $W(G, T)(R)$ for general $k$-algebras $R$ is involved, but at least for subfields $k \leq k' \leq k^{\text{sep}}$ we can describe it as follows. Since $N_G(T)$ and $C_G(T)$ are schemes over $k$, the set $N_G(T)(k^{\text{sep}})$ comes equipped with an action of $\text{Gal}_k$ preserving $C_G(T)(k^{\text{sep}})$; hence we also obtain an action of $\text{Gal}_k$ on $W(G, T)(k^{\text{sep}})$. Then

$$W(G, T)(k') = W(G, T)(k^{\text{sep}})^{\text{Gal}(k^{\text{sep}}/k')}.$$  

A nice discussion of this point, which is a special case of what is known as Galois descent, is contained in [Mum99, §II.4]. We wish to discuss these concepts in the special case where $G = \text{GL}_n$. Before beginning, we record the following definition:

**Definition 1.23.** A reductive group $G$ is said to be split if there exists a maximal torus of $G$ that is split.

If $T \leq \text{GL}_n$ is the torus of diagonal matrices then $T$ is a split maximal torus, and hence $\text{GL}_n$ is split. For any field $k'/k$, the Weyl group
W(GL_n, T)(k') can be identified with the group of permutation matrices and hence in this case W(GL_n, T)(k') is S_n, the symmetric group on n letters. More generally:

**Lemma 1.7.3** If T is a maximal split torus of a reductive group G over k then the group scheme W(G, T) is constant. In particular, for all fields k'/k one has

\[ W(G, T)(k') = N_G(T)(k')/C_G(T)(k'). \]

We emphasize that we are not assuming G is split in this lemma; thus T need not be a maximal torus.

**Proof.** See [Bor91, Chapter V, Theorem 21.2] (or [Mil17, Proposition 21.1] for an easier special case).

In general, if L/k is an étale k-algebra of rank n (for example, a field extension of degree n) then choosing a basis for L we obtain an embedding

\[ \text{Res}_{L/k} \mathbb{G}_m \rightarrow \text{GL}_n. \]

In particular, if L/k is Galois then W(GL_n, T)(k) \cong \text{Gal}(L/k). Every maximal torus in GL_n arises in this manner for some L/k (see Exercise 1.11).

After spending all this time discussing tori, we give one important motivation for introducing them. Suppose that G is a split reductive group and that T \unlhd G is a split maximal torus over k. Given any representation

\[ r : G \rightarrow \text{GL}_V, \]

it follows from the fact that T is abelian and reductive that we can decompose V as

\[ V = \bigoplus_{\alpha \in X^*(T)} V_\alpha \tag{1.13} \]

where for k-algebras R

\[ V_\alpha \otimes R = \{ v \in V \otimes R : r(t).v = \alpha(t).v \text{ for all } t \in T(R) \}. \tag{1.14} \]

The \( \alpha \) occurring in this decomposition are known as **weights** and the \( V_\alpha \) are known as **weight spaces**. The dimension \( \text{dim} V_\alpha \) is known as the **multiplicity** of the weight \( \alpha \).

One has the following basic theorem:

**Theorem 1.7.4** Suppose that G is a split reductive group and that V is a representation of G as above. The collection of weights \( \alpha \) together with the weight multiplicities \( \text{dim} V_\alpha \) determine the representation V up to isomorphism.
It is not hard to see that the set of weights is invariant under the natural action of $W(G, T)(k)$ on $X^*(T)$. Though the action does not preserve individual weights, it does preserve the multiplicities of weights.

One can place a partial ordering on $X^*(T)$ such that each representation of $G$ admits a highest weight with respect to the ordering, unique up to the action of $W(T, G)(k)$. Moreover, one can characterize the set of highest weights that can occur. This is known as Cartan-Weyl highest weight theory (see [Mil17], Chapter 22 for instance).

The key fact to take from this discussion is that the representation theory of $G$ is determined by the restriction of the representation to a sufficiently large subgroup (in this case a maximal torus). Happily, the representation theory of a maximal torus is fairly simple and can be studied via linear algebra. The idea of studying representations by studying their restriction to subgroups is an incredibly useful one; it will come up in various guises throughout this book.

### 1.8 Root data

Let $G$ be a split reductive group over a field $k$ and let $T \leq G$ be a split maximal torus. Our next goal is to associate to such a pair $(G, T)$ a root datum $\Psi(G, T) = (X^*(T), X_*(T), \Phi(G, T), \Phi^i(G, T))$. The root datum is a refinement of the related notion of a root system (also defined below) that Demazure introduced to systematically keep track of the central torus of $G$ [DG74, Expose XXII]. The root datum characterizes $G$, and in fact Demazure proves that it characterizes $G$ even in the case where $k$ is replaced by $\mathbb{Z}$. This in some sense completed work of Chevalley who proved the analogous statement for semisimple groups over fields [Che58].

Let $\mathfrak{g}$ denote the Lie algebra of $G$. As we saw in §1.6, one has an adjoint representation

$$\text{Ad} : G \rightarrow \text{GL}_\mathfrak{g}, \quad (1.15)$$

For example, when $G = \text{GL}_n$ this is the usual action of $\text{GL}_n$ on the space of $n \times n$ matrices $\mathfrak{gl}_n$ by conjugation.

By restriction we obtain an action of $T$ on $\mathfrak{g}$ which we decompose into weight spaces as in (1.13). For a character $\alpha \in X^*(T)$, let $\mathfrak{g}_\alpha$ be the corresponding weight space. Thus for $k$-algebras $R$,

$$\mathfrak{g}_\alpha \otimes_k R := \{ X \in \mathfrak{g} \otimes_k R \mid \text{Ad}(t)X = \alpha(t)X \text{ for all } t \in T(R) \}. \quad (1.16)$$

**Definition 1.24.** The nontrivial $\alpha \in X^*(T)$ such that $\mathfrak{g}_\alpha \neq 0$ are called the roots of $T$ in $G$. We let $\Phi(G, T)$ be the (finite) set of all such roots $\alpha$, and call the corresponding $\mathfrak{g}_\alpha$ root spaces.
Just as in (1.13) we have a decomposition

$$g = t \oplus \bigoplus_{\alpha \in \Phi(G,T)} g_{\alpha}$$  \hspace{1cm} (1.17)

where $t := \text{Lie } T$. It turns out that each of the root spaces $g_{\alpha}$ are one dimensional.

One turns the set of roots $\Phi(G,T)$ into a combinatorial gadget as follows. Let

$$V := \langle \Phi(G,T) \rangle \otimes_{\mathbb{Z}} \mathbb{R},$$

where $\langle \Phi(G,T) \rangle \subset X^* (T)$ denotes the ($\mathbb{Z}$-linear) span of $\Phi(G,T)$. The pair $(\Phi(G,T), V)$ satisfies remarkable symmetry properties that are axiomatized in the following:

**Definition 1.25.** Let $V$ be a finite dimensional $\mathbb{R}$-vector space, and $\Phi$ a subset of $V$. We say that $(\Phi, V)$ is a root system if the following three conditions are satisfied:

(a) $\Phi$ is finite, does not contain 0, and spans $V$;
(b) For each $\alpha \in \Phi$ there exists a reflection $s_{\alpha}$ relative to $\alpha$ (i.e. an involution of $V$ with $s_{\alpha}(\alpha) = -\alpha$ and restricting to the identity on a subspace of $V$ of codimension 1) such that $s_{\alpha}(\Phi) = \Phi$;
(c) For every $\alpha, \beta \in \Phi$, $s_{\alpha}(\beta) - \beta$ is an integer multiple of $\alpha$.

A root system $(\Phi, V)$ is said to be of rank $\dim_{\mathbb{R}} V$, and is said to be reduced if for each $\alpha \in \Phi$, $\pm \alpha$ are the only multiples of $\alpha$ in $\Phi$.

We will not need the notion until §1.9, but a subset

$$\Delta \subset \Phi$$ \hspace{1cm} (1.18)

is a base or a set of simple roots if it is a basis of $V$ and each $\alpha \in \Phi$ can be uniquely expressed as

$$\alpha = \sum_{\beta \in \Delta} c_{\beta} \beta,$$

where either $c_{\beta} \geq 0$ for all $\beta \in \Delta$ or $c_{\beta} \leq 0$ for all $\beta \in \Delta$. Any root system admits a base. We define

$$\Phi^+ \subset \Phi \quad (\text{resp. } \Phi^- \subset \Phi)$$

to be the set of roots expressible as a sum of positive (resp. negative) linear combinations of a nonempty subset of elements in the base. Then $\Phi = \Phi^+ \coprod \Phi^-$. Conversely, suppose that we are given a subset $\Phi^+ \subset \Phi$ such that

(a) for every $\alpha \in \Phi$, exactly one of $\pm \alpha$ is in $\Phi^+$;
(b) if $\alpha, \beta \in \Phi^+$ satisfy $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Phi^+$.
We then call $\Phi^+$ a set of positive roots. Setting $\Phi^- = \{-\alpha : \alpha \in \Phi^+\}$ we have $\Phi = \Phi^+ \coprod \Phi^-$. We then say an element of $\Phi^+$ is simple with respect to $\Phi$ if it cannot be written as a sum of two elements of $\Phi^+$. The set of elements that are simple with respect to $\Phi^+$ then forms a base (or in other words a set of simple roots). Thus the notion of a base and a set of positive roots are equivalent concepts. We refer to [Hum78, §10] for proofs of the assertions in this paragraph.

The Weyl group of $(\Phi, V)$ is the subgroup of $GL(V)$ generated by the reflections $s_\alpha$:

$$W(\Phi, V) := \langle s_\alpha : \alpha \in \Phi \rangle \subseteq GL(V).$$

The following is [Mil17, Corollary 21.38]:

**Proposition 1.8.1** If

$$(\Phi, V) = (\Phi(G, T), (\Phi(G, T)) \otimes \mathbb{Z} \otimes \mathbb{R})$$

is the root system associated with the split maximal torus $T \leq G$ over $k$ then $(\Phi, V)$ is reduced and

$$W(\Phi, V) \cong W(G, T)(k).$$

We observe that $\Phi(G, T)$ depends only on the Lie algebra of $G$. In particular, if $G$ is semisimple, it does not change if we replace $G$ by the adjoint group $G^{ad} := G/Z_G$. Thus if we wish to classify reductive groups it does not provide enough information. We thus incorporate more information from $G$ in order to create an invariant, called the root datum, which determines $G$ up to isomorphism. We refer to [Mil17, §21.c] for more details on the discussion that follows.

For $\alpha \in \Phi(G, T)$, let $T_\alpha$ be the maximal subtorus of $\ker(\alpha) < T$ and let

$$G_\alpha := C_G(T_\alpha) \quad (1.19)$$

be its centralizer. Then $G_\alpha$ is a split reductive group and $T < G_\alpha$ is a maximal torus. The derived group $G_\alpha$ has rank 1 (and hence has Lie algebra isomorphic to $\mathfrak{sl}_2$). One has

$$\mathrm{Lie} G_\alpha = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}.$$

There is a unique unipotent subgroup

$$N_\alpha < G_\alpha \quad (1.20)$$

normalized by $T$ with Lie algebra $\mathfrak{g}_\alpha$. The group $N_\alpha$ is called the root group of $\alpha$. One has that $N_\alpha \cong \mathbb{G}_a$.

The Weyl group $W(G_\alpha, T)$ is the constant group scheme with

$$W(G_\alpha, T)(k) = \langle s_\alpha \rangle \cong \mathbb{Z}/2.$$
The reflection $s_\alpha$ can be described as follows. Given $(\alpha, \beta) \in X^*(T) \times X_*(T)$ we have a homomorphism $\alpha \circ \beta : \mathbb{G}_m \to \mathbb{G}_m$. Thus there is a perfect pairing
\[
\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}
\] (1.21)
given stipulating that
\[
\alpha \circ \beta(a) = a^{(\alpha,\beta)}
\]
where $a \in R^\times$ for a $k$-algebra $R$. There is a unique $\alpha^\vee \in X_*(T)$ such that
\[
s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha
\]
for all $x \in X^*(T)$. The element $\alpha^\vee \in X_*(T)$ is called the coroot associated to $\alpha$. One has that $\langle \alpha, \alpha^\vee \rangle = 2$. Let
\[
\Phi^\vee(G,T) = \{ \alpha^\vee : \alpha \in \Phi(G,T) \} \quad \text{and} \quad V^\vee := \langle \Phi^\vee(G,T) \rangle \otimes_{\mathbb{Z}} \mathbb{R}.
\]
Then one has the following lemma:

**Lemma 1.8.2** The pair $(\Phi^\vee(G,T), V^\vee)$ is a root system.

The pair $(\Phi^\vee(G,T), V^\vee)$ is called the dual root system of $(\Phi(G,T), V)$.

Theorem 1.8.3 below is the fundamental result that the quadruple
\[
\Psi(G,T) = (X^*(T), X_*(T), \Phi(G,T), \Phi^\vee(G,T))
\]
attached to $T \leq G$ contains enough information to characterize $G$ among all split reductive $k$-groups. To make this precise, suppose we have quadruple $(X,Y,\Phi,\Phi^\vee)$ consisting of a pair of free abelian groups $X, Y$ with a perfect pairing $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{Z}$, finite subset $\Phi \subset X$ and $\Phi^\vee \subset Y$ and a bijective correspondence
\[
\Phi \longrightarrow \Phi^\vee \\
\alpha \longmapsto \alpha^\vee.
\]
For each $\alpha \in \Phi$, we let $s_\alpha : X \to X$ and $s_{\alpha^\vee} : Y \to Y$ be the endomorphism given by
\[
s_\alpha(x) := x - \langle x, \alpha^\vee \rangle \alpha, \\
s_{\alpha^\vee}(y) := y - \langle \alpha, y \rangle \alpha^\vee
\]
for any $x \in X$ and any $y \in Y$.

**Definition 1.26.** The quadruple $(X,Y,\Phi,\Phi^\vee)$ is a root datum if
\begin{itemize}
  \item[(a)] for each $\alpha \in \Phi$, $\langle \alpha, \alpha^\vee \rangle = 2$ and
  \item[(b)] for each $\alpha \in \Phi$, $s_\alpha(\Phi) \subseteq \Phi$ and $s_{\alpha^\vee}(\Phi^\vee) \subseteq \Phi^\vee$.
\end{itemize}

We say that a root datum is reduced if for each $\alpha \in \Phi$ the only multiplies of $\alpha$ that are in $\Phi$ are $\pm \alpha$. 
1.8 Root data

For more information about root data, we recommend [Mil17, Appendix C]. One can find a proof in [Mil17, Corollary 21.12] that \( \Psi(G, T) \) is a reduced root datum.

**Definition 1.27.** A central isogeny of root data

\[
(X', Y', \Phi', \Phi^\vee) \to (X, Y, \Phi, \Phi^\vee)
\]

is an injective morphism \( f : X' \to X \) with finite cokernal such that there is a bijection \( \iota : \Phi \to \Phi' \) satisfying

\[
f(\iota(\alpha)) = \alpha \quad \text{and} \quad f^\vee(\alpha^\vee) = \iota(\alpha)^\vee
\]

for all \( \alpha \in \Phi \). Here \( f^\vee : Y \to Y' \) is the dual of \( f \). A central isogeny is an isomorphism of root data if \( f \) is an isomorphism of \( \mathbb{Z} \)-modules.

This allows us to define the category

\[
\text{RRD}
\]

of reduced root data; the objects are reduced root data and the morphisms are central isogenies of reduced root data.

The definition of a central isogeny of root data is motivated by the definition of a central isogeny of affine algebraic groups. For the moment, we drop our assumption that \( G \) is reductive.

**Definition 1.28.** A central isogeny of connected affine algebraic groups \( G \) over a field \( k \) is a surjective homomorphism with finite kernel contained in the center of \( G \).

Here when we say a “finite kernel” we mean that the kernel is a finite group scheme (see [Mil17, Chapter 11]).

For more information on the statements in this paragraph, we refer to [Mil17, §23.f]. A semisimple group \( G \) is simply connected if any central isogeny \( H \to G \) is an isomorphism, and adjoint if any central isogeny \( G \to H \) is an isomorphism. Any semisimple group \( G \) sits in a sequence of central isogenies

\[
G^{sc} \to G \to G^{ad} := G/Z_G
\]

where \( G^{sc} \) is simply connected and \( G^{ad} \) is adjoint. If \( G \) is split the groups \( G^{sc} \) and \( G^{ad} \) are uniquely determined by the root system of \( G \). If \( G \to H \) is a central isogeny between two semisimple \( k \)-groups then \( G \) and \( H \) have the same root system, but not necessarily the same root datum. In general, simply connected groups have the largest center out of the set of semisimple groups with the same root system and adjoint groups have trivial center. For example, \( \text{SL}_n \) is simply connected and \( \text{PGL}_n \) is adjoint. The quotient map \( \text{SL}_n \to \text{PGL}_n \) is a central isogeny.
We now return to the case where \( G \) is a split reductive group over \( k \). We note that an isomorphism of reductive groups over \( k \) gives rise to an isomorphism of their underlying root data. Indeed, if \( G \to G' \) is an isomorphism of reductive groups over \( k \), then upon precomposing the isomorphism with an inner automorphism of \( G \) and postcomposing with an inner automorphism of \( G' \), we can assume that the isomorphism maps a given maximal torus \( T \) of \( G \) to a given maximal torus \( T' \) of \( G' \), and hence induces an isomorphism of the underlying root data. A similar statement is true for general central isogenies [Mil17, Proposition 23.5].

Let

\[
\text{Spl}_k
\]

be the category whose objects are pairs \((G, T)\) where \( G \) is a split reductive group over \( k \) and \( T \leq G \) is a split maximal torus. The morphisms \((G, T) \to (G', T')\) are central isogenies taking \( T \) to \( T' \) up to equivalence, where we say two morphisms are equivalent if they differ by conjugation by an element of \((T'/Z_G)(k)\). A morphism \((G, T) \to (G', T')\) in \( \text{Spl}_k \) induces a central isogeny \( \Psi(G', T') \to \Psi(G, T) \) [Mil17, Proposition 23.5]. The following result not only implies that every root datum comes from an affine algebraic group, but also implies that every central isogeny of root data arises from a central isogeny of reductive groups [Mil17, Theorem 23.25]:

**Theorem 1.8.3 (Chevalley, Demazure)** Consider the category of pairs \((G, T)\) consisting of a split reductive group \( G \) over \( k \) and a split maximal torus \( T \leq G \). Then

\[
\begin{align*}
\text{Spl}_k & \to \text{RRD} \\
(G, T) & \mapsto \Psi(G, T)
\end{align*}
\]

is a contravariant equivalence of categories.

It is interesting to note that though \( \text{Spl}_k \) a priori depends on \( k \), \( \text{RRD} \) does not. In particular, if \( k'/k \) is an extension of fields, then we obtain a functor \( \text{Spl}_k \to \text{Spl}_{k'} \) by base change that commutes with the functors to \( \text{RRD} \):
Theorem 1.8.3 tells us that if we can classify root data, then we can classify split reductive groups. In fact, Killing obtained a classification of the underlying root systems (up to some mistakes) in [Kil88, Kil90]. Killing’s work was revisited and corrected in E. Cartan’s thesis. There are many references for the classification, including [Mil17, Appendix C.g]. One can essentially reduce the classification of root data to the classification of root systems as in [Mil17, Appendix C.f].

If \((X \hookrightarrow Y \hookrightarrow \_ \hookrightarrow \_ )\) is a root datum, then so is \((Y \hookrightarrow X \hookrightarrow \_ \hookrightarrow \_ )\). It is known as the dual root datum to \((X \hookrightarrow Y \hookrightarrow \_ \hookrightarrow \_ )\). If \((G \hookrightarrow T)\) then the root datum dual to \(\Phi(G, T)\) gives rise to a pair \((\hat{G}, \hat{T}) \in \text{Spl}_C\) called the complex dual of \((G, T)\) (or sometimes simply \(G\)). We note that there is an isomorphism

\[
W(G, T)(k) \cong \overline{\text{W}(\hat{G}, \hat{T})(\mathbb{C})}
\]

\[
s_\alpha \mapsto s_\alpha^\vee.
\]

The formation of the complex dual yields a contravariant equivalence of categories

\[
\text{Spl}_k \rightarrow \text{Spl}_C
\]

\[
(G, T) \mapsto (\hat{G}, \hat{T}).
\]

This implies, in particular, that if \(G\) is adjoint then \(\hat{G}\) is simply connected and if \(G\) is simply connected then \(\hat{G}\) is adjoint. The complex dual \(\hat{G}\) is a key component in the definition of Langlands dual group (see §7.3). We also note that if \(k'/k\) is an extension of fields then the dual groups of \((G, T)\) and \((G_k, T_k)\) may be canonically identified in view of the commutativity of the diagram (1.24).

Certainly, not every morphism of a reductive group is a central isogeny. One might ask if one could define in a natural way to extend the notion of a morphism of root data, and thereby use root data to classify morphisms between reductive groups. We are unaware of such a definition. However, it is the case that a great deal of information about morphisms between reductive groups can be deduced by considering root data. A systematic account of this for classical groups is given in Dynkin’s work [Dyn52].

We now record arguably the most basic example of a root datum:

**Example 1.12.** Let \(G = \text{GL}_n\). For a \(k\)-algebra \(R\), let

\[
T(R) := \left\{ \begin{pmatrix} t_1 & \cdots & t_n \\ \end{pmatrix} \mid t_i \in R^\times \right\}
\]
be the group of diagonal matrices in $\text{GL}_n(R)$. Then $T$ is a split maximal torus in $G$. The groups of characters and of cocharacters of $T$ are both isomorphic to $\mathbb{Z}^n$ via

$$(k_1, \ldots, k_n) \mapsto \left(\begin{array}{c} t_1 \\ \vdots \\ t_n \end{array}\right) \mapsto t_1^{k_1} \cdots t_n^{k_n},$$

and

$$(k_1, \ldots, k_n) \mapsto \left(\begin{array}{c} t_1^{k_1} \\ \vdots \\ t_n^{k_n} \end{array}\right),$$

respectively. Note that with these identifications, the natural pairing $(\ ,\ ) : X^*(T) \times X_*(T) \rightarrow \mathbb{Z}$ corresponds to the standard “inner product” in $\mathbb{Z}^n$. The roots of $G$ relative to $T$ are the characters $e_{ij}$:

$$e_{ij} : \left(\begin{array}{c} t_1 \\ \vdots \\ t_n \end{array}\right) \mapsto t_j t_i^{-1}$$

for every pair of integers $(i, j)$, $1 \leq i, j \leq n$ with $i \neq j$, and the corresponding root spaces $\mathfrak{gl}_{e_{ij}}$ are the linear span of the $n \times n$ matrix $X_{ij}$ with all entries zero except the $ij$-th entry and a 1 in the $ij$-th entry. The coroot $\check{e}_{ij}$ associated with $e_{ij}$ is the map sending $t$ to the diagonal matrix with $t$ in the $i$th entry and $t^{-1}$ in the $j$th entry and 1 in all other entries. The root group of $e_{ij}$ is

$$N_{e_{ij}}(R) := I_n + R e_{ij}. \quad (1.28)$$

### 1.9 Parabolic subgroups

We assume in this section that $G$ is a reductive group over a field $k$ with algebraic closure $\overline{k}$.

**Definition 1.29.** A subgroup $B \leq G$ is a **Borel subgroup** if $B_\overline{k}$ is a maximal smooth connected solvable subgroup of $G_\overline{k}$. A smooth subgroup $P \leq G$ is a **parabolic subgroup** if $P_\overline{k}$ contains a Borel subgroup of $G_\overline{k}$.

The group $G_\overline{k}$ trivially has Borel subgroups, and hence $G$ always has at least one parabolic subgroup, namely $G$ itself. A **proper** parabolic subgroup is a parabolic subgroup of $G$ not equal to $G$. In general, a reductive group need not have proper parabolic subgroups. For example, if $D$ is a division algebra with center $k$ with $\dim_k D > 1$ and $G$ is the affine algebraic group defined by

$$G(R) = (D \otimes_k R)\times$$

for a $k$-algebra $R$ then $G$ does not have a proper parabolic subgroup (see Exercise 1.12).

It is useful to isolate a weakening of the notion of a split reductive group:
Definition 1.30. A reductive group $G$ is said to be **quasi-split** if it contains a Borel subgroup.

One can show that $G$ is quasi-split if it is split [Spr09, Proposition 16.2.2], but that the converse is not true. Indeed, take $G$ to be the unitary group $U(1,1)$ over the real numbers whose points in an $\mathbb{R}$-algebra $R$ are

$$G(R) = \{g \in \text{GL}_2(\mathbb{C} \otimes_R R) : \overline{g}^t (1 \ 1) g = (1 \ 1)\}$$

where the bar denotes complex conjugation. Then the subgroup of upper triangular matrices in $G$ is a Borel subgroup of $G$. Thus $G$ is quasi-split. It is not, however, split.

From the optic of finite dimensional representation theory, the behavior of a split reductive group is essentially as simple as a group over an algebraically closed field (compare §1.7). Quasi-split groups are a little more technical to handle, but the existence of the Borel subgroup makes the theory not much more difficult. The fact that a general reductive group does not have a Borel subgroup (over the base field) creates more substantial problems. In this case, one has to do with a **minimal parabolic subgroup**. Of course, in the quasi-split case, a minimal parabolic subgroup is simply a Borel subgroup.

Basic facts about parabolic subgroups come up constantly in the theory of automorphic representations. For example, they are used to understand the structure of adelic quotients at infinity (see §2.7) and are used to describe the representation theory of a reductive group inductively (see §4.9, §8.2 and Chapter 10). Thus we record some of the basic structural facts about the set of parabolic subgroups of $G$ in this section. One reference is [Bor91, §20-21].

It is convenient to start with split tori in $G$. If there is no split torus contained in $G$ then $G$ is said to be **anisotropic**. Otherwise $G$ is said to be **isotropic**. The derived (sub)group $G^{\text{der}}$ is anisotropic if and only if the only parabolic subgroup of $G$ is $G$ itself. If $G$ is isotropic then there exists a maximal split torus $T \leq G$ unique up to conjugation. Just as in the case when $G$ is split (discussed in §1.8), we can decompose $\mathfrak{g}$ under the adjoint action (1.15) into eigenspaces under $T$:

$$\mathfrak{g} := \mathfrak{m} \oplus \bigoplus_{\alpha \in \Phi(G,T)} \mathfrak{g}_\alpha$$  \hspace{1cm} (1.30)

where $\mathfrak{g}_\alpha$ is defined as in (1.16). Here $\Phi(G,T)$ is the set of nonzero weights occurring in the decomposition above, and $\mathfrak{m}$ is the 0-eigenspace. The set $\Phi(G,T)$ is usually referred to as the set of relative roots (or more precisely, roots relative to $k$ as opposed to $\overline{k}$), but we will avoid this terminology because we will later discuss relative trace formulae, which have nothing to do with this notion of relative. The set $\Phi(G,T) \subset X^*(T) \otimes_\mathbb{R} \mathbb{R}$ is a root system, but it is not in general reduced. Just as in the split case, for each $\alpha \in \Phi(G,T)$ there is a unique unipotent subgroup
normalized by $T$ with Lie algebra $\mathfrak{g}_\alpha$ [Mil17, 25.19]. The group $N_\alpha$ is known as the root group of $\alpha$. The group $N_\alpha$ is not isomorphic to $\mathbb{G}_a$ in general, but it does admit a subnormal series with successive quotients isomorphic to $\mathbb{G}_a$ [Mil17, 25.19] (in positive characteristic, this is stronger than the assertion that $N_\alpha$ is unipotent).

Just as all the maximal split tori in $G$ are conjugate under $G(k)$, the minimal parabolic subgroups are also conjugate under $G(k)$. Moreover, every minimal parabolic subgroup contains the centralizer $C_G(T)$ of a maximal split torus $T \leq G$. Thus to describe all minimal parabolic subgroups, it suffices to fix a maximal split torus $T \leq G$ and describe all minimal parabolic subgroups containing $C_G(T)$. Let $\Delta \subset \Phi(G,T)$ be a base. Then we can consider the set of positive roots $\Phi^+$ with respect to $\Delta$. Then there is a unique minimal parabolic subgroup $P_0$ containing $C_G(T)$ whose unipotent radical $N_0 < P_0$ satisfies

$$\text{Lie } N_0 = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha.$$  

$\square$

**Theorem 1.9.1** The correspondence described above defines a bijection

$$\{\text{bases } \Delta \subseteq \Phi(G,T)\} \overset{\sim}{\longrightarrow} \{\text{minimal parabolic subgroups } P_0 \geq C_G(T)\}.$$

If we fix a minimal parabolic subgroup $P_0$, then the parabolic subgroups containing $P_0$ are called standard. Every parabolic subgroup is conjugate under $G(k)$ to a unique standard parabolic subgroup, and two standard parabolic subgroups are $G(\bar{k})$-conjugate if and only if they are equal. Thus to describe all parabolic subgroups of $G$, it suffices to fix a minimal parabolic subgroup $P_0$ of $G$ and describe all standard parabolic subgroups (with respect to $P_0$).

To accomplish this, let $J \subseteq \Delta$ be a subset. Let

$$\Phi(J) := \mathbb{Z}J \cap \Phi(G,T).$$  

(1.32)

There is a unique parabolic subgroup $P_J \geq P_0$ with unipotent radical $N_J$ such

$$\text{Lie } N_J = \bigoplus_{\alpha \in \Phi^+ - (\Phi(J) \cap \Phi^+)} \mathfrak{g}_\alpha.$$  

For the proof of the following theorem, see [Bor91, Proposition 21.12]:

**Theorem 1.9.2** There is a bijective correspondence

$$\{J \subseteq \Delta\} \overset{\sim}{\longrightarrow} \{\text{standard parabolic subgroups of } G\}$$

$$J \longmapsto P_J.$$
We observe that the parabolic subgroup corresponding to $\emptyset$ is $P_0$ and the parabolic subgroup corresponding to $\Delta$ is $G$. For each $J \subset \Delta$, let

$$M_J$$

be the subgroup of $G$ generated by $C_G(T)$ and the $G_{\alpha}$ with $\alpha \in \Phi(J)$ (see (1.19) for the definition of $G_{\alpha}$). The following is [Spr09, Lemma 15.4.5]:

**Lemma 1.9.3** The subgroup $M_J$ of $G$ is a Levi subgroup of $P_J$. □

The two bijections in theorems 1.9.1 and 1.9.2 allow us to define the notion of a parabolic subgroup opposite to a given parabolic subgroup. More precisely, suppose we are given a standard parabolic subgroup $P \leq G$ with unipotent radical $N$ containing the centralizer of a maximal split torus $T$ of $G$, i.e., $C_G(T) \leq P \leq G$. Then one has an **opposite parabolic** $P^-$ constructed as follows: If $P = P_0$ is minimal, then we take $P_0^-$ to be the minimal parabolic subgroup attached to the base

$$-\Delta := \{ \alpha \in \Phi : -\alpha \in \Delta \}.$$

If $P = P_J$, then we define $P^-$ to be the parabolic subgroup containing the minimal parabolic subgroup $P_0^-$ that is attached to the subset

$$-J := \{ \alpha \in \Phi : -\alpha \in J \} \subseteq -\Delta.$$

Still assuming $P = P_J$, we have

$$P_J \cap P^- = M_J.$$

The unipotent radical of $P^-$ is usually denoted $N^-$. We collect the next two propositions for our use in §10.1. The first is an immediate consequence of [Spr09, Corollary 8.4.4]:

**Proposition 1.9.4** Let $P$ be a parabolic subgroup of $G$ and let $T \leq P$ be a maximal torus. Then there is at most one Levi subgroup of $P$ containing $T$. □

**Proposition 1.9.5** Let $P_0 \leq G$ be a minimal parabolic subgroup and let $M_0 \leq P_0$ be a Levi subgroup. If $P_0 \leq P \leq G$ then there is a unique Levi subgroup $M \leq P$ such that $M_0 \leq M$. □

**Proof.** Let $T \leq M_0$ be a maximal split torus and let $T' \leq M_0$ be a maximal torus containing $T$. Use $T$ to define $\Phi(G,T)$ and $P_0$ to define a set of simple roots $\Delta$. Thus the groups $M_J$ and $P_J$ are defined for all $J \subseteq \Delta$. We observe that $M_0$ and $C_G(T) = M_0$ both contain $T'$, hence $M_0 = M_0$ by Proposition 1.9.4.

We have $P = P_J$ for some subset $J \subseteq \Phi$ by Theorem 1.9.2, and hence $M := M_J$ is a Levi subgroup by Lemma 1.9.3. This proves existence. Clearly
Example 1.13. The subgroup \( B \leq \text{GL}_n \) of upper triangular matrices is a Borel subgroup. Throughout this book when we speak of standard parabolic subgroups in \( \text{GL}_n \) we will mean parabolic subgroups that are standard with respect to this choice of Borel subgroup. The base of \( \Phi(G, T) \) and set of positive roots associated to \( B \) are

\[
\Delta := \{ e_{i(i+1)} : 1 \leq i \leq n - 1 \} \quad \text{and} \quad \Phi^+ := \{ e_{ij} : i < j \},
\]

respectively. Here \( e_{ij} \) is defined as (1.27). There are bijections

\[
\left\{ n_1, \ldots, n_d \in \mathbb{Z}_{>0} : \sum_{i=1}^d n_i = n \right\} \sim 2^{(1, \ldots, n-1)} \sim 2^\Delta,
\]

where \( 2^X \) is the set of subsets of \( X \). Here the first bijection sends \( (n_1, \ldots, n_d) \) to \( \Delta - \{ n_1, n_1 + n_2, \ldots, n_1 + \cdots + n_{d-1} \} \) and the second is induced by the bijection \( \{ 1, \ldots, n-1 \} \sim \Delta \) sending \( i \) to \( e_{i(i+1)} \). Thus the standard parabolic subgroups of \( \text{GL}_n \) correspond bijectively to ordered tuples of positive integers \( n_1, \ldots, n_d \) such that \( \sum_{i=1}^d n_i = n \). The corresponding parabolic subgroup is the product \( MB \) where

\[
M(R) := \left\{ \begin{pmatrix} x_1 & & \\
 & \ddots & \\
& & x_d \end{pmatrix} : x_i \in \text{GL}_{n_i}(R) \right\}.
\]

We refer to this parabolic subgroup as the standard parabolic subgroup of type \((n_1, \ldots, n_d)\). The subgroup \( M(R) \) is referred to as the **standard Levi subgroup** of type \((n_1, \ldots, n_d)\).

Finally, it is sometimes useful to generalize the notion of roots still further. If \( T \) is any split torus in \( G \), we can decompose \( \mathfrak{g} \) into eigenspaces as before. We let \( \Phi(G, T) \) be the set of nonzero eigenspaces. Suppose that \( P \leq G \) is a parabolic subgroup and that \( T \leq P \) is a maximal split torus in the center of \( M \). In this case, the set of roots need not be a root system [Kna86, §V.5]. However, we can still define a partition \( \Phi^+ \sqcup \Phi^- = \Phi(G, T) \) such that \( \Phi^+ \) is the set of roots contained in the unipotent radical of \( P \) and \( \Phi^- \) is the set of roots contained in the unipotent radical of the opposite parabolic \( P^- \). The set \( \Phi^+ \) is the **set of positive roots defined by** \( P \). We will also require the notion of a positive Weyl chamber in this context. Let \( \Lambda \subset X^*(T) \otimes \mathbb{Z} \mathbb{R} \) denote the union of the hyperplanes on which an element of \( \Phi(G, T) \) vanishes. A **Weyl chamber** is a connected component of \( X^*(T) \otimes \mathbb{Z} \mathbb{R} - \Lambda \). The **positive Weyl chamber attached to** \( P \) is the unique Weyl chamber whose elements are positive \( \mathbb{R} \)-linear combinations of the positive roots defined by \( P \).
1.9 Parabolic subgroups

Exercises

1.1. Prove that a morphism of affine schemes induces a continuous map of the underlying topological spaces. If \( I \) is an ideal of \( A \), prove that the map of topological spaces induced by the morphism

\[
\text{Spec}(A/I) \rightarrow \text{Spec}(A)
\]

is a closed embedding with image \( V(I) \). Here \( V(I) \) is defined as (1.6).

1.2. Assume that \( X = \text{Spec}(\mathbb{C}[t_1, \ldots, t_n]/(f_1, \ldots, f_{n-d})) \) is a smooth scheme over the complex numbers. Show that \( X(\mathbb{C}) \) may be identified with a smooth complex manifold, namely the zero locus of the \( f_i \).

1.3. Let \( X \) and \( Y \) be affine schemes over the ring \( k \). Prove that giving a natural transformation of functors \( X \rightarrow Y \) is equivalent to giving a morphism of \( k \)-algebras \( \mathcal{O}(Y) \rightarrow \mathcal{O}(X) \). Deduce that \( \text{Spec} \) induces an anti-equivalence of categories from the category of \( k \)-algebras to the category of affine \( k \)-schemes.

1.4. Give examples of affine schemes of finite type over \( \mathbb{C} \) that are nonreduced, reducible, and reduced and irreducible.

1.5. Prove that if \( X \rightarrow Y \) is a closed immersion of affine schemes over \( k \) then \( X(R) \rightarrow Y(R) \) is injective for all \( k \)-algebras \( R \).

1.6. Let \( X, Y, Z \) be affine \( k \)-schemes equipped with morphisms \( f : X \rightarrow Y \) and \( g : Z \rightarrow Y \). Prove that

\[
(X \times_Y Z)(R) = X(R) \times_Y Z(R)
\]

for \( k \)-algebras \( R \).

1.7. Let \( G \) be an affine scheme over \( k \). Let \( \text{Id} : G \rightarrow G \) denote the identity morphism, let

\[
p_i : G \times G \rightarrow G
\]
denote the two projections, and let

\[
diag : G \rightarrow G \times G
\]
denote the diagonal map. We say that \( G \) is a group object in the category of affine \( k \)-schemes if there exist morphisms of affine \( k \)-schemes

\[
m : G \times G \rightarrow G,
\]

\[
e : \text{Spec}(k) \rightarrow G,
\]

\[
i : G \rightarrow G
\]
such that the following diagrams commute:
Here the unlabeled diagonal arrows are the canonical isomorphisms and the unlabeled vertical arrows are the structural morphism. Prove that $G$ is an affine group scheme over $k$ if and only if it is a group object in the category of affine $k$-schemes.

1.8. For $\mathbb{R}$-algebras $R$, define

$$U_n(R) := \{ g \in \text{GL}_n(\mathbb{C} \otimes \mathbb{R}) : gg^l = I_n \}$$

where the bar denotes the action of complex conjugation. Show that $U_n$ is an affine algebraic group over $\mathbb{R}$, that $U_n(\mathbb{R})$ is compact, and that $U_n(\mathbb{C}) \cong \text{GL}_n(\mathbb{C})$.

The group $U_n$ is called the **definite unitary group** over $\mathbb{R}$.

1.9. Let $\epsilon : \mathcal{O}(G) \to k$ be the coidentity defined as (1.10). Prove that

$$\mathcal{O}(G)/\ker(\epsilon)^2 = k \oplus \ker(\epsilon)/\ker(\epsilon)^2$$

as $k$-algebras.

1.10. Assume $k$ is a field of characteristic not 2 and $J \in \text{GL}_n(k)$ is an invertible symmetric or skew-symmetric matrix. For $k$-algebra $R$, define

$$G(R) := \{ g \in \text{GL}_n(R) : g^t J g = J \}.$$ 

Prove that $G$ is an affine algebraic group. Moreover, show that

$$\text{Lie} G = \{ X \in \mathfrak{gl}_n : X^t J + J X = 0 \}.$$

1.11. Let $k$ be a perfect field. View $\text{GL}_n$ as a reductive group over $k$. Prove that the set of conjugacy classes of maximal tori $T \leq \text{GL}_n$ is in natural bijection with étale $k$-algebras of degree $n$. 
1.12. Let $D$ be a division algebra over a field $k$ and $G$ be the affine algebraic group defined by

$$G(R) = (D \otimes_k R)^\times$$

for $k$-algebras $R$. Prove $G$ has no proper parabolic subgroups (over $k$).

1.13. Show that the complex dual of $\text{GL}_n$ is $\text{GL}_n\mathbb{C}$.

1.14. Compute the complex duals of $\text{SL}_n$ and $\text{PGL}_n$.

1.15. Let $P \leq G$ be a parabolic subgroup of a reductive group $G$ with Levi decomposition $P = MN$. Let $Q$ be a parabolic subgroup of $M$. Prove that $QN$ is a parabolic subgroup of $G$. 

Chapter 2
Adeles

Adeles make life possible.

J. Arthur

Abstract We review the adele ring and recall the basic properties of the adelic points of affine schemes.

2.1 Adeles

The arithmetic objects of interest in this book are constructed using global fields and their adele rings. We briefly summarize the construction of the adeles in this section; references include \cite{CF86, Neu99, RV99}.

Definition 2.1. A \textbf{global field} $F$ is a field which is a finite extension of $\mathbb{Q}$ or of $\mathbb{F}_p(t)$ for some prime $p$. Global fields over $\mathbb{Q}$ are called \textbf{number fields} while global fields over $\mathbb{F}_p(t)$ are called \textbf{function fields}.

To each global field $F$, one can associate an adele ring $\mathbb{A}_F$. Before defining this ring, we introduce the related notion of a valuation (or a finite place) of a global field.

Definition 2.2. Let $F$ be a field. A \textbf{(nonarchimedean) valuation} on $F$ is a map

$$v : F \longrightarrow \mathbb{R} \cup \{\infty\}$$

such that for all $a, b \in F$, $v$ satisfies the following:

(a) $v(a) = \infty$ if and only if $a = 0$.
(b) $v(a) + v(b) = v(ab)$.
(c) $v(a + b) \geq \min(v(a), v(b))$. 

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These axioms are designed so that if one picks $0 < \alpha < 1$ then

$$| \cdot |_v : F \rightarrow \mathbb{R}_{\geq 0}$$

$$x \mapsto \alpha^{v(x)}$$

(2.1)

is a nonarchimedean absolute value on $F$ in the sense of the following definition:

**Definition 2.3.** Let $F$ be a field. An absolute value

$$| \cdot |_v : F \rightarrow \mathbb{R}_{\geq 0}$$

is a function satisfying the following axioms:

(a) $|a|_v = 0$ if and only if $a = 0$.
(b) $|ab|_v = |a|_v |b|_v$.
(c) $|a + b|_v \leq |a|_v + |b|_v$.

It is nonarchimedean if it satisfies the following strengthening of (c):

(c') $|a + b|_v \leq \max(|a|_v, |b|_v)$.

If $| \cdot |_v$ satisfies (c) but not (c') we say $| \cdot |_v$ is archimedean.

We point out that (c') implies (c), but not conversely. We often implicitly exclude the trivial absolute value given by $|a|_v = 1$ for all $a \in F^\times$.

An absolute value $| \cdot |_v$ induces a metric on $F$ known as the $v$-adic metric. The completion of $F$ with respect to this metric is denoted $F_v$. This completion is a local field, that is, a field equipped with a nontrivial absolute value that is locally compact with respect to the induced metric. All local fields arise as the completion of some global field [Lor08, §25, Theorem 2].

**Definition 2.4.** A place of a global field $F$ is an equivalence class of absolute values, where two absolute values are said to be equivalent if they induce the same topology on $F$. A place is (non)archimedean if it consists of (non)archimedean absolute values.

We now describe these places and fix representative absolute values in each place. These representative absolute values are said to be the normalized absolute values. The places of a global field $F$ fall into two categories: the finite and infinite places.

The finite places are in bijection with the prime ideals of $\mathcal{O}_F$. The place $v$ associated to a prime $p$ of $\mathcal{O}_F$ is the equivalence class of an absolute value attached to the valuation

$$v(x) := \max\{ k \in \mathbb{Z} : x \in p^k \mathcal{O}_F \}.$$ 

The normalized absolute value in this equivalence class is

$$|x|_v = q_v^{-v(x)}$$
where \( q_v := |O_F/p| \). These valuations are all nonarchimedean.

The infinite places of a number field are all archimedean. They are indexed by embeddings \( \tau : F \hookrightarrow \mathbb{C} \) up to complex conjugation; the associated normalized absolute value is

\[
|x|_\tau := \begin{cases} 
|\tau(x)| & \text{if } \tau(F) \subseteq \mathbb{R}, \\
\tau(x)\overline{\tau(x)} & \text{if } \tau(F) \not\subseteq \mathbb{R}.
\end{cases}
\]

Here on the right the \(| \cdot |\) denotes the usual absolute value on \( \mathbb{R} \) and the bar denotes complex conjugation. Notice that in the complex case (i.e. where \( \tau(F) \not\subseteq \mathbb{R} \)), this is the square of the usual absolute value.

The infinite places of a function field are those attached to extensions of the absolute value

\[
\left|\frac{f(t)}{g(t)}\right|_{\infty} := p^{\deg f - \deg g}
\]

on \( \mathbb{F}_p(t) \). If \( F_v \) is the completion of \( F \) with respect to some valuation, the associated normalized absolute value is

\[
|x|_v := |N_{F_v/F_p}((t^{-1}))x|_{\infty}^{1/[F_v:F_p((t^{-1}))]}.
\]

Here we are using the fact that the completion of \( \mathbb{F}_p(t) \) at the infinite place is \( \mathbb{F}_p((t^{-1})) \). These valuations are nonarchimedean, in contrast to the number field case. Henceforth we will always take \(| \cdot |_v\) to be the unique normalized absolute value attached to \( v \).

For any place \( v \), write \( F_v \) for the completion of \( F \) with respect to some choice of absolute value associated to \( v \). Since the absolute values corresponding to \( v \) all induce the same topology on \( F \), the field \( F_v \) is independent of this choice. If \( v \) is finite, then the ring of integers of \( F_v \) is

\[
O_{F_v} = \{ x \in F_v : |x|_v \leq 1 \};
\]

it is a local ring with a unique maximal ideal

\[
\varpi_v O_{F_v} := \{ x \in F_v : |x|_v < 1 \}.
\]

Here \( \varpi_v \) is a uniformizer for \( F_v \), that is, a generator for the maximal ideal of \( O_{F_v} \).

Let us make these constructions explicit when \( F = \mathbb{Q} \). If \( p \in \mathbb{Z} \) is a prime, then completing \( \mathbb{Q} \) at the \( p \)-adic absolute value gives rise to the local field \( \mathbb{Q}_p \). Its ring of integers is \( \mathbb{Z}_p \) and the maximal ideal is \( p\mathbb{Z}_p \). The residue field is \( \mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z} \cong \mathbb{F}_p \), so the normalized absolute value is just the usual \( p \)-adic norm. There is only one infinite place of \( \mathbb{Q} \), denoted \( \infty \), and \( \mathbb{Q}_\infty \cong \mathbb{R} \). The normalized archimedean norm \(| \cdot |_\infty\) is the usual Euclidean norm on \( \mathbb{R} \).

The normalized absolute values above are chosen so that the following product formula holds:
Proposition 2.1.1  For \( x \in F^\times \), one has that
\[
\prod_v |x|_v = 1
\]
where the product is over all places \( v \) of \( F \). \( \square \)

Definition 2.5. Let \( F \) be a global field. The ring of adeles of \( F \), denoted by \( \mathbb{A}_F \), is the restricted direct product of the completions \( F_v \) with respect to the rings of integers \( \mathcal{O}_{F_v} \). In other words,
\[
\mathbb{A}_F = \left\{ (x_v) \in \prod_v F_v : x_v \in \mathcal{O}_{F_v} \text{ for all but finitely many places } v \right\}. \tag{2.2}
\]
The restricted product is usually denoted by a prime:
\[
\mathbb{A}'_F = \prod_v' F_v.
\]
Note that \( \mathbb{A}_F \) is a subring of the full product \( \prod_v F_v \). If \( S \) is a finite set of places of \( F \) then we write
\[
\mathbb{A}^S_F = \prod_{v \in S}' F_v := \left\{ (x_v) \in \prod_{v \in S} F_v : x_v \in \mathcal{O}_{F_v} \text{ for all but finitely many places } v \right\}
\]
and
\[
F_S := \mathbb{A}_{F,S} = \prod_{v \in S} F_v.
\]
Thus we may identify \( F_S \times \mathbb{A}^S_F = \mathbb{A}_F \).

For any finite sets of places \( S' \subseteq S \), we set
\[
F^{S'}_S := \prod_{v \in S - S'} F_v. \tag{2.3}
\]

We endow \( \mathbb{A}_F \) with the **restricted product topology**. This is defined by stipulating that open sets are the empty set together with all unions of sets of the form
\[
U_S = \prod_{v \in S} \mathcal{O}_{F_v}
\]
where \( S \) is a finite set of places of \( F \) including the infinite places and \( U_S \subseteq F_S \) is an open set. We give \( \mathbb{A}^S_F = 0 \times \mathbb{A}_F^S \subset \mathbb{A}_F \) the subspace topology and \( F_S \) the product topology. Then the identification \( F_S \times \mathbb{A}^S_F = \mathbb{A}_F \) becomes an isomorphism of topological rings. We will often use the following useful notation: if \( S \) is a finite set of places of \( F \) including the infinite places.
\[ \hat{O}_F^S := \prod_{v \notin S} O_{F_v}. \]

Let \( O_F^S \) denote the ring of \( S \)-integers of \( F \), that is, elements of \( F \) that are integral outside of \( S \). The notation \( \hat{O}_F^S \) is justified by the fact that \( \hat{O}_F^S \) is naturally isomorphic to the profinite completion of \( O_F^S \).

The topology on \( \mathbb{A}_F \) is not the same as the topology induced on \( \mathbb{A}_F \) by regarding it as a subset of the direct product \( \prod_v F_v \). While \( \prod_v F_v \) is not locally compact, one has the following result for \( \mathbb{A}_F \):

**Proposition 2.1.2** The adele ring \( \mathbb{A}_F \) of a global field \( F \) is a locally compact topological ring.

Here, and throughout this book, we take the convention that a locally compact space is Hausdorff.

**Proof.** We prove that \( \mathbb{A}_F \) is locally compact and leave the other details to the reader. For any finite set \( S \) of places of \( F \) including the infinite places, the subset

\[ \prod_{v \in S} F_v \times \hat{O}_F^S \]

is an open subring of \( \mathbb{A}_F \) for which the induced topology coincides with the product topology. The open subring \( (2.5) \) is locally compact by Tychonoff's theorem. Every \( x \in \mathbb{A}_F \) is contained in a set of the form \( (2.5) \), which shows that \( \mathbb{A}_F \) is locally compact. 

We note that the set of infinite places of \( F \) is often denoted by \( \infty \), and one often writes \( v|\infty \) or \( v \nmid \infty \) as shorthand for “\( v \) is an infinite place of \( F \)” and “\( v \) is a finite place of \( F \)” respectively. As an example of this notation, the ring \( \mathbb{A}_F^\infty \) is known as the finite adeles.

There is a natural diagonal embedding \( F \hookrightarrow \mathbb{A}_F \) given by sending an element of \( F \) to all of its completions. We often identify \( F \) with its image under the diagonal embedding. The quotient \( \mathbb{Q} \setminus \mathbb{A}_Q \) can be regarded as a generalization of \( \mathbb{Z} \setminus \mathbb{R} \). Indeed, one has a homeomorphism

\[ \mathbb{Z} \setminus \mathbb{R} \rightarrow \mathbb{Q} \setminus \mathbb{A}_Q / \hat{\mathbb{Z}} \]

\[ \mathbb{Z} + r \hookrightarrow \mathbb{Z} + (r,0) + \hat{\mathbb{Z}} \]

where we have identified

\[ \hat{\mathbb{Z}} := \prod_p \mathbb{Z}_p \]

with the subgroup \( \{0\} \times \hat{\mathbb{Z}} \) of \( \mathbb{A}_Q \) and \( (r,0) \) denotes the adele that is \( r \) at the infinite place and 0 at the finite places.

The subgroup \( \mathbb{Z} < \mathbb{R} \) is closed and discrete. The same is true of \( F < \mathbb{A}_F \):

**Lemma 2.1.3** The image of \( F \) in \( \mathbb{A}_F \) under the diagonal embedding is closed and discrete.
Proof. Choose \( x \in F^\times \). We will construct a neighborhood of \( x \) in \( \mathbb{A}_F \) containing no other element of \( F \). Since \( \mathbb{A}_F \) is a topological group under addition this will suffice to complete the proof.

For each finite place \( v \) of \( F \), let \( n_v = v(x) \), so that \( x \in \varpi_v^{n_v} \mathcal{O}_F \), but \( x \notin \varpi_v^{n_v+1} \mathcal{O}_F \) for all \( v \). Note that \( n_v = 0 \) for all but finitely many places. For each infinite place \( w \), let

\[
U_w := \left\{ y \in F_w : |y - x|_w < \prod_{v \mid \infty} |x|_v^{-1/|\infty|} \right\}.
\]

Here \(|\infty|\) is the number of infinite places of \( F \). Consider the open subset of \( \mathbb{A}_F \) defined by

\[
U = \prod_{v \mid \infty} U_v \times \prod_{v \mid \infty} \varpi_v^{n_v} \mathcal{O}_F.
\]

By construction, \( x \in U \). Suppose that \( y \in U \cap F \). Then \( |x - y|_v \leq |x|_v \) for all finite places \( v \). Thus

\[
\prod_{v} |x - y|_v \leq \prod_{v \mid \infty} |x|_v \times \prod_{v \mid \infty} |x - y|_v < 1.
\]

By the product formula, we conclude that \( x = y \). \( \Box \)

Given Lemma 2.1.3 the following theorem can be surprising the first time one sees it:

**Theorem 2.1.4 (Strong Approximation)** If \( S \) is any finite nonempty set of places of \( F \) then \( F \) is dense in \( \mathbb{A}_F^S \).

Thus omitting one place is enough to move \( F \) from being discrete to being dense. The proof can be found in any standard reference, see [Cas67, §15] for example.

**Example 2.1.** To understand Theorem 2.1.4, it is illustrative to consider the case of a real quadratic field \( F = \mathbb{Q}(\sqrt{d}) \). Suppose that \( d \) is square free such that \( d \equiv 3 \pmod{4} \); thus \( \mathcal{O}_F = \mathbb{Z}[\sqrt{d}] \). There are two infinite places \( \sigma_{\pm} \), both real, that are characterized by

\[
\sigma_{\pm}(\sqrt{d}) = \pm \sqrt{d}.
\]

Let \( \varepsilon > 0 \) and consider the open set \((-\varepsilon, \varepsilon) \times \hat{\mathcal{O}}_F \) in \( F_{\sigma_-} \times \mathbb{A}_F^\infty \). Theorem 2.1.4 implies that there are infinitely many elements of \( F \) in this open set, which is equivalent to the statement that there are infinitely many elements of \( \alpha \in \mathbb{Z}[\sqrt{d}] \) satisfying \( |\sigma_{-}(\alpha)| < \varepsilon \). This can be checked independently of Theorem 2.1.4 by observing that \( \sqrt{d} \) is irrational.

We close this section by remarking that one can construct analogues of the finite adeles in more general situations, for instance, schemes of finite type.
2.2 Adelic points of affine schemes

Weil and Grothendieck both gave approaches (under different hypotheses) to topologizing the points of schemes of finite type over a topological ring (for example, $k_F$). Conrad gave a beautiful exposition and elaboration in [Con12b]. We lift Theorem 2.2.1 below and its proof from loc. cit. with little modification.

The naïve idea is as follows. Given a topological ring $R$, $R^n$ inherits a natural topology (this is a special case of a fiber product topology). Given an affine scheme $X$, we choose a closed immersion $X \to \mathbb{G}_a^n$. This induces an injection

$$X(R) \hookrightarrow R^n.$$ 

Since we have endowed $R^n$ with a topology already, we can then give $X(R)$ the subspace topology. The problem with this method is that it is not clear that the topology on $X(R)$ is canonical, and it is also not clear that it is functorial with respect to morphisms $X \to Y$ of affine schemes.

Conrad reverses this naïve approach. He abstracts axiomatic properties we would like the topology on $X(R)$ to have, proves that these properties uniquely characterize a topology on $X(R)$ (if the topology exists) and then proves such a topology exists.

**Theorem 2.2.1** Let $R$ be a topological ring. There exists a unique way to topologize $X(R)$ for all affine schemes $X$ of finite type over $R$ such that

(a) the topology is functorial in $X$; that is if $X \to Y$ is a morphism of affine schemes of finite type over $R$, then the induced map on points $X(R) \to Y(R)$ is continuous;

(b) the topology is compatible with fibre products; this means that if $X \to Y$ and $Z \to Y$ are morphisms of affine schemes, all of finite type over $R$, then the topology on $(X \times_Y Z)(R)$ is the fibre product topology;

(c) closed immersions of schemes $X \to Y$ induce topological embeddings $X(R) \hookrightarrow Y(R)$;

(d) if $X = \text{Spec}(R[t])$ then $X(R)$ is homeomorphic with $R$ under the natural identification $X(R) \cong R$.

If $R$ is Hausdorff or locally compact, then so is $X(R)$. Moreover, if $R$ is Hausdorff then closed immersions induce closed embeddings, not just topological embeddings.

As above, we take the convention that locally compact spaces are Hausdorff. Assume that $R$ is Hausdorff. In the sequel, if we need to distinguish between

over the ring of integers of a global field [Hub91]. The construction is quite a bit more involved than that given above.
the Zariski topology on \( X \) and the topology on \( X(R) \) provided by the theorem 
above, we will refer to the topology on \( X(R) \) as the \textbf{Hausdorff topology} 
(the Zariski topology is rarely Hausdorff).

\textit{Proof.} Let \( X \) be an affine \( R \)-scheme of finite type. Pick an \( R \)-algebra isomorphism 
\[ A := \mathcal{O}(X) \cong R[t_1, \ldots, t_n]/I \] 
for an ideal \( I \), and identify \( X(R) \) with the subset of \( R^n \) on which the elements 
of \( I \) (thought of as polynomials on \( R^n \)) vanish.

We start with uniqueness. By our assumption on compatibility with fiber 
products, the natural bijection 
\[ \text{Spec}(R[t_1, \ldots, t_n]) \to R^n \] 
is a homeomorphism provided that we give the right hand side the product 
topology. By assumption, this induces a topological embedding \( X(R) \to R^n \). 
This completes the proof of uniqueness and also shows that \( X(R) \) is 
Hausdorff if \( R \) is Hausdorff. If \( R \) is Hausdorff, then \( 0 \in R \) is closed, so viewing \( X(R) \) 
as the vanishing locus of \( f \in I \) (viewed as polynomials on \( R^n \)), we see that 
\( X(R) \) is closed. Thus if \( R \) is locally compact then \( X(R) \) is as well.

We now prove existence. Consider the direct product \( R^A \) of \( A \) copies of \( R \), 
or more formally the set of all set-theoretic maps from \( A \) to \( R \). Since \( R \) is a topological ring, \( R^A \) comes equipped with the direct product topology. 
Note that there is a canonical and tautological injection 
\[ X(R) = \text{Hom}(A, R) \to R^A. \] 
We claim that the topology defined using (2.6) as above is the same as the 
subspace topology defined by the canonical injection \( X(R) \to R^A \), so it is 
independent of the choice of (2.6). Let \( a_1, \ldots, a_n \in A \) correspond to \( t_1 \) 
(mod \( I \)), \ldots, \( t_n \) (mod \( I \)) via (2.6), so the injection \( X(R) \to R^n \) defined by 
(2.6) is the composition of the natural injection \( X(R) \to R^A \) and the map 
\( R^A \to R^n \) given by projection to the factors indexed by \( (a_1, \ldots, a_n) \). Therefore 
every open set in \( X(R) \) (with respect to the topology induced by (2.6)) is 
the inverse image of an open set in \( R^A \) because \( R^A \to R^n \) is continuous. Since 
every element of \( A \) is an \( R \)-polynomial in \( a_1, \ldots, a_n \) and \( R \) is a topological 
ring, it follows that the map \( X(R) \to R^A \) is also continuous. Thus \( X(R) \) has 
been given the subspace topology from \( R^A \). This completes the proof of the 
claim. It also implies that the formation of the topology on \( X(R) \) is functorial 
(i.e. morphisms of affine schemes induce continuous maps on \( R \)-points).

Consider a closed immersion 
\[ i : X := \text{Spec}(A) \hookrightarrow \text{Spec}(A') =: X' \] 
corresponding to a surjective \( R \)-algebra map \( h : A' \to A \). The map
is visibly a topological embedding; it topologically identifies $R^A$ with the subset of $R^{A'}$ cut out by a collection of equalities among components. Moreover $j$ is a closed embedding when $R$ is Hausdorff. We have

$$X'(R) \cap j(R^A) = j(X(R))$$

because a set theoretic map $A \to R$ is an $R$-algebra homomorphism if and only if its composition with $h$ is an $R$-algebra map. Hence $i : X(R) \to X'(R)$ is an embedding of topolgival spaces and it is a closed embedding when $R$ is Hausdorff. By forming products of closed immersions into affine spaces $\mathbb{G}_a^n$, we see that

$$(X \times_{\text{Spec}(R)} X')(R) \to X(R) \times X'(R)$$

is a topological isomorphism via reduction to the trivial special case when $X$ and $X'$ are affine spaces.

Finally, we prove that for given maps $X \to Y$ and $Z \to Y$ between affine $R$-schemes, the bijection $(X \times_Y Z)(R) \to X(R) \times_Y Z(R)$ is a topological isomorphism. Consider the (tautological) isomorphism

$$X \times_Y Z \cong (X \times_R Z) \times_{Y \times_R Y} Y$$

and its topological counterpart. The product map

$$\mathcal{O}(Y) \otimes_R \mathcal{O}(Y) \to \mathcal{O}(Y)$$

is surjective and hence

$$Y \to Y \times_R Y$$

is a closed immersion (i.e. affine schemes are separated).

Since we have already checked compatibility with fiber products over $\text{Spec}(R)$, we see that we are reduced to the case in which one of the maps defining the fiber product is a closed immersion. We have already proven that closed immersions yield topological embeddings, so we deduce compatibility with fiber products.

From now on in this book, whenever $R$ is a topological ring and $X$ is an affine $R$-scheme of finite type we give $X(R)$ the topology described in Theorem 2.2.1.

We record the following easy consequence of Theorem 2.2.1:

**Proposition 2.2.2** If $R$ is a topological ring and $G$ is an affine group scheme of finite type over $R$, then $G(R)$ is a topological group.

**Proof.** The morphisms $G \times G \to G$ and $G \to G$ given by multiplication and the inverse, respectively, induce continuous maps $G(R) \times G(R) \to G(R)$ and $G(R) \to G(R)$ by Theorem 2.2.1. □
2.3 Relationship with restricted direct products

In practice, we will be interested in the topology on $G(A_F)$ given by Theorem 2.2.1 when $G$ is an affine algebraic group over the global field $F$. Let us discuss this in more detail, starting with $G = \text{GL}_n$. Consider the affine $\mathbb{Z}$-scheme of $n \times n$ matrices. There is a closed immersion $\text{GL}_n \to M_n \times \mathbb{G}_a$ given on points in a $\mathbb{Z}$-algebra $R$ by

$$\text{GL}_n(R) \to M_n(R) \times \mathbb{G}_a(R)$$

$$g \mapsto (g, \text{det}^{-1} g).$$

Thus for any topological ring $R$, the topology on $\text{GL}_n(R)$ afforded by Theorem 2.2.1 is the subspace topology if we view $\text{GL}_n(R)$ as a subspace of $M_n(R) \times \mathbb{G}_a(R)$ via (2.8). For example, if $v$ is an archimedean place of $F$, then by this recipe $\text{GL}_n(F_v)$ acquires its usual topology.

To describe the topology for nonarchimedean $v$, we recall that to describe a topology on a space, it suffices to give a neighborhood base for every point $x$, that is, a set of open neighborhoods $x \in U_\alpha$ such that every neighborhood of $x$ contains some $U_\alpha$. In a topological group, it suffices to just give a neighborhood base of the identity, since we can take translates of this neighborhood base to be neighborhood bases at every other point.

Now if $w_v$ is a uniformizer for $\mathcal{O}_{F_v}$, then

$$\{(I_n + w_v^k M_n(\mathcal{O}_{F_v})) \times (1 + w_v^k \mathcal{O}_{F_v}) : k \in \mathbb{Z}_{\geq 1}\}$$

forms a neighborhood base of the point $(I_n, 1) \in M_n(\mathcal{O}_{F_v}) \times F_v$. It follows that

$$\{I_n + w_v^k M_n(\mathcal{O}_{F_v}) : k \in \mathbb{Z}_{\geq 1}\}$$

forms a neighborhood base of the identity in $\text{GL}_n(F_v)$. Note that this is the same topology we would obtain if we just gave $\text{GL}_n(\mathcal{O}_{F_v}) \subset M_n(\mathcal{O}_{F_v})$ the subspace topology.

On the other hand, if $S$ is any finite set of places of $F$ including the infinite places then

$$\left\{(I_n + m M_n(\mathcal{O}_F^S)) \times (1 + m \mathcal{O}_F^S) : m \in \mathcal{O}_F^S\right\}$$

forms a neighborhood basis for $(I_n, 1) \in M_n(\mathcal{A}_F^S) \times \mathcal{A}_F^S$. Here $m$ runs over proper ideals of $\mathcal{O}_F^S$ and
2.3 Relationship with restricted direct products

\[ mM_n(\mathcal{O}_F^S) := \prod_{v \notin S} \omega_v^{\nu(m)} M_n(\mathcal{O}_{F_v}). \]

If we intersect one of these neighborhoods with the image of \( \text{GL}_n(\mathbb{A}_F^S) \) under (2.8) then we obtain

\[ \prod_{v \notin S} (I_n + mM_n(\mathcal{O}_{F_v})) \times \prod_{v \notin S} \text{GL}_n(\mathcal{O}_{F_v}). \]

Here \( mM_n(\mathcal{O}_{F_v}) = \omega_v^{\nu(m)} M_n(\mathcal{O}_{F_v}) \). This is not the same as the topology obtained by giving \( \text{GL}_n(\mathbb{A}_F^S) \subset M_n(\mathbb{A}_F^S) \) the subset topology.

Finally, for any set \( S \) of places of \( F \) including the infinite places, it is not hard to see by modifying this argument that one has a topological isomorphism of locally compact groups

\[ \text{GL}_n(\mathbb{A}_F) = \text{GL}_n(\mathbb{F}_S) \times \text{GL}_n(\mathbb{A}_F^S). \]

If \( F \) is a local field and \( G \) is an affine algebraic group over \( F \), to define the topology on \( G(F) \) we choose an embedding \( G \hookrightarrow \text{GL}_n \) and give \( G(F) \) the subspace topology. Theorem 2.2.1 tells us that this topology is independent of the choice of embedding \( G \hookrightarrow \text{GL}_n \). If \( F \) is global instead we similarly obtain the topology on \( G(\mathbb{A}_F) \).

A helpful way to organize these comments is to introduce the notion of a restricted direct product of topological spaces. This notion is implicit in the definition of the topology on \( \mathbb{A}_F \) given in §2.1. Let \( \{X_\alpha\}_{\alpha \in A} \) be a set of locally compact topological spaces indexed by a countable set \( A \), and for all \( \alpha \) outside a finite subset \( S_0 \) of \( A \), let \( K_\alpha \subseteq X_\alpha \) be a compact open subset of \( X_\alpha \). Then the **restricted direct product** of the \( X_\alpha \) with respect to the \( K_\alpha \) is the set

\[ X := \prod_{\alpha \in A} X_\alpha := \left\{(x_\alpha) \in \prod_{\alpha \in A} X_\alpha : x_\alpha \in K_\alpha \text{ for almost all } \alpha \in A - S_0 \right\}. \]

Here, and throughout this book, “almost all” means all but finitely many.

We give \( X \) a topology by declaring a subset of \( X \) to be open if it is either empty or a union of sets of the form

\[ U \times \prod_{\alpha \in A - S} K_\alpha \]

where \( S \) is a finite subset of \( A \) including \( S_0 \) and

\[ U \subseteq X_S := \prod_{\alpha \in S} X_\alpha. \]
is an open subset. One verifies that this is indeed a topology and with respect
to this topology, $X$ is locally compact. Note that if the $X_\alpha$ are topological
groups and the $K_\alpha$ are topological subgroups then $X$ is again a topological
group. We also note that the topology does not change if we replace $S_0$ by
any finite subset of $A$ containing $S_0$.

Now assume that $G$ is an affine algebraic group over the global field $F$. Choose a faithful representation

$$G \rightarrow \text{GL}_n.$$ 

Identify $G$ with its image in $\text{GL}_n$ and define, for all $v \nmid \infty$,

$$K_v := G(F_v) \cap \text{GL}_n(\mathcal{O}_{F_v}).$$  \hspace{1cm} (2.9)

Then $K_v$ is a compact open subgroup of $G(F_v)$. Our discussion of the adelic
topology on $G(\mathbb{A}_F)$ above can be summarized as follows:

**Proposition 2.3.1** One has an isomorphism of topological groups

$$G(\mathbb{A}_F) \rightarrow \prod_v G(F_v)$$

$$(g_v) \mapsto (g_v)$$

where the restricted direct product is defined with respect to the subgroups $K_v$. \hfill \Box$

In fact, in most references, $G(\mathbb{A}_F)$ is defined using Proposition 2.3.1. This
has the advantage of being concrete, but it makes it awkward to rigorously
prove that the topology satisfies good functorial properties.

2.4 Hyperspecial subgroups and models

Let $G$ be an affine algebraic group over a nonarchimedean local field $F$ and
let $G \rightarrow \text{GL}_n$ be a faithful representation. Identify $G$ with its image in $\text{GL}_n$.
In the previous section, we made use of the fact that $K := G(F) \cap \text{GL}_n(\mathcal{O}_F)$
is a compact open subgroup of $G(F)$. This construction is actually algebraic
in nature in the sense that $K$ is the $\mathcal{O}_F$-points of an affine group scheme over $\mathcal{O}_F$. We explain this in more detail in this section and use it as motivation
to introduce the notion of a hyperspecial subgroup. We go into some detail,
and this requires some algebraic geometry over local rings. For most of this
book the reader can manage with an understanding of Definition 2.8 and the
statement of Corollary 2.4.9. Our presentation in this section was influenced
by [Yu].

We start with a Dedekind domain $\mathfrak{o}$ with fraction field $F$. For example,
$\mathfrak{o}$ could be the ring of $S$-integers $\mathcal{O}_F^S$ of a global field $F$ for a finite set $S$ of
places of $F$ including the infinite places or the ring of integers of a local field. Let $Z$ be an affine scheme over $\mathfrak{o}$. The **generic fiber** of $Z$ is $Z_F$. If $F$ is local and $\mathfrak{o}$ is its ring of integers $\mathcal{O}_F$ then $\mathfrak{o}$ has a unique prime ideal $\mathfrak{p}$; in this case we let $k = \mathfrak{o}/\mathfrak{p}$ be the residue field. The scheme $Z_k$ is known as the **special fiber** of $Z$.

Suppose we are given an affine scheme $Y$ over $F$; often it is useful to consider schemes $\mathcal{Y}$ that have $Y$ as their generic fiber and satisfy certain desiderata:

**Definition 2.6.** A model $\mathcal{Y}$ of $Y$ over $\mathfrak{o}$ is an affine scheme of finite type over $\mathfrak{o} = \text{Spec}(A)$ where $A \subseteq \mathcal{O}(Y)$ is an $\mathfrak{o}$-algebra and $A \otimes_\mathfrak{o} F = \mathcal{O}(Y)$.

The assertion that an affine $\mathfrak{o}$-scheme $\text{Spec}(A)$ is flat over $\mathfrak{o}$ in the usual sense is equivalent to the assertion that $A$ is flat as an $\mathfrak{o}$-module, which is equivalent to the assertion that $A$ is torsion-free as an $\mathfrak{o}$-module since $\mathfrak{o}$ is a Dedekind domain. In the situation of the definition, $\mathcal{O}(Y)$ may be regarded as an $F$-module by forgetting the algebra structure and $A$ is an $\mathfrak{o}$-submodule of this $F$-module. In particular, $A$ is torsion-free as an $\mathfrak{o}$-module. Thus we see that models are flat over $\mathfrak{o}$. Suppose conversely we are given an affine scheme of finite type $\mathcal{Y} = \text{Spec}(A)$ over $\mathfrak{o}$ together with an isomorphism $Y \cong \mathcal{Y}_F$. Assume moreover that $\mathcal{Y}$ is flat, which is to say that $A$ is flat as an $\mathfrak{o}$-module. Then the map $A \to \mathcal{O}(Y)$ induced by the isomorphism $Y \cong \mathcal{Y}_F$ is injective and $\mathcal{Y}$ is a model of $Y$.

An affine $\mathfrak{o}$-scheme $\mathcal{Y}$ is **smooth** if it is flat and of finite type and for all algebraically closed fields $\overline{k}$ that admit a nonzero $\mathfrak{o}$-algebra morphism $\mathfrak{o} \to \overline{k}$, the fiber $\mathcal{Y}_{\overline{k}}$ is smooth. Since models are $\mathfrak{o}$-schemes, it makes sense to speak of smooth models.

When $G$ is a group scheme we always assume that models $G$ of $G$ are again group schemes and that the generic fiber of the multiplication map $G \times G \to G$ and the inversion map $G \to G$ are the multiplication map and inversion map on $G$, respectively. In brief, we assume that the group scheme structures on $G$ and $G$ are compatible.

Sometimes there are obvious models for affine schemes. For example, we can view $\text{GL}_n$ as an affine group scheme over $\mathfrak{o}$, and it is clearly a model of its generic fiber $\text{GL}_n,F$. However, there are many other models of $\text{GL}_n$ (see Exercise 2.8 for an example).

We now describe a particular type of model of a reductive group. Assume that $F$ is a nonarchimedean local field.

**Definition 2.7.** A reductive group $G$ over a nonarchimedean local field $F$ is **unramified** if it is quasi-split and there is a finite degree unramified extension $E/F$ such that $G_E$ is split.

The following theorem is foundational for the theory of automorphic representations. It is an amalgamation of several results of Bruhat and Tits, see [Mac17, §1.2] for precise references.
Theorem 2.4.1 Let $F$ be a nonarchimedean local field. The reductive group $G$ over $F$ is unramified if and only if there exists a model $\mathcal{G}$ of $G$ over $\mathcal{O}_F$ such that the special fiber of $\mathcal{G}$ is reductive. If $\mathcal{G}$ is a model of $G$ over $\mathcal{O}_F$ such that the special fiber of $\mathcal{G}$ is reductive then the subgroup $\mathcal{G}(\mathcal{O}_F) \leq G(F)$ is a maximal compact subgroup of $G(F)$.

Definition 2.8. A subgroup of $G(F)$ of the form $\mathcal{G}(\mathcal{O}_F)$ for a model $\mathcal{G}$ of $G$ over $\mathcal{O}_F$ with reductive special fiber is called a hyperspecial subgroup.

We observe that the group scheme $\mathcal{G}$ in the definition of a hyperspecial subgroup is smooth.

Example 2.2. When $G = \text{GL}_n$, it is clear that $\text{GL}_n(\mathcal{O}_F)$ is a hyperspecial subgroup of $\text{GL}_n(F)$. It turns out that all maximal compact subgroups of $\text{GL}_n(F)$ are conjugate to $\text{GL}_n(\mathcal{O}_F)$ [Ser06, Chapter IV, Appendix 1].

Though the theory is not as pleasant for ramified $G$, we still have the following theorem:

Theorem 2.4.2 For any reductive group $G$ over $F$, every compact subgroup of $G(F)$ is contained in a maximal compact subgroup. Every maximal compact subgroup $K \leq G(F)$ is of the form $\mathcal{G}(\mathcal{O}_F)$ where $\mathcal{G}$ is a smooth model of $G$. Maximal compact subgroups of $G(F)$ are open.

Proof. See [Tit79, §3.2] for the assertion that every compact subgroup of $G(F)$ is contained in a maximal compact subgroup. The second assertion follows from [Tit79, §3.2, 3.4.1]. The final assertion follows from the second assertion and Exercise 2.2.

We briefly discuss conjugacy classes of hyperspecial subgroups. The group $G$ acts on $\mathcal{G}$ by conjugation, and the action factors through the center $Z_G$. Hence the group $(G/Z_G)(F)$ acts on $G(F)$. In general, orbits under this action are larger than the orbits of $G(F)$ on itself by conjugation; for a discussion of group actions and orbits in a general context, we refer to §17.1. The set of hyperspecial subgroups is permuted transitively by the action of $(G/Z_G)(F)$ [Tit79, §2.5] (see also [Con14, Theorem 7.2.16] when $G$ is split). However not all hyperspecial subgroups are conjugate under $G(F)$ in general. This plays a direct role in the definition of unramified $L$-packets (see below Definition 12.4).

Now suppose that we are given an unramified reductive group $G$ over a nonarchimedean local field $F$. One might ask how one ought to go about finding a hyperspecial subgroup of $G(F)$. One method proceeds via constructing schematic closures as we now explain. We again revert to the assumptions at the beginning of this section, so $\mathfrak{o}$ is a Dedekind domain with fraction field $F$. Let $Y$ be an affine scheme of finite type over $F$ and let $\mathcal{Y}$ be a model of $Y$. Suppose in addition we are given a closed immersion of $F$-schemes

$$X \longrightarrow Y; \quad (2.10)$$
we identify $X$ with a closed subscheme of $Y$. Let
\[ A := \text{Im}(O(Y) \rightarrow O(Y) \rightarrow O(X)). \]

Since $O(Y) \rightarrow O(Y)$ is injective and (2.10) is a closed immersion, we see that $X := \text{Spec}(A)$ comes equipped with a closed immersion $X \rightarrow Y$ and $XF = X$. We leave the proof of the following lemma as an exercise (see Exercise 2.4).

**Lemma 2.4.3** The scheme $X$ is a model of $X$ and
\[ X(\mathfrak{o}) = X(F) \cap Y(\mathfrak{o}). \]

The scheme $X$ earns its moniker as a schematic closure via a universal property (see Exercise 2.5).

**Lemma 2.4.4** Any affine scheme of finite type over $F$ admits a model. In particular, any affine algebraic group $G$ over $F$ admits a model $\mathcal{G}$ that is again a group scheme such that the generic fiber of the multiplication map $\mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ and the inversion map $\mathcal{G} \rightarrow \mathcal{G}$ are the multiplication map and inversion map on $G$.

**Proof.** The map $\mathfrak{o}^n \rightarrow F^n$ is clearly injective, hence $G^n_{nF}$, viewed as a scheme over $\mathfrak{o}$, is a model of $G^n_{nF}$. If $X$ is an affine scheme of finite type over $F$ then choose a closed immersion $X \rightarrow G^n_{nF}$. Its schematic closure $X$ in $G^n_{nF}$ is then a model of $X$ by Lemma 2.4.3.

Similarly, if we view $\text{GL}_n$ as an affine group scheme over $\mathfrak{o}$ then then the condition that $O(\text{GL}_n) \rightarrow O(\text{GL}_nF)$ is injective is clearly satisfied. Thus $\text{GL}_n$ is a model of $\text{GL}_n$. Using Theorem 1.5.1 choose a faithful representation $G \rightarrow \text{GL}_nF$. Let $\mathcal{G}$ be the schematic closure of $G$ in $\text{GL}_n$. It follows from Exercise 2.6 that it is a closed subgroup scheme of $\text{GL}_n$ with generic fiber $G$. \qed

In the special case where $F$ is a local field and $\mathfrak{o} = O_F$ is its ring of integers, the procedure in the proof yields
\[ \mathcal{G}(O_F) := G(F) \cap \text{GL}_n(O_F), \]

which we denoted by $K$ earlier. Thus in our construction of the restricted direct topology from the previous section, we implicitly used schematic closures.

Now suppose that $F$ is global and that we are given a faithful representation $G \rightarrow \text{GL}_n$ of a reductive group $G$. We can then take the schematic closure $\mathcal{G}$ of $G$ in $\text{GL}_nO_F$. This is a model of $G$ by the proof of Lemma 2.4.4. This model, or in fact any model, gives rise to hyperspecial subgroups for almost all $v$:
**Proposition 2.4.5** Let \( S \) be a finite set of places of \( F \) including the infinite places and let \( G \) be a reductive group over \( F \). Let \( G \) be a model of \( G \) over \( \mathcal{O}_F^S \). For almost all \( v \not\in S \), the subgroup \( G(\mathcal{O}_F^v) \) is hyperspecial.

This proposition and variants of it are extremely useful in applications so we indicate the proof after giving some preparation.

See [Poo17, Theorem 3.2.1] for the following lemma and a variety of similar results:

**Lemma 2.4.6** Let \( X \) be a smooth affine \( F \)-scheme, \( S \) be a finite set of places of \( F \) including the infinite places, and let \( \mathcal{X} \) be a model of \( X \) over \( \mathcal{O}_F^S \). Then there is a finite set \( S' \supseteq S \) such that \( \mathcal{X}_{\mathcal{O}_F^{S'}} \) is smooth over \( \mathcal{O}_F^{S'} \).

We now indicate the proof of Proposition 2.4.5:

**Proof of Proposition 2.4.5:** The scheme \( \mathcal{G}_{\mathcal{O}_F^{S'}} \) is smooth over \( \mathcal{O}_F^{S'} \) for a sufficiently large finite set \( S' \supseteq S \) by Lemma 2.4.6. Thus the proposition follows from [Con14, Propositions 3.1.9 and 3.1.12].

Models give us another perspective on the restricted direct product isomorphism in Proposition 2.3.1. Let \( X \) be an affine scheme of finite type over \( F \). Suppose that we have a model \( \mathcal{X} \) of \( X \) over \( \mathcal{O}_F^S \), where \( S \) is a finite set of places including the infinite places. The following is a refinement of Proposition 2.3.1:

**Proposition 2.4.7** One has a homeomorphism

\[
\pi : X(\mathbb{A}_F) \rightarrow \prod_v X(F_v)
\]

where the restricted direct product is defined with respect to the subsets \( X(\mathcal{O}_F^v) \). The set on the right, its topology, and the isomorphism are independent of the choice of model, and are unchanged if we replace \( S \) by \( S' \) (and \( \mathcal{X} \) by \( \mathcal{X}_{\mathcal{O}_F^{S'}} \)) for any set of places \( S' \supseteq S \). If \( X \) is an affine algebraic group then the homeomorphism is a group isomorphism.

Here when \( X \) is an affine algebraic group we assume (per our convention) that the model \( \mathcal{X} \) is a group scheme and that the group scheme structures on \( \mathcal{X} \) and \( X \) are compatible.

**Proof.** The homeomorphism assertion of the proposition is true for \( \mathcal{X} = \mathbb{G}_m^n \) by part (b) of Theorem 2.2.1 and the definition of the topology on \( \mathbb{A}_F \) given in §2.1.

We now reduce the homeomorphism assertion of the proposition to this special case. Since \( \mathcal{X} \) is of finite type we have a closed immersion

\[
t : \mathcal{X} \hookrightarrow \mathbb{G}_m^n
\]
over $\mathcal{O}_F^S$. The scheme $\iota(X)$ is then the schematic closure of $\iota(X)$ in $\mathbb{G}_m^n$. By Theorem 2.2.1 the generic fiber of this map induces a closed embedding

$$X(\mathbb{A}_F) \hookrightarrow \mathbb{A}_F^n \rightarrow \prod_v F_v^n$$

(2.11)

where the homeomorphism is given by the special case of the proposition proved at the beginning of the proof. Since $\iota(X(\mathcal{O}_F^v)) = \mathcal{O}_F^v \cap \iota(X(F_v))$ for all nonarchimedean $v$ by Lemma 2.4.3, the image of (2.11) is $\prod_v X(F_v)$.

This implies the isomorphism assertion of the proposition. The fact that the isomorphism is a group isomorphism when $X$ is a group scheme is obvious.

The independence of the restricted direct product and the isomorphism under replacing $S$ by a larger set $S'$ is clear.

Now, suppose that $X'$ is another model of $X$. Then $\mathcal{O}(X) \otimes_{\mathcal{O}_F} F = \mathcal{O}(X') \otimes_{\mathcal{O}_F} F = \mathcal{O}(X)$, and $\mathcal{O}(X)$ and $\mathcal{O}(X')$ are of finite type over $\mathcal{O}_F^S$. Hence for a sufficiently large finite set of places $S' \supset S$, we have

$$\mathcal{O}(X') \otimes_{\mathcal{O}_F^S} \mathcal{O}_{S'}^S = \mathcal{O}(X) \otimes_{\mathcal{O}_F^S} \mathcal{O}_{S'}^S.$$ (2.12)

This is equivalent to the assertion that the identity morphism $X_F = X_F \rightarrow X_F = X'_F$ extends to an isomorphism

$$\mathcal{X}_{\mathcal{O}_F^S} \cong \mathcal{X}_{S'}^S$$

for a sufficiently large set of places $S' \supset S$ (see [Poo17, Theorem 3.2.1] for many useful generalizations and variants). In particular the subsets $X(\mathcal{O}_F^v)$ and $X'(\mathcal{O}_F^v)$ of $X(F_v)$ are equal for $v \notin S'$. The assertion on the independence of the choice of model follows. 

\begin{corollary} Let $S$ be a finite set of places of $F$ including the infinite places, let $G$ be a smooth affine algebraic group over $F$, and let $K^S \leq G(\mathbb{A}_F^S)$ be a compact open subgroup. Let $\mathcal{G}$ be a model of $G$ over $\mathcal{O}_F^S$. Then there is a finite set of places $S' \supset S$ such that $K^S = K^S_{S'} K_{S'}^S$ where

$$K^S_{S'} \leq G(F_{S'}^S) \quad \text{and} \quad \mathcal{G}(\mathcal{O}_{S'}^S) = K_{S'}^S.$$ \end{corollary}

\begin{corollary} Let $S$ be a finite set of places of $F$ including the infinite places, let $G$ be a reductive group over $F$, and let $K^S \leq G(\mathbb{A}_F^S)$ be a compact open subgroup. Then the projection $K_v$ of $K^S$ to $G(F_v)$ is hyperspecial for all but finitely many $v$.

\begin{proof} Choose a model $\mathcal{G}$ of $G$ over $\mathcal{O}_F$ using Lemma 2.4.4. Then $\mathcal{G}(\mathcal{O}_{F_v}) = K_v$ for all but finitely many $v$ by Corollary 2.4.8. Thus the corollary follows from Proposition 2.4.5. \end{proof}
We end the section by discussing a possible point of confusion related to Proposition 2.4.7. Suppose we are given a reductive group \( G \) and a collection of hyperspecial subgroups \( K_v \leq G(F_v) \) for almost every \( v \). Using the fact that all hyperspecial subgroups of \( G(F_v) \) are in the same \( (G/Z_G)(F_v) \)-orbit, we can again construct an isomorphism

\[
G(\mathbb{A}_F) \xrightarrow{\sim} \prod_v' G(F_v)
\]

where the double prime is to indicate that we are taking the restricted direct product with respect to the hyperspecial subgroups \( K_v \). Now the isomorphism in Proposition 2.4.7 has good functorial properties, in particular, it commutes with the obvious embeddings of \( G(F) \) into \( G(\mathbb{A}_F) \) and \( \prod_v G(F_v) \). This is impossible to arrange in general for (2.13):

**Example 2.3.** Let

\[
K_p = \begin{pmatrix} p & 1 \\ 1 & 1 \end{pmatrix} \text{SL}_2(\mathbb{Z}_p) \begin{pmatrix} p^{-1} & 1 \\ 1 & 1 \end{pmatrix}.
\]

Then \( K_p \leq \text{SL}_2(\mathbb{Q}_p) \) is hyperspecial and is not conjugate in \( \text{SL}_2(\mathbb{Q}_p) \) to \( \text{SL}_2(\mathbb{Z}_p) \) [Con14, Example 7.2.15]. We observe that for all \( p \), the element \( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \text{SL}_2(\mathbb{Q}) \) is not in \( K_p \).

Thus we will always decompose \( X(\mathbb{A}_F) \) as in Proposition 2.4.7, not with respect to some “exotic” collection of models chosen place by place.

### 2.5 Approximation in affine algebraic groups

For a global field \( F \) and a nonempty finite set \( S \) of places of \( F \), the image of \( F \) under the diagonal embeddings \( F \to F_S \) is dense. This is fairly easy to prove and can be viewed as a generalization of the Chinese remainder theorem. Moreover, the image of \( F \) in \( \mathbb{A}_F^S \) under the diagonal embedding is dense by Theorem 2.1.4. If \( X \) is an affine scheme of finite type over \( F \) and \( S \) is a finite set of places of \( F \) then one can ask if a similar phenomenon occurs:

**Definition 2.9.** Let \( S \) be a nonempty finite set of places of \( F \). The affine scheme \( X \) satisfies **weak approximation** with respect to \( S \) if the image of the diagonal embedding \( X(F) \to X(F_S) \) is dense. It satisfies **strong approximation** with respect to \( S \) if \( X(F) \) is dense in \( X(\mathbb{A}_F^S) \).

Here when we speak of density we are of course using the canonical topologies on \( X(F_S) \) and \( X(\mathbb{A}_F^S) \) afforded by Theorem 2.2.1.

Despite the relative ease of proving that \( \mathbb{G}_m \) satisfies weak and strong approximation, establishing whether or not a general affine scheme enjoys these properties is very difficult. In general, they do not hold. We refer to [Har04] for a more detailed discussion.
2.5 Approximation in affine algebraic groups

Our goal in this section is to state when weak and strong approximation hold in settings related to affine algebraic groups. To simplify the discussion we often assume that $F$ is a number field; additional complications come up in the general case. Our primary reference is [PR94, Chapter 7]. We start with the following proposition, the proof of which we leave as an exercise:

**Proposition 2.5.1** Let $G$ be a connected affine algebraic group over a number field $F$ with Levi decomposition $G = MN$. Then $G$ admits weak (resp. strong) approximation with respect to a nonempty finite set $S$ of places of $F$ if and only if $M$ does.

Thus in the number field case, studying strong and weak approximation of connected affine algebraic groups is equivalent to studying it for the smaller set of reductive groups. It turns out that under suitable restrictions on the set of places $S$, weak approximation always holds:

**Theorem 2.5.2** Let $G$ be a connected affine algebraic group over a number field $F$. There is a finite set $S_0$ of finite places of $F$ such that $G$ has weak approximation with respect to any nonempty finite set of places of $F$ not containing $S_0$.

We refer to [PR94, §7.3] for the proof.

If we wish to guarantee that we can take $S_0$ to be empty, we need to assume additional arithmetic conditions on the group $G$. To state them, one needs the notions of simply connected and adjoint semisimple groups from §1.8. For the following theorem, we refer again to [PR94, §7.3]:

**Theorem 2.5.3** A simply connected or adjoint semisimple group $G$ over a number field $F$ admits weak approximation with respect to any nonempty finite set of places $F$.

We finish our discussion of weak approximation by considering the case where $X$ is not an affine algebraic group, but a particular type of homogeneous space. Let $G$ be a reductive group over a number field $F$ and let $H \triangleleft G$ be a reductive subgroup. Then there is a natural action of $G$ on $X := G/H := \text{Spec}(O(G)^H)$.

For the definition of $O(G)^H$, we refer to (17.8). If $\overline{F}$ is an algebraic closure of $F$ and $F \leq L \leq \overline{F}$ is a subfield then

$$X(L) = (G(\overline{F})/H(\overline{F}))^{\text{Gal}(\overline{F}/L)}$$

with the natural Galois action (see Proposition 17.1.6). For more details on affine algebraic group actions we refer to §17.1.

To state weak approximation theorems in this context, we recall that an algebraic torus $T$ is quasi-trivial or induced if $X^*(T)$ is a permutation $\text{Gal}(\overline{F}/F)$-module. We record the following theorem of Borovoi [Bor09, Theorem 3.12], which generalizes Theorem 2.5.2:
Theorem 2.5.4 Let $G$ be a reductive group such that $G/G^{\text{der}}$ is quasi-trivial and $G^{\text{der}}$ is simply connected. Assume that $H$ is a connected reductive subgroup. There is a finite set $S_0$ of finite places of the number field $F$ such that $G/H$ has weak approximation with respect to any nonempty finite set of places of $F$ not containing $S_0$. □

In particular, $G/H$ always has weak approximation with respect to $\infty$.

Similarly we have a generalization of Theorem 2.5.3:

Theorem 2.5.5 Assume that $G$ is a reductive group such that $G/G^{\text{der}}$ is quasi-trivial and $G^{\text{der}}$ is simply connected. Let $H$ be a connected reductive subgroup such that $H/H^{\text{der}}$ splits over a cyclic extension of a number field $F$. Then $G/H$ satisfies weak approximation with respect to any nonempty finite set of places of $F$. □

This is [Bor09, Corollary 3.14]. We note that Borovoi obtains more precise results and actually works in a more general context where $G$ and $H$ are not necessarily reductive. We have restricted to the situation above for simplicity.

We now turn to a discussion of strong approximation. A connected affine algebraic group $G$ over a global field $F$ is almost simple if $\text{Lie } G$ is a simple Lie algebra, that is, a Lie algebra with no proper ideals. The basic result on strong approximation is the following. For the proof, see [Pra77]:

Theorem 2.5.6 Let $G$ be a connected affine algebraic group over a global field $F$ such that $G$ is almost simple and let $S$ be a nonempty finite set of places of $F$. If $G$ is simply connected and $G(F_S)$ is noncompact then $G$ satisfies strong approximation with respect to $S$. □

At least in the number field case, there is a converse to this theorem, see [PR94, Theorem 7.12]. We point out in particular that the semisimplicity assumption is necessary. Even in the case $G = \text{GL}_1$ strong approximation is false in general. Indeed,

$$F^\times \backslash (\mathbb{A}^\times_F)\backslash \hat{\mathcal{O}}^\times_F$$

is the class group of $F$ (see (2.14) below). This is a finite group, but it is often nontrivial.

Here is an analogue of Theorem 2.5.6 for certain homogeneous spaces [Rap14, Proposition 2.4]:

Theorem 2.5.7 Assume that $H \leq G$ are almost simple affine algebraic groups over a global field $F$ that are simply connected and semisimple. Assume moreover that $G/F$ is almost simple. Let $S$ be a nonempty finite set of places of $F$. Then $G/H$ satisfies strong approximation with respect to $S$ if and only if $(G/H)(F_S)$ is noncompact. □
2.6 The adelic quotient

Let $G$ be an affine algebraic group over a global field $F$ (so we do not assume $G$ to be reductive or connected). Then the subgroup $G(F) \leq G(\mathbb{A}_F)$ is discrete and closed (compare Exercise 2.2) and we can consider the quotient $G(F) \backslash G(\mathbb{A}_F)$. In this section, we state some basic facts on the topology of $G(F) \backslash G(\mathbb{A}_F)$. More precise results will be explained in the following section.

The first result we state is the following (see [Con12a] for the proof):

**Theorem 2.6.1 (Borel, Conrad, Oesterlé, Prasad)** For any finite set $S$ of places of $F$ containing the infinite places and for any compact open subgroup $K^S \leq G(\mathbb{A}_F)$, the quotient

$$G(F) \backslash G(\mathbb{A}_F)/K^S$$

is finite.

We note that one does not even have to assume that $G$ is smooth (although this is automatic in the characteristic zero case, see Theorem 1.5.2). Theorem 2.6.1 is known as **finiteness of class numbers** for affine algebraic groups. It is a strict generalization of the assertion that the class group of $F$ is finite. To see this, recall that the class group $\text{Cl}_F$ sits in an exact sequence

$$1 \to \text{Prin}(O_F) \to \text{Div}(O_F) \to \text{Cl}_F \to 1$$

where $\text{Prin}(O_F)$ is the group of principal divisors of $O_F$ and $\text{Div}(O_F)$ is the group of divisors of $O_F$ (which can be identified with the group of fractional ideals). There is a surjection

$$\text{Div}(O_F) \to (\mathbb{A}_F^\times)^\times/\hat{O}_F^\times$$

$$m \mapsto (mO_F^\times).$$

It induces a bijection

$$\text{Cl}_F \to F^\times/(\mathbb{A}_F^\times)^\times/\hat{O}_F^\times. \quad (2.14)$$

Hence taking $G = G_1$ and $K^\infty := \hat{O}_F^\times$ in Theorem 2.6.1 implies the finiteness of the class group of $F$. Interestingly, if we view the class group as the Picard group of $\text{Spec}(O_F)$, that is, the group of isomorphism classes of line bundles on $\text{Spec}(O_F)$, one is naturally led to Weil’s description of

$$\text{GL}_n(F) \backslash \text{GL}_n(\mathbb{A}_F^\times)/\text{GL}_n(\hat{O}_F)$$

in terms of vector bundles. We refer to [Gai03] for an introduction to what can be gained from this perspective in the function field case.

It is not hard to see that in general the quotient $G(F) \backslash G(\mathbb{A}_F)$ itself is infinite. However, we could ask when the quotient is compact or finite volume
with respect to a suitable measure (see §3.6), and there is a complete answer to this question which we recall in Theorem 2.6.3 below. It is useful to first eliminate a trivial obstruction to the quotient $G(F) \backslash G(\mathbb{A}_F)$ being compact or finite volume.

The units $F^\times$ of $F$ embed into $\mathbb{A}^\times_F$ diagonally as a discrete closed subgroup. The idelic norm

$$|\cdot| := \prod_v |\cdot|_v : F^\times \backslash \mathbb{A}^\times_F \longrightarrow \mathbb{R}_>0,$$

is continuous and nontrivial. In fact, its image is $\mathbb{R}_>0$ if $F$ is a number field

$$R_99, \text{Theorem } 5.14.$$  

Thus $F^\times \backslash \mathbb{A}^\times_F$ is noncompact and moreover is of infinite volume with respect to the Haar measure (for more on Haar measures, see §3.2 below). An analogous phenomenon occurs for other groups, and this motivates the following definition:

$$G(\mathbb{A}_F)^1 := \bigcap_{\chi \in \mathbb{X}^*(G)} \ker(|\cdot| \circ \chi : G(\mathbb{A}_F) \longrightarrow \mathbb{R}_>0).$$

Note that $G(F)$ is contained in $G(\mathbb{A}_F)^1$ in virtue of the product formula stated in Proposition 2.1.1. Moreover, $G(F)$ is discrete and closed in $G(\mathbb{A}_F)^1$ (see Exercise 2.2).

We now define a useful subgroup

$$A_G < G(F_\infty) < G(\mathbb{A}_F).$$

When $F$ is a number field, we let $A_G$ be the identity component of the $\mathbb{R}$-points of the greatest $\mathbb{Q}$-split torus in $\text{Res}_{F/\mathbb{Q}}Z_G$. When $F$ is a function field of characteristic $p$, temporarily write $Z$ for the largest $\mathbb{F}_p(t)$-split torus in

$$\text{Res}_{F/\mathbb{F}_p(t)}Z_G.$$

Then if $Z$ has rank $d$, there is an isomorphism $Z \cong G_m^d$ of tori over $\mathbb{F}_p(t)$ inducing an isomorphism $Z(\mathbb{F}_p(t)_{\infty}) \rightarrow (\mathbb{F}_p((t^{-1}))^\times)^d$. We let $A_G < Z(\mathbb{F}_p(t)_{\infty})$ be the inverse image of the subgroup $(t^Z)^d$.

The groups $A_G$ and $G(\mathbb{A}_F)^1$ may seem mysterious at first blush, but this is illusory. For example, when $G = G_m$ over a number field $F$, the group $A_{G_m}$ is just the copy of $\mathbb{R}_>0$ embedded diagonally in $F^\times_\infty < \mathbb{A}_F^\times$ and

$$A_{\text{GL}_n} = \{aI_n : a \in A_{G_m}\},$$

where $I_n \in \text{GL}_n(\mathbb{A}_F)$ is the identity matrix. Similarly

$$\text{GL}_n(\mathbb{A}_F)^1 = \{g \in \text{GL}_n(\mathbb{A}_F) : |\text{det} g| = 1\}.$$  

We leave the proof of the following lemma as Exercise 2.15:
Lemma 2.6.2 If $G$ is reductive, or more generally has neutral component that is the direct product of a reductive and a unipotent group, then $A_G(G\mathbb{A}_F)^1$ has finite index in $G(G\mathbb{A}_F)$ and $A_G\cap G(G\mathbb{A}_F)^1 = 1$. If $F$ is a number field, then $A_G(G\mathbb{A}_F)^1 = G(G\mathbb{A}_F)$.

The conclusions of Lemma 2.6.2 are false for a general affine algebraic group $G$ over $F$ (see Exercise 2.17).

The adelic quotient of $G$ is the quotient

$$[G] := A_G(G(F)\backslash G(G\mathbb{A}_F)). \quad (2.18)$$

We warn the reader that in the literature, $[G]$ often refers to $G(F)\backslash G(G\mathbb{A}_F)$, but the convention (2.18) is more convenient for the purposes of this book.

The canonical map $G(A_F)^1 \to A_G(G\mathbb{A}_F)$ induces a continuous map

$$G(F)\backslash G(G\mathbb{A}_F)^1 \to [G] \quad (2.19)$$

that intertwines the right actions of $G(G\mathbb{A}_F)^1$. Under the assumptions of Lemma 2.6.2, if $F$ is a number field then (2.19) is a homeomorphism and if $F$ is a function field there are only finitely many $G(G\mathbb{A}_F)^1$-orbits in the codomain of (2.19). Thus under the assumptions of Lemma 2.6.2, for many purposes one can work with either $A_G(G(F)\backslash G(G\mathbb{A}_F))$ or $G(F)\backslash G(G\mathbb{A}_F)^1$. For any affine algebraic group $G$ over $F$, the group $G(G\mathbb{A}_F)^1$ is unimodular by Lemma 3.6.4 below, and hence admits a Haar measure (the notion of a Haar measure and a unimodular group will be discussed in §3.2).

The basic topological and measure theoretic properties of $G(F)\backslash G(G\mathbb{A}_F)^1$ are summarized in the following theorem:

**Theorem 2.6.3 (Borel, Conrad, Harder, Oesterlé)** Assume that $G$ is smooth and connected. The quotient $G(F)\backslash G(G\mathbb{A}_F)^1$ has finite volume with respect to the measure induced by a Haar measure on $G(G\mathbb{A}_F)^1$. It is compact if and only if for every $F$-split torus $T \leq G$, one has that $T_F \leq R(G_F)$.

**Proof.** The first assertion is [Con12a, Theorem 1.3.6]. The “only if” portion of the last assertion is straightforward. The “if” portion of the last assertion is part of [Con12a, Theorem A.5.5].

For comparison with the classical theory of automorphic forms, it is convenient to relate the adelic quotient to locally symmetric spaces (see Chapter 6). For this purpose, if $K^\infty \leq G(G\mathbb{A}_F^\infty)$ is a compact open subgroup let

$$h := h(K^\infty) = \lvert G(F)\backslash G(G\mathbb{A}_F^\infty)/K^\infty \rvert.$$ 

By Theorem 2.6.1, $h < \infty$. Let $t_1, \ldots, t_h$ denote a set of representatives for $G(F)\backslash G(G\mathbb{A}_F^\infty)/K^\infty$. We then have a homeomorphism

$$\prod_{i=1}^h \Gamma_i(K^\infty)\backslash G(F) \to G(F)\backslash G(G\mathbb{A}_F)/K^\infty \quad (2.20)$$
given on the \( i \)th component by
\[
\Gamma_i(K^\infty) g_\infty \mapsto G(F) g_\infty t_i K^\infty,
\]
where
\[
\Gamma_i(K^\infty) := G(F) \cap t_i K^\infty t_i^{-1}
\]
which is viewed as a subgroup of \( G(F_\infty) \) via the map \( G(F) \to G(F_\infty) \). Notice that the \( \Gamma_i(K^\infty) \) are discrete subgroups of \( G(F) \) and they are moreover arithmetic in the following sense:

**Definition 2.10.** Let \( G \leq \text{GL}_n \) be a linear algebraic group. A subgroup \( \Gamma \leq G(F) \) is **arithmetic** if it is commensurable with \( G(F) \cap \text{GL}_n(\mathbb{Z}) \).

Here one says that two subgroups \( \Gamma_1 \) and \( \Gamma_2 \) of a group are **commensurable** if \( \Gamma_1 \cap \Gamma_2 \) is of finite index in both \( \Gamma_1 \) and \( \Gamma_2 \).

The notion of arithmeticity does not depend on the choice of representation \( G \leq \text{GL}_n \) (see Exercise 2.13). The subgroups of \( G(F) \) that are intersections of a compact open subgroup of \( G(\hat{\mathbb{A}}^\infty) \) and \( G(F) \) are known as **congruence subgroups**. Not every arithmetic group is a congruence subgroup in general, although for some groups \( G \), this is the case. For more information, the reader can consult the literature on the so-called congruence subgroup problem.

We end this section by considering the special case where \( G = \text{GL}_2 \). Then \( K^\infty = \text{GL}_2(\hat{\mathbb{Z}}) \) is a maximal compact open subgroup, where \( \hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p \) is the profinite completion of \( \mathbb{Z} \). Moreover, if we denote by
\[
K_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\hat{\mathbb{Z}}) : N|c \right\} \quad \text{and} \quad \Gamma_0(N) := K_0(N) \cap \text{GL}_2(\mathbb{Z})
\]
then (2.20) becomes
\[
\Gamma_0(N) \text{GL}_2(\mathbb{R})/K_\infty = \text{GL}_2(\mathbb{Q}) \text{GL}_2(\hat{\mathbb{A}}_\mathbb{Q})/K_0(N).
\]
If we let \( K_\infty = \text{SO}_2(\mathbb{R}) \) then we have a bijection
\[
\text{A}_{\text{GL}_2} \text{GL}_2(\mathbb{R})/K_\infty \longrightarrow \mathbb{C} - \mathbb{R}
\]
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{ai+b}{ci+d}.
\]
Hence
\[
\Gamma_0(N) \mathbb{C} - \mathbb{R} \longrightarrow \text{GL}_2(\mathbb{Q}) \text{A}_{\text{GL}_2} \text{GL}_2(\hat{\mathbb{A}}_\mathbb{Q})/K_\infty K_0(N)
\]
\[
= \text{GL}_2(\mathbb{Q}) \mathbb{C} - \mathbb{R} \times \text{GL}_2(\hat{\mathbb{A}}_{\mathbb{Q}}) K_0(N),
\]
where on the left \( \Gamma_0(N) \) acts via Möbius transformations:
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d}.
\]
2.7 Reduction theory

Reduction theory allows us to control the adelic quotient \([G]\) when it is non-compact. To be more precise about this, assume that \(G\) is a reductive group over a global field \(F\). In the number field case, this assumption can be weakened to allow for arbitrary connected affine algebraic groups over \(F\) at the cost of introducing more notation [PR94, §4.3].

The following theorem provides what is known as the Iwasawa decomposition:

**Theorem 2.7.1** Let \(G\) be a reductive group over a global field \(F\) and let \(P \leq G\) be a parabolic subgroup. There exists a maximal compact subgroup \(K \leq G(\mathbb{A}_F)\) such that

\[
G(\mathbb{A}_F) = P(\mathbb{A}_F)K.
\] (2.21)

We sketch the proof of this theorem in Appendix A.

Ideally one would like to use the Iwasawa decomposition to give a well-behaved fundamental domain for the action of \(G(F)\) on \(G(\mathbb{A}_F)\). Describing such fundamental domains is a difficult task in general. Fortunately for many purposes, it is unnecessary because one can make do with approximate fundamental domains known as Siegel sets.

We now prepare to give the definition. Assume that \(P\) is a minimal parabolic subgroup of \(G\) with unipotent radical \(N\). Fix a Levi subgroup \(M \leq P\). Let \(T_0\) be a maximal split torus of \(G^\text{der}\) contained in \(M\) and let \(Z_0\) be the maximal split torus of \(Z_G\). Then \(T := T_0Z_0\) is a maximal split torus of \(G\). As mentioned in §1.9, the set of nonzero weights of \(T\) acting on Lie \(P\) is a set of positive roots with respect to a unique base \(\Delta \subset \Phi(G,T)\) for the set of roots of \(T\) in \(G\). We choose a maximal compact subgroup \(K \leq G(\mathbb{A}_F)\) so that \(G(\mathbb{A}_F) = P(\mathbb{A}_F)K\) using Theorem 2.7.1.

A **Siegel set** in \(G(\mathbb{A}_F)^1\) is a set of the form

\[
\mathcal{S}(t) := \mathcal{S}_\Omega(t) = \Omega A_{T_0}(t)K,
\] (2.22)

where \(\Omega \subset P(\mathbb{A}_F) \cap G(\mathbb{A}_F)^1\) is a compact subset, \(t \in \mathbb{R}_{>0}\), and

\[
A_{T_0}(t) := \{ x \in A_{T_0} : |\alpha(x)|_{\infty} \geq t \text{ for all } \alpha \in \Delta \}.
\] (2.23)

Here we are using the fact that \(\alpha(x) \in \mathbb{G}_m(F_{\infty}) = F_{\infty}^\times\) and

\[
|\cdot|_{\infty} := \prod_v |\cdot|_v.
\]

The main result of reduction theory is the following:

**Theorem 2.7.2 (Reduction theory)** There exists a Siegel set \(\mathcal{S}(t)\) such that
\[ G(F) S(t) = G(A_F)^1. \]

**Proof.** See [Spr94]. To aid the reader, we note that in loc. cit. the group \( G(F) \) acts on the right. Thus one has to take an inverse to move \( G(F) \) to the left, and this changes the inequality in the analogue of \( A_{T_0}(t) \) in loc. cit. to that we used in defining \( A_{T_0}(t) \).

By the theorem, the set \( S(t) \) can indeed be thought of as an approximate Siegel set, as every element of \( G(A_F)^1 \) is in the \( G(F) \)-orbit of an element of \( S(t) \). However, two elements of \( A_G S(t) \) can be in the same \( G(F) \)-orbit; this is already the case for \( G = GL_2 \) and \( F = \mathbb{Q} \) (see Exercise 2.18).

To understand what is going on in Theorem 2.7.2, it is useful to consider what is arguably the simplest nontrivial case, that when \( G = GL_2 \) and \( F = \mathbb{Q} \). Take \( K = O_2(\mathbb{R})GL_2(\mathbb{Z}) \) and let \( B = P \) be the Borel subgroup of upper triangular matrices (since it is a Borel subgroup, it is a minimal parabolic subgroup). Let \( T_0 \leq SL_2 \) and \( T \leq GL_2 \) be the maximal tori of diagonal matrices. Then \( \Delta \) consists of the single root

\[ \alpha : T_0(\mathbb{R}) \to R^\times \]

\[ (t_1 \ t_2) \mapsto t_1 t_2^{-1} \]

for \( \mathbb{Q} \)-algebras \( R \). In this setting, reduction theory tells us that there is a compact subset \( \Omega \leq B(A_\mathbb{Q}) \) such that

\[ GL_2(A_\mathbb{Q}) = GL_2(\mathbb{Q}) A_{GL_2} \Omega A_{T_0}(t) K \]

for small enough \( t \in \mathbb{R}_{>0} \). This is the content of Exercise 2.18. We remark that it is this example that gives reduction theory its name. The key result in the reduction theory of positive definite binary quadratic forms over \( \mathbb{Q} \) is the assertion that every form of a given discriminant is representable by a reduced form. A slight strengthening of Theorem 2.7.2 in the special case considered above implies this result.

We end this section by explaining how reduction theory is used in practice. Let \( \varphi : [G] \to \mathbb{C} \) be a continuous function. Reduction theory implies that if we can control the restriction of \( \varphi \) to \( A_{T_0}(t) \) then we can control the function on all of \([G] \). This observation motivates growth conditions we place on functions on \([G]\) (see Lemma 6.3.1) and plays a key role in convergence arguments. See \( \S 9.4 \), \( \S 9.5 \), and the proof of Corollary 14.3.4 for instance.

**Exercises**

2.1. Prove Proposition 2.1.1.
2.2. Let $R \to R'$ be a continuous map of topological rings and let $X$ be an affine scheme of finite type over $R$. Show that $X(R) \to X(R')$ is continuous. Show moreover that if $R \to R'$ is a
(a) topological embedding
(b) open topological embedding
(c) closed topological embedding
(d) topological embedding onto a discrete subset
then so is $X(R) \to X(R')$.

2.3. Let $G$ be an affine algebraic group over a local field $F$, let $\rho : G \to \text{GL}_n$ be a faithful representation, and let $K \leq \text{GL}_n(F)$ be a compact open subgroup. Prove that $\rho(G(F)) \cap K$ is a compact open subgroup of $\rho(G(F))$.

2.4. Prove Lemma 2.4.3.

2.5. Let $\mathfrak{o}$ be a Dedekind domain with fraction field $F$ and let $Y$ be a flat affine scheme of finite type over $\mathfrak{o}$. Suppose that $X \to Y := Y_F$ is a closed immersion. Let $\mathcal{X}$ be the schematic closure of $X$ in $Y$. Let $Z$ be a flat affine scheme of finite type over $\mathfrak{o}$. For any closed immersion $Z \to Y$ whose generic fiber induces an isomorphism $Z_F \to X$, there is a unique closed immersion $X \to Z$ such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{X} & \to & Y \\
\downarrow & & \\
Z & \to & Y
\end{array}
\]

2.6. Let $\mathfrak{o}$ be a Dedekind domain with fraction field $F$. Let $G$ be a flat group scheme over $\mathfrak{o}$ of finite type with generic fiber $G := G_F$. Let $H$ be a group scheme over $F$ equipped with a morphism of group schemes $H \to G$ that is a closed immersion. Show that the schematic closure $\mathcal{H}$ of $H$ in $G$ is a closed subgroup scheme of $G$.

2.7. Let $S$ be a non-empty finite set of places of a global field $F$. Prove that $F$ is dense in $F_S$.

2.8. Let $m > 1$ be an integer. Let

$$A := \mathbb{Z}[x_{ij}, t_{ij}, y_{1 \leq i, j \leq n}/(\det(x_{ij})y - 1, \{x_{ij} - 1 + mt_{ij} : 1 \leq i, j \leq n\})].$$

Show that $G := \text{Spec}(A)$ is a model of $\text{GL}_n$ over $\mathbb{Z}$ and that

$$G(\mathbb{Z}) = \{g \in \text{GL}_n(\mathbb{Z}) : g \equiv I_n \pmod{mM_n(\mathbb{Z})}\}.$$

2.9. Let $X_1, X_2$ be affine schemes of finite type over a global field $F$. Show that $X_1$ and $X_2$ admit weak (resp. strong) approximation with respect to a finite set $S$ of places $F$ if and only if $X_1 \times X_2$ does. Deduce that if $G$ is an affine
algebraic group over a number field \( F \) then \( G \) satisfies weak (resp. strong) approximation with respect to a finite set \( S \) of places \( F \) if and only if a Levi subgroup of \( G \) does.

2.10. Let \( F \) be a global field. Show that \( \text{GL}_n \) admits weak approximation with respect to any nonempty finite set of the places of \( F \).

2.11. Assume that the affine algebraic group \( G \) over the global field \( F \) satisfies strong approximation with respect to a nonempty finite set \( S \) of places of \( F \). Show that

\[
|G(F)\backslash G(\mathbb{A}_F^S)/K^S| = 1
\]

for any compact open subgroup \( K^S \leq G(\mathbb{A}_F^S) \).

2.12. Let \( F \) be a number field. Show that \( \text{GL}_n \) does not admit strong approximation with respect to the set of infinite places of \( F \) if the class number of \( F \) is not 1.

2.13. For \( i = 1, 2 \), let \( \rho_i : G \to \text{GL}_{n_i} \) be a pair of faithful representations of the affine algebraic group \( G \) over a number field \( F \). Let \( \mathcal{G}_i \) be the schematic closure of \( \rho_i(G) \) in \( \text{GL}_{n_i}(\mathcal{O}_F) \). Show that a subgroup \( \Gamma \leq G(F) \) is commensurable with \( \mathcal{G}_1(\mathcal{O}_F) \) if and only if it is commensurable with \( \mathcal{G}_2(\mathcal{O}_F) \).

2.14. Let \( B_n \leq \text{GL}_n \) be the Borel subgroup of upper triangular matrices. Show that

\[
\text{GL}_n(\mathbb{R}) = B_n(\mathbb{R})\text{O}_n(\mathbb{R}) \quad \text{and} \quad \text{GL}_n(\mathbb{C}) = B_n(\mathbb{C})\text{U}_n(\mathbb{R})
\]

and that for a nonarchimedean local field \( F \),

\[
\text{GL}_n(F) = B_n(F)\text{GL}_n(\mathcal{O}_F).
\]

Here for \( \mathbb{R} \)-algebras \( R \),

\[
\text{O}_n(R) := \{ g \in \text{GL}_n(R) : gg^t = I_n \}
\]

and

\[
\text{U}_n(R) := \{ g \in \text{GL}_n(\mathbb{C} \otimes \mathbb{R} R) : gg^t = I_n \}.
\]

Deduce the adelic Iwasawa decomposition for \( \text{GL}_n(\mathbb{A}_F) \) for global fields \( F \).

2.15. Prove Lemma 2.6.2.

2.16. Show that if \( F \) is a function field then there is a strict inequality

\[
AGL_2(\mathbb{A}_F)^1 < \text{GL}_2(\mathbb{A}_F).
\]

2.17. Give an example of a connected smooth affine algebraic group \( G \) over a global \( F \) such that \( AGG(\mathbb{A}_F)^1 \) is not of finite index in \( G(\mathbb{A}_F) \).
2.18. Let $T_0 \leq \text{SL}_2$ be the maximal torus of diagonal matrices and let
\[ \Omega = \{(1, x) : x \in [-\frac{1}{2}, \frac{1}{2}]\}. \]
Show that $\text{GL}_2(A_{\mathbb{Q}}) = \text{GL}_2(\mathbb{Q})A_{\text{GL}_2} \Omega A_{T_0}(t)O_2(\mathbb{R})\text{GL}_2(\mathbb{Z})$ for $t$ sufficiently small. Show, on the other hand, that if $t$ is chosen so that
\[ \text{GL}_2(A_{\mathbb{Q}}) = \text{GL}_2(\mathbb{Q})A_{\text{GL}_2} \Omega A_{T_0}(t)O_2(\mathbb{R})\text{GL}_2(\mathbb{Z}) \]
then there are at least two elements of
\[ A_{\text{GL}_2} \setminus A_{\text{GL}_2} A_{T_0}(t)O_2(\mathbb{R})\text{GL}_2(\mathbb{Z})/O_2(\mathbb{R})\text{GL}_2(\mathbb{Z}) \]
that are in the same $\text{GL}_2(\mathbb{Q})$-orbit.

2.19. For each integer $N > 1$, let $\Gamma(N) \leq \text{GL}_n(\mathbb{Z})$ be the kernel of the reduction map
\[ \text{GL}_n(\mathbb{Z}) \rightarrow \text{GL}_n(\mathbb{Z}/N). \]
Prove that a subgroup of $\text{GL}_n(\mathbb{Q})$ is a congruence subgroup if and only if it contains $\Gamma(N)$ for some $N$ as a subgroup of finite index.
Chapter 3
Automorphic Representations

Abstract In this chapter we give a definition of an automorphic representa-
tion in the category of unitary representations of the adelic points of an affine
algebraic group. This definition will subsequently be refined, by enlarging the
category of representations considered to the category of admissible repre-
sentations, and then later by replacing representations of the archimedean
component of the adelic group by \((g, K)\)-modules.

3.1 Representations of locally compact groups

Our goal in this chapter is to give a definition of an automorphic representa-
tion that is completely natural from the point of view of trace formulæ.
This definition will subsequently be refined in Chapter 6 (see Definitions 6.6
and 6.8).

We will start by introducing some basic abstract representation theory
(i.e. representation theory that requires no more structure than a locally
compact group). Our basic reference is [Fol95]. Throughout this chapter,
we let \(G\) be a locally compact (Hausdorff) topological group (for example
\(G = H(\mathbb{A}_F)\) for \(H\) an affine algebraic group over a global field \(F\)). Let \(V\) be
a topological vector space over \(\mathbb{C}\). Following [Rud91], we take the convention
that topological vector spaces are Hausdorff. Let

\[
\text{End}(V)
\]
be the space of all continuous linear maps from $V$ to itself. We let $\text{GL}(V) \subset \text{End}(V)$ be the group of invertible continuous linear maps with continuous inverse.

Often we will be concerned with the case where $V$ is a Hilbert space, that is, a complex inner product space complete with respect to the metric induced by the inner product. We always assume that our Hilbert spaces are separable, that is, that they admit an orthonormal basis (in the Hilbert space sense) that is countable. The inner product will be denoted $(\cdot, \cdot)$ and the norm by $\|\cdot\|_2$. In this case, we let $\text{U}(V)$ be the subgroup of $\text{GL}(V)$ preserving $(\cdot, \cdot)$.

**Definition 3.1.** A representation $(\pi, V)$ of $G$ is a group action

$$G \times V \rightarrow V$$

$$(g, \varphi) \mapsto \pi(g)\varphi$$

such that the map $G \times V \rightarrow V$ is (jointly) continuous. In particular, $(\pi, V)$ defines a group homomorphism $\pi : G \rightarrow \text{GL}(V)$. If $V$ is a Hilbert space and $\pi$ maps $G$ to $\text{U}(V)$, that is,

$$(\pi(g)\varphi_1, \pi(g)\varphi_2) = (\varphi_1, \varphi_2)$$

for all $\varphi_1, \varphi_2 \in V$ and $g \in G$, then $\pi$ is said to be unitary.

Often one writes $\pi$ (resp. $V$) for $(\pi, V)$ if $V$ (resp. $\pi$) is understood. If we wish to specify that $\pi$ acts on a $\mathcal{D}$-space (e.g. Fréchet, Hilbert, locally convex) we often refer to $\pi$ as a $\mathcal{D}$-representation.

A morphism of representations (or a $G$-intertwining map) is a morphism of topological vector spaces that is $G$-equivariant; it is an isomorphism if the underlying morphism of topological vector spaces is an isomorphism. If two representations are isomorphic then they are also called equivalent. Two unitary representations $(\pi_1, V_1)$ and $(\pi_2, V_2)$ are unitarily equivalent if there is a $G$-equivariant isomorphism of Hilbert spaces $V_1 \rightarrow V_2$. Thus unitary equivalence implies equivalence. The converse is also true under suitable assumptions by Theorem 4.4.6 and Proposition 5.3.6.

A subrepresentation of $(\pi, V)$ is a closed subspace $W \leq V$ that is preserved by $\pi$, a quotient of $(\pi, V)$ is a topological vector space $W$ equipped with an action of $G$ and a $G$-equivariant continuous linear surjection $V \rightarrow W$, and a subquotient of $(\pi, V)$ is a subrepresentation of a quotient of $(\pi, V)$.

We will mostly be interested in unitary representations, and we need language to describe when a representation that is not originally presented as a unitary representation actually “is” unitary. First, we say that a representation on a Hilbert space is unitarizable if it is equivalent to a unitary representation. Sometimes the natural representations to study are not on Hilbert spaces, or even complete spaces (see Chapter 5 for example). However, one can still discuss unitarity in this situation as follows: A pre-Hilbert
space is a complex inner product space that admits a countable orthonormal basis. Thus a Hilbert space is simply a pre-Hilbert space that is complete with respect to the metric induced by the inner product. We can still define \( U(V) \) as before. A representation \( \pi \) of \( G \) on a pre-Hilbert space \( V \) extends by continuity to the completion \( \overline{V} \) of \( V \) (which is a Hilbert space). With this in mind, we say that a representation \( \pi \) on a pre-Hilbert space \( V \) is pre-unitary if \( \pi(G) \subseteq U(V) \). Thus for every pre-unitary representation of \( G \), we obtain, by continuity, a canonical unitary representation of \( G \) on \( \overline{V} \).

Finally, we say that a representation is pre-unitarizable if it is equivalent to a pre-unitary representation. Of course, in practice, these distinctions are often ignored, and one sometimes calls a unitarizable representation, or even a pre-unitarizable representation, simply a unitary representation.

We now discuss the basic means of constructing unitary representations. Assume one has a continuous right action

\[
X \times G \rightarrow X
\]

\[
(x, g) \mapsto xg
\]

where \( X \) is a locally compact (Hausdorff) space equipped with a right \( G \)-invariant Radon measure \( dx \). The Radon measure gives rise to an \( \mathcal{L}^2 \)-space, namely the space \( \mathcal{L}^2(X, dx) \) of square integrable complex valued functions on \( X \) endowed with the inner product

\[
(\varphi_1, \varphi_2) := \int_X \overline{\varphi_1(x)} \varphi_2(x) dx.
\]

We obtain a left action of \( G \) on \( \mathcal{L}^2(X, dx) \) via

\[
R(g)\varphi(x) := \varphi(xg).
\]

We refer to the action of \( G \) on \( \mathcal{L}^2(X, dx) \) just constructed as the natural action or the regular action.

**Theorem 3.1.1** The action of \( G \) on \( \mathcal{L}^2(X, dx) \) defined as in (3.3) is continuous. It defines a unitary representation of \( G \).

For the proof, we follow \[Fol95, Proposition 2.41\]:

**Proof.** We first observe that for any \( \varphi \in \mathcal{L}^2(X, dx) \) one has that

\[
\int_X |\varphi(xg)|^2 dx = \int_X |\varphi(x)|^2 dx
\]

since \( dx \) is right \( G \)-invariant. Thus \( R(g) \) is an isometry of \( \mathcal{L}^2(X, dx) \) for all \( g \in G \). Let

\[
C_c(X) < \mathcal{L}^2(X, dx)
\]

be the subspace of compactly supported continuous functions. Let \( U \subset G \) be an open neighborhood of the identity \( 1 \in G \) with compact closure such that
$g \in U$ if and only if $g^{-1} \in U$. Let $\varphi \in C_c(X)$, and let $W = \text{supp}(\varphi)U \subset X$. Then $W$ is relatively compact, and $R(g)\varphi$ is supported in $W$ if $g \in U$. Thus
\[ \|R(g)\varphi - \varphi\|_2 \leq \text{vol}(W)^{1/2}\|R(g)\varphi - \varphi\|_\infty. \]
The right hand side goes to zero as $g \to 1$ (see Exercise 3.1).

Since $X$ is locally compact and $dx$ is a Radon measure, $C_c(X)$ is dense in $L^2(X, dx)$. Thus given $\varphi_1 \in L^2(X, dx)$ and $\varepsilon > 0$, we can choose $\varphi_2 \in C_c(X)$ such that $\|\varphi_1 - \varphi_2\|_2 < \varepsilon$. We have already checked that $R(g)$ is an isometry of $L^2(X, dx)$, so for all $g \in G$ we have that
\[ \|R(g)\varphi_1 - \varphi_1\|_2 \leq \|R(g)(\varphi_1 - \varphi_2)\|_2 + \|R(g)\varphi_2 - \varphi_2\|_2 + \|\varphi_2 - \varphi_1\|_2 \]
\[ \leq 2\varepsilon + \|R(g)\varphi_2 - \varphi_2\|_2. \]
Since $\|R(g)\varphi_2 - \varphi_2\|_2 \to 0$ as $g \to 1$, we deduce that for each $\varphi \in L^2(X, dx)$ the map
\[ G \to L^2(X, dx) \]
\[ g \mapsto R(g)\varphi \]
is continuous. We now prove that the map
\[ G \times L^2(X, dx) \to L^2(X, dx) \]
\[ (g, \varphi) \mapsto R(g)\varphi \]
is jointly continuous. We have
\[ \|R(g_1)\varphi_1 - R(g_2)\varphi_2\|_2 = \|\varphi_1 - R(g_1^{-1}g_2)\varphi_2\|_2 \]
\[ \leq \|\varphi_1 - \varphi_2\|_2 + \|\varphi_2 - R(g_1^{-1}g_2)\varphi_2\|_2 \]
since $R(g_1)$ is an isometry of $L^2(X, dx)$. We have already shown that the map $g \to R(g)\varphi_2$ is continuous, so if we choose $g_1$ sufficiently close to $g_2$ and $\varphi_1$ sufficiently close to $\varphi_2$ then the quantity above can be made to be arbitrarily small, proving joint continuity.

### 3.2 Haar measures on locally compact groups

To give more concrete examples of representations, it is useful to know the notion of Haar measure. If $G$ is a locally compact group then there exists a positive Radon measure $d_r g$ on $G$ that is right invariant under the action of $G$:
\[ \int_G f(gx)d_r g = \int_G f(g)d_r g \quad (3.4) \]
for all $x \in G$ and $f \in C_c(G)$ (see [Fol95, Theorem 2.10]).

Moreover, this measure is unique up to multiplication by an element of $\mathbb{R}_{>0}$. A **right Haar measure** is a choice of such a measure. There is also a left invariant positive Radon measure, again unique up to multiplication by $\mathbb{R}_{>0}$. Such a measure is known as a **left Haar measure**. Indeed given a right Haar measure $d_r g$, we obtain a left Haar measure $d_l g$ by assigning to a measurable set $X \subseteq G$ the measure of $\{x^{-1} : x \in X\}$ with respect to $d_r g$. More briefly, $d_l g = d_r (g^{-1})$.

The existence of the right Haar measure gives via Theorem 3.1.1 a representation of $G$ on $L^2(G) = L^2(G, d_r g)$ called the **regular representation**.

**Definition 3.2.** A locally compact group $G$ is **unimodular** if any right Haar measure is also a left Haar measure.

We note that an abelian group is trivially unimodular. Moreover a compact (Hausdorff) group is unimodular (see Exercise 3.2) and discrete groups are unimodular.

**Example 3.1.** The points of Borel subgroups are, in general, not unimodular. For example, if $B \leq \text{GL}_2$ is the Borel subgroup of upper triangular matrices then for any local field $F$ with normalized absolute value $|\cdot|$ we can write

$$B(F) = \left\{ \begin{pmatrix} a & b \\ 1 & t \end{pmatrix} : a, b \in F^\times \text{ and } t \in F \right\}.$$ 

With respect to this decomposition, one can take

$$d_l g := \frac{dadbdt}{|a||b|} \quad \text{and} \quad d_r g := \frac{dadbdt}{|b|^2},$$ 

where $da$, $db$ and $dt$ are Haar measures on the additive group of $F$ as in §3.5.

The failure of a group to be unimodular is measured by a quasi-character. Assume that $d_r g$ is a Haar measure on $G$ and let $h \in G$. The map that sends a measurable set $X \subseteq G$ to the measure of $hX$ with respect to $d_r g$ is another right Haar measure denoted $d_r (hg)$ (we use several obvious variants of this notation below). Since Haar measures are unique up to scaling, there exists a $\delta_G(h) \in \mathbb{R}_{>0}$ such that

$$d_r (hg) = \delta_G(h)d_r g.$$ 

The map

$$\delta_G : G \longrightarrow \mathbb{R}_{>0}$$

is obviously a homomorphism. It is known as the **modular quasi-character**. One can alternately define $\delta_G(g)$ via the relation

$$d_r (g^{-1}) = \delta_G(g)d_l g$$

(3.7)
(compare Exercise 3.3). Since \( d_r(g^{-1}) \) is a right Haar measure, we see that \( \delta_G \) is trivial if and only if \( G \) is unimodular.

**Remark 3.1.** Some references define the modular quasi-character to be the multiplicative inverse of \( \delta_G \). In particular, in \([\text{Fol95}], \Delta(g) := \delta_G^{-1}(g) \). The choice of normalization we use in this book seems to be consistent with most of the literature in automorphic representation theory; one motivation to define things this way is provided by Proposition 3.6.1.

The following proposition provides a useful means of decomposing Haar measures:

**Proposition 3.2.1** Suppose that \( S \) and \( T \) are closed subgroups of \( G \) with compact intersection and that the product map \( S \times T \rightarrow G \) is open with image of full measure with respect to \( d_r g \). Then one can normalize the left and right Haar measures on \( S \) and \( T \) so that for all \( f \in C_c(G) \) one has

\[
\int_G f(g) d_r g = \int_{S \times T} f(st) \frac{\delta_T(t)}{\delta_G(t)} d_r sd_r t = \int_{S \times T} f(st) \frac{\delta_G(t)}{\delta_G(t)} d_r sd_r t.
\]

In particular, if \( G \) is unimodular, then

\[
\int_G f(g) d_g = \int_{S \times T} f(st) d_s d_r t.
\]

**Proof.** The first equality is \([\text{Kna86}, \text{Theorem 8.32}] \) in the case where \( G \) is a Lie group. The proof is the same in general. We also note that in the notation of loc. cit. one has \( \Delta_G := \delta_G, \Delta_S := \delta_S, \text{ and } \Delta_T := \delta_T \). The second equality follows from Exercise 3.3. \( \square \)

We will also require a slight generalization of the notion of a Haar measure. Let \( G \) be a locally compact group and let \( H \leq G \) be a closed subgroup. The following theorem characterizes when \( H \setminus G \) admits a right \( G \)-invariant Radon measure. See \([\text{Fol95}, \text{Theorem 2.49}] \) for a proof:

**Theorem 3.2.2** There is a right \( G \)-invariant Radon measure \( \mu \) on \( H \setminus G \) if and only if \( \delta_G|_H = \delta_H \). In this case, \( \mu \) is unique up to a constant factor, and for a suitable choice of right Haar measures \( d_g \) (resp. \( d_h \)) on \( G \) (resp. on \( H \)), one has that

\[
\int_G f(g) d_g = \int_{H \setminus G} \left( \int_H f(hg) d_h \right) d\mu(Hg)
\]

for \( f \in C_c(G) \). \( \square \)

The measure \( d\mu \) in the theorem will be denoted by \( \frac{d_g}{dn} \).
3.3 Gelfand-Pettis integrals

In the following section, we will require integration with values in topological vector spaces so we pause to quickly state the relevant definitions and results. Let $X$ be a locally compact (Hausdorff) topological space and let $\mu$ be a regular Borel measure on $X$. Assume that the continuous dual of the topological vector space $V$ separates points. Given a continuous function $f : X \to V$, the Gelfand-Pettis integral
\[
\int_X f(x) d\mu(x)
\]
is an element of $V$ such that
\[
\lambda \left( \int_X f(x) d\mu(x) \right) = \int_X \lambda(f(x)) d\mu(x)
\]
for all continuous $\mathbb{C}$-linear functionals $\lambda : V \to \mathbb{C}$. Since the continuous dual of $V$ separates points, if $\int_X f(x) d\mu(x)$ exists, it is unique.

A topological vector space $V$ is **quasi-complete** if every bounded Cauchy net converges (thus complete topological vector spaces are quasi-complete). The following theorem is a rephrasing of [Gar18, Corollary 14.1.2]:

**Theorem 3.3.1** Assume $V$ is locally convex and quasi-complete. For continuous and compactly supported function $f : X \to V$, the Gelfand-Pettis integral $\int_X f(x) d\mu(x)$ exists. The vector
\[
\int_X f(x) d\mu(x)
\]
lies in $\int_{\text{supp}(f)} d\mu(x)$ times the closed convex hull of $f(X)$. \hfill \qed

**Corollary 3.3.2** Assume that the continuous dual of $V$ separates points and that the Gelfand-Pettis integral $\int_X f(x) d\mu(x)$ exists. If $\nu : V \to \mathbb{R}_{\geq 0}$ is a continuous seminorm then
\[
\nu \left( \int_X f(x) d\mu(x) \right) \leq \int_X \nu(f(x)) d\mu(x).
\]

**Proof.** Let $\lambda$ be the unique linear functional on the 1-dimensional $\mathbb{C}$-vector subspace of $V$ spanned by $\int_X f(x) d\mu(x)$ such that
\[
\lambda \left( \int_X f(x) d\mu(x) \right) = \nu \left( \int_X f(x) d\mu(x) \right).
\]
By the Hahn-Banach theorem, there is an extension of \( \lambda \) to a linear functional on \( V \) (still denoted by \( \lambda \)) such that \( |\lambda(\varphi)| \leq \nu(\varphi) \) for all \( \varphi \in V \). Since \( \nu \) is continuous, \( \lambda : V \to \mathbb{C} \) is continuous. Thus

\[
\nu \left( \int_X f(x) d\mu(x) \right) = \lambda \left( \int_X f(x) d\mu(x) \right) = \int_X \lambda(f(x)) d\mu(x) \\
\leq \int_X |\lambda(f(x))| d\mu(x) \\
\leq \int_X \nu(f(x)) d\mu(x).
\]

Many topological vector spaces of interest are locally convex and quasi-complete. This is true, for example, of Hilbert spaces, Banach spaces and Fréchet spaces. Every topological vector space we encounter in this book will be at least a subspace of a locally convex and quasi-complete vector space with one exception. When dealing with totally disconnected groups (see Chapter 5), we will have occasion to work with topological vector spaces \( V \) given the discrete topology. In this case, the relevant integrals reduce to finite sums by Lemma 5.2.1, so they can be interpreted either directly or in terms of Gelfand-Pettis integrals.

3.4 Convolution algebras of test functions

To any locally compact group \( G \), we can associate the space of integrable functions

\[
L^1(G) := \left\{ d_{\tau}g\text{-measurable } f : G \to \mathbb{C} : \int_G |f(g)| d_{\tau}g < \infty \right\}
\]

and the subspace of compactly supported continuous functions \( C_c(G) \). This space is endowed with the structure of a \( \mathbb{C} \)-algebra with product

\[
(f_1 * f_2)(g) := \int_G f_1(gh^{-1}) f_2(h) d\tau h.
\]  

(3.8)

This product is associative (see Exercise 3.17).

Example 3.2. When \( G = \mathbb{R} \), the convolution \( f_1 * f_2 \) is exactly the usual convolution from Fourier analysis.

Given a representation \( \pi : G \to \text{GL}(V) \), if \( f \in C_c(G) \) then one obtains a linear map
3.5 Haar measures on local fields

\[ \pi(f) : V \to V \]
\[ \varphi \mapsto \int_{G} f(g) \pi(g) \varphi \, dg \]

under mild assumptions on the topological vector space \( V \) (see Theorem 3.3.1). Here the integral is to be understood as a Gelfand-Pettis integral as in §3.3.

We observe that if \( \pi \) is unitary then \( \pi(f) \) is bounded, and in fact in the unitary case, it is well-defined for all \( f \in L^1(G) \) (see Exercise 3.18). The definition (3.8) is chosen so that
\[ \pi(f_1 \ast f_2) = \pi(f_1) \circ \pi(f_2) \]
(see Exercise 3.19). Thus, in particular, representations \((\pi, V)\) of \( G \) give rise to algebra representations of \( C_c(G) \), that is, \( \mathbb{C} \)-algebra homomorphisms
\[ C_c(G) \to \text{End}(V). \]

3.5 Haar measures on local fields

Let \( F \) be a local field, that is, a field equipped with an absolute value which is locally compact with respect to the topology induced by the absolute value (see §2.1). All infinite locally compact fields arise as the completion of some global field [Lö08, §25, Theorem 2] with respect to some absolute value. In this section, we define Haar measures on the additive group of \( F \).

Assume first that \( F \) is archimedean, so \( F \) is isomorphic to \( \mathbb{R} \) or \( \mathbb{C} \). The standard Haar measure on \( \mathbb{R} \) is the usual Lebesgue measure, and twice the standard Haar measure on \( \mathbb{C} \). If \( z = x + iy \) with \( x, y \in \mathbb{R} \), then this Haar measure is \(|dz \wedge d\bar{z}| = 2dx \wedge dy\) where \( dx \) and \( dy \) are the Lebesgue measure on \( \mathbb{R} \).

Let \( F \) be a nonarchimedean local field with normalized absolute value \(|\cdot|\) and ring of integers \( \mathcal{O}_F \). Let \( \varpi \) be a uniformizer and set \( \varpi^{-1} := |\varpi| \). Since the additive group of \( F \) is a locally compact topological group, it admits a Haar measure \( dx \) (see §3.2). The set \( \mathcal{O}_F \subset F \), being compact, has finite measure, so to specify the Haar measure, it suffices to specify the measure \( dx(\mathcal{O}_F) \) of \( \mathcal{O}_F \). We often assume \( dx(\mathcal{O}_F) = 1 \), but sometimes other normalizations are convenient.

Open sets in \( F \) of the form \( a + \varpi^k \mathcal{O}_F \) where \( a \in F \) and \( k \in \mathbb{Z}_{>0} \) form a neighborhood base. Using additivity and invariance of the Haar measure, one can compute the measure of such sets:

**Lemma 3.5.1** If \( dx \) is a Haar measure on \( F \) then
\[ dx(a + \varpi^k \mathcal{O}_F) = \varpi^{-k} dx(\mathcal{O}_F). \]
Thus for every local field $F$, we have a Haar measure $dx$. Via the usual procedure, we then obtain a product measure on any $F$-vector space. It is again a Haar measure with respect to addition.

Given a Haar measure $dx$ on $F$, we obtain a Haar measure

$$d^\times x := \frac{dx}{|x|}$$

on the unit group $F^\times$. With this normalization in the nonarchimedean case $d^\times x(O_F^\times) = (1 - q^{-1})dx(O_F)$ (see Exercise 3.7). Another normalization we will often use in the nonarchimedean case is $d^\times x = \frac{dx}{(1 - q^{-1})|x|}$, so that $d^\times x(O_F^\times) = dx(O_F)$.

Similarly, we can define a Haar measure on $GL_n(F)$ by viewing it as an (open) subset of the $n$ by $n$ matrices $\mathfrak{gl}_n(F)$ and then taking the measure

$$dX \frac{1}{|\det X|^n}$$

where $dX$ is the (additive) Haar measure on $\mathfrak{gl}_n(F)$ (see Exercise 3.11). This construction will be generalized to treat the $F$-points of affine algebraic groups in the following section. We observe that this description of the Haar measure implies that $GL_n(F)$ is unimodular. This is also a special case of Corollary 3.6.2 below.

### 3.6 Haar measures on the points of affine algebraic groups

Let $G$ be a smooth affine algebraic group over a global field $F$. It is useful to more explicitly describe the Haar measures on $G(R)$ for a locally compact topological $F$-algebras $R$ using differential forms. We will do this in the current section.

Let $\mathfrak{g}^*$ denote the linear dual of $\mathfrak{g}$, which we may identify with the cotangent space of $G$ at the origin. Let $n = \dim_F G$. Then there is a nonzero top dimensional left invariant differential form

$$\omega \in \wedge^n \mathfrak{g}^*$$

which is unique up to multiplication by an element of $F^\times$. For every place $v$ of $F$, we have an injection

$$\wedge^n \mathfrak{g}^* \hookrightarrow \wedge^n \mathfrak{g}^*_{F_v}.$$
and we continue to denote by $\omega_\ell$ the image of $\omega_{\ell'}$. This allows us to define for each $v$ a Radon measure

$$C_c(G(F_v)) \rightarrow \mathbb{C}$$

$$f \mapsto \int_{G(F_v)} f(g)|\omega_\ell|_v(g).$$

Since $\omega_\ell$ is left $G(F_v)$-invariant, the measure $|\omega_\ell|_v$ is left $G(F_v)$-invariant, and hence is a left Haar measure. This is explained in more detail in the archimedean case in [Kna86, Chapter VIII.2] and in the nonarchimedean case in [Oes84, §2].

**Proposition 3.6.1** One has that

$$\delta_{G(F_v)}(g) = |\det(\text{Ad}(g) : g \otimes_F F_v \rightarrow g \otimes_F F_v)|_v.$$  \hfill (3.9)

**Proof.** One has that

$$\text{Ad}(g)\omega_\ell = c(g)\omega_\ell$$

where

$$c(g) := \det(\text{Ad}(g) : g \otimes_F F_v \rightarrow g \otimes_F F_v).$$

We write this in terms of Haar measures:

$$|\omega_\ell|_v(hg^{-1}) = |\omega_\ell|_v(ghg^{-1}) = |c(g)|_v|\omega_\ell|_v(h).$$

On the other hand

$$d_\ell(hg^{-1}) = \delta_{G(F_v)}(g)d_\ell(h)$$

for any left Haar measure $d_\ell$ on $G(F_v)$ by Exercise 3.3. We deduce that $|c(g)|_v = \delta_{G(F_v)}(g)$. \hfill $\Box$

**Corollary 3.6.2** If $G^o$ is reductive, then $G(F_v)$ is unimodular.

**Proof.** Assume first that $G$ is connected. Let $Z_G$ be the center of $G$ and $G^{\text{der}}$ its derived group. By (1.8) we then have a commutative diagram

$$\begin{array}{ccc}
Z_G^o \times G^{\text{der}} & \overset{a}{\longrightarrow} & \mathbb{G}_m \\
\downarrow b & & \downarrow \text{Id} \\
G & \overset{c}{\longrightarrow} & \mathbb{G}_m
\end{array}$$

where $a$ and $c$ are given on points by

$$a(z, g) := \det(\text{Ad}(zg) : g \otimes_F F_v \rightarrow g \otimes_F F_v),$$

$$c(g) := \det(\text{Ad}(g) : g \otimes_F F_v \rightarrow g \otimes_F F_v),$$

the left vertical map $b$ is the product map, and the right vertical map is the identity. The map $a$ is trivial on $Z_G^o$ and it is trivial on $G^{\text{der}}$ since $X^*(G^{\text{der}})$
is trivial. Indeed, the kernel of a character in $X^*(G^\text{der})$ is a normal subgroup $N \leq G^\text{der}$ such that $G^\text{der}/N$ is commutative, hence $N = G^\text{der}$. Since $b$ is a quotient map, $c$ is trivial as well, and we deduce the corollary when $G$ is connected.

For the general case we observe that Proposition 3.6.1 implies that $\delta_{G(F_v)}|_{G^0(F_v)} = \delta_{G(F_v)}$, which we have just shown is trivial. Thus $\delta_{G(F_v)}$ is trivial on the finite index subgroup $G^0(F_v)$ of $G(F_v)$. It follows that $\delta_{G(F_v)}$ itself is trivial.

Proposition 3.2.1 provides us with a description of the left Haar measures $d_L g_v$ on $G(F_v)$ for every $v$. We similarly obtain right Haar measures $d_R g_v := d_L(g_v^{-1})$. One now wishes to obtain left and right Haar measures on $G(A_F)$.

To do this, let $S$ be a finite set of places of $F$ and let

$$K^S = \prod_{v \not\in S} K_v \leq G(A^S_F)$$

be a compact open subgroup. We then define a left (resp. right) Haar measure on $G(A_F)$ via

$$d_L g = \prod_v d_L g_v \quad \text{and} \quad d_R g = \prod_v d_R g_v,$$  \hspace{1cm} (3.10)

where we assume that $d_L g_v(K_v) = d_R g_v(K_v) = 1$ for all $v \not\in S$. Then it is easy to see that the measures $d_L g, d_R g$ so defined are left and right Haar measures on $G(A_F)$, respectively (see Exercise 3.6). If $G$ is reductive we can apply an analogous procedure and obtain left and right Haar measures on $A_G \backslash G(A_F)$.

We can now prove the following lemma:

**Lemma 3.6.3** Any right Haar measure on $G(A_F)$ is left invariant under $G(F)$.

**Proof.** Since the right Haar measure is constructed as the product of local Haar measures, one has that

$$\delta_{G(A_F)}(g) = \prod_v \delta_{G(F_v)}(g_v) = |\det (\text{Ad}(g) : \mathfrak{g} \otimes F \mathbb{A}_F \longrightarrow \mathfrak{g} \otimes F \mathbb{A}_F)|.$$  \hspace{1cm} (3.11)

By the product formula, we deduce that $\delta_{G(A_F)}(\gamma) = 1$ for $\gamma \in G(F)$. \qed

The proof here amounted to observing that $\delta_{G(A_F)}$ is trivial on $G(F)$. In fact there is a much larger subgroup of $G(A_F)$ on which the modular quasi-character is trivial, namely $G(A_F)^1$, defined in (2.16):

**Lemma 3.6.4** The group $G(A_F)^1$ is unimodular. If $G^0$ is reductive, then $G(A_F)$ is unimodular.

**Proof.** The mapping
defines an element of $X^*(G)$, the character group of $G$. Since the modular quasi-character of $G(\mathbb{A}_F)$ is just this character followed by the idelic norm by (3.11), we deduce that $G(\mathbb{A}_F)^1$ is unimodular.

The modular quasi-character of $G(\mathbb{A}_F)$ is the product of the local modular quasi-characters by (3.11). Therefore if $G$ is reductive we deduce that $G(\mathbb{A}_F)$ is unimodular by Corollary 3.6.2.

Now $G(F)$ is a discrete subgroup of $G(\mathbb{A}_F)$ and $G(\mathbb{A}_F)^1$. It is in particular closed in either of these groups. By Lemma 3.6.3, the restriction of the modular quasi-character of $G(\mathbb{A}_F)$ to $G(F)$ is trivial, and the modular quasi-character of any discrete group is trivial. Thus by Theorem 3.2.2, $d_r g$ induces a right $G(\mathbb{A}_F)$-invariant Radon measure on $G(F) \backslash G(\mathbb{A}_F)$.

This quotient has finite measure if and only if $G(\mathbb{A}_F) = G(\mathbb{A}_F)^1$ by Theorem 2.6.3. Similarly, any right Haar measure on $A_G \backslash G(\mathbb{A}_F)$ induces a right $A_G \backslash G(\mathbb{A}_F)$-invariant Radon measure on $[G] := A_G G(F) \backslash G(\mathbb{A}_F)$, defined as in (2.18).

The way we obtained the Haar measure on $G(\mathbb{A}_F)$ is a little unnatural. It would be more natural to just take the product $\prod_v |\omega|_v(C_v)$ directly. However, for arbitrary compact subsets $C = \bigcap_v C_v \subset G(\mathbb{A}_F)$

$\prod_v |\omega|_v(C_v)$

need not converge. This is already the case for $G = \text{GL}_1$.

Assume $G$ is reductive. One can modify this procedure by introducing a family of positive real numbers $\lambda_v$ such that for all compact subsets $C = \bigcap_v C_v \subset G(\mathbb{A}_F)$

$\prod_v \lambda_v |\omega|_v(C_v)$

is convergent. In this way one obtains the so-called Tamagawa measure on $G(\mathbb{A}_F)$. Upon choosing a suitable measure on $A_G$, we obtain the Tamagawa measure on $[G]$. The measure of $[G]$ with respect to the Tamagawa measure is called the Tamagawa number of $G$. This number is intimately connected with the arithmetic of $G$. We refer to the reader to [Kot88] and the references therein for more details.
3.7 Automorphic representations in the $L^2$-sense

Now assume that $G$ is an affine algebraic group over a global field $F$. Consider the quotient
$$[G] := \mathcal{A}_G G(F) \backslash G(\mathbb{A}_F)$$
as in (2.18).

A right Haar measure on $\mathcal{A}_G G(\mathbb{A}_F)$ induces a right $G(\mathbb{A}_F)$-invariant Radon measure on $[G]$. Thus we have a well-defined Hilbert space
$$L^2([G])$$
with inner product defined as (3.2) (but with $X$ replaced by $[G]$ and $dx$ replaced by a right $\mathcal{A}_G G(\mathbb{A}_F)$-invariant Radon measure).

**Definition 3.3.** An automorphic representation of $\mathcal{A}_G G(\mathbb{A}_F)$ in the $L^2$-sense is an irreducible unitary representation $\pi$ of $\mathcal{A}_G G(\mathbb{A}_F)$ that is equivalent to a subquotient of $L^2([G])$.

In fact, subquotients are automatically subrepresentations since the representation of $\mathcal{A}_G G(\mathbb{A}_F)$ on $L^2([G])$ is unitary (see the proof of Theorem 6.6.4 below). This should be regarded as a defect of the definition above, because in general $L^2([G])$ cannot be written as a direct sum of its irreducible subrepresentations (see §10.4). This is one of several reasons why a refined definition of an automorphic representation is necessary. We give this refinement in Chapter 6 below. Until Chapter 6, we will refer to automorphic representations in the $L^2$-sense simply as automorphic representations.

Note that, contrary to the usual convention, we have not assumed that $G$ is reductive. Even without this assumption, the preceding definition makes sense. It is only when one starts discussing admissible representations that the assumption of reductivity is really necessary. A concrete example of this is given right after [BS98, Proposition 3.2.11]. The nonreductive case has not received as much attention as the reductive case, but see [Sli84].

We end this chapter with §3.8 through §3.11. Sections 3.8 through 3.10 discuss how one can decompose unitary representations of locally compact groups in general, and §3.11 discusses why we have restricted our attention to affine algebraic groups instead of certain natural larger classes of groups. All of these sections can be omitted on a first reading.

Before delving into these sections, we note that the definition of an automorphic representation leaves much to be desired. It does not illuminate the place of automorphic representations in arithmetic and geometry and it is unclear how one would go about studying these objects. In the subsequent two chapters, we will develop the algebras of operators on automorphic representations that are used to study them. They are refinements of the algebra $L^1(\mathcal{A}_G G(\mathbb{A}_F))$ discussed in §3.4 above. For a number field $F$, the decomposition
naturally gives rise to a decomposition of $L^1(A_{G \setminus G} \times G(\mathbb{A}_F^{\infty}))$; the first factor is the archimedean factor which will be discussed in Chapter 4 and the second factor is the nonarchimedean factor which will be discussed in chapters 5 and 8.

This gives us tools to study automorphic representations, but it still does not shed much light into how they appear in geometry and arithmetic. The link with geometry will be made precise in Chapter 15. The primary link with arithmetic is much more profound, subtle, and conjectural; it is the content of the Langlands functoriality conjectures, explained in Chapter 12.

3.8 Decomposition of representations

If $V$ is a finite dimensional complex representation of a finite group $G$, then it is completely reducible. In other words, it admits a direct sum decomposition

$$V = \oplus_i V_i$$

where the $V_i$ are irreducible representations of $G$ (see Exercise 3.15). Thus a subquotient of $V$ in this case is just a subrepresentation. When $G$ is no longer a finite group, these sorts of decompositions may no longer hold (see Exercise 3.16).

However, there is a very general theory which guarantees some sort of decomposition exists if we restrict our attention to unitary representations. First, let us briefly discuss the notion of a direct integral. We follow the exposition of [Fol95, §7.4]. Let $X$ be a measurable space. A field of Hilbert spaces over $X$ is a collection of Hilbert spaces $\{V_x\}_{x \in X}$ indexed by $x \in X$. We have an obvious projection map

$$p : \prod_{x \in X} V_x \to X$$

sending everything in $V_x \hookrightarrow \prod_{x \in X} V_x$ to $x$. The inner product on $V_x$ will be denoted by $\langle \cdot, \cdot \rangle_x$ and the induced norm by $\|\cdot\|_x$. A section or vector field is a map

$$s : X \to \prod_{x \in X} V_x$$

such that with $(p \circ s)(x) = x$.

These notions are uninteresting unless one imposes some measurability condition on the fibers of $p$ and on sections $s$. This is introduced as follows. A measurable field of Hilbert spaces over $X$ is a pair $\{(V_x)_{x \in X}, \{\epsilon_j\}\)$
consisting of a field of Hilbert spaces \( \{ V_x \}_{x \in X} \) together with a sequence of sections \( e_1, e_2, \ldots \) such that

(a) \( x \mapsto (e_i(x), e_j(x))_x \) is measurable for all \( i, j \),
(b) for each \( x \in X \), the linear span of \( e_1(x), e_2(x), \ldots \) is dense in \( V_x \).

Given a measurable field of Hilbert spaces over \( X \), a section \( s \) is called measurable if \( x \mapsto h e_i(x) \mapsto e_j(x) i x \) is measurable on \( X \) for each \( j \). For simplicity, in the sequel we will drop the \( \{ e_j \} \) from notation, despite the fact that it is an essential part of the data defining the measurable field of vector spaces.

Suppose that \( \{ V_x \}_{x \in X} \) is a measurable field of Hilbert spaces over \( X \) and suppose that \( \mu \) is a measure on \( X \). For measurable sections \( s \mapsto s' \) define

\[
(s, s')_\mu := \int_X (s(x), s'(x))_x d\mu(x) \tag{3.12}
\]

and \( \| s \|_\mu := (s, s)^{1/2}_\mu \). Then the direct integral of \( \{ V_x \}_{x \in X} \) with respect to \( \mu \) is

\[
\int_X \oplus V_x d\mu(x) := \{ \text{measurable sections } s : \| s \|_\mu < \infty \}. \tag{3.13}
\]

As usual we identify sections that agree almost everywhere with respect to \( \mu \). The space (3.13) is again a Hilbert space with inner product \( (\cdot, \cdot)_\mu \).

For each \( x \in X \), let \( \text{End}(V_x) \) denote the \( \mathbb{C} \)-vector space of continuous linear operators from \( V_x \) to itself. A field of operators on a measurable field of Hilbert spaces \( \{ V_x \}_{x \in X} \) is an element

\[
(A_x) \in \prod_{x \in X} \text{End}(V_x).
\]

The field of operators is said to be measurable if \( x \mapsto A_x(s(x)) \) is a measurable section whenever \( s \) is a measurable section. If the essential supremum

\[
eq \sup \| A_x \|_x
\]

is finite, then we have a bounded operator

\[
\int_X A_x d\mu(x) : \int_X \oplus V_x d\mu(x) \rightarrow \int_X \oplus V_x d\mu(x). \tag{3.14}
\]

It sends a section \( s \) to the section \( x \mapsto A_x(s(x)) \). The operator (3.14) is called the direct integral of \( (A_x) \). For example, a diagonal operator is an operator of the form

\[
\int_X \oplus V_x d\mu(x) \rightarrow \int_X \oplus V_x d\mu(x) \\
\quad s \mapsto (x \mapsto \phi(x)s(x)) \tag{3.15}
\]
for $\phi \in L^\infty(\mu)$.

Now let $G$ be a locally compact (Hausdorff) second countable group (for example, $G(\mathbb{A}_F)$ or $A_G \backslash G(\mathbb{A}_F)$ for some affine algebraic group $G$ over a global field $F$). A **measurable field of representations** on $\{V_x\}_{x \in X}$ is a collection of continuous representations

$$\pi_x : G \times V_x \rightarrow V_x$$

indexed by $x \in X$ such that

- (a) $(\pi_x(g))$ is a measurable field of operators for each $g \in G$,
- (b) $\sup_{x \in X} \|\pi_x(g)\|_x < \infty$ for each $g \in G$.

We say the measurable field of representations is **unitary** if $\pi_x$ is unitary for all $x \in X$. In this case, condition (b) above is automatically satisfied. For any measurable field of representations on $\{V_x\}_{x \in X}$ and $g \in G$, the integral

$$\int_X \pi_x(g) d\mu(x)$$

is defined. Hence we obtain a representation of $G$ on $\int_X V_x d\mu(x)$ in the usual sense. This is the direct integral of the measurable field of representations, or more briefly, the **direct integral of the representations**.

Two motivating examples to keep in mind are the right regular representation of $\mathbb{R}$ on $L^2(\mathbb{Z} \cap \mathbb{R})$ and $L^2(\mathbb{R})$. Fourier analysis tells us that

$$L^2(\mathbb{Z} \cap \mathbb{R}) = \bigoplus_{n \in \mathbb{Z}} V_n \quad \text{and} \quad L^2(\mathbb{R}) = \int_{t \in \mathbb{R}} V_t dt. \quad (3.16)$$

Here $V_t$ is the 1-dimensional $\mathbb{C}$-vector space on which $\mathbb{R}$ acts via $x \mapsto e^{2\pi itx}$. The decomposition of $L^2(\mathbb{Z} \cap \mathbb{R})$ is said to be **discrete** and the decomposition of $L^2(\mathbb{R})$ is said to be **continuous**: this will be discussed in greater generality in §3.10 below.

Decompositions like the decomposition of $L^2(\mathbb{R})$ abound in the theory of automorphic forms (compare §10.4). This allows us to explain both why working with subquotients is sometimes necessary, and why working with subquotients in an $L^2$-setting is somewhat unsatisfactory. One would like to consider all of the $V_t$ occurring in (3.16) as “pieces” of the representation of $\mathbb{R}$ on $L^2(\mathbb{R})$, but they are not subquotients:

**Lemma 3.8.1** For every $t \in \mathbb{R}$, there is a continuous linear $\mathbb{R}$-intertwining map

$$L^1(\mathbb{R}) \rightarrow V_t.$$

There is no nontrivial continuous linear $\mathbb{R}$-intertwining map

$$V_t \rightarrow L^p(\mathbb{R}) \quad \text{for} \quad p \in \{1, 2\}.$$
Moreover, there is no nontrivial \( R \)-intertwining map from \( V_t \) to any quotient of \( L^2(\mathbb{R}) \).

Proof. For each \( t \), one has a continuous \( R \)-intertwining linear map

\[
L^1(\mathbb{R}) \rightarrow V_t \\
f \mapsto \int_{\mathbb{R}} f(x) e^{-2\pi itx} dx.
\]

Let \( p \in \{1, 2\} \). An \( R \)-intertwining map

\[
V_t \rightarrow L^p(\mathbb{R})
\]

is either zero or has image contained in the \( \mathbb{C} \)-span of \( x \mapsto e^{2\pi itx} \). On the other hand \( x \mapsto e^{2\pi itx} \) is not in \( L^p(\mathbb{R}) \), so we deduce that any linear \( R \)-intertwining map \( V_t \rightarrow L^p(\mathbb{R}) \) is zero.

Suppose that \( V_t \) is a subquotient of \( L^2(\mathbb{R}) \). Thus there is a continuous surjective \( R \)-intertwining map \( L^2(\mathbb{R}) \rightarrow W \) and a continuous injective \( R \)-intertwining map \( V_t \rightarrow W \). By the Riesz representation theorem, this implies that there is a continuous injective intertwining map

\[
V_t \rightarrow W \rightarrow L^2(\mathbb{R}),
\]

where the second arrow is the adjoint of \( L^2(\mathbb{R}) \rightarrow W \). This contradicts the fact, proven above, that there is no nontrivial \( R \)-intertwining map \( V_t \rightarrow L^2(\mathbb{R}) \).

\( \square \)

For any second countable locally compact group \( G \) and unitary representation \((\pi, V)\) of \( G \), one can obtain a direct integral decomposition [Fol95, Theorem 7.37] satisfying certain desiderata. Unfortunately the direct integral decomposition is fairly badly behaved for general locally compact topological groups. If one restricts to the class of groups that are of type I then the situation is much more pleasant. We will discuss this in §3.10.

### 3.9 The Fell topology

To describe the refined decomposition that exists for type I groups \( G \), it is useful to define the **unitary dual** \( \widehat{G} \) of \( G \). Our basic reference is [Fol95, §7.2]. As a set, \( \widehat{G} \) is the set of unitary equivalence classes of unitary representations. The unitary dual \( \widehat{G} \) is equipped with the **Fell topology**. The procedure for defining this topology is very similar to the procedure for defining the Zariski topology as in §1.2. To define this topology, we recall from §3.4 every representation \((\pi, V)\) of \( G \) gives rise to an algebra homomorphism \( C_c(G) \rightarrow \text{End}(V) \) which extends to \( L^1(G) \) if \( \pi \) is unitary. We can extend this
homomorphism still further by taking a suitable completion as follows. For each irreducible unitary representation \( \pi \), denote by \([\pi]\) its class in \( \hat{G} \). We set
\[
\| f \|_* = \sup_{[\pi] \in \hat{G}} \| \pi(f) \|
\]  
(3.17)
where \( \| \pi(f) \| \) denotes the operator norm of \( \pi(f) \) acting on the space of \( \pi \). By considering the trivial representation we see that \( \| f \|_* \leq \| f \|_1 \) where \( \| \cdot \|_1 \) is the norm on \( L^1(G) \). The function \( \| \cdot \|_* \) is obviously a seminorm and it is in fact a norm \([\text{Fol95}, \text{Corollary 7.2}]\). We define \( C^*(G) \) to be the completion of \( L^1(G) \) with respect to \( \| \cdot \|_* \). The convolution operation extends by continuity to \( C^*(G) \) and endows it with the structure of an algebra. The algebra \( C^*(G) \) is called the \( C^* \)-algebra of the group \( G \). Thus for each unitary representation \((\pi, V)\) of \( G \), we obtain a representation of algebras
\[
C^*(G) \to \text{End}(V).
\]
This is the motivating example of a \( C^* \)-algebra, but we will not give the (elementary) definition of a \( C^* \)-algebra, referring instead to \([\text{Fol95}, \S 1.1]\).

Now given a unitary representation \( \pi \) of \( G \), we obtain a closed 2-sided ideal
\[
\ker(\pi) = \{ f \in C^*(G) : \pi(f) = 0 \}
\]
of \( C^*(G) \). If \( \pi \) is irreducible, this ideal is called a \textbf{primitive ideal} of \( C^*(G) \). In fact it depends only on the equivalence class of \( \pi \) in the category of unitary representations. The set of all primitive ideals in \( C^*(G) \) is called \( \text{Prim}(G) \):
\[
\text{Prim}(G) := \{ \ker(\pi) : [\pi] \in \hat{G} \}.
\]
If we think of this as the analogue of the prime spectrum of a ring, then proceeding as follows is reasonable: For any subset \( S \subset \text{Prim}(G) \), we let
\[
V(S) := \left\{ p \in \text{Prim}(G) : p \supset \bigcap_{q \in S} q \right\}.
\]
By convention, \( V(\emptyset) = \emptyset \). Then one checks that the \( V(S) \) just defined form the closed sets of a topology on \( \text{Prim}(G) \), called the \textbf{hull-kernel} or \textbf{Jacobson topology}. Now we have a natural map
\[
\hat{G} \to \text{Prim}(G)
\]
\([\pi] \mapsto \ker(\pi)
\]
and we can pull back the topology on \( \text{Prim}(G) \) to obtain a topology on \( \hat{G} \). In other words, the open sets in \( \hat{G} \) are just the inverse images of open sets in \( \text{Prim}(G) \). This is the \textbf{Fell topology}. Since the Fell topology is a topology, the set of Borel sets on \( \hat{G} \) is defined; this gives \( \hat{G} \) the structure of a measurable
space. When speaking of measures and measurable functions on $\Gamma$, we will always mean measures and measurable functions with respect the $\sigma$-algebra of Borel sets with respect to the Fell topology.

### 3.10 Type I groups

We have not (and will not) given the definition of a type I group. However, we can give a characterization using the following theorem [Fol95, Theorem 7.6]:

**Theorem 3.10.1** If $G$ is a second countable locally compact group then the following are equivalent:

(a) The group $G$ is of type I.
(b) The Fell topology on $\Gamma$ is $T_0$: for all distinct $x, y \in \Gamma$, there is an open set $U \subset \Gamma$ such that $x \in U$ and $y \not\in U$ or $y \in U$ and $x \not\in U$.
(c) The map $\Gamma \to \text{Prim}(G)$ is injective.

Fortunately the groups of primary concern to us in this book are of type I:

**Theorem 3.10.2** If $G$ is an affine algebraic group over a characteristic zero local field $F$ then $G(F)$ is of type I.

**Proof.** See [Sli84, Théorème 1.2.3].

One can find a proof of the following in [Clo07, Appendix]:

**Theorem 3.10.3** If $G$ is a reductive group over a number field $F$ then $G(\mathbb{A}_F)$ is of type I.

It should be pointed out that if $G$ is not reductive then $G(\mathbb{A}_F)$ need not be of type I [Moo65, §7].

For $\pi \in \Gamma$, let $V_\pi$ be the space of (a particular realization of) $\pi$. For type I groups, the following theorem describes the decomposition of a unitary representation into irreducible subquotients (see [Clo07, Theorem 3.3]):

**Theorem 3.10.4** Let $(\rho, V)$ be a unitary representation of the type I group $G$. Then there exists a Borel measurable multiplicity function

$$
\begin{align*}
\Gamma &\to \mathbb{Z}_{\geq 1} \\
\pi &\mapsto m(\pi)
\end{align*}
$$

and a positive measure $\mu$ on $\Gamma$ such that

$$(\rho, V) = \int_{\Gamma} \left( \bigoplus_{\pi} V_\pi ight) m(\pi) \, d\mu(\pi).$$
The decomposition is unique up to changing the measures and multiplicity functions on sets of measure zero.

Of course in practice it would be better to have an explicit description of the decomposition of Theorem 3.10.4 for naturally occurring representations \( V \). Langlands’ profound and foundational work on Eisenstein series provides such a decomposition in our primary case of interest, namely when \( V = L^2([G]) \). We will return to this point in \( \S 10.4 \) below.

### 3.11 Why affine groups?

In the preceding discussion, we have always assumed that \( G \) is an affine algebraic group. There are two possible avenues for generalization. First, we could look at covering groups of \( G(\mathbb{A}_F) \) that are not algebraic. There is a great deal of beautiful theory that has developed around this topic and it is starting to become systematic. However, due to the authors’ lack of experience with the theory and additional complications that come up, we have decided not to treat this topic.

If we forgo covering groups, we could still generalize from affine groups to algebraic groups that are not necessarily affine. To get some idea of how this enlarges the set of groups in question, we recall that a general algebraic group \( G \) over a number field \( \mathbb{F} \) (not necessarily affine) has a unique linear algebraic subgroup \( G_a \subseteq G \) such that \( G_a := G/G_a \) is an abelian variety by a famous theorem of Chevalley [Con02].

Now consider an abelian variety \( A \) over a number field \( \mathbb{F} \). In [Con12b] one finds a proof of a result of Weil stating that \( A(\mathbb{A}_F) \) can still be defined as a topological space. In fact it is compact [Con12b, Theorem 4.4]. In particular, if \( A(F) \) is infinite, it cannot be simultaneously discrete and closed in \( A(\mathbb{A}_F) \).

Thus \( A(\mathbb{A}_F) \) and the adelic quotient \( A(F) \backslash A(\mathbb{A}_F) \) are very different from their analogues in the affine case. We do not know whether or not it is sensible to pursue automorphic forms in this setting.

### Exercises

3.1. In the situation of the proof of Theorem 3.1.1, prove that if \( \varphi \in C_c(X) \) then \( \| R(g) \varphi - \varphi \|_\infty \to 0 \) as \( g \to 1 \).

3.2. Prove that compact (Hausdorff) topological groups are unimodular.

3.3. Show that for a locally compact topological group \( G \) the following are valid:

\[
d_r(hg) = \delta_G(h)d_r(g),
\]
\[ \begin{align*}
d_r(gh) &= \delta_G^{-1}(h)d_{r}g, \\
d_r(g^{-1}) &= \delta_G^{-1}(g)d_{r}g, \\
d_{l}(g^{-1}) &= \delta_G(g)d_{l}g.
\end{align*} \]

Show in addition that if the measures \( d_{r}g \) and \( d_{l}g \) are normalized appropriately then

\[ d_r g = \delta_G(g)d_g. \]

3.4. Prove Lemma 3.5.1.

3.5. Let \( G \) be a reductive group over a global field \( F \), let \( P \leq G \) be a parabolic subgroup, and let \( K \leq G(\mathbb{A}_F) \) be a maximal compact subgroup such that the Iwasawa decomposition

\[ G(\mathbb{A}_F) = P(\mathbb{A}_F)K \]

holds (see Theorem 2.7.1). Show that the Haar measures on \( G(\mathbb{A}_F) \), \( P(\mathbb{A}_F) \), and \( K \) can be normalized so that

\[ dg = d_{l}pd_{k}. \]

3.6. Let \( K^\infty \leq G(\mathbb{A}_F^\infty) \) be a compact open subgroup and let \( \Omega \subset G(\mathbb{A}_F^\infty) \) be a compact set. Show there is a finite set \( S \) of places of \( F \) including the infinite places such that the projection of \( \Omega \) to \( G(\mathbb{A}_F^S) \) is contained \( K^S \), the projection of \( K^\infty \) to \( G(\mathbb{A}_F^S) \). Deduce that the products (3.10) are left and right Haar measures on \( G(\mathbb{A}_F) \), respectively.

3.7. Let \( F \) be a nonarchimedean local field, let \( dx \) be a Haar measure on \( F \), and let \( d^\times x = \frac{dx}{|x|} \). Show that

\[ (1 - q^{-1})dx(O_F) = d^\times x(O_F^\times) \]

where \( q \) is the order of the residue field of \( O_F \).

3.8. Let \( F \) be a nonarchimedean local field and let \( dg \) be the Haar measure on \( GL_2(F) \) such that \( dg(GL_2(O_F)) = 1 \). For \( n \geq 0 \), compute

\[ dg(\{ g \in gl_2(O_F) \cap GL_2(F) : |\det g| = q^{-n}\}). \]

3.9. Suppose that \( G \) is a reductive group over a global field \( F \) and that \( A_G \neq 1 \). Prove that \( G(F) \backslash G(\mathbb{A}_F) \) has infinite volume with respect to the measure induced by a Haar measure on \( G(\mathbb{A}_F) \).

3.10. Suppose that \( G \) is a reductive group over a global field \( F \). Show that \( [G] \) has finite volume using reduction theory of Theorem 2.7.2.

3.11. Let \( F \) be a local field with absolute value \( |\cdot| \). Prove that
3.11 Why affine groups?

\[ d(x_{ij}) = \frac{\wedge_{i,j} dx_{ij}}{|\det(x_{ij})|^n} \]

is a (right and left) Haar measure on \( GL_n(F) \).

3.12. Let \( N \) be a unipotent group over a local field \( F \). Prove that \( N(F) \) is unimodular.

3.13. Let \( N \) be a unipotent group over a global field \( F \). Prove that \( N(\mathbb{A}_F) \) is unimodular.

3.14. Let \( F \) be a global field and let \( R \) be an \( F \)-algebra. For \((x_{ij}) \in GL_2(R)\) let

\[ \omega = \frac{dx_{11} \wedge dx_{21} \wedge dx_{12} \wedge dx_{22}}{(x_{11}x_{22} - x_{21}x_{12})^2} \]

viewed as a left-invariant top-dimensional differential form on \( G \). For a nonarchimedean place \( v \) of \( F \), compute

\[ |\omega|_v(GL_2(O_{F_v})) \]

Prove that

\[ \prod_v |\omega|_v(GL_2(O_{F_v})) \]

does not converge but that

\[ \prod_v (1 - q_v^{-1})|\omega|_v(GL_2(O_{F_v})) \]

does converge.

3.15. Prove that a representation of a finite group \( G \) on a finite dimensional complex vector space is completely reducible; that is, it decomposes into a direct sum of irreducible subrepresentations.

3.16. Let \( B \leq GL_2 \) be the Borel subgroup of upper triangular matrices. Prove that the 2-dimensional representation given by the inclusion

\[ B(\mathbb{C}) \hookrightarrow GL_2(\mathbb{C}) \]

is not completely reducible; that is, it does not decompose into a direct sum of irreducible subrepresentations.

3.17. Let \( G \) be a locally compact group. Prove that the convolution product on \( C_c(G) \) defined as (3.8) is associative.

3.18. If \( \pi : G \to GL(V) \) is a unitary representation of \( G \) then show that \( \pi(f) \in \text{End}(V) \) for all \( f \in L^1(G) \).

3.19. Let \( G \) be a locally compact group and let \( f_1, f_2 \in C_c(G) \). Prove that for any representation \( \pi \) of \( G \), one has \( \pi(f_1 \ast f_2) = \pi(f_1) \circ \pi(f_2) \).
Chapter 4
Archimedean Representation Theory

I have often pondered over the roles of knowledge or experience, on the one hand, and imagination or intuition, on the other, in the process of discovery. I believe that there is a certain fundamental conflict between the two, and knowledge, by advocating caution, tends to inhibit the flight of imagination. Therefore, a certain naivete, unburdened by conventional wisdom, can sometimes be a positive asset.

attributed to Harish-Chandra by Langlands

Abstract In this chapter, we introduce the main players in the representation theory of real Lie groups. In particular we define admissible representations, \((g, K)\)-modules and infinitesimal characters. The chapter ends with a brief discussion of the Langlands classification.

4.1 The passage between analysis and algebra

For the moment, let \(F\) be a number field and let \(G\) be an affine algebraic group over \(F\). Consider a Hilbert space representation \(V\) of

\[
G(F_{\infty}) = \prod_{v|\infty} G(F_v).
\]
Since we are interested in automorphic representations, the example one should keep in mind, given Definition 3.3, is $L^2([G])$. The fact that $V$ is generally not a space of smooth functions is often inconvenient. It is also not consonant with the classical theory of automorphic forms (to be discussed in Chapter 6). Indeed, the classical theory defines automorphic forms to be certain smooth, even real analytic functions satisfying specific differential equations.

The goal of this chapter is to explain how one passes from the abstract representation $V$ to a space of smooth functions that determines the original space under suitable hypotheses (see Theorem 4.4.6). An added benefit of explaining this process is that it allows us to pass back and forth from the analytic theory of representations of $G(F_\infty)$ to the algebraic theory of $(\mathfrak{g}, K)$-modules (to be defined in §4.4). Sections 4.2, 4.3, and 4.4 establish crucial vocabulary for the remainder of the book. At the end of §4.4 we will use this vocabulary to describe the remaining sections of the chapter, which could be omitted on a first reading and then referred back to as needed.

Before continuing, we note that much of [Bum97, §2.4] is reproduced here without essential change. The presentation in loc. cit. seemed nearly perfect so we did not feel the need to modify it. We have of course added material that frames these basic results in a form suitable for our purposes later in the book.

There is a substantial literature dealing solely with archimedean representation theory, and we have passed over much of it in silence. We cannot, however, omit explicit mention of Harish-Chandra. Anyone studying reductive Lie groups must at some point walk the roads laid by him. His original papers are still the only reference for certain results. We mention that Knapp [Kna02] and Wallach [Wal88, Wal92] have written lucid introductions to the theory in complementary styles.

### 4.2 Smooth vectors

In this section, $G$ denotes an affine algebraic group over the archimedean local field $F$. Let $\mathfrak{g}$ denote its Lie algebra. If $F$ is real then $G(F)$ is a real Lie group and $\mathfrak{g}$ is a real Lie algebra. When $F$ is complex we use the canonical identification

$$\text{Res}_{F/R} G(\mathbb{R}) = G(F)$$

to view $G(F)$ as a real Lie group as well. Similarly when $F$ is complex then we identify the complex Lie algebra $\mathfrak{g}$ with the real Lie algebra $\text{Lie} \text{Res}_{F/R} G$ by viewing $\mathfrak{g}$ as a real vector space via the inclusion $\mathbb{R} \hookrightarrow \mathbb{C}$. In this setting the algebraic definition of the Lie algebra $\mathfrak{g}$ given in §1.6 may be identified with the tangent space at the identity of the real Lie group $G(F)$. Indeed,
This follows from the discussion in [EH00, §VI.1.3] identifying two notions of the tangent space in algebraic geometry together with the following lemma:

**Lemma 4.2.1** Let $X$ be a smooth affine scheme of finite type over $\mathbb{R}$. If $X(\mathbb{R})$ is nonempty, it is a union of smooth manifolds. The tangent space of $x \in X(\mathbb{R})$ in the sense of algebraic geometry may be canonically identified with the tangent space in the sense of differential geometry.

**Proof.** The result follows from the discussion of tangent spaces in [Mum99, §III.4].

This allows us to reduce the representation theory of $G(F)$ to the representation theory of real Lie groups.

Recall that there exists an exponential map

$$\exp : \mathfrak{g} \to G(F).$$

In the case where $G = \text{GL}_n \mathbb{C}$ the Lie algebra $\mathfrak{gl}_n \mathbb{C}$ is the complex vector space of $n \times n$ matrices. The exponential is simply the matrix exponential in this case:

$$\exp(X) := \sum_{i=0}^{\infty} \frac{X^i}{i!}.$$

In general one chooses a faithful representation $G \hookrightarrow \text{GL}_n$ (which exists by Theorem 1.5.1). This induces an injection

$$\mathfrak{g} \hookrightarrow \mathfrak{gl}_n \mathbb{F} \subseteq \mathfrak{gl}_n \mathbb{C}$$

and the exponential on $\mathfrak{g}$ is just obtained by restriction. Of course, there is an intrinsic definition of the exponential map [Kna02, Proposition 1.84], and using it one proves that the exponential map is independent of the choice of representation.

Let $(\pi, V)$ be a Hilbert space representation of $G(F)$. Given $\varphi \in V$ and $X \in \mathfrak{g}$ we write

$$\pi(X)\varphi = \frac{d}{dt} \pi(\exp(tX))\varphi \bigg|_{t=0}$$

$$= \lim_{s \to 0} \pi(\exp(sX))\varphi - \varphi$$

if the limit exists. We will sometimes simply write $X\varphi$ for $\pi(X)\varphi$. We say that a vector $\varphi \in V$ is $C^1$ if for all $X \in \mathfrak{g}$, the derivative $X\varphi$ is defined. We define $C^j$ inductively by stipulating that $\varphi \in V$ is $C^j$ if $\varphi$ is $C^{j-1}$ and $X\varphi$ is $C^{j-1}$ for all $X \in \mathfrak{g}$. A vector $\varphi \in V$ is $C^\infty$ if it is $C^j$ for all $j \geq 1$.

**Definition 4.1.** A vector $\varphi \in V$ is said to be smooth if $\varphi$ is $C^\infty$. The subspace of smooth vectors is denoted by $V_{\text{sm}} \subseteq V$. 
Let
\[ C^\infty(G(F), V) \]
denote the space of smooth functions on \( G(F) \) with values in the Hilbert space \( V \).

**Theorem 4.2.2** Let \( \varphi \in V \). The function \( g \mapsto \pi(g)\varphi \) is in \( C^\infty(G(F), V) \) if and only if \( \varphi \in V_{sm} \).

**Proof.** See [God15, §25-26], especially [God15, Theorem 39]. \( \square \)

**Lemma 4.2.3** The space \( V_{sm} \) is invariant under \( G(F) \).

**Proof.** Let \( g \in G(F) \) and let \( X \in \mathfrak{g} \). An elementary calculation implies
\[ g^{-1} \exp(X)g = \exp(g^{-1}Xg). \]
Thus we have that
\[
X(\pi(g)\varphi) = \lim_{s \to 0} \frac{\pi(\exp(sX))\pi(g)\varphi - \pi(g)\varphi}{s} = \lim_{s \to 0} \frac{\pi(g\exp(sg^{-1}Xg))\varphi - \pi(g)\varphi}{s} = \pi(g) \left( \lim_{s \to 0} \frac{\pi(\exp(s\text{Ad}(g^{-1}X))\varphi - \varphi)}{s} \right).
\]
Here \( \text{Ad}(g^{-1})X = g^{-1}Xg \). The limit exists if \( \varphi \) is \( C^1 \). This implies that \( \pi(g)\varphi \) is \( C^1 \). One shows that if \( \varphi \) is \( C^j \), then \( \pi(g)\varphi \) is \( C^j \) for all \( j \) by induction. \( \square \)

**Lemma 4.2.4** Let \( (\pi, V) \) be a Hilbert space representation of \( G(F) \). Then the action of \( \mathfrak{g} \) on \( V_{sm} \) defined above is a Lie algebra representation of \( \mathfrak{g} \) on \( V_{sm} \).

Here a representation \( \mathfrak{g} \) on \( V_{sm} \) is just a Lie algebra morphism \( \mathfrak{g} \to \text{End}(V_{sm}) \); there is no continuity condition imposed.

**Proof.** Recall that \( C^\infty(G(F)) \) is a representation of the Lie algebra \( \mathfrak{g} \) where the action is given by sending \( X \in \mathfrak{g} \) to \( dX \), that is, differentiation in the direction of \( X \). The strategy of the proof is to reduce to this case.

Let \( \varphi_0 \in V \). We claim that one has a \( \mathfrak{g} \)-equivariant map
\[
I := I_{\varphi_0} : V_{sm} \to C^\infty(G(F)) \\
\varphi \mapsto (g \mapsto \langle \pi(g)\varphi, \varphi_0 \rangle),
\]
where the pairing is the pairing on the Hilbert space \( V \). To prove that \( I \) is an intertwining map, it suffices to verify that
\[
(dX \circ I)\varphi(g) = (I \circ X)\varphi(g).
\]
For this, we compute that
\[
\left. \frac{d}{dt} I(g \exp(tX)) \right|_{t=0} = \left. \frac{d}{dt} \langle \pi(g) \pi(\exp(tX)) \varphi, \varphi_0 \rangle \right|_{t=0} \\
= \langle \pi(g) X \varphi, \varphi_0 \rangle \\
= (I \circ X) \varphi(g).
\]

Since we are assuming that \( \varphi \) is smooth, we see that \( I \) is smooth as well.

In order to verify that \( V_{sm} \) is a representation of \( g \), we must check that
\[
X(Y \varphi) - Y(X \varphi) = [X, Y] \varphi
\]  
(4.1)
for all \( X, Y \in \mathfrak{g} \) and all \( \varphi \in V_{sm} \). By the Riesz representation theorem, the map
\[
V \longrightarrow \text{Hom}_{\text{cont}}(V, \mathbb{C}) \\
\varphi \longmapsto \langle \cdot, \varphi \rangle
\]
is bijective and antilinear. Here, as usual, \( \text{Hom}_{\text{cont}}(V, \mathbb{C}) \) denotes the space of continuous homomorphisms. Thus it suffices to prove that (4.1) holds after pairing with \( \varphi_0 \) for all \( \varphi_0 \in V \), in other words, that
\[
I_{\varphi_0}(X(Y \varphi)) - I_{\varphi_0}(Y(X \varphi)) = I_{\varphi_0}([X, Y] \varphi)
\]
for all \( \varphi \) and \( \varphi_0 \). This is a consequence of the fact that \( C^\infty(G(F)) \) is a representation of \( \mathfrak{g} \) as recalled at the beginning of the proof.

Thus \( V_{sm} \) affords a representation of \( G(F) \) and of \( \mathfrak{g} \). Note that so far we do not even know if \( V_{sm} \) is nonzero; it ought to be large in order for this notion to be useful. Fortunately, it is indeed large. To make this precise, if \( f \in C^\infty_c(G(F)) \) then define
\[
\pi(f) \varphi = \int_{G(F)} f(g) \pi(g) \varphi d_r g
\]
where \( d_r \) is a right Haar measure. Let \( d_l g = \delta_{G(F)}^{-1}(g) d_r g \). It is a left Haar measure (see Exercise 3.3).

**Proposition 4.2.5** If \( f \in C^\infty_c(G(F)) \) and \( \varphi \in V \) then \( \pi(f) \varphi \in V_{sm} \). Moreover, the space \( V_{sm} \) is dense in \( V \).

**Proof.** For \( X \in \mathfrak{g} \) let
\[
f_X(g) = \left. \frac{d}{dt} f(\exp(-tX)g) \right|_{t=0}.
\]  
(4.2)
Then \( f_X \in C^\infty_c(G(F)) \). We apply the Leibniz integral rule to obtain
\[
\int_{G(F)} \frac{f_X(g)}{\delta_{G(F)}(g)} \pi(g) \varphi \, d_r g = \frac{d}{dt} \int_{G(F)} f(\exp(-tX)g) \pi(g) \varphi \, d_r g \bigg|_{t=0}
\]
\[
= \frac{d}{dt} \int_{G(F)} f(g) \pi(\exp(tX)g) \varphi \, d_r g \bigg|_{t=0}
\]
\[
= \frac{d}{dt} \pi(\exp(tX)) \pi(f \delta_{G(F)}^{-1}) \varphi \bigg|_{t=0}
\]
\[
= X \pi(f \delta_{G(F)}^{-1}) \varphi.
\] (4.3)

Since \( f \) is arbitrary, this implies that \( \pi(f) \varphi \in C^1 \). By induction we see that \( \pi(f) \varphi \in \mathcal{V}_{sm} \).

For the second claim, let \( \varepsilon > 0 \). The action map

\[
G(F) \times V \to V
\]
\[
(g, \varphi) \mapsto \pi(g) \varphi
\]
is continuous. This implies that for all \( \varepsilon > 0 \) there exists a neighborhood \( U \subseteq G(F) \) of the identity such that \( \| \pi(g) \varphi - \varphi \|_2 < \varepsilon \) for all \( g \in U \). Choose a nonnegative function \( f \in C^\infty_c(G(F)) \) supported in \( U \) such that

\[
\int_{G(F)} f(g) \, d_r g = 1.
\]

Then by Corollary 3.3.2

\[
\| \pi(f) \varphi - \varphi \|_2 = \left\| \int_{G(F)} f(g) (\pi(g) \varphi - \varphi) \, d_r g \right\|_2 \leq \int_{G(F)} f(g) \| \pi(g) \varphi - \varphi \|_2 \, d_r g \leq \varepsilon
\]
which implies that \( \pi(f) \varphi \) is as close to \( \varphi \) as we wish. Hence \( \mathcal{V}_{sm} \) is dense. \( \square \)

The subspace of \( \mathcal{V}_{sm} \) spanned by vectors of the form \( \pi(f) \varphi \) for some \( f \in C^\infty_c(G(F)) \) is known as the \textbf{Gårding subspace}. Note that what we actually proved was that the Gårding subspace is contained in \( \mathcal{V}_{sm} \) and is dense in \( V \). In fact, one has the following [DM78]:

\textbf{Theorem 4.2.6} (Dixmier-Malliavin theorem) \textit{The Gårding subspace is equal to } \( \mathcal{V}_{sm} \). \( \square \)

Another result from the same paper, known as the Dixmier-Malliavin lemma, also comes up often in practice:

\textbf{Theorem 4.2.7} (Dixmier-Malliavin lemma) \textit{Let } \( f \in C^\infty_c(G(F)) \). \textit{Then it can be written as a finite sum of convolutions:}

\[
f = \sum_i h_{i1} * h_{i2}
\]
4.3 Restriction to compact subgroups

The representation theory of compact groups is much simpler than the representation theory of noncompact groups. For example, any irreducible representation of a compact group is unitarizable and finite dimensional (see Theorem 4.3.3 below). Because of this, a profitable strategy in representation theory is to analyze the restriction of a given representation to compact subgroups. We will apply this strategy to reductive groups over the reals in the following section. In this section we prepare by briefly discussing what we require from the representation theory of general locally compact groups.

Let $G$ be a locally compact (Hausdorff) topological group and let $K \subseteq G$ be a compact subgroup.

**Lemma 4.3.1** Let $\pi$ be a representation of $G$ on a Hilbert space $V$. There exists a Hermitian inner product $(\cdot, \cdot) : V \times V \to \mathbb{C}$ which gives the same topology as the given pairing on $V$ but with respect to which $\pi|_K$ is unitary:

$$(\pi(k)\varphi_1, \pi(k)\varphi_2) = (\varphi_1, \varphi_2)$$

for all $k \in K$ and $\varphi_1, \varphi_2 \in V$.

**Proof.** Let $(\cdot, \cdot)$ denote the original Hilbert space pairing and $\|\cdot\|_2$ the original norm. Define

$$(\varphi_1, \varphi_2) = \int_K (\pi(k)\varphi_1, \pi(k)\varphi_2) dk.$$

It is $K$-invariant by construction so we need only check the claim about the topology. The maps $K \to \mathbb{C}$ given by

$$k \mapsto (\pi(k)\varphi, \pi(k)\varphi) = \|\pi(k)\varphi\|_2^2$$

for $\varphi \in V$ are continuous on $K$, hence bounded. Thus the family of continuous operators $\{\pi(k) : k \in K\}$ is uniformly pointwise bounded on $V$, hence uniformly bounded in operator norm by the uniform boundedness principle. In particular, there is some nonzero constant $\alpha_0 \in \mathbb{R}_{>1}$ such that $\|\pi(k)\varphi\|_2 < \alpha_0\|\varphi\|_2$ for all $k \in K$. We can likewise find a similar bound for $\{\pi(k^{-1}) : k \in K\}$. Thus there is an $\alpha \in \mathbb{R}_{>1}$ such that $\alpha^{-1}\|\varphi\|_2 \leq \|\pi(k)\varphi\|_2 \leq \alpha\|\varphi\|_2$ for all $k \in K$. From this we find that

$$\|\varphi\|_2^2 := \int_K (\pi(k)\varphi, \pi(k)\varphi) dk$$

satisfies

$$\alpha^{-2}\text{meas}(K)\|\varphi\|_2^2 < \|\varphi\|_{2,\text{new}}^2 < \alpha^2\text{meas}(K)\|\varphi\|_2^2.$$
This implies the result. \( \square \)

We have a representation

\[
K \times K \times L^2(K) \to L^2(K)
((k_1, k_2), f) \mapsto (k \mapsto f(k_1^{-1} k k_2)).
\]

(4.4)

It turns out that every representation of \( K \) can be embedded into the restriction of this representation to one copy of \( K \) (see Theorem 4.3.3). To prove this, one uses the following definition:

**Definition 4.2.** Let \((\pi, V)\) be a continuous representation of \( G \) on a Hermitian vector space \( V \). A **matrix coefficient** of \( \pi \) is a function of the form

\[
m : G \to \mathbb{C}
g \mapsto (\pi(g) \varphi_1, \varphi_2)
\]

for some \( \varphi_1, \varphi_2 \in V \).

Irreducible representations of \( K \) can be recovered from their matrix coefficients:

**Proposition 4.3.2** Suppose \((\pi_1, V_1)\) and \((\pi_2, V_2)\) are two representations of \( K \) with \( \pi_2 \) unitary. If there exist matrix coefficients \( m_1, m_2 \) for \( \pi_1, \pi_2 \) respectively that are not orthogonal in \( L^2(K) \) then there exists a nontrivial intertwining operator \( I : V_1 \to V_2 \).

**Proof.** Write \((\ , \)_i\) for the Hermitian pairing on \( V_i \). Let \( x_1, y_1 \in V_1 \) and \( x_2, y_2 \in V_2 \) be such that

\[
\int_K (\pi_1(k)x_1, y_1)(\overline{\pi_2(k)x_2, y_2})dk \neq 0.
\]

Let

\[
I(\varphi) = \int_K (\pi_1(k)\varphi, y_1)\overline{\pi_2(k^{-1})y_2}dk.
\]

We claim that \( I \) is an intertwining operator

\[
I : V_1 \to V_2.
\]

Indeed, for all \( g \in K \) and \( \varphi \in V_1 \),

\[
\pi_2(g)I(\varphi) = \int_K (\pi_1(k)\varphi, y_1)\overline{\pi_2(gk^{-1})y_2}dk.
\]

If we change variables \( k \mapsto kg \) this becomes \( I(\pi_1(g)\varphi) \) and so \( I \) is an intertwining operator. We now verify that \( I \) is nonzero. One has that
4.3 Restriction to compact subgroups

\[(I(x_1), x_2)_2 = \left( \int_K (\pi_1(k)x_1, y_1)\pi_2(k^{-1})y_2dk, x_2 \right)_2 \]
\[= \int_K (\pi_1(k)x_1, y_1)(\pi_2(k^{-1})y_2, x_2)_2dk \]
\[= \int_K (\pi_1(k)x_1, y_1)(\pi_2(k)x_2, y_2)_2dk \quad (V_2 \text{ is unitary}) \]
\[\neq 0 \]
by our assumption above. Thus \(I\) is nonzero. \(\square\)

Let \(\hat{K}\) be the set of equivalence classes of irreducible representations of \(K\). Since all irreducible representations of \(K\) are unitarizable by Lemma 4.3.1, Exercise 4.3 below implies this is consistent with the notation of §3.9. For \(\pi \in \hat{K}\) we write \((\pi, V_\pi)\) for a representative of the equivalence class which we assume is unitary. This is no loss of generality by Lemma 4.3.1. It gives rise to a representation
\[
\text{End}(V_\pi) \cong \pi^\vee \otimes \pi
\]
of \(K \times K\). There is an obvious inner product on \(\text{End}(V_\pi)\) given by
\[
\text{End}(V_\pi) \times \text{End}(V_\pi) \rightarrow \mathbb{C}
\[(S, T) \mapsto \langle S, T \rangle := \text{tr}(ST^*)\]
where the \(*\) denotes the adjoint. Here we are using the fact that \(V_\pi\) is necessarily finite dimensional, which is part (a) of the following theorem:

**Theorem 4.3.3** Let \(K\) be a compact Hausdorff topological group. Then

(a) Any irreducible representation of \(K\) is finite dimensional.
(b) If \((\pi, V)\) is a unitary representation of \(K\), then \(V\) decomposes into a (completed) Hilbert space direct sum of irreducible unitary subrepresentations.
(c) (Peter-Weyl) There is an isomorphism
\[
L^2(K) \xrightarrow{\sim} \bigoplus_{\pi \in \hat{K}} \text{End}(V_\pi)
\[f \mapsto (\pi(f) : V_\pi \rightarrow V_\pi)\]
that is a unitary equivalence of representations of \(K \times K\).

Here the \(\oplus\) in (c) denotes the (completed) Hilbert space direct sum.

*Proof.* Assertion (a), (b) and (c) are [DE09, Theorem 7.2.4], [DE09, Theorem 7.3.2] and [DE09, Theorem 7.2.3]. \(\square\)

We observe that the Peter-Weyl theorem (part (c) of Theorem 4.3.3 above) implies that the span of the matrix coefficients of \(\pi\) in \(L^2(K)\) is the inverse image of \(\text{End}(V_\pi)\) in \(L^2(K)\).
If we assume that $K$ is a Lie group and admits a faithful representation then the proof of Theorem 4.3.3 is not difficult (see Exercise 4.2). This case suffices for the purposes of this book. However, when $K$ is a Lie group, the usual proof that $K$ admits a faithful representation uses the Peter-Weyl theorem [Kna02, Corollary 4.22].

4.4 ($g$, $K$)-modules

Let $G$ be an affine algebraic group over the archimedean field $F$ and let $K \leq G(F)$ be a maximal compact subgroup. Our goals in this section are to introduce the notion of admissibility, define ($g$, $K$)-modules, and attach a ($g$, $K$)-module to each admissible representation. With these concepts in hand, at the end of the section we will describe the content of the remainder of the chapter.

Before we begin in earnest, let us state a structural result which implies in particular that the choice of $K$ is not too serious:

**Theorem 4.4.1** If $G$ is an affine algebraic group over the archimedean field $F$ then $G(F)$ has finitely many components in the Hausdorff topology. Moreover any compact subgroup of $G(F)$ is contained in a maximal compact subgroup, and any two maximal compact subgroups are conjugate (even by an element of the neutral component of $G(F)$ in the Hausdorff topology).

**Proof.** For the first assertion, when $F = \mathbb{R}$ we refer to [BCR98, Theorem 2.3.6] which states that any affine $\mathbb{R}$-scheme of finite type has finitely many components in the Hausdorff topology. If $F = \mathbb{C}$ we replace $G$ by Res$_{\mathbb{C}/\mathbb{R}}G$ to reduce to the case when $F = \mathbb{R}$. For the second assertion, see [Bor98a, Chapter VII, Theorem 1.2].

Let $(\pi, V)$ be a representation of $K$. For each irreducible representation $\sigma$ of $K$, let $V(\sigma) \leq V$ be the sum over all (finite dimensional) subrepresentations of $V$ that are equivalent to $\sigma$:

$$V(\sigma) = \langle \varphi \in V : \langle \pi(k)\varphi : k \in K \rangle \cong \sigma \rangle.$$  

Here the brackets denote the $\mathbb{C}$-span. This is the $\sigma$-isotypical subspace. A vector in $V$ (resp. a subspace of $V$) is said to have $K$-type $\sigma$ if it is an element of $V(\sigma)$ (resp. a subspace of $V(\sigma)$). The space $V(\sigma)$ depends only on the equivalence class of $\sigma$.

**Definition 4.3.** Consider the algebraic direct sum

$$V_{\text{fin}} := \bigoplus_{\sigma \in \hat{K}} V(\sigma).$$

An element of $V_{\text{fin}}$ is a $K$-finite vector and $V_{\text{fin}}$ is the space of $K$-finite vectors.
Here, as in the previous section, $\hat{K}$ is the set of all equivalence classes of irreducible representations of $K$. We will later show in Proposition 4.4.3 that if $V$ is a Hilbert space, then $V_{\text{fin}}$ is dense in $V$, and hence $V$ is the Hilbert space direct sum of the $V(\sigma)$.

**Definition 4.4.** A representation $V$ of $G(F)$ is **admissible** if for each $\sigma \in \hat{K}$, the dimension of $V(\sigma)$ is finite.

Here to define $V(\sigma)$ we are regarding $V$ as a representation of $K$ via restriction. The notion of admissibility isolates a subcategory of the category of representations of $G(F)$ that is closed under many natural operators in representation theory (for example parabolic induction, see §4.9). This category properly contains the subcategory of unitary representations [Wal88, Theorem 3.4.10]:

**Theorem 4.4.2 (Harish-Chandra)** If $G$ is reductive then irreducible unitary representations of $G(F)$ are admissible.

**Remark 4.1.** Unitary representations of nonreductive groups are not in general admissible. A precise discussion of this phenomenon together with examples that are relevant to classical automorphic forms is contained in [BS98, §3.2].

Let $\mathfrak{k} = \text{Lie } K$ be the Lie algebra of the real Lie group $K$. The basic tool used to analyze admissible representations is the notion of a $(\mathfrak{g}, K)$-module:

**Definition 4.5.** A $(\mathfrak{g}, K)$-module is a vector space $V$ with a representation $\pi$ of $\mathfrak{g}$ and $K$ which satisfy the following:

(a) The space $V$ is a countable algebraic direct sum $V = \bigoplus V_i$ with each $V_i$ a finite dimensional $K$-invariant vector space.

(b) For $X \in \mathfrak{k}$ and $\varphi \in V$, we have that

$$\pi(X)\varphi = X\varphi = \frac{d}{dt} \pi(\exp(tX))\varphi \bigg|_{t=0} = \lim_{s \to 0} \frac{\pi(\exp(sX))\varphi - \varphi}{s}.$$

In particular, the limit on the right exists.

(c) For $k \in K$ and $X \in \mathfrak{g}$, we have $\pi(k)\pi(X)\pi(k^{-1})\varphi = \pi(\text{Ad}(k)X)\varphi$.

We say that the $(\mathfrak{g}, K)$-module is **admissible** if $V(\sigma)$ is finite dimensional for all $\sigma \in \hat{K}$.

A morphism of $(\mathfrak{g}, K)$-modules is simply a vector space morphism equivariant with respect to the action of $\mathfrak{g}$ and $K$, and an isomorphism is a morphism that induces an isomorphism of the underlying vector spaces. One important difference between $(\mathfrak{g}, K)$-modules and Hilbert space representations is that we impose no topology on the former. In particular, a submodule of a $(\mathfrak{g}, K)$-module $V$ is just a vector subspace of $V$ invariant under the actions of $\mathfrak{g}$ and $K$. Since we have not required $V$ to be topological, we do not require
the subspace to be closed. A \((g, K)\)-module \(V\) is \textit{irreducible} if it admits no nonzero subspace invariant under the action of \(g\) and \(K\).

The following proposition states that an admissible representation of \(G(F)\) yields a \((g, K)\)-module:

\textbf{Proposition 4.4.3} Let \((\pi, V)\) be a Hilbert space representation of \(G(F)\). Let \(V_{\text{fin}} \subseteq V\) be the space of \(K\)-finite vectors. Then \(V_{\text{fin}} \cap V_{\text{sm}}\) is dense in \(V\) and invariant under the action of \(g\). If \(V\) is admissible, then \(V_{\text{fin}} \subseteq V_{\text{sm}}\).

Thus the space of \(K\)-finite vectors in an admissible Hilbert space representation of \(G(F)\) is in a natural manner an admissible \((g, K)\)-module. We remark that every admissible \((g, K)\)-module is canonically the space of \(K\)-finite vectors of an admissible representation of \(G(F)\) on a smooth Fréchet space of moderate growth by a theorem of Casselman and Wallach \([\text{Cas89}]\) \([\text{Wal92}, \text{Chapter 11}]\).

Before giving the proof of Proposition 4.4.3 we state a lemma:

\textbf{Lemma 4.4.4} Let \(k = \text{Lie } K\). The following are equivalent:

(a) The vector \(\varphi \in V\) is \(K\)-finite.

(b) The space \(\langle \pi(k)\varphi \mid k \in K \rangle\) is finite dimensional.

If \(\varphi\) is smooth, then this is equivalent to

(c) The space \(\langle \pi(X)\varphi \mid X \in \mathfrak{k} \rangle\) is finite dimensional. \(\square\)

Thus \(K\)-finiteness can be detected using the Lie algebra of \(K\). We leave the proof of Lemma 4.4.4 as an exercise (see Exercise 4.4). Assuming the lemma, we prove Proposition 4.4.3:

\textit{Proof of Proposition 4.4.3:} We assume without loss of generality that \(\pi|_K\) is unitary. Write

\[ V_0 := V_{\text{sm}} \cap V_{\text{fin}}. \]

We first prove that \(V_0\) is dense in \(V\). Let \(U\) be a neighborhood of 1 in \(G(F)\) and let \(\varepsilon > 0\). Suppose that \(f\) is a nonnegative smooth function on \(G(F)\) with support in \(KU\) such that

\[ \int_{G(F)} f(g) d\gamma g = 1 \quad \text{and} \quad \int_{G(F) - U} f(g) d\gamma g < \varepsilon. \quad (4.5) \]

By making \(U\) and \(\varepsilon\) sufficiently small, we can make \(\pi(f)\varphi\) as close as we like to \(\varphi\) for all \(\varphi \in V\) by an analogue of the argument in the proof of Proposition 4.2.5.

It therefore suffices to show that for arbitrary \(U\) and \(\varepsilon > 0\), we can choose an \(f\) satisfying (4.5) such that \(\pi(f)\varphi\) is \(K\)-finite. To construct such an \(f\), let \(U_1 \subset G(F)\) and \(W \subset K\) be neighborhoods of 1 such that \(WU_1 \subset U\), and let \(f_1\) be a nonnegative smooth function supported in \(U_1\) such that \(\int_{G(F)} f_1(g) d\gamma g = 1\). By the Peter-Weyl theorem, there exists a matrix coefficient \(f_0\) of a finite
dimensional representation of $K$ that is nonnegative such that $\int_K f_0(k)\,dk = 1$ and $\int_{K-W} f_0(k)\,dk < \varepsilon$. Let

$$f(g) := \int_K f_0(k) f_1(k^{-1}g)\,dk.$$  

Clearly, $f$ has support contained in $KU_1 \subset KU$ and $\int_{G(F)} f(g)\,d_r g = 1$. Moreover, since $WU_1 \subset U$, if $k \in K$ is such that there exists $g \in G(F) - U$ with $f_1(k^{-1}g) \neq 0$ then $k \notin W$. Indeed, otherwise we would have

$$g = k(k^{-1}g) \in WU_1 \subset U.$$  

Therefore

$$\int_{G(F)-U} f(g)\,d_r g \leq \int_{G(F)-U} \int_K f_0(k) f_1(k^{-1}g)\,dk\,d_r g$$

$$= \int_{G(F)-U} \int_{K-W} |f_0(k)||f_1(k^{-1}g)|\,dk\,d_r g$$

$$\leq \int_{K-W} |f_0(k)| \int_{G(F)} f_1(k^{-1}g)\,d_r g\,dk$$

$$= \int_{K-W} f_0(k)\,dk < \varepsilon.$$  

Thus $f$ satisfies (4.5). Here we have used the fact that $\delta_{G(F)}(k) = 1$ because $K$ is compact.

We now show that $\pi(f)\varphi$ is $K$-finite. Let $\rho$ be a finite dimensional unitary representation of which $f_0$ is a matrix coefficient. Thus $f_0(k) = (\rho(k)\xi, \zeta)$ for some $\xi, \zeta$ in the space of $\rho$. Then for $k_1 \in K$,

$$f(k_1^{-1}g) = \int_K f_0(k) f_1(k^{-1}k_1^{-1}g)\,dk$$

$$= \int_K (\rho(k)\xi, \zeta) f_1(k^{-1}k_1^{-1}g)\,dk$$

$$= \int_K \rho(k^{-1}) \rho(k)\xi, \zeta) f_1(k^{-1}g)\,dk$$

$$= \int_K \rho(k)\xi, \rho(k_1)\zeta) f_1(k^{-1}g)\,dk.$$  

Therefore the linear span of the functions $f(k_1^{-1}g)$ is contained in the linear span of the functions

$$g \mapsto \int_K (\rho(k)\xi, \zeta) f_1(k^{-1}g)\,dk.$$
for varying $\xi, \zeta$ in the space of $\rho$, and this space is finite dimensional. Thus the space spanned by the vectors

$$\pi(k_1)\pi(f)\varphi = \int_{G(F)} f(g)\pi(k_1 g)\varphi d_r g = \int_{G(F)} f(k_1^{-1} g)\pi(g)\varphi d_r g$$

as $k_1$ varies over $K$ is finite dimensional, so $\pi(f)\varphi \in V_{\text{fin}}$. Moreover, $\pi(f)\varphi$ is smooth for any vector $\varphi$ by Proposition 4.2.5. It follows that $V_0$ is dense in $V_{\text{sm}}$, which is dense in $V$.

Next we prove that $V_{\text{fin}} \subseteq V_{\text{sm}}$ if $V$ is admissible. First observe that $V_0$ is $K$-invariant. Let $\sigma$ be an irreducible unitary representation of $K$. Then $V_0(\sigma) \subseteq V(\sigma)$. Since $V_{\text{fin}}$ is an algebraic direct sum of the $V(\tau)$, it suffices to show that $V_0(\sigma) = V(\sigma)$.

Since $V(\sigma)$ is finite dimensional by admissibility, $V_0(\sigma)$ admits a well-defined orthogonal complement in $V(\sigma)$ (this is the only part of the proof where admissibility is used). If $\varphi$ is in this orthogonal complement then $\varphi$ is orthogonal to all of $V_0$, because it is orthogonal to $V(\tau)$ for every $\tau \neq \sigma$. Therefore $\varphi = 0$, since $V_0$ is dense. This establishes that $V_0(\sigma) = V(\sigma)$, and hence $V_0 = V_{\text{fin}} \subseteq V_{\text{sm}}$.

Finally, we must show that $V_0$ is invariant under $\mathfrak{g}$. Let $\varphi \in V_0$ and let $W$ be the span of $\varphi$ under $\mathfrak{k} := \text{Lie } K$. It is finite dimensional by Lemma 4.4.4. Let

$$W_1 := \langle Y \varphi \mid Y \in \mathfrak{g} \text{ and } \varphi \in W \rangle,$$

which is again finite dimensional. We claim that $W_1$ is invariant under the action of $\mathfrak{k}$.

Indeed, if $X \in \mathfrak{k}$, and $Y \varphi \in W_1$, then $X(Y \varphi) = [X, Y] \varphi + Y(X \varphi)$, which is an element of $W_1$. Therefore the elements of $W_1$ are $K$-finite by Lemma 4.4.4, and hence $Y \varphi$ is $K$-finite for all $Y \in \mathfrak{g}$. □

We can detect irreducibility of admissible Hilbert space representations using $(\mathfrak{g}, K)$-modules [Wal88, Theorem 3.4.12]:

**Theorem 4.4.5** Assume $G$ is reductive. An admissible Hilbert space representation $(\pi, V)$ of $G(F)$ is irreducible if and only if the underlying $(\mathfrak{g}, K)$-module is irreducible. □

Two admissible representations are **infinitesimally equivalent** if their underlying $(\mathfrak{g}, K)$-modules are isomorphic. Certainly equivalent representations are infinitesimally equivalent, but the converse is not true in general [Bum97, Exercise 2.6.1]. However, the situation is better if we assume that both representations are unitary [Wal88, Theorem 3.4.11]:

**Theorem 4.4.6** Assume $G$ is reductive. A unitary representation of $G(F)$ is irreducible if and only if the underlying $(\mathfrak{g}, K)$-module is irreducible. Two infinitesimally equivalent irreducible unitary representations of $G(F)$ are unitarily equivalent. □
Now that we have introduced the notions of admissibility and the \((g, K)\)-module of an admissible representation, we can begin to make use of them. One manifestation of the power of admissibility is that it allows us to understand the infinite dimensional space \(V\) in terms of finite dimensional subspaces. An example of this is given in §4.5.

There is a useful invariant attached to any irreducible \((g, K)\)-module called an \textbf{infinitesimal character}; its construction will be discussed in §4.6. We then give an example of how \((g, K)\)-modules can be computed and classified explicitly in the simplest nontrivial case in §4.7.

The real power of admissibility is that one can classify admissible representations. Thus the description of the unitary dual \(\widehat{G(F)}\) is reduced to the question of which admissible representations are unitary. This is still very difficult, and at the present time, the unitary dual has not been computed for general reductive groups.

The classification of admissible representation (which has several forms) is known as the \textbf{Langlands classification}. Langlands used it to establish the local Langlands correspondence for archimedean fields. Historically, this was the first case of the local Langlands conjecture proven (see Chapter 12 for more on the local Langlands correspondence). We will review the Langlands classification in §4.9 after stating some prerequisites involving tempered and discrete series representations in §4.8.

### 4.5 Hecke algebras with \(K\)-types

As above \(K \leq G(F)\) is a maximal compact subgroup. Let \(\sigma\) be an irreducible representation of \(K\). For \(k \in K\), let

\[
e_{\sigma}(k) := \dim(\sigma)^{-1}\text{tr}(\sigma)(k^{-1}).
\]  

This is a smooth function on \(K\). For any Hilbert space representation \((\pi, V)\) of \(G(F)\), the operator \(\pi(e_{\sigma})\) is defined:

\[
\pi(e_{\sigma}) : V \longrightarrow V(\sigma) \quad \varphi \longmapsto \int_{K} e_{\sigma}(k)\pi(k)\varphi dk.
\]  

Here we normalize the Haar measure \(dk\) so that the measure of \(K\) is 1. These operators are idempotents:

\[
\pi(e_{\sigma})\pi(e_{\sigma'}) = \begin{cases} 
\text{Id} & \text{if } \sigma \cong \sigma', \\
0 & \text{otherwise}.
\end{cases}
\]
One also can define the left and right convolution of $e_\sigma$ with any element $f \in C_\infty^c(G(F))$:

\[
(f * e_\sigma)(g) := \int_K f(gk^{-1})e_\sigma(k)dk, \quad (4.9)
\]

\[
(e_\sigma * f)(g) := \int_K e_\sigma(k^{-1})f(kg)dk. \quad (4.10)
\]

These are again elements of $C_\infty^c(G(F))$.

For any finite subset $\mathcal{E}$ of the unitary dual $\hat{K}$ of $K$, we define

\[
e_\mathcal{E} := \sum_{\sigma \in \mathcal{E}} e_\sigma \quad (4.11)
\]

and the algebraic direct sum

\[
V(\mathcal{E}) := \bigoplus_{\sigma \in \mathcal{E}} V(\sigma) \leq V. \quad (4.12)
\]

**Definition 4.6.** The **Hecke algebra of $K$-type** $\mathcal{E}$ is

\[
C_\infty^c(G(F), \mathcal{E}) := e_\mathcal{E} * C_\infty^c(G(F)) * e_\mathcal{E}. \quad (4.13)
\]

We observe that $V(\mathcal{E})$ is naturally a representation of $C_\infty^c(G(F), \mathcal{E})$. We invite the reader to compare this with the nonarchimedean setting discussed in §5.3. The algebra $C_\infty^c(G(F), \mathcal{E})$ is often called the spherical Hecke algebra of type $\mathcal{E}$, but we will avoid this terminology because it is usually reserved for the special case where $\mathcal{E}$ is 1-element set consisting of the trivial representation.

If $(\pi, V)$ is admissible, then $V(\mathcal{E})$ is finite dimensional for all $\mathcal{E}$. This observation is quite useful in practice as it allows us to reduce certain questions about the infinite dimensional representation $V$ of $G(F)$ to the family of finite dimensional representations of the algebras $C_\infty^c(G(F), \mathcal{E})$. We give some examples in the remainder of this section.

First, we state a very general version of Schur’s lemma. Let $V$ be a vector space and let $\Lambda \subseteq \text{End}(V)$ be a subset of its set of endomorphisms. We say that $\Lambda$ **acts irreducibly** if for any subspace $W$ of $V$ such that $\Lambda W \leq W$, either $W = 0$ or $W = V$. For a proof of the following version of Schur’s lemma, see [Wal88, §0.5.2]:

**Lemma 4.5.1** Assume that $V$ is a vector space of countable dimension and that $\Lambda \subseteq \text{End}(V)$ acts irreducibly. If $L \in \text{End}(V)$ commutes with every element of $\Lambda$ then $L$ is a scalar multiple of the identity operator. \(\square\)

We also require the following version of the **Jacobson density theorem**:
Theorem 4.5.2 Let $V$ be a vector space of countable dimension and let $A \subset \text{End}(V)$ be a $\mathbb{C}$-subalgebra (possibly without identity) that acts irreducibly. Then for any $\varphi_1, \ldots, \varphi_n, \varphi'_1, \ldots, \varphi'_n \in V$ with $\varphi_1, \ldots, \varphi_n$ linearly independent, there is an $a \in A$ such that $a\varphi_i = \varphi'_i$ for all $i$. \qed

We warn the reader that the Jacobson density theorem is usually stated for modules over rings with identity. For a version that avoids the identity assumption, we refer to [Jac64, §II.2].

Proposition 4.5.3 An admissible Hilbert space representation $(\pi, V)$ of $G(F)$ is irreducible if and only if $V(\Xi)$ is an irreducible $C^\infty_c(G(F), \Xi)$-module for all $\Xi$.

Proof. Suppose $V$ is reducible, that is, $V = V_1 \oplus V_2$, where the $V_i$ are some nonzero closed subrepresentations of $G(F)$. Then $V(\Xi) = V_1(\Xi) \oplus V_2(\Xi)$ as $C^\infty_c(G(F), \Xi)$-modules for all finite subsets $\Xi \subset \hat{K}$. By Proposition 4.4.3 $V_1(\Xi)$ and $V_2(\Xi)$ are nonzero for $\Xi$ large enough.

Conversely, suppose $V$ is irreducible, and suppose that $V(\Xi) = V_1 \oplus V_2$ as $C^\infty_c(G(F), \Xi)$-modules for some $\Xi \subset \hat{K}$. Here $V_1$ and $V_2$ are two subspaces of $V(\Xi)$, which is finite dimensional by the admissibility of $V$. If $V_1 \neq 0$ then

$$V_1 = C^\infty_c(G(F), \Xi)V_1 = e_\Xi * C^\infty_c(G(F)) * e_\Xi V_1 = e_\Xi * C^\infty_c(G(F))V_1,$$

where we have used Theorem 4.5.2 for the first equality. It is clear that $e_\Xi * C^\infty_c(G(F))V_1$ is a closed subspace of $V(\Xi)$. If $e_\Xi * C^\infty_c(G(F))V_1$ is a proper subspace of $V(\Xi)$ then its inverse image under

$$e_\Xi : V \longrightarrow V(\Xi)$$

is a proper closed $G(F)$-invariant subspace of $V$, contradicting our irreducibility assumption. Thus $e_\Xi * C^\infty_c(G(F))V_1 = V(\Xi)$. We deduce from this and (4.14) that $V_1 = V(\Xi)$ and hence $V_2 = 0$. \qed

Combining Lemma 4.5.1 and Proposition 4.5.3 we obtain the following corollary:

Corollary 4.5.4 Assume $(\pi, V)$ is an irreducible admissible Hilbert space representation of $G(F)$. If $L \in \text{End}(V_{\text{fin}})$ commutes with the action of $C^\infty_c(G(F), \Xi)$ for all finite subsets $\Xi \subset \hat{K}$ then it is a scalar multiple of the identity operator. \qed
The following proposition will be used in the proof of Proposition 16.2.5. It is an immediate consequence of Theorem 4.5.2 and Proposition 4.5.3:

**Proposition 4.5.5** Let \((\pi, V)\) be an admissible Hilbert space representation of \(G(F)\) and let \(\Xi \subset \hat{K}\) be finite. Let \(\varphi_1, \ldots, \varphi_n, \varphi'_1, \ldots, \varphi'_n \in \oplus_{\sigma \in \Xi} V(\sigma)\) with \(\varphi_1, \ldots, \varphi_n\) linearly independent. Then there exists an \(f \in C^\infty_c(G(F), \Xi)\) such that \(\pi(f)\varphi_i = \varphi'_i\) for all \(i\). \(\square\)

### 4.6 Infinitesimal characters

Let \(g\) be a Lie algebra over an archimedean field \(F\). We let

\[
g^C := g \otimes_{\mathbb{R}} \mathbb{C}.
\]

We use this odd notation as a reminder that if \(F\) is complex then \(g^C \neq g\), whereas \(g_\mathbb{C} := g \otimes_{\mathbb{C}} \mathbb{C}\) is isomorphic to \(g\) in this case.

We denote by \(U(g)\) the universal enveloping algebra of \(g^C\). This is a unital associative algebra. Any associative algebra becomes a Lie algebra with bracket \([X, Y] = XY - YX\), and thus we can regard \(U(g)\) as a Lie algebra. The universal enveloping algebra is equipped with an injective Lie algebra morphism

\[
g^C \rightarrow U(g)
\]

by the Poincaré-Birkhoff-Witt theorem. This morphism is universal in that given any Lie algebra homomorphism \(g^C \rightarrow A\), where \(A\) is a unital associative algebra (equipped with the Lie algebra structure mentioned above), there is a unique morphism \(U(g) \rightarrow A\) such that the following diagram commutes:

\[
\begin{array}{ccc}
g^C & \rightarrow & U(g) \\
\downarrow & & \downarrow \\
A
\end{array}
\]

We denote by \(Z(g)\) the center of \(U(g)\).

We now present the explicit description of \(Z(g)\) for reductive \(g\) afforded by the Harish-Chandra isomorphism. Assume that \(g\) is reductive and let \(t \leq g\) be a Cartan subalgebra. For example, we could choose a reductive algebraic group \(G\) over \(F\) with Lie algebra \(g\) and a maximal torus \(T \leq G\) with Lie algebra \(t\). We then obtain a root system

\[
\Phi := \Phi((\text{Res}_{F/\mathbb{R}} G)_C, (\text{Res}_{F/\mathbb{R}} T)_C)
\]

as in §1.8. Let \(\Delta \subset \Phi\) be a base and let \(\Phi^+ \subset \Phi\) be the associated set of positive roots. Let \(n^+ < g^C\) be the subalgebra spanned by the roots \(\alpha \in \Phi^+\) and let \(n^- < g^C\) be the subspace spanned by the roots \(\alpha \in \Phi^- = \Phi - \Phi^+\). By
the Poincaré-Birkoff-Witt theorem, we have a decomposition

\[ U(g) = U(t) \oplus (n^- U(g) + U(g)n^+) \]

and hence a quotient map of associative algebras \( Z(g) \to U(t) \) [Wal88, §3.2.2]. There is a morphism \( U(t) \to U(t) \) induced by the universal property of universal enveloping algebras from the map

\[
t^C \longrightarrow U(t) \\
X \mapsto X - \rho(X),
\]

where

\[
\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \tag{4.15}
\]

is the half-sum of positive roots. Composing the maps described above, we obtain the **Harish-Chandra homomorphism**

\[
\gamma : Z(g) \longrightarrow U(t) \longrightarrow U(t).
\]

This is our first time explicitly mentioning \( \rho \), the ubiquitous half-sum of positive roots. It is incorporated into the definition to make it compatible with parabolic induction (see §4.9).

Let \( W := W(\text{Res}_{F/R}G, \text{Res}_{F/R}T)(\mathbb{C}) \) denote the Weyl group of \( \text{Res}_{F/R}T \) in \( \text{Res}_{F/R}G \); it acts on \( t^C \) and hence \( U(t) \). For the proof of the following theorem of Harish-Chandra, see [Wal88, Theorem 3.2.3]:

**Theorem 4.6.1**  The linear map \( \gamma \) is an algebra homomorphism, is independent of the choice of base \( \Delta \), has image in \( U(t)^W \), and induces an isomorphism

\[
\gamma : Z(g) \longrightarrow U(t)^W.
\]

\( \Box \)

The theorem allows us, in particular, to explicitly describe the characters of \( Z(g) \) as we now explain. Suppose we are given \( \lambda \in (t^C)'^* \), the \( \mathbb{C} \)-linear dual of \( t^C \). The linear map \( \lambda : t^C \to \mathbb{C} \) is tautologically a Lie algebra map because \( t^C \) is commutative. By the universal property of universal enveloping algebras, this extends uniquely to an algebra morphism \( \lambda : U(t) \to \mathbb{C} \). We let

\[
\chi_\lambda := \lambda|_{U(t)^W} \circ \gamma : Z(g) \longrightarrow \mathbb{C}.
\]

This is an algebra morphism (i.e. a character) from \( Z(g) \) to \( \mathbb{C} \). For the proof of the following proposition, see [Wal88, Theorem 3.2.4]:

**Proposition 4.6.2**  Every algebra morphism from \( Z(g) \) to \( \mathbb{C} \) is of the form \( \chi_\lambda \) for some \( \lambda \in (t^C)'^* \). Two \( \lambda \) define the same \( \chi_\lambda \) if and only if they are in the same orbit under \( W \).

\( \Box \)
A $g$-module $V$ is said to admit an **infinitesimal character** if $Z(g)$ acts via a character on $V$. Infinitesimal characters are a very useful invariant that is surprisingly closely tied to arithmetic (see §12.8). By Theorem 4.6.1 and Proposition 4.6.2 such a character can be identified with an element of $(t^C)\vee$.

If $V$ is a $(g, K)$-module then it is a module under $U(g)$. It is not hard to see that the induced action of $Z(g)$ on $V$ commutes with the action of $g$ and $K$ [Kna86, Proposition 3.8]. Thus using Lemma 4.5.1 we obtain the following corollary:

**Corollary 4.6.3** Let $V$ be an irreducible admissible $(g, K)$-module and let $L : V \to V$ be a $C$-linear map commuting with the actions of $g$ and $K$. Then $L$ is a scalar multiple of identity operator. In particular, any irreducible admissible $(g, K)$-module admits an infinitesimal character. 

\[ \square \]

### 4.7 Classification of $(g, K)$-modules for $GL_2\mathbb{R}$

In this section, we state the classification of admissible irreducible $(g, K)$-modules when $$g := \mathfrak{gl}_2 \quad \text{and} \quad K := O_2(\mathbb{R}).$$

We refer to [Bum97, §2.5] for proofs and many more details.

Define

$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Z := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (4.16)$$

These elements form a basis for $g$. If we set

$$\Delta = -\frac{1}{4}(H^2 + 2XY + 2YX)$$

then $Z(g) = \langle Z, \Delta \rangle$ (see Exercise 4.9). The reason for the normalization of $\Delta$ is due to its connection with the Laplace-Beltrami operator on the complex upper half plane (see [Bum97, Proposition 2.2.5]). Finally we set

$$w_\theta := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},$$

For every $(s, \mu) \in \mathbb{C}^2$, consider the character

$$\chi : Z(g) \to \mathbb{C}$$

$$Z \mapsto \mu,$$

$$\Delta \mapsto s(1 - s).$$

Then for each $\varepsilon \in \{0, 1\}$ one has a $(g, K)$-module
4.8 Matrix coefficients

\((\pi, V) = (\pi_{s, \mu, \varepsilon}, V_{s, \mu, \varepsilon})\)

with infinitesimal character \(\chi\) constructed as follows. The space is

\[ V = \bigoplus_{\ell \equiv \varepsilon \pmod{2}} \mathbb{C}v_\ell. \]

The action is given by

(a) \(\pi(ug)v_\ell = e^{it\theta}v_\ell\) and \(\pi \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) v_\ell = v_{-\ell},\)

(b) \(Xv_\ell = \left( s + \frac{\ell}{2} \right) v_{\ell+2},\)

(c) \(Yv_\ell = \left( s - \frac{\ell}{2} \right) v_{\ell-2},\)

(d) \(\Delta v_\ell = s(1-s)v_\ell,\)

(e) \(Zv_\ell = \mu v_\ell.\)

This is an admissible \((\mathfrak{g}, K)\)-module. It is irreducible unless \(s = \frac{k}{2}\) where \(k\) is an integer congruent to \(\varepsilon\) modulo 2. If \(k \geq 1\) then \(\pi_{s, \mu, \varepsilon}\) has a unique irreducible infinite dimensional subrepresentation denoted by \(\pi_k\). The set of \(K\)-types of \(\pi_k\) consists of the \(v_\ell\) with \(|\ell| \geq k\). The quotient by this is a finite dimensional representation, and all irreducible finite dimensional representations are obtained in this manner.

If \(\pi_{s, \mu, \varepsilon}\) is irreducible it is known as an irreducible \textbf{principal series} representation. If it is reducible, \(\pi_k\) is known as a \textbf{discrete series} representation if \(k \neq 1\) and a \textbf{limit of discrete series} if \(k = 1\). This terminology should make more sense after the next two sections. It is clear (upon considering \(K\)-types, for example) that if \(\pi_{k_1} \cong \pi_{k_2}\) then \(k_1 = k_2\), and that an irreducible representation cannot be both in the principal series and in the discrete series. Moreover, it is easy to see that if

\[ \pi_{s_1, \mu_1, \varepsilon_1} \cong \pi_{s_2, \mu_2, \varepsilon_2} \]

then \(\mu_1 = \mu_2, \varepsilon_1 = \varepsilon_2,\) and either \(s_1 = s_2\) or \(s_1 = 1 - s_2.\)

The following is the classification result mentioned in the title of this section:

\textbf{Theorem 4.7.1} Every infinite dimensional irreducible admissible \((\mathfrak{g}, K)\)-module is isomorphic to \(\pi_{s, \mu, \varepsilon}\) for some \(s, \mu, \varepsilon\) or \(\pi_k\) for some \(k \geq 1.\)

4.8 Matrix coefficients

Let \(G\) be a reductive group over the local field \(F\). For the moment, we do not assume that \(F\) is archimedean. Let \((\pi, V)\) be a representation of \(G(F)\). Recall the notion of a matrix coefficient of \(\pi\) from Definition 4.2 above. In this section we explain how matrix coefficients can be used to isolate important classes of representations of \(G(F)\).
We have already seen in §2.6 that the center of \( G \) often causes analytic problems for somewhat trivial reasons. One often circumvents this difficulty by defining classes of representations based on their restriction to appropriate subgroups. In analogy with (2.16), we let

\[
G(F)^1 := \bigcap_{\chi \in X^*(G)} \ker (| \cdot | \circ \chi : G(F) \to \mathbb{R}_{>0}).
\]  

(4.17)

**Definition 4.7.** An irreducible representation \((\pi, V)\) of \( G(F) \) is **square integrable** (resp. **tempered**) if its matrix coefficients lie in \( L^2(G(F)) \) (resp. \( L^{2+\varepsilon}(G(F)) \) for any \( \varepsilon > 0 \)). It is **essentially square integrable** (resp. **essentially tempered**) if the matrix coefficients of \( \pi|_{G(F)^1} \) lie in \( L^2(G(F)^1) \) (resp. \( L^{2+\varepsilon}(G(F)^1) \) for any \( \varepsilon > 0 \)).

(Essentially) square integrable representations are also known as (essentially) **discrete series representations**. We concentrate on square integrable and tempered representations in the following discussion, but similar results hold for essentially square integrable and essentially tempered representations.

Discrete series representations can be realized as subrepresentations of \( L^2(G(F)) \), and hence are always unitarizable (see [Dix77, Chapter 14] for details). When \( F \) is archimedean, a theorem of Harish-Chandra (Theorem 12.3.1) characterizes when \( G(F) \) admits essentially square integrable representations. Consider the nonarchimedean case. In this setting, a **supercuspidal representation** is a representation whose matrix coefficients are compactly supported modulo the center (see Definition 8.1). Such representations are clearly essentially square integrable. When the characteristic of \( F \) is zero then supercuspidal representations of \( G(F) \) exist for every reductive \( G \) [BP16c].

Irreducible admissible tempered representations are infinitesimally equivalent to unitary representations in the archimedean case [Kna86, Theorem 8.53] and equivalent to unitary representations in the nonarchimedean case [Sil79, Corollary 4.5.13]. We note that, in many references including [Sil79, §4.5], one finds a more technical definition of temperedness. Once one knows that tempered representations are unitary, one can prove that the more technical definition (which we have not given and will not give) is equivalent to the easily stated one above using the arguments of [CHH88].

Now that we have defined these classes of representations, we ought to comment briefly on their significance. We concentrate on the notion of temperedness. Let \( \pi \) be an irreducible admissible representation of \( G(F) \) and let \( B_\pi \) be an orthonormal basis of the space of \( \pi \). Then for \( f \in C^\infty_c(G(F)) \), the sum

\[
\Theta_{\pi}(f) := \sum_{\varphi \in B_\pi} \langle \pi(f)\varphi, \varphi \rangle
\]

converges absolutely (this is obvious in the nonarchimedean case, see §8.5, and in the archimedean case it is proven in [Kna86, Theorem 10.2]). Technically
speaking we have not defined admissibility in the nonarchimedean case; it is
defined in §5.3. The linear functional

$$\Theta_\pi : C^\infty_c(G(F)) \rightarrow \mathbb{C}$$

$$f \mapsto \Theta_\pi(f)$$

is known as the character of \( \pi \).

In the context of Fourier theory on \( \mathbb{R} \), tempered distributions are precisely
those which admit a Fourier transform; in other words, they are analytically
fairly well-behaved. In the archimedean case, tempered representations are
precisely those whose characters define a tempered distribution on a cer-
tain Schwartz space containing \( C^\infty_c(G(F)) \) (see [Kna02, Theorem 12.23] and
[Sil79, §4.5]). There is also a version of the Plancherel theorem for \( G(F) \) relating
spaces of functions on \( G(F) \) and tempered representations of \( G(F) \). See
[Wal92, Theorem 13.4.1] in the archimedean case and [Wal03] in the nonar-
chimedean case.

Finally the condition of temperedness is also a representation theorectic
encapsulation of how well-behaved automorphic \( L \)-functions are. This will be
discussed in the context of the Ramanujan conjecture in Conjecture 10.6.4.

4.9 The Langlands classification

The Langlands classification gives a description of representations of reductive
groups over archimedean local fields in terms of tempered representations
of Levi subgroups. We state it in this section. There is a refinement where
the tempered representations are replaced by discrete series representations
and limits of discrete series representations. We will touch on this in §10.5
in the case \( G = \text{GL}_n \). For the general case, we refer to [Kna86, §XIV.17].

Let \( G \) be a reductive group over the archimedean local field \( F \) and let \( P \leq
G \) be a parabolic subgroup. It admits a Levi decomposition as in Theorem
1.5.4:

$$P = MN,$$

where \( M \) is a Levi subgroup of \( P \) and \( N \) is its unipotent radical. We choose a
maximal compact subgroup \( K \leq G(F) \) such that the Iwasawa decomposition
\( G(F) = P(F)K \) holds (see Appendix A).

We let \( A_M \) be the identity component in the real topology of the largest
\( \mathbb{R} \)-split torus in the center of \( \text{Res}_{F/\mathbb{R}} M \) and we let

$$M(F)^1 := \bigcap_{\chi \in X^*(M)} \ker (| \cdot | \circ \chi : M(F) \rightarrow \mathbb{R}_{>0}),$$

where \( X^*(M) \) is the group of characters of \( M \) as in §1.7. Then \( A_M M(F)^1 =
M(F) \) and the product is direct. The Langlands decomposition of \( P(F) \)
is
\[ P(F) = A_M M(F)^1 N(F). \]
If we let \( a_M := \text{Hom}(X^*(M), \mathbb{R}) \) then \( a_M \) can be identified with \( \text{Lie} A_M \). We then have a map
\[ H_M : M(F) \rightarrow a_M \]
defined by
\[ \langle H_M(g), \lambda \rangle = \log |\lambda(g)| \]
for \( \lambda \in X^*(M) \) and \( M(F)^1 = \ker H_M \). If we identify \( M \) with the maximal reductive quotient of \( P \) then pullback along the quotient map \( P \rightarrow M \) induces an isomorphism
\[ X^*(M) \rightarrow X^*(P). \]
This allows us to extend \( H_M \) to a map \( H_P : P(F) \rightarrow a_M \).
For each \( \lambda \in a_M^* : = \text{Hom}(a_M, \mathbb{C}) \), we therefore obtain a character
\[ P(F) \rightarrow \mathbb{C}^\times \]
\[ p \mapsto e^{(H_P(p), \lambda)}. \]
(4.19)
Identify \( a_M^* = X^*(M) \otimes_{\mathbb{Z}} \mathbb{R} \). We have projections
\[ \text{Re} : a_M^* \rightarrow a_M^* \quad \text{and} \quad \text{Im} : a_M^* \rightarrow i a_M^* \]
defined in the obvious manner. We let \( \rho \) denote the half-sum of positive roots defined by the parabolic subgroup \( P \) (see \S 1.9 and (4.15)). For each Hilbert representation \( (\sigma, V) \) of \( M(F) \), one can form the representation \( (\sigma, \lambda) \) of \( P(F) \) by extending \( \sigma \) trivially to \( P(F) \) and twisting by the character \( e^{(H_P(\cdot), \lambda)} \). We can then form the (normalized) induced representation
\[ I(\sigma, \lambda) \]
of \( G(F) \). A dense subspace of the space of \( I(\sigma, \lambda) \) consists of continuous functions \( \varphi : G(F) \rightarrow V \) such that
\[ \varphi(nmg) = e^{(H_P(m), \lambda+\rho)} \sigma(m) \varphi(g). \]
(4.21)
This space is equipped with an inner product
\[ \langle \varphi_1, \varphi_2 \rangle = \int_K \varphi_1(k) \overline{\varphi_2}(k) dk. \]
The whole space of \( I(\sigma, \lambda) \) is the completion with respect to this inner product. Finally the action of \( G(F) \) is given by
\[ I(\sigma, \lambda)(g) \varphi(x) := \varphi(xg). \]
The $\rho$ is incorporated so that if $\sigma \otimes \langle H_\rho(\cdot), \lambda \rangle$ is unitary then so is $I(\sigma, \lambda)$ [Kna86, §VII.1]. This also explains the parenthetic “normalized” mentioned above.

The following is known as the subquotient theorem (see [Kna86, Theorem 7.24]):

**Theorem 4.9.1** If $\sigma$ is unitary and tempered and $\text{Re}(\lambda)$ lies in the positive Weyl chamber then $I(\sigma, \lambda)$ admits a unique irreducible quotient $J(\sigma, \lambda)$. □

The quotient $J(\sigma, \lambda)$ is known as the Langlands quotient and $\sigma, \lambda$ are known as Langlands data.

Fix a minimal parabolic subgroup $P_0 \leq G$. This allows us to speak about standard parabolic subgroups. We can now give the Langlands classification [Kna86, Theorem 8.54]:

**Theorem 4.9.2** Every irreducible admissible representation of $G(F)$ is isomorphic to some $J(\sigma, \lambda)$. Moreover, if we insist that the parabolic subgroup defining $J(\sigma, \lambda)$ is standard, fix a Levi decomposition $P = MN$ of each standard parabolic subgroup, assume $\sigma$ is trivial on $A_M$, and stipulate that $\text{Re}(\lambda)$ is in the positive Weyl chamber then every irreducible admissible representation of $G(F)$ is isomorphic to a $J(\sigma, \lambda)$ that is unique up to replacing $\sigma$ by another representation of $M(F)$ equivalent to $\sigma$. □

We observe that, in both Theorem 4.9.1 and Theorem 4.9.2, the assumption that $\text{Re}(\lambda)$ is in the positive Weyl chamber is automatically satisfied when $P = G$. There is also a version of the Langlands classification in the nonarchimedean setting. For this we refer to §8.4.

**Exercises**

4.1. Let $T$ be a torus over $\mathbb{R}$ and let $K \leq T(\mathbb{R})$ be a maximal compact subgroup. Show that there exists an isomorphism between the $\mathbb{C}$-vector space of $K$-finite functions in $C_\infty^c(T(\mathbb{R}))$ and $C_\infty^c(A_T) \otimes_\mathbb{C} \mathbb{C}(K)$ where $A_T$ is defined as in (2.17). Here by $\mathbb{C}(K)$ we mean the free vector space on the isomorphism classes of irreducible unitary representations of $K$.

4.2. Let $K$ be a compact Lie group that admits a faithful representation $K \to \text{GL}_n(\mathbb{C})$. We can then view $K$ as a subset of $\text{gl}_n(\mathbb{C})$ which in turn can be viewed as a real vector space. Call a function on $K$ a polynomial if it is the restriction to $K$ of a polynomial on this real vector space. Show that every polynomial function on $K$ is the matrix coefficient of a finite dimensional representation of $K$, and deduce that the matrix coefficients of finite dimensional unitary representations of $K$ are dense in $C(K)$ and $L^p(K)$ for all $1 \leq p \leq \infty$ (use the Stone-Weierstrass theorem). Deduce the Peter-Weyl theorem in Theorem 4.3.3.
4.3. Let \((\pi_1, V_1)\) and \((\pi_2, V_2)\) be two irreducible unitary representations of a compact (Hausdorff) group \(K\). Prove that if \(V_1\) and \(V_2\) are equivalent then they are unitarily equivalent.

4.4. Prove Lemma 4.4.4.

4.5. Let \((\pi, V)\) be an irreducible unitary representation of \(G(F)\) where \(G\) is a reductive group over an archimedean local field \(F\) and \(V\) is a Hilbert space. Show that there is a quasi-character \(\omega_{\pi} : Z_G(F) \to \mathbb{C}^\times\) such that \(\pi(z)\) acts via \(\omega_{\pi}(z)\) on \(V\) for all \(z \in Z_G(F)\). Here, as usual, \(Z_G\) is the center of \(G\). The quasi-character \(\omega_{\pi}\) is called the central quasi-character of \(\pi\). Show that if \(\chi : G(F) \to \mathbb{C}^\times\) is a quasi-character, then the central quasi-character of \(\pi \otimes \chi\) is \(\chi |_{Z_G(F)} \omega_{\pi}\).

4.6. Give an example of a reductive group \(G\) over \(\mathbb{R}\) and an irreducible admissible representation of \(G(\mathbb{R})\) that is not unitary.

4.7. Let \(g\) be a semisimple Lie algebra over an archimedean field \(F\) and let \(\mathfrak{g}^\mathbb{C} := g \otimes_{\mathbb{R}} \mathbb{C}\). Let \(B\) be the Killing form on \(g\) and denote also by \(B\) its extension to \(g^\mathbb{C}\). It is nondegenerate. Let \(\{X_i\}\) denote a basis for \(g^\mathbb{C}\) and denote by \(X^i\) the dual basis with respect to \(B\). The element

\[
\Delta := \sum_i X_i X^i \in U(g)
\]

is known as the Casimir element. Prove that \(\Delta\) is independent of the choice of basis and that \(\Delta \in Z(g)\).

4.8. Prove that the image of \(e_{\sigma}\) is contained in \(V(\sigma)\) as asserted in (4.7).

4.9. Prove that \(Z(gl_2) = \langle Z, \Delta \rangle\) in the notation of §4.7.

4.10. Prove that the space \(V_{s,\mu,\varepsilon}\) of §4.7 is indeed a \((\mathfrak{g}, K)\)-module with the given action.

4.11. In the notation of §4.7, prove that if \(\pi_{k_1} \cong \pi_{k_2}\) then \(k_1 = k_2\), and that if

\[
\pi_{s_1,\mu_1,\varepsilon_1} \cong \pi_{s_2,\mu_2,\varepsilon_2}
\]

then \(\mu_1 = \mu_2, \varepsilon_1 = \varepsilon_2\), and either \(s_1 = s_2\) or \(s_1 = 1 - s_2\).

4.12. Prove that all finite dimensional \((gl_2, O_2(\mathbb{R}))\)-modules are quotients (resp. subrepresentations) of \(V_{s,\mu,\varepsilon}\) for some \(s, \mu, \varepsilon\), and identify the \(V_{s,\mu,\varepsilon}\) of which they are quotients (resp. subrepresentations).

4.13. Let \(G_1\) and \(G_2\) be reductive groups over an archimedean field \(F\) with Lie algebras \(\mathfrak{g}_1\) and \(\mathfrak{g}_2\), respectively. For each \(i\) let \(K_i \leq G_i(F)\) be a maximal compact subgroup. Suppose that for each \(i\) we are given irreducible admissible \((\mathfrak{g}_i, K_i)\)-modules \((\pi_i, V_i)\). Prove that the algebraic tensor product...
4.9 The Langlands classification

\[(\pi_1 \otimes \pi_2, V_1 \otimes V_2)\]

is an irreducible admissible \((g_1 \times g_2, K_1 \times K_2)\)-module. Prove that any irreducible admissible \((g_1 \times g_2, K_1 \times K_2)\)-module is isomorphic to

\[(\pi_1 \otimes \pi_2, V_1 \otimes V_2)\]

for some irreducible admissible \((g_i, K_i)\)-modules \((\pi_i, V_i)\) and that the \((\pi_i, V_i)\) are uniquely determined up to isomorphism by \((\pi_1 \otimes \pi_2, V_1 \otimes V_2)\).
Chapter 5
Representations of Totally Disconnected Groups

I will tell you a false proof. But like every fairy tale, it has a kernel of truth.

D. Kazhdan

Abstract In this chapter, our goal is to develop enough of the representation theory of locally compact totally disconnected groups (or td-groups for short) to state a refined definition of an automorphic representation in the next chapter. At the end of the chapter, we prove Flath’s theorem. This fundamental result implies that automorphic representations of a global field can be factored, in a suitable sense, along the places of the field.

5.1 Totally disconnected groups

We start with the following definition, following [Car79]:

Definition 5.1. A Hausdorff topological group $G$ is of td-type or simply a td-group if every neighborhood of the identity contains a compact open subgroup.

Here the letters td stand for totally disconnected. A td-group is indeed totally disconnected, which explains the terminology (see Exercise 5.1). A td-group is locally compact (see Exercise 5.1), so the theory of §3.1 is applicable.

Our basic examples of td-groups are given by the next lemma. It follows readily from the definition of the topology on the points of an affine algebraic group in §2.2:

Lemma 5.1.1 Let $G$ be an affine algebraic group over a nonarchimedean local field $F$. Then $G(F)$ is of td-type. Similarly, if $F$ is a global field and $S$ is a finite set of places of $F$ (including the archimedean places if $F$ is a number field), then $G(\mathbb{A}_F^S)$ is of td-type. \qed
Thus the topology on a td-group is very different from that of a connected Lie group. In fact, in stark contrast to the defining property of td-group, a connected Lie group has no small subgroups in a sense made precise in the following lemma:

**Lemma 5.1.2** Let $G$ be a connected Lie group. Then there is a neighborhood $U$ of the identity $1$ so that for all $g \in U - \{1\}$ there is an integer $N$ depending on $g$ such that $g^N \notin U$.

*Proof.* It is a standard result that there are neighborhoods $V_0$ of $0$ in $\text{Lie}G$ and $U_1$ of $1$ in $G$ such that the exponential map $\exp : V_0 \to U_1$ is a diffeomorphism. Let $U = \exp \left( \frac{1}{2} V_0 \right)$. Then if $g = \exp \left( \frac{1}{2} v \right) \in U - \{1\}$ for some $v \in V_0$ one has

$$g^n = \exp \left( \frac{1}{2} v \right) \ldots \exp \left( \frac{1}{2} v \right).$$

Choosing $n$ large enough that $\frac{n}{2} v \notin V_0$ we deduce the result. \(\square\)

Our aim in this chapter is to use the special topological properties of td-groups to refine our understanding of their representation theory. We deal with the basic theory in this chapter, including the local Hecke algebra (see §5.2), the definition of admissible representations in this context (see §5.3), and Flath’s theorem on decomposition of admissible representations (Theorem 5.7.1). The reader will notice that in this section we essentially make no use of the fact that the groups in question arise as the points of reductive groups in nonarchimedean local fields. The (deeper) part of the theory that requires this structure is relegated to Chapter 8.

Unless otherwise specified, for the remainder of this chapter, we let $G$ be of td-type.

### 5.2 Smooth functions on td-groups

The first step in understanding the representation theory of a td-group is to define what we mean by a smooth function. As usual $C(G)$ (resp. $C_c(G)$) denotes the complex vector space of continuous (resp. compactly supported continuous) functions on $G$.

**Definition 5.2.** A function $f : G \to \mathbb{C}$ is **smooth** if it is locally constant. The complex vector subspace of $C(G)$ (resp. $C_c(G)$) consisting of smooth functions is denoted by $C^\infty(G)$ (resp. $C^\infty_c(G)$).

It turns out that $C^\infty_c(G)$ is preserved under convolution, defined as in §3.4. Thus $C^\infty_c(G)$ is an algebra under convolution. It is known as the **Hecke algebra** of $G$. If $K \leq G$ is a compact open subgroup, then we let
denote the subalgebra of functions that are right and left $K$-invariant.

**Lemma 5.2.1** Any element $f \in C_c^\infty(G)$ is in $C_c^\infty(G \sslash K)$ for some compact open subgroup $K$. If $f \in C_c^\infty(G \sslash K)$, then $f$ is a finite $\mathbb{C}$-linear combination of elements of the form

$$1_{KgK}$$

for $g \in G$. Here $1_X$ denotes the characteristic function of a set $X$.

**Proof.** By local constancy of $f \hookrightarrow$ for every $g$ in the support of $f$, supp$(f)$, we can choose a compact open subgroup $K(g) \leq G$ so that $f$ is constant on $gK(g)$. Then

$$\{gK(g) : g \in \text{supp}(f)\}$$

is an open cover of supp$(f)$, so it admits a finite subcover

$$\{g_iK(g_i) : 1 \leq i \leq n\}$$

since $f$ has compact support. Let

$$K_r := \bigcap_{i=1}^n K(g_i);$$

it is a finite intersection of compact open subgroups so it is itself a compact open subgroup. We see that $f$ is right $K_r$-invariant. Similarly we can find a compact open subgroup $K_t \leq G$ such that $f$ is left $K_t$-invariant, and letting $K := K_t \cap K_r$ we see that $f \in C_c^\infty(G \sslash K)$.

Assume $f \in C_c^\infty(G \sslash K)$. Then

$$\{KgK : g \in \text{supp}(f)\}$$

is an open cover of supp$(f)$, which therefore admits a finite subcover, say $\{Kg_iK : 1 \leq i \leq n\}$. Since $f$ is constant on $Kg_iK$ for each $g \in G$, we deduce that

$$f = \sum_{i=1}^n f(g_i) 1_{Kg_iK}.$$ 

\[\square\]

Note that $C_c^\infty(G \sslash K) \leq C_c^\infty(G \sslash K')$ if $K \leq K'$. Thus, in view of the lemma, to explicitly describe the convolution operation on $C_c^\infty(G)$ it suffices to describe

$$1_{K\alpha K} * 1_{K\beta K}$$

for a fixed compact open subgroup $K \leq G$ and $\alpha, \beta \in G$. Assume for simplicity that $G$ is unimodular. We can write
where both disjoint unions are finite and \( \alpha_i, \beta_j \in G \). We then have

\[
\mathbb{1}_{K\alpha K} \ast \mathbb{1}_{K\beta K} = \text{meas}_{dg}(K) \sum_{i,j} \mathbb{1}_{K\alpha_i \beta_j K}
\]

(5.1)

where \( dg \) is the Haar measure on \( G \) used to define the convolution.

**Example 5.1.** All compact open subgroups of \( \text{GL}_n(\mathbb{A}_F^\infty) \) are of the form

\[
K_S \prod_{p \in S} \text{GL}_n(\mathbb{Z}_p)
\]

for a finite set \( S \) of finite primes and a compact open subgroup \( K_S \) of \( \text{GL}_n(\mathbb{Q}_S) \). The subgroup \( \text{GL}_n(\hat{\mathbb{Z}}) = \prod_p \text{GL}_n(\mathbb{Z}_p) \leq \text{GL}_n(\mathbb{A}_F^\infty) \) is a maximal compact open subgroup, and all maximal compact open subgroups are conjugate to this maximal compact open subgroup [Ser06, Chapter IV, Appendix 1]. Examples of nonmaximal compact open subgroups are given by the kernel of the reduction map

\[
\text{GL}_n(\hat{\mathbb{Z}}) \to \text{GL}_n(\mathbb{Z}/m)
\]

for integers \( m \).

Let \( G \) be an affine algebraic group over \( F \) and assume that we have chosen a model for \( G \) over \( \mathcal{O}_F \) which we denote by the same letter by abuse of notation (see §2.4 for models). If \( g \in G(\mathbb{A}_F^\infty) \), then \( g = g_S g^S \) where \( g_S \in G(F_S^\infty) \) and \( g^S \in G(\hat{\mathcal{O}}_F^S) \) for some finite set \( S \) of places of \( F \) including the infinite places. If \( K^\infty \leq G(\mathbb{A}_F^\infty) \) is a compact open subgroup, then upon enlarging \( S \) we can assume that \( K^S = G(\hat{\mathcal{O}}_F^S) \). For such a choice of \( S \) we have

\[
\mathbb{1}_{K^\infty g K^\infty} = \mathbb{1}_{K_S g_S K_S} \otimes \mathbb{1}_{K^S}.
\]

As we will explain in §5.7 below, this reduces the study of \( C^\infty_c(G(\mathbb{A}_F^\infty)) \) to the study of the local Hecke algebras

\[
C^\infty_c(G(F_v))
\]

as \( v \) varies over nonarchimedean places of \( F \).

### 5.3 Smooth and admissible representations

For the following definition, we temporarily lift our running assumption that representations are continuous:
Definition 5.3. A representation $(\pi, V)$ of $G$ on a complex vector space $V$ is \textbf{smooth} if the stabilizer of any vector in $V$ is open in $G$.

Equivalently, $(\pi, V)$ is smooth if and only if

$$V = \bigcup_{K \leq G} V^K,$$

where the superscript denotes the subspace of fixed vectors and where the union is over all compact open subgroups $K \leq G$.

In Definition 5.3, we have not made any continuity assumption or placed any topology on $V$. However, we can rephrase the condition of smoothness topologically:

Lemma 5.3.1. A representation $(\pi, V)$ of $G$ on a complex vector space $V$ is smooth if and only if the action map

$$G \times V \to V$$

is continuous when $V$ is given the discrete topology.

With this in mind, we again enforce our running assumption that representations are continuous.

We note that the smooth representations of a given td-group form a category where morphisms are simply $G$-equivariant $\mathbb{C}$-linear maps. The category is even abelian.

It is sometimes useful to rephrase the condition of smoothness in terms of the associated representation of the algebra $C_c^\infty(G)$. For this, we recall that a module $M$ for an algebra $A$ is \textbf{nondegenerate} if every element of $M$ can be written as a finite sum

$$a_1 m_1 + \cdots + a_n m_n$$

for some $(a_i, m_i) \in A \times M$. Of course, this is trivially true if $A$ contains an identity, but we will be applying this concept when $A = C_c^\infty(G)$, which has an identity if and only if $G$ is compact. However, $C_c^\infty(G)$ always has approximate identities in the following sense. For each compact open subgroup $K \leq G$, let

$$e_K := \frac{1}{\operatorname{meas}_d(K)} \mathbb{1}_K,$$

where $\mathbb{1}_K$ is the characteristic function of $K$. Then $e_K$ is the identity element of the algebra

$$C_c^\infty(G \sslash K) = e_K * C_c^\infty(G) * e_K.$$

If $(\pi, V)$ is a smooth representation of $G$, then it is naturally a nondegenerate $C_c^\infty(G)$-module and $C_c^\infty(G \sslash K)$ preserves

$$V^K = e_K V.$$
This observation is in fact the key to the proof of the following lemma, which we leave as an exercise (see Exercise 5.11):

**Lemma 5.3.2** There is an equivalence of categories between nondegenerate $C_c^\infty(G)$-modules and smooth representations of $G$. \[\square\]

Using this equivalence, we prove the following irreducibility criterion:

**Proposition 5.3.3** A smooth representation $(\pi, V)$ of $G$ is irreducible if and only if $V^K$ is an irreducible $C_c^\infty(G \parallel K)$-module for all compact open subgroups $K \leq G$.

**Proof.** Suppose $V$ is not irreducible. Let $V' < V$ be a proper subrepresentation. Choose $\varphi \in V - V'$. Since $V$ is smooth, $\varphi \in V^K$ for some $K$ and $\varphi \notin V'^K$. Hence $V^K$ is not irreducible as a $C_c^\infty(G \parallel K)$-module. Conversely, suppose $V$ is irreducible, and suppose that

$$V_1 \leq V^K$$

is a $C_c^\infty(G \parallel K)$-submodule for some compact open subgroup $K \leq G$. If $V_1 \neq 0$ then by irreducibility of $V$ one has that

$$V_1 = C_c^\infty(G \parallel K)V_1 = e_K * C_c^\infty(G) * e_K V_1 = e_K * C_c^\infty(G)V_1 = e_K V = V^K$$

which proves the claim. \[\square\]

We now come to the most important definition of the chapter:

**Definition 5.4.** A representation $(\pi, V)$ of $G$ is **admissible** if it is smooth and $V^K$ is finite dimensional for every compact open subgroup $K \leq G$. A $C_c^\infty(G)$-module $V$ is **admissible** if it is nondegenerate and $e_K V$ is finite dimensional for all compact open subgroups $K \leq G$.

It is immediate that a representation $(\pi, V)$ of $G$ is admissible if and only if its associated $C_c^\infty(G)$-module is admissible.

Jacquet and Langlands introduced this definition in their classic work [JL70], though some indication of the definition in the real case appeared in work of Harish-Chandra [HC53]. The importance of the definition is that it isolates exactly the correct category of representations to study, and at the same time eliminates extraneous topological assumptions such as the presence of a Hilbert or pre-Hilbert space structure on $V$. One indication that this is the correct category is the following theorem, which is proven as part of Theorem 8.3.5 below:

**Theorem 5.3.4** Let $F$ be a nonarchimedean local field. Assume that $G$ is the $F$-points of a reductive group over $F$. Then an irreducible smooth representation of $G$ is admissible. \[\square\]
Another indication that admissible representations are the correct category


to study is the fact that unitary representations of $G$ give rise to admissible


representations. To explain this, we require some preparation. Recall that


in the archimedean setting of §4.4 we passed from Hilbert representations
to their subspace of $K$-finite vectors ($K$ a maximal compact subgroup of
the relevant group) in order to define a notion of admissibility for them. We
require a similar process here, but it is slightly simpler. Given a Hilbert space
representation $V$ of $G$, we let


$$V_{sm} := \bigcup_{K} V^K \leq V$$


where the union is over all compact open subgroups $K \leq G$. Then $V_{sm} \leq V$
is evidently a smooth subrepresentation of $G$. We say that the Hilbert space
representation $(\pi, V)$ is admissible if $V_{sm}$ is admissible. We note that in this
case $V_{sm}$ is a pre-Hilbert representation. As a warning, if $V$ is an infinite
dimensional Hilbert representation of $G$ then $V_{sm} \neq V$, so $V$ itself is not even
smooth, let alone admissible (see Exercise 5.4).


Lemma 5.3.5 Let $(\pi, V)$ be a Hilbert space representation of $G$. The subspace $V_{sm} \leq V$ is dense and $G$-invariant. □


We leave the proof as an exercise (see Exercise 5.12). We also have the fol-


lowing analogue of Theorem 4.4.6:


Proposition 5.3.6 Let $G$ be the $F$-points of a reductive group over a nonar-


chimedean local field $F$. If $(\pi, V)$ is a unitary representation of $G$ then $V$ is


irreducible if and only if $V_{sm}$ is irreducible. If $(\pi_1, V_1)$ and $(\pi_2, V_2)$ are irre-


ducible unitary representations of $G$ and their spaces of smooth vectors $V_{1sm}$
and $V_{2sm}$ are equivalent, then $(\pi_1, V_1)$ and $(\pi_2, V_2)$ are unitarily equivalent.


We give the proof of Proposition 5.3.6 in §5.4. Finally, we have the following
nonarchimedean analogue of Theorem 4.4.2:


Theorem 5.3.7 (Harish-Chandra, Bernstein) Let $G$ be the $F$-points of


a reductive group over a nonarchimedean local field $F$. Then all irreducible


unitary representations of $G$ are admissible. □


The proof was reduced to the special case of square integrable representations
in [HC70b] (which invoked earlier work of Godement, see loc. cit.). The square
integrable case was treated in [Ber74]. In fact, if $G$ is as in Theorem 5.3.7,
Bernstein proves that the dimension of $V^K$ is bounded by a constant that
depends only on $G$ and $K$ (in particular, it is independent of the irreducible
unitary representation $V$).

We end the section by discussing the relationship between the definitions of
an admissible representation in the archimedean and nonarchimedean case.
Clearly the nonarchimedean definition makes no sense in the archimedean
setting because of the no small subgroups phenomenon (Lemma 5.1.2). However, the definition in the archimedean case (Definition 4.4) does make sense in the nonarchimedean case and is equivalent to the definition we gave above. Let $K \leq G$ be a compact open subgroup. For an irreducible representation $\sigma$ of $K$ and a representation $(\pi, V)$ of $G$, let

$$V(\sigma) = \langle \varphi \in V : \langle \pi(k)\varphi : k \in K \rangle \cong \sigma \rangle$$

be the $\sigma$-isotypic subspace. Here we are regarding $V$ as a representation of $K$ by restriction.

**Proposition 5.3.8** Let $K \leq G$ be a compact open subgroup. A representation $(\pi, V)$ of $G$ is smooth if and only if

$$V = \bigoplus_{\sigma \in \widehat{K}} V(\sigma). \quad (5.3)$$

It is admissible if and only if $\dim_{\mathbb{C}} V(\sigma) < \infty$ for all $\sigma$.

Here, as in §4.3, $\widehat{K}$ is the set of equivalence classes of irreducible representations of $K$.

*Proof.* It is clear that if (5.3) holds then $V$ is smooth. For any $V$ we have a canonical map

$$\bigoplus_{\sigma \in \widehat{K}} V(\sigma) \rightarrow V. \quad (5.4)$$

It is injective by orthogonality of characters of representations of $K$. In other words the internal sum of the subspaces $V(\sigma)$ in $V$ is direct. If $\varphi \in V$ is smooth then let $K_0 \leq K$ be a subgroup fixing $\varphi$. We can and do assume that $K_0$ is normal. Then

$$\varphi \in \bigoplus_{\sigma \in \widehat{K}/K_0} V(\sigma).$$

Thus (5.4) is surjective if $V$ is smooth. The first assertion of the proposition follows.

As for the second assertion, every element of $V(\sigma)$ is fixed by the kernel of $\sigma$. Thus if $V$ is admissible then $\dim_{\mathbb{C}} V(\sigma) < \infty$ for all $\sigma$. Suppose conversely that $\dim_{\mathbb{C}} V(\sigma) < \infty$ for all $\sigma$. It suffices to show that $\dim_{\mathbb{C}} V^{K_0} < \infty$ for all normal compact open subgroups $K_0 \leq K$. But

$$V^{K_0} = \bigoplus_{\sigma \in \widehat{K}/K_0} V(\sigma),$$

which is finite dimensional. \(\square\)

There is an important difference between Proposition 5.3.8 and the decomposition of the space of $K$-finite vectors in the archimedean case. Assume for
this paragraph that $G$ is a reductive group over an archimedean field $F$ and that $(\pi, V)$ is a Hilbert space representation of $G(F)$. In this setting it is only the proper subspace of finite vectors $V_{\text{fin}} < V_{\text{sm}}$ that admits an algebraic direct sum decomposition as in Proposition 5.3.8.

Given Proposition 5.3.8, one might ask why we did not define admissibility in the nonarchimedean case using the obvious analogue of the definition in the archimedean case. The reason is that the definition we gave in the nonarchimedean case is simply more convenient; there is no need to keep track of isotypic components.

### 5.4 Contragredients

In this section, we use the proof of Proposition 5.3.6 as a convenient excuse for introducing the useful concept of the contragredient of a smooth representation.

Let $(\pi, V)$ be a smooth representation of $G$. Then there is an action of $G$ on the space $\text{Hom}(V, \mathbb{C})$ of $\mathbb{C}$-linear functionals on $V$ given by

$$G \times \text{Hom}(V, \mathbb{C}) \rightarrow \text{Hom}(V, \mathbb{C})$$

$$(g, \lambda) \mapsto \lambda \circ \pi(g^{-1}).$$

A linear functional $\lambda : V \rightarrow \mathbb{C}$ is smooth if $\lambda$ is fixed by some compact open subgroup $K \leq G$. The smooth dual $V^\vee \subseteq \text{Hom}(V, \mathbb{C})$ is the subspace of smooth linear functionals. The action (5.5) preserves $V^\vee$ and affords a smooth representation $(\pi^\vee, V^\vee)$ of $G$.

**Definition 5.5.** The contragredient representation $(\pi^\vee, V^\vee)$ is the representation of $G$ on the smooth dual $V^\vee$ given above.

We note that $\vee$ is a contravariant functor from the category of admissible representations of $G$ to itself. Indeed, Proposition 5.3.8 implies that

$$V^\vee = \bigoplus_{\sigma \in \hat{K}} V(\sigma)^\vee.$$

We now define a variant of the contragredient representation. For any $\mathbb{C}$-vector space $V$, let

$$V^c := \mathbb{C} \otimes_{(a \rightarrow \pi)} \mathbb{C} V.$$

This is the tensor product with respect to the identity map from $\mathbb{C}$ to $\mathbb{C}$ on the left and the complex conjugate of the scaling action of $\mathbb{C}$ on $V$ on the right. Thus

$$ab \otimes v = b \otimes \overline{av}$$

for $a, b \in \mathbb{C}$ and $v \in V$. The representation
$(\pi^*, V^*) := (\pi^V, (V^V)^c)$

is known as the **Hermitian contragredient**. The map $\ast$ again defines a contravariant functor from the category of admissible representations of $G$ to itself. If $V$ is admissible then there are natural isomorphisms

$$V \rightarrow (V^V)^V \quad \text{and} \quad V \rightarrow (V^*)^*$$

(see Exercise 5.13).

The following lemma is [Lau96, Lemma D.6.3]:

**Lemma 5.4.1** Assume that $G$ is second countable. Let $(\pi, V)$ be an admissible representation of $G$. There is a canonical bijection between the set of $G$-invariant, definite, Hermitian inner products on $V$ and the set of isomorphisms

$$\iota : V \rightarrow V^*$$

of smooth representations such that $\iota^* = \iota$. If $(\pi, V)$ is irreducible, then any two $G$-invariant, definite (resp. positive definite) Hermitian inner products on $V$ are $\mathbb{R}$ (resp. $\mathbb{R}_{>0}$) multiples of each other.

Here we use the canonical isomorphism $(V^*)^* = V$ to regard $\iota^*$ as a map from $V$ to $V^*$.

**Proof.** The last claim follows from Schur’s lemma (in the form of Exercise 5.6) and the first claim. Suppose that we have a $G$-invariant, definite, Hermitian inner product $(,)$ on $V$ that is $\mathbb{C}$-linear in the first variable and $\mathbb{C}$-antilinear in the second. We then obtain a $G$-equivariant morphism

$$\iota : V \rightarrow V^*$$

$$\varphi_0 \mapsto (\varphi \mapsto (\varphi, \varphi_0)).$$

(5.6)

It is clearly injective. Now if $K \leq G$ is a compact open subgroup, then $\dim_{\mathbb{C}}(V^*)^K = \dim_{\mathbb{C}}(V^V)^K = \dim_{\mathbb{C}}V^K < \infty$. Thus for all compact open subgroups $K \leq G$, we have that (5.6) induces an isomorphism

$$V^K \rightarrow (V^*)^K.$$

Hence by the smoothness of $V$ and $V^*$, the injection (5.6) is an isomorphism. Since $(\varphi_1, \varphi_2) = (\varphi_2, \varphi_1)$, we deduce that $\iota = \iota^\ast$. Conversely if $\iota : V \rightarrow V^*$ is an isomorphism of admissible representations satisfying $\iota = \iota^\ast$, we can associate to $\iota$ the $G$-invariant Hermitian inner product

$$(\varphi_1, \varphi_2) := \iota(\varphi_2)(\varphi_1).$$

We use this lemma to prove Proposition 5.3.6:
Proof of Proposition 5.3.6: Assume first that \((\pi, V)\) is a unitary representation of \(G\). Suppose that \(V\) is reducible. Since \(V\) is unitary this implies that \(V\) is a direct sum of closed \(G\)-invariant subspaces \(V = V_1 \oplus V_2\). The space of smooth vectors in any Hilbert space representation is dense by Lemma 5.3.5 and we deduce that \(V_{\text{sm}}\) is reducible.

Conversely suppose \((\pi, V)\) is irreducible. Let \(\varphi \in V_{\text{sm}} - \{0\}\) and let

\[
W = \langle \pi(g)\varphi : g \in G \rangle.
\]

Let \(\overline{W}\) denote the closure of \(W\) in \(V\). Since \(V\) is unitary, \(W\) is preunitary, and hence the action of \(G\) on \(W\) extends to \(\overline{W}\). Thus \(\overline{W} = V\) by irreducibility of \(V\). We claim that \(\overline{W}_{\text{sm}} = W\). (5.7)

Since \(\overline{W}\) is unitary it is admissible by Theorem 5.3.7, so it suffices to verify that \(\overline{W}_K = W^K\) for all compact open subgroups \(K \leq G\). For \(w \in \overline{W}_K\) choose a sequence \(\{w_n\} \subset W\) such that \(w_n \to w\) as \(n \to \infty\). Then \(e_K w_n \to e_K w = w\) as \(n \to \infty\) which implies that \(W^K\) is dense in \(\overline{W}_K\). Since \(W^K\) and \(\overline{W}_K\) are both finite dimensional \(\mathbb{C}\)-vector spaces by admissibility, we deduce that \(W^K = \overline{W}_K\) which in turn implies (5.7). Thus

\[
V_{\text{sm}} = \overline{W}_{\text{sm}} = W = \langle \pi(g)\varphi : g \in G \rangle.
\]

Since \(\varphi\) was an arbitrary nonzero element of \(V_{\text{sm}}\) we deduce that \(V_{\text{sm}}\) is irreducible.

Suppose that \((\pi_1, V_1)\) and \((\pi_2, V_2)\) are irreducible unitary representations of \(G\) with invariant pairings \((\ , \)_1, \((\ , \)_2 and that we are given a \(G\)-equivariant isomorphism \(\Phi : V_{1\text{sm}} \to V_{2\text{sm}}\). We have a diagram

\[
\begin{array}{ccc}
V_{1\text{sm}} & \overset{\iota_1}{\longrightarrow} & V_{1\text{sm}}^* \\
\downarrow \Phi & & \downarrow \Phi^* \\
V_{2\text{sm}} & \overset{\iota_2}{\longrightarrow} & V_{2\text{sm}}^*
\end{array}
\]

where \(\iota_i\) is induced by \((\ , \)_i\) as in Lemma 5.4.1. Since

\[\Phi^* \circ \iota_2 \circ \Phi = \Phi^* \circ \iota_1 \circ \Phi = (\Phi^* \circ \iota_2 \circ \Phi)^*,\]

the isomorphisms \(\Phi^* \circ \iota_2 \circ \Phi\) and \(\iota_1\) both correspond to \(G\)-invariant, definite Hermitian inner products on \(V_{1\text{sm}}\). Hence by Lemma 5.4.1 the diagram commutes after possibly multiplying \(\iota_1\) by an element of \(\mathbb{R}^\times\). After this normalization, the commutativity of the diagram implies

\[
(x, y)_1 = \iota_1(y)(x) = \Phi^* \circ \iota_2 \circ \Phi(y)(x) = \iota_2 \circ \Phi(y)(\Phi(x)) = (\Phi(x), \Phi(y))_2.
\]
In other words after our normalization, \( \Phi : V_{1sm} \to V_{2sm} \) is an isometric isomorphism. Since \( V_{im} \) is dense in \( V_i \), this implies that \( \Phi \) extends by continuity to an isometric isomorphism \( \Phi : V_1 \to V_2 \). \( \square \)

### 5.5 The unramified Hecke algebra

Let \( G \) be a reductive group over a nonarchimedean local field \( F \). Recall from §2.4 that \( G \) is **unramified** if \( G \) is quasi-split and split over an unramified extension of \( F \). In this case, there exist hyperspecial subgroups \( K \leq G(F) \), unique up to the action of conjugation by \((G/Z_G)(F)\) (see §2.4).

**Definition 5.6.** If \( G \) is unramified and \( K \leq G(F) \) is a hyperspecial subgroup, then

\[
C_c^\infty(\Gamma \setminus G(F) \cap K)
\]

is known as the unramified Hecke algebra or spherical Hecke algebra. Here, as before, \( C_c^\infty(\Gamma \setminus G(F) \cap K) \) is the subalgebra of functions invariant on the left and right under \( K \). It would be more correct to call this the \( K \)-unramified Hecke algebra, but the \( K \) is usually fixed and omitted from the notation. Similar remarks apply to the definitions below.

The following is arguably the most important fact about this algebra:

**Theorem 5.5.1** The unramified Hecke algebra \( C_c^\infty(\Gamma \setminus G(F) \cap K) \) is commutative. \( \square \)

One way to prove this is via the Satake isomorphism (see Theorem 7.2.1). In special cases, this can also be proven using a trick due to Gelfand (see Exercise 5.14). Let us describe the algebra in more detail in a special case:

**Example 5.2.** Let \( G = \text{GL}_n \), viewed as a group over \( \mathbb{Q}_p \). A hyperspecial subgroup is \( \text{GL}_n(\mathbb{Z}_p) \). The unramified Hecke algebra in this case is

\[
C_c^\infty(\text{GL}_n(\mathbb{Q}_p) \cap \text{GL}_n(\mathbb{Z}_p)).
\]

Let

\[
\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n
\]

and let \( T \) be the maximal torus of diagonal matrices in \( \text{GL}_n \). Then \( \lambda \) defines a cocharacter \( \lambda : \mathbb{G}_m \to T \) given on points by

\[
\lambda(x) = \left( x^{\lambda_1} \atop \cdots \atop x^{\lambda_n} \right).
\]

Consider

\[
\{ \mathbf{1}_{\text{GL}_n(\mathbb{Q}_p) \lambda(\mathbb{Z}_p) \text{GL}_n(\mathbb{Z}_p)} : \lambda_1 \geq \cdots \geq \lambda_n \}.
\]
The Smith normal form for matrices over \( \mathbb{Q}_p \) from the theory of elementary divisors implies that the set above is a \( \mathbb{C} \)-vector space basis for the unramified Hecke algebra.

**Definition 5.7.** Assume that \( G \) is unramified with hyperspecial subgroup \( K \leq G(F) \). An irreducible smooth representation \((\pi, V)\) of \( G(F) \) is **\( K \)-unramified**, or **\( K \)-spherical**, if \( V^K \neq 0 \).

By Theorem 5.3.4 an irreducible smooth representation is admissible, so unramified representations are admissible.

As mentioned earlier, hyperspecial subgroups \( K \leq G(F) \) are only unique up to the action of conjugation by \((G/Z_G)(F), \not= G(F)\). Thus the property of a representation being \( K \)-unramified depends on the choice of \( K \) in general. For better or worse, the relevant \( K \) is often omitted from notation, and one speaks simply of **unramified representations** or **spherical representations**. We also point out that using the terminology “unramified” for Hecke algebras and representations is preferred. This is to avoid confusion when applying the theory of spherical varieties (see §14.4).

An important consequence of Theorem 5.5.1 is that the unramified line in an unramified representation is unique. Let \( K \leq G(F) \) be a hyperspecial subgroup.

**Corollary 5.5.2** Let \((\pi, V)\) be an irreducible admissible \( K \)-unramified representation of \( G(F) \). Then \( \dim_{\mathbb{C}} V^K = 1 \).

**Proof.** Since \( V \) is irreducible, if \( V^K \neq 0 \) then it must be an irreducible representation of the commutative algebra \( C^\infty_c(G(F) \parallel K) \) by Proposition 5.3.3. \( \square \)

For more information on unramified representations, see Chapter 7.

### 5.6 Restricted tensor products of modules

In the representation theory of compact or affine algebraic groups, one quickly reduces the representation theory of products of groups to representation theory of the individual factors, thus simplifying problems significantly. We now explain how to accomplish this in the context of automorphic representations. The problem is that \( G(\mathbb{A}_F) \) is a restricted direct product

\[
G(\mathbb{A}_F) \cong \prod_v G(F_v),
\]

not a direct product. Thus we should not expect a representation of \( G(\mathbb{A}_F) \) to decompose into a tensor product of representations in the usual sense. Instead one defines the notion of restricted tensor products of representations.
We make this precise in the current section, and then prove in Theorem 5.7.1 that automorphic representations indeed factor into restricted tensor products indexed by the places in $F$.

We start by defining a restricted tensor product of vector spaces. Let $\Xi$ be a countable set, let $\Xi_0 \subseteq \Xi$ be a finite subset, let

$$\{W_v : v \in \Xi\}$$

be a family of $\mathbb{C}$-vector spaces and for each $v \in \Xi - \Xi_0$ let $\varphi_{0v} \in W_v - 0$. For all sets

$$\Xi_0 \subseteq S \subseteq \Xi$$

of finite cardinality, set $W_S := \bigotimes_{v \in S} W_v$. If $S \subseteq S'$ there is a map

$$W_S \rightarrow W_{S'}$$

$$\bigotimes_{v \in S} \varphi_v \mapsto \bigotimes_{v \in S'} \varphi_v \otimes (\bigotimes_{v \in S' - S} \varphi_{0v}).$$

Consider the vector space

$$W := \bigotimes' W_v := \lim_{\rightarrow S} W_S,$$

where the transition maps are given by (5.8). This is the restricted tensor product of the $W_v$ with respect to the $\varphi_{0v}$. Thus $W$ is the set of

$$\bigotimes' \varphi_v \in \bigotimes' W_v$$

such that $\varphi_v = \varphi_{0v}$ for all but finitely many $v \in \Xi$. We note that if we are given, for each $v \in \Xi$, a $\mathbb{C}$-linear map

$$B_v : W_v \rightarrow W_v$$

such that $B_v(\varphi_{0v}) = \varphi_{0v}$ for all but finitely many $v \in \Xi$ then this gives a $\mathbb{C}$-linear map

$$B = \bigotimes' B_v : W \rightarrow W$$

$$\bigotimes' \varphi_v \mapsto \bigotimes' B_v(\varphi_v).$$

We now define a restricted tensor product of algebras. Suppose we are given $\mathbb{C}$-algebras (not necessarily with unit) $\{A_v : v \in \Xi\}$ and idempotents $a_{0v} \in A_v$ for all $v \in \Xi - \Xi_0$. If $S \subseteq S'$ there is a $\mathbb{C}$-linear map

$$A_S \rightarrow A_{S'}$$

$$\bigotimes_{v \in S} a_v \mapsto \bigotimes_{v \in S} a_v \otimes (\bigotimes_{v \in S' - S} a_{0v}).$$

It is moreover a morphism of algebras because the $a_{0v}$ are idempotents. Consider the algebra
where the transition maps are given by (5.9). This is the restricted tensor product of the $A_v$ with respect to the $a_{0v}$. Finally, if $W_v$ is an $A_v$-module for all $v \in \mathfrak{F}$ such that $a_{0v} \varphi_{0v} = \varphi_{0v}$ for all but finitely many $v$, then $W$ is an $A$-module. The isomorphism class of $W$ as an $A$-module in general depends on the choice of $\{ \varphi_{0v} \}$. However, if we replace the $\varphi_{0v}$ by nonzero scalar multiples we obtain isomorphic $A$-modules.

An easy example of this construction is the ring of polynomials in infinitely many variables. It can be given the structure of a restricted tensor product of algebras

$$
\mathbb{C}[X_1, X_2, \ldots] = \otimes_v [\mathbb{C}[X_1]]
$$

where we take $a_{0i}$ to be the identity in $\mathbb{C}[X_i]$.

Let $G$ be an affine algebraic group over a global field $F$ and let $\infty$ be the set of infinite places. Let $S$ be a finite set of places of $F$ including the infinite places. Choose a model of $G$ over $\mathcal{O}_F^S$ which we again denote by $G$ by abuse of notation (see §2.4). For $v \not\in S$ write $K_v := G(\mathcal{O}_F^v)$; it is a compact open subgroup of $G(F_v)$. Let $\Xi_0$ be a finite set of places of $F$ including $S$. If $G$ is reductive, then upon enlarging $\Xi_0$, if necessary, we can and do assume that if $v \not\in \Xi_0$ then $K_v$ is hyperspecial (see Proposition 2.4.5). By Proposition 2.4.7 we have an isomorphism

$$
G(\mathbb{A}_F^\infty) \sim \prod_{v \not\in \Xi_0} G(F_v) \tag{5.10}
$$

where the restricted direct product is with respect to $K_v$ for $v \not\in \Xi_0$. We recall from Proposition 2.4.7 that the isomorphism and the set on the right are in fact independent of the choice of model.

Choose right Haar measures $d_r g_v$ on $G(F_v)$ for all $v \not\in \Xi_0$ and assume that they are normalized so that

$$
\prod_{v \not\in \Xi_0} \int_{K_v} d_r g_v < \infty.
$$

For example, we could assume that $\int_{K_v} d_r g_v = 1$ for all but finitely many $v$ as in (3.10). Then one can define the Haar measure $d'_r \otimes d_r g_v$ on $\prod_{v \not\in \Xi_0} G(F_v)$ in the obvious manner: it is the unique Haar measure such that all relatively compact measurable sets

$$
\prod_{v \not\in \Xi_0} \Omega_v \subseteq \prod_{v \not\in \Xi_0} G(F_v)
$$

have measure
\[
\prod_{v \neq \infty} \int_{Q_v} d_r g_v.
\]

We write \(d_r g\) for the Haar measure on \(G(\mathbb{A}_F^\infty)\) obtained from \(\otimes'_v \int d_r g_v\) using the isomorphism (5.10).

Given that we have normalized the Haar measures in this manner, we obtain an isomorphism

\[
C_c^\infty(G(\mathbb{A}_F^\infty)) \rightarrow \otimes'_v C_c^\infty(G(F_v))
\]  

(5.11)

where the restricted tensor product on the right is defined with respect to the idempotents

\[
e_{K_v} := \frac{1}{\meas_{d_r g_v}(K_v)} \mathbb{1}_{K_v}.
\]  

(5.12)

Every compact open subgroup of \(G(\mathbb{A}_F^\infty)\) contains a finite index subgroup of the form \(K' = \prod_v K_v\) where \(K_v = G(O_{F_v})\) for all but finitely many \(v\). The isomorphism (5.11) is the unique \(\mathbb{C}\)-linear map sending \(\mathbb{1}_{K'gK'}\) to \(\prod_v \mathbb{1}_{K_v \cdot g \cdot K_v}\) for all \(g \in G(\mathbb{A}_F^\infty)\) and all \(K'\). Since changing the model of \(G_F\) affects only finitely many \(K_v\) (see the proof of Proposition 2.4.7), the isomorphism (5.11) and the algebra \(\otimes'_v C_c^\infty(G(F_v))\) are independent of the choice of model.

### 5.7 Flath’s theorem

Flath’s theorem implies that every automorphic representation can be factored into components indexed by the places of the global field over which the representation is defined. This fact is fundamental to the study of automorphic representations. In this section, we state and prove Flath’s result.

Let \(G\) be a reductive group over a global field \(F\). As in \(\S 5.6\), we choose a model of \(G\) over \(O_F^S\) for a sufficiently large finite set \(S\) of places of \(F\) including the infinite places and let \(K_v := G(O_{F_v})\) for \(v \notin S\). A \(C_c^\infty(G(\mathbb{A}_F^\infty))\)-module \(W\) is factorizable if

(a) there are admissible representations \(W_v\) of \(G(F_v)\) for all \(v \nmid \infty\) such that \(\dim C_{K_v} W_v = 1\) for all but finitely many \(v\),
(b) there is a \(\mathbb{C}\)-linear isomorphism

\[
W \rightarrow \otimes'_v W_v,
\]  

(5.13)

where the restricted tensor product is with respect to nonzero elements \(\varphi_v \in W_v^{K_v}\) for all but finitely many \(v\), and
(c) the diagram
Here the left vertical arrow is given by the product of (5.11) and (5.13), the right vertical arrow is given by (5.13), and the horizontal arrows are the action maps. In view of the assumption that \( \dim_{\mathbb{C}} W^K_v = 1 \), the module \( \otimes'_{\varphi \in \mathcal{K}} W_v \) only depends on the choice of the \( \varphi_0v \) up to isomorphism. We observe that \( W \) is admissible. It is irreducible if and only if the \( W_v \) are irreducible for all \( v \). If assumptions (a), (b), and (c) above are valid we write

\[
W \otimes'_{\varphi \in \mathcal{K}} W_v.
\]

**Theorem 5.7.1 (Flath)** Every irreducible admissible \( C^\infty_c(G(\mathbb{A}_F)) \)-module \( W \) is factorizable. If \( W \otimes'_{\varphi \in \mathcal{K}} W_v \) then the \( C^\infty_c(G(F_v)) \)-modules \( W_v \) are uniquely determined by \( W \) up to isomorphism.

We start with the following easy case of Theorem 5.7.1:

**Theorem 5.7.2** Let \( G_1 \) and \( G_2 \) be of td-type and let \( G = G_1 \times G_2 \).

(a) If \( V_i \) is an irreducible admissible representation of \( G_i \) then \( V_1 \otimes V_2 \) is an irreducible admissible representation of \( G \).

(b) If \( V \) is an irreducible admissible representation of \( G \) then there exists irreducible admissible representations \( V_i \) of \( G_i \) such that \( V \cong V_1 \otimes V_2 \).

Moreover, the isomorphism classes of the \( V_i \) are uniquely determined by \( V \) up to isomorphism.

**Proof.** We first prove (a). It is easy to see that \( V_1 \otimes V_2 \) is admissible. We must check that it is irreducible. By the irreducibility criterion in Proposition 5.3.3, for every compact open subgroup \( K_1 \times K_2 \leq G_1 \times G_2 \), the finite dimensional \( C^\infty_c(G_i \// K_i) \)-modules \( V_i^{K_i} \) are irreducible. Since the \( V_i^{K_i} \) are finite dimensional, it is easy to see that \( V_1^{K_1} \otimes V_2^{K_2} \) is irreducible as a module under

\[
C^\infty_c(G_1 \// K_1) \otimes C^\infty_c(G_2 \// K_2).
\]

But

\[
C^\infty_c(G_1 \times G_2 \// K_1 \times K_2) = C^\infty_c(G_1 \// K_1) \otimes C^\infty_c(G_2 \// K_2),
\]

\[
(V_1 \otimes V_2)^{K_1 \times K_2} = V_1^{K_1} \otimes V_2^{K_2},
\]

so \( V_1 \otimes V_2 \) is irreducible by Proposition 5.3.3.

For (b), let \( V \) be an irreducible admissible representation of \( G \). Choose \( K = K_1 \times K_2 \) such that \( V^K \neq 0 \) (this is possible by smoothness). Then
since $V^K$ is finite dimensional, there exist finite dimensional irreducible $C^\infty_c(G_i//K_i)$-modules $V_i(K_i)$, unique up to isomorphism, and an isomorphism of $C^\infty_c(G//K)$ modules $V^K \rightarrow V_1(K_1) \otimes V_2(K_2)$. Varying $K$, we obtain a decomposition

$$V \cong V_1 \otimes V_2$$

as $C^\infty_c(G) \cong C^\infty_c(G \times G_2)$-modules, where

$$V_1 := \lim_{K_1} V_1(K_1) \quad \text{and} \quad V_2 := \lim_{K_2} V_2(K_2).$$

The $V_i$ are evidently admissible, and moreover they are irreducible by Proposition 5.3.3. They are uniquely determined by $V$ up to isomorphism since the $V_i(K_i)$ are uniquely determined by $V$ up to isomorphism.

We now prove Flath’s theorem:

**Proof of Theorem 5.7.1:** Let $G$ be a reductive group over a global field $F$. Let $S$ be a finite set of places of $F$ including the infinite places. Use (5.11) to identify $C^\infty_c(G(A_1/F))$ and $\otimes_{v \in S} C^\infty_c(G(F_v))$ for the proof. We then have a well-defined subalgebra

$$A_S := C^\infty_c(G(A_1/F)) \otimes e_{K,S} \subseteq C^\infty_c(G(A_1/F)),$$

where $e_{K,S} = \otimes_{v \in S} e_{K_v}$.

By Corollary 5.5.2 and Theorem 5.7.2, as a representation of $A_S$ we have an isomorphism

$$W^K_S \cong \otimes_{v \in S} W_v \otimes W^S,$$

where $W^S$ is a 1-dimensional $C$-vector space on which $e_{K,S}$ acts trivially.

Hence, by admissibility,

$$W = \bigcup_S W^K_S \cong \lim_{S} \otimes_{v \in S} W_v \otimes W^S \quad (5.14)$$

with respect to the obvious transition maps explained in §5.6. On the other hand, (5.11) induces an identification

$$C^\infty_c(G(A_1^\infty)) = \bigcup_S A_S = \lim_{S} A_S \quad (5.15)$$

where again the direct limit is taken with respect to the obvious transition maps in §5.6. It is clear that (5.14) is equivariant with respect to (5.15). The uniqueness assertion follows from Theorem 5.7.2.
Exercises

Throughout these exercises $G$ is a group of td-type and $(\pi, V)$ is a representation of $G$.

5.1. Prove that a group of td-type is locally compact and totally disconnected.

5.2. Assume that $(\pi, V)$ is smooth. Prove that $(\pi, V)$ is admissible if and only if $V^U$ is finite dimensional for any open subgroup $U \leq G$. Here $V^U \leq V$ denotes the subspace of $U$-fixed vectors.

5.3. Define

$$V_{\text{sm}} := \bigcup_K V^K$$

where the union is over all compact open subgroups $K \leq G$. Prove that $V_{\text{sm}}$ is preserved by $G$ and is a smooth subrepresentation of $G$. The subspace $V_{\text{sm}} \leq V$ is the space of smooth vectors in $V$ (in the current setting of td-type groups).

5.4. Suppose that $V$ is an admissible Hilbert space representation. Assume that $G$ is second countable. Define $V_{\text{sm}}$ as in Exercise 5.3 above. Show that $V_{\text{sm}}$ admits a countable algebraic (also known as Hamel) basis. Show that if $V$ is infinite dimensional then $V_{\text{sm}} \not\subseteq V$.

5.5. Suppose that $V$ is admissible and pre-unitary, that is, there is a non-degenerate inner product on $V$ that is preserved by $G$. Prove that every $G$-invariant subspace $V_1 \leq V$ admits a complement, that is, a $G$-invariant subspace $V_2 \leq V$ such that $V_1 \oplus V_2 = V$.

5.6. Assume that $G$ is second countable and $V$ is smooth and irreducible. Show that any linear map $L : V \to V$ commuting with the action of $G$ is a scalar multiple of the identity map.

5.7. Assume that $G$ is second countable and $V$ is smooth and irreducible. Let $\{\varphi_1, \ldots, \varphi_n\}, \{\varphi'_1, \ldots, \varphi'_n\}$ be two sets of elements of $V$ with $\varphi_1, \ldots, \varphi_n$ linearly independent. Then there is an $f \in C_c^\infty(G(F))$ such that

$$\pi(f)\varphi_i = \varphi'_i$$

for all $1 \leq i \leq n$.

5.8. Assume that $G$ is second countable and $V$ is smooth and irreducible. Let $Z_G$ be the center of $G$. Show that there is a quasi-character $\omega_\pi : Z_G \to \mathbb{C}^\times$ such that $\pi(z)$ acts via $\omega_\pi(z)$ on $V$ for all $z \in Z_G$. The quasi-character $\omega_\pi$ is called the central quasi-character of $\pi$. Show that if $\chi : G \to \mathbb{C}^\times$ is a quasi-character, then the central quasi-character of $\pi \otimes \chi$ is $\chi|_{Z_G} \omega_\pi$. 

5.9. Let $p$ be a prime number in $\mathbb{Z}$. Prove that there are no constants $c_1, c_2 \in \mathbb{R}_{>0}$ such that
\[ c_1 |\cdot|_\infty \leq |\cdot|_p \leq c_2 |\cdot|_\infty, \]
where $|\cdot|_\infty$ denotes the usual archimedean norm on $\mathbb{Q}$ and $|\cdot|_p$ denotes the $p$-adic norm.

5.10. Prove the equality (5.1).

5.11. Prove Lemma 5.3.2.

5.12. Prove Lemma 5.3.5.

5.13. Prove that if $V$ is an admissible representation of a td-type group then there are canonical isomorphisms $V \rightarrow (V^\vee)^\vee$ and $V \rightarrow (V^*)^*$.

5.14. Let $F$ be a nonarchimedean local field. Consider the spherical Hecke algebra $C_c^\infty(GL_n(F) \backslash GL_n(O_F))$. For $f \in C_c^\infty(GL_n(F))$ let $f^\dagger(g) := f(g^t)$. Show that for every $f_1, f_2 \in C_c^\infty(GL_n(F))$,
\[ (f_1 * f_2)^\dagger(g) = (f_2^\dagger * f_1^\dagger)(g). \]
Show, on the other hand, that $f^\dagger = f$ for $f \in C_c^\infty(GL_n(F) \backslash GL_n(O_F))$. Conclude that $C_c^\infty(GL_n(F) \backslash GL_n(O_F))$ is commutative.
Chapter 6
Automorphic Forms

Discovery is the privilege of the child: the child who has no fear of being once again wrong, of looking like an idiot, of not being serious, of not doing things like everyone else.

Alexander Grothendieck

Abstract In this chapter we give a new definition of automorphic representations. We relate the new definition to classical automorphic forms on locally symmetric spaces and explain the relationship with the old definition.

6.1 Smooth functions

Throughout this chapter, we let $G$ be a reductive group over a global field $F$. Our goal in this chapter is to give a new definition of an automorphic representation and explain its relation to the old definition. There are at least three reasons for introducing a new definition. The first is that it is easier to relate the new definition to spaces of automorphic forms on locally symmetric spaces (in the number field case). The second is that the new definition allows us to move from the category of Hilbert space representations to the category of admissible representations. The last and most compelling reason is that representations constructed from Eisenstein series (which may not even be unitary) become automorphic representations under the new definition. We refer to Chapter 10 for more information about Eisenstein series.

As preparation, in the current section, we define precisely what one means by a smooth function on $G(\mathbb{A}_F)$ and $A_G \backslash G(\mathbb{A}_F)$. There is an obvious isomorphism

$$A_G \backslash G(\mathbb{A}_F) \cong A_G \backslash G(F_\infty) \times G(\mathbb{A}_F^\infty).$$
We define
\[ C^\infty(G(\mathbb{A}_F)) := C^\infty(G(F_\infty)) \otimes C^\infty(G(\mathbb{A}_F^\infty)), \]
\[ C^\infty_c(G(\mathbb{A}_F)) := C^\infty_c(G(F_\infty)) \otimes C^\infty_c(G(\mathbb{A}_F^\infty)), \]
\[ C^\infty_c(A_G \backslash G(\mathbb{A}_F)) := C^\infty_c(A_G \backslash G(F_\infty)) \otimes C^\infty_c(G(\mathbb{A}_F^\infty)), \]
\[ C^\infty(A_G \backslash G(\mathbb{A}_F)) := C^\infty(A_G \backslash G(F_\infty)) \otimes C^\infty_c(G(\mathbb{A}_F^\infty)), \]
where the tensor products are algebraic.

### 6.2 Classical automorphic forms

For this section, \( F \) is a number field, \( K_\infty \leq G(F_\infty) \) is a maximal compact subgroup, and \( K_\infty \leq G(\mathbb{A}_F^\infty) \) is a compact open subgroup. Recall the homeomorphism
\[ G(F_\backslash G(\mathbb{A}_F)/K_\infty \twoheadrightarrow \prod_i \Gamma_i(K_\infty) \backslash G(F_\infty) \]

of (2.20). By Theorem 2.6.1, there are finitely many indices in this disjoint union. The important point for the current discussion is not the precise definition of the \( \Gamma_i(K_\infty) \), but just that they are discrete arithmetic subgroups of \( G(F_\infty) \) such that the quotient \( A_G \Gamma_i(K_\infty) \backslash G(F_\infty) \) has finite volume (see Theorem 2.6.3). Clearly automorphic representations are related to functions on the left side. In this section and the following we make this relationship precise. We work only at infinity in the current section and then pass to the adelic setting in the next.

Let
\[ \iota' : G \longrightarrow \text{GL}_n \]
be a closed embedding and let \( \iota : G \rightarrow \text{SL}_{2n} \) be the embedding defined by
\[ \iota(g) := \begin{pmatrix} \iota'(g) & \iota'(g^{-1})t \\ \iota'(g^{-1}) & \iota(g) \end{pmatrix}. \]

For \( g \in G(F_\infty) \) we then define the norm
\[ \|g\| := \|g\|_\iota = \prod_{v|\infty} \sup_{1 \leq i,j \leq 2n} |\iota(g)_{ij}|_v. \]

To see why we work with \( \iota \) instead of \( \iota' \), consider the case where \( G = \mathbb{G}_m, F = \mathbb{Q} \), and \( \iota' : \mathbb{G}_m \rightarrow \text{GL}_1 \) is the identity. Then for \( t \in \mathbb{G}_m(\mathbb{R}) = \mathbb{R}^\times \) the norm \( \|t\| \) is bounded if and only if \( t \in \mathbb{R}^\times \) is bounded and bounded away
from zero. In this special case if we defined $\|g\|$ only using $\iota'$ then the norm would be bounded, but not bounded away from zero.

Let

$$g := \oplus_{v|\infty} g_v \quad \text{where} \quad g_v = \text{Lie Res}_{F_v} R G_{F_v}$$

and let $K_\infty \leq G(F_\infty)$ be a maximal compact subgroup. Let $U(g)$ be the universal enveloping algebra of $g^\mathbb{C}$ and $Z(g)$ its center (see $\S$4.6).

**Definition 6.1.** A function

$$\varphi : G(F_\infty) \longrightarrow \mathbb{C}$$

is of **moderate growth** or **slowly increasing** if there are constants $c, r \in \mathbb{R}_{>0}$ such that

$$|\varphi(g)| \leq c\|g\|^r.$$

It is of **uniform moderate growth** if it is smooth and there exists an $r \in \mathbb{R}_{>0}$ such that for all $X \in U(g)$

$$|X \varphi(g)| \leq c_X \|g\|^r$$

for some $c_X \in \mathbb{R}_{>0}$.

It turns out that different choices of $\iota$ lead to norms that are equivalent in a sense made precise by Exercise 6.1. In particular, the notion of moderate growth and uniform moderate growth is independent of the choice of $\iota$.

For the definition of automorphic forms, we will require the notion of $K_\infty$-finite functions and $Z(g)$-finite functions.

**Definition 6.2.** Let $V$ be a $Z(g)$-module. A vector $\varphi \in V$ is $Z(g)$-**finite** if $Z(g)\varphi$ is finite dimensional, or equivalently, if there is an ideal $J \leq Z(g)$ with $\dim_{\mathbb{C}} Z(g)/J < \infty$ that annihilates $\varphi$.

At the moment we are interested in the case $V = C^\infty(G(F_\infty))$. We say that a function $\varphi \in C^\infty(G(F_\infty))$ is $Z(g)$-finite if it is a $Z(g)$-finite vector in $C^\infty(G(F_\infty))$. Similarly, a function $\varphi \in C^\infty(G(F_\infty))$ is $K_\infty$-finite if it is a $K_\infty$-finite vector in $C^\infty(G(F_\infty))$ in the sense of Definition 4.3.

We now finally come to the notion of an automorphic form:

**Definition 6.3.** Let $\Gamma \leq G(F)$ be an arithmetic subgroup. A smooth function $\varphi : G(F_\infty) \to \mathbb{C}$ of moderate growth is an **automorphic form** for $\Gamma$ if it is left $\Gamma$-invariant, $K_\infty$-finite, and $Z(g)$-finite. We denote the space of automorphic forms on $\Gamma$ by $A(\Gamma)$.

We will now show that automorphic forms are automatically of uniform moderate growth. To prove this requires a preparatory result that is of independent interest. We observe that the right action of $G(F_\infty)$ on $G(F_\infty)$ induces an action of $G(F_\infty)$ on $C^\infty(G(F_\infty))$. We will refrain from calling this
a representation because we do not wish to discuss topology. In any case, we 
have an induced action of $K_\infty$ and $Z(\mathfrak{g})$ on $C^\infty(G(F_\infty))$. This is the action 
implicit in the following theorem of Harish-Chandra [HC66, §8, Theorem 1]:

**Theorem 6.2.1** For any function $\varphi : G(F_\infty) \to \mathbb{C}$ that is $K_\infty$-finite and $Z(\mathfrak{g})$-finite, there is an $f \in C_c^\infty(G(F_\infty))$ satisfying $f(kxk^{-1}) = f(x)$ for all $k \in K_\infty$ such that

$$R(f)\varphi(g) := \int_{G(F_\infty)} f(h)\varphi(gh)dh = \varphi(g).$$

The support of $f$ can be taken to lie in any neighborhood of 1 in $G(F_\infty)$. □

**Corollary 6.2.2** Any automorphic form is of uniform moderate growth.

**Proof.** Let $\varphi : G(F_\infty) \to \mathbb{C}$ be an automorphic form. Choose $f$ as in Theorem 6.2.1. Then for $X \in \mathfrak{g}$, we have

$$X\varphi = X(R(f)\varphi) = R(f_X)\varphi$$

where $f_X \in C_c^\infty(G(F_\infty))$ is defined as in (4.2) and we have used (4.3). By induction on the degree of an element of $U(\mathfrak{g})$ we see that for any $X \in U(\mathfrak{g})$ there is an $f_X \in C_c^\infty(G(F_\infty))$ such that

$$X\varphi = R(f_X)\varphi.$$

Assume that

$$|\varphi(g)| \leq c\|g\|^{r}$$

for all $g$. Then

$$|R(f_X)\varphi(g)| \leq c \int_{G(F_\infty)} |f_X(h)| \cdot \|gh\|^{r}dh \leq c_X\|g\|^{r}$$

for some $c_X \in \mathbb{R}_{>0}$ (see Exercise 6.2). □

Often it is convenient to isolate the subspace that is annihilated by a particular ideal $J \leq Z(\mathfrak{g})$. It is denoted $\mathcal{A}(\Gamma, J)$. This space decomposes still further:

$$\mathcal{A}(\Gamma, J) := \oplus_{\sigma} \mathcal{A}(\Gamma, J, \sigma) \quad (6.6)$$

where $\mathcal{A}(\Gamma, J, \sigma) := \mathcal{A}(\Gamma, J)(\sigma)$ in the notation of §4.4. We record the following important result of Harish-Chandra [HC68]:

**Theorem 6.2.3 (Harish-Chandra)** For each ideal $J \leq Z(\mathfrak{g})$ with

$$\dim_{\mathbb{C}} Z(\mathfrak{g})/J < \infty,$$
the space $A(\Gamma, J)$ is an admissible $(g, K_\infty)$-module. In particular, $A(\Gamma, J, \sigma)$ is finite dimensional for each $\sigma$.

As pointed out in [BJ79, §1.7], this is stated for semisimple groups in [HC68], but the proof given there is valid for general reductive groups.

One might ask about the relationship between $A(\Gamma)$ and $L^2(A_G \Gamma \backslash G(F_\infty))$. There is no obvious relationship between them except under more stringent assumptions. For example, Eisenstein series (which will be discussed in Chapter 10) provide examples of elements of $A(\Gamma)$ that need not even be square integrable. It turns out to be more convenient to explain the relationship between the so-called cuspidal subspaces of $A(\Gamma)$ and $L^2(A_G \Gamma \backslash G(F_\infty))$. We defer the discussion of this connection to §6.5.

### 6.3 Adelic automorphic forms over number fields

Let $\iota : G \to \text{SL}_{2n}$ be the embedding of (6.3). In this section we assume that $F$ is a number field. Let $g \in G(\mathbb{A}_F)$. For a place $v$ of $F$ let

$$\|g\|_v := \|g\|_{\iota,v} = \sup_{1 \leq i, j \leq 2n} |\iota(g)_{ij}|_v$$

(6.7)

and for a set of places $S$ of $F$ (finite or infinite) let

$$\|g\|_S := \prod_{v \in S} \|g\|_v.$$  

(6.8)

If $S$ is the set of all places of $F$ then we omit it from notation. We observe that since $\iota(g) \in \text{SL}_{2n}(\mathbb{A}_F)$ we have $\iota(g_v) \in \text{SL}_{2n}(\mathcal{O}_F)$ for all but finitely many $v$ and hence $\|g\|_v = 1$ for all but finitely many $v$. In particular $\|g\|$ is well-defined.

As in the archimedean setting, we have a definition of a function of moderate growth:

**Definition 6.4.** A function

$$\varphi : G(\mathbb{A}_F) \to \mathbb{C}$$

is of **moderate growth** or **slowly increasing** if there are constants $c, r \in \mathbb{R}_{>0}$ such that

$$|\varphi(g)| \leq c \|g\|^r.$$  

A function $\varphi$ is of **uniform moderate growth** if it is smooth and there exists an $r \in \mathbb{R}_{>0}$ such that for all $X \in U(g)$

$$|X \varphi(g)| \leq c_X \|g\|^r$$

for some $c_X \in \mathbb{R}_{>0}$. 
Here $X\varphi$ is defined as follows: Any smooth $\varphi$ can be written as a finite sum of vectors of the form $\varphi_{\infty} \otimes \varphi_{\infty}$ with $(\varphi_{\infty}, \varphi_{\infty}) \in C^\infty(G(F) \times C^\infty(G(\mathbb{A}_F^\infty))$.

One defines

$$X(\varphi_{\infty} \otimes \varphi_{\infty}) := X\varphi_{\infty} \otimes \varphi_{\infty}$$

and extends linearly.

Recall that reduction theory allows us to control the adelic quotient

$$[G] := A_GG(F) \backslash G(\mathbb{A}_F)$$

given as in (2.18) by a Siegel set

$$\Omega A_{T_0}(t)K$$

defined as in (2.22). It is often convenient to phase the notion of moderate growth in terms of Siegel sets:

**Lemma 6.3.1** A smooth function

$$\varphi : [G] \longrightarrow \mathbb{C}$$

is of moderate growth if and only if, for all Siegel sets as above, there are constants $c, r \in \mathbb{R}_{\geq 0}$ (depending on $\Omega$ and $t$) such that

$$|\varphi(gs\kappa)| \leq c\alpha(s)^r$$

for all $(g, s, k) \in \Omega \times A_{T_0}(t) \times K$ and $\alpha \in \Delta$.

For a proof of this lemma, we refer to [Bor07, §5.4].

By definition, a $(g, K_\infty) \times G(\mathbb{A}_F^\infty)$-module $(\pi, V)$ is a complex vector space $V$ that is both a $(g, K_\infty)$-module and a $G(\mathbb{A}_F^\infty)$-module such that the two actions commute. We denote the two action maps by $\pi$. It is admissible if for each compact open subgroup $K_\infty \leq G(\mathbb{A}_F^\infty)$, the space of fixed vectors $V^{K_\infty}$ is admissible as a $(g, K_\infty)$-module. A morphism of $(g, K_\infty) \times G(\mathbb{A}_F^\infty)$-modules is a complex linear map commuting with the actions of $(g, K_\infty)$ and $G(\mathbb{A}_F^\infty)$ (no continuity condition is assumed). We define an irreducible $(g, K_\infty) \times G(\mathbb{A}_F^\infty)$-module to be one that admits no nonzero proper invariant submodule. There is no topology on $V$, so we do not assume that the submodule is closed.

Let $K = K_\infty K_\infty < G(\mathbb{A}_F)$ where $K_\infty \leq G(F)_{\infty}$ is a maximal compact subgroup and $K_\infty < G(\mathbb{A}_F^\infty)$ is a compact open subgroup. We say that a function $\varphi : G(\mathbb{A}_F) \rightarrow \mathbb{C}$ is $K$-finite if the span of translates

$$\{ x \mapsto \varphi(xk) : k \in K \}$$

is finite dimensional. An alternate way of phrasing this condition is given in Exercise 6.4.
Definition 6.5. A smooth function
\[ \varphi : G(\mathbb{A}_F) \rightarrow \mathbb{C} \]
of moderate growth is an **automorphic form** on \( G \) if it is left \( G(F) \)-invariant, \( K \)-finite, and \( Z(\mathfrak{g}) \)-finite. The \( \mathbb{C} \)-vector space of automorphic forms is denoted by \( \mathcal{A}(G) \) or \( \mathcal{A} \) if \( G \) is understood.

The definition of an automorphic form depends on \( K_1 \), but by Exercise 6.4, it does not depend on the choice of compact open subgroup \( K_\infty \leq G(\mathbb{A}_F) \).

The space \( \mathcal{A} \) is equipped with an action of \( (\mathfrak{g} \hookrightarrow K_1) \) and an action of \( G(\mathbb{A}_1 F) \) by right translations:
\[ (\mathfrak{g} \hookrightarrow K_1) \times G(\mathbb{A}_F) \times \mathcal{A} \rightarrow \mathcal{A} \]
\[ (X, k, g, \varphi) \mapsto (x \mapsto X \varphi(x kg)) \]  \hspace{1cm} (6.9)

It is important to note that the action of \( G(\mathbb{A}_F) \) does not extend to an action of \( G(\mathbb{A}_F) \). Indeed, if we let \( \varphi \in \mathcal{A} \) and let \( g \in G(F_\infty) \), then \( x \mapsto \varphi(x g) \) is smooth, but in general it is only finite under \( gK_\infty g^{-1} \), not \( K_\infty \).

If \( \sigma \) is an irreducible representation of \( K_\infty \), we denote by \( \mathcal{A}(\sigma) \) the subspace of vectors with \( K_\infty \)-type \( \sigma \), and if \( J \leq Z(\mathfrak{g}) \) is an ideal with \( \dim_{\mathbb{C}} Z(\mathfrak{g})/J < \infty \), then the subspace of vectors annihilated by \( J \) is denoted by \( \mathcal{A}(J) \) (resp. \( \mathcal{A}(J, \sigma) \)). We have
\[ \mathcal{A} := \bigcup J \mathcal{A}(J) \]  \hspace{1cm} (6.10)
where the sum is over ideals of \( Z(\mathfrak{g}) \) such that \( \dim_{\mathbb{C}} Z(\mathfrak{g})/J < \infty \). Moreover
\[ \mathcal{A}(J) = \bigoplus_{\sigma \in K_\infty} \mathcal{A}(J, \sigma) \]  \hspace{1cm} (6.11)
where the algebraic direct sum is over equivalence classes of irreducible representations of \( K_\infty \). The action of \( G(\mathbb{A}_F^\infty) \) preserves these spaces, and the action of \( (\mathfrak{g}, K_\infty) \) preserves \( \mathcal{A}(J) \) (but of course not \( \mathcal{A}(J, \sigma) \) in general).

The following is the adelic analogue of Theorem 6.2.3:

**Theorem 6.3.2** Let \( J \leq Z(\mathfrak{g}) \) be an ideal with \( \dim_{\mathbb{C}} Z(\mathfrak{g})/J < \infty \). Then \( \mathcal{A}(J) \) is an admissible \( (\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty) \)-module.

**Proof.** We are to show that \( \mathcal{A}(J, \sigma)_{K_\infty} \) is finite dimensional for each \( K_\infty \)-type \( \sigma \) and each compact open subgroup \( K_\infty \leq G(\mathbb{A}_F^\infty) \). We start by recalling the description of the adelic quotient given by (2.20). Letting \( t_1, \ldots, t_h \) denote a set of representatives for the finite set \( G(F) \backslash G(\mathbb{A}_F) / K_\infty \), one has a homeomorphism
\[ \prod_{i=1}^h \Gamma_i(K_\infty \backslash G(F_\infty)) \rightarrow G(F) \backslash G(\mathbb{A}_F) / K_\infty \]
given on the $i$th component by $\Gamma_i(K^\infty) g \mapsto G(F) g_\infty t_i K^\infty$ where $\Gamma_i(K^\infty) = G(F) \cap t_i K^\infty t_i^{-1}$. By pullback we obtain an isomorphism of $\mathbb{C}$-vector spaces

$$A(J)^{K^\infty} \xrightarrow{\sim} \bigoplus_{i=1}^h A(\Gamma_i(K^\infty), J)$$

where $\varphi \mapsto (x_i \mapsto \varphi(x_i))$

which intertwines the natural action of $(\mathfrak{g}, K_\infty)$ on both sides. We can now deduce the theorem by applying Theorem 6.2.3.

In view of the previous theorem the following definition is reasonable:

**Definition 6.6.** An **automorphic representation** of $G(\mathbb{A}_F)$ is an irreducible admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$-module isomorphic to a subquotient of $A$.

Two automorphic representations are isomorphic if their underlying $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$-modules are isomorphic. The actions of $Z(\mathfrak{g})$ and $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$ on $A$ commute. Thus by (6.10) every automorphic representation is in fact a subquotient of $A(J)$ for some $J$.

We defined automorphic representations in the $L^2$-sense previously in Definition 3.3. We will discuss the precise relationship between the two definitions in §6.6 below. Automorphic representation in the $L^2$-sense, defined as subquotients of $L^2([G])$, always turn out to be subrepresentations (see the proof of Theorem 6.6.4 below). We warn the reader that there are certainly automorphic representations that are most naturally described as subquotients, and not subrepresentations (see §10.6, for example). We do not know whether or not every automorphic representation is in fact a subrepresentation (and not merely a subquotient) of $A$.

Let us end the section by discussing factorization of admissible representations. We leave the proof of the following lemma as an exercise (see Exercise 6.6).

**Lemma 6.3.3** Every irreducible admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$-module $(\pi, V)$ is an algebraic tensor product

$$(\pi, V) = (\pi_\infty \otimes \pi^\infty, V_\infty \otimes V^\infty)$$

where $(\pi_\infty, V_\infty)$ is an irreducible admissible $(\mathfrak{g}, K_\infty)$-module and $(\pi^\infty, V^\infty)$ is an irreducible admissible representation of $G(\mathbb{A}_F^\infty)$. The isomorphism classes of $(\pi_\infty, V_\infty)$ and $(\pi^\infty, V^\infty)$ are uniquely determined by $\pi$. □

Assume that $K_\infty = \prod_{v|\infty} K_v$ is a maximal compact subgroup of $G(F_\infty)$ and $K^\infty = \prod_{v|\infty} K_v$ is a compact open subgroup of $G(\mathbb{A}_F^\infty)$.

**Theorem 6.3.4** Every irreducible admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$-module $(\pi, V)$ admits a factorization
where \((\pi_v, V_v)\) is an irreducible admissible \((\mathfrak{g}_v, K_v)\)-module for \(v|\infty\) and is an irreducible admissible representation of \(G(F_v)\) for \(v \nmid \infty\). The restricted tensor product is taken with respect to a choice of nonzero vectors \(\varphi_v \in V_v^K\) for all but finitely many places \(v\). The factors \((\pi_v, V_v)\) are uniquely determined by \((\pi, V)\) up to isomorphism.

**Proof.** By Lemma 6.3.3 we have

\[
(\pi, V) = (\pi_{v|\infty} \otimes_v \pi_{v|\infty} V_v) \otimes (\otimes_{v|\infty} \pi_v, \otimes_{v|\infty} V_v)
\]

with notation as in that lemma. One has

\[
(\pi_{v|\infty}, V_{v|\infty}) \cong (\otimes_v \pi_v, \otimes_v V_v)
\]

by Exercise 4.13 and

\[
(\pi_{v|\infty}, V_{v|\infty}) \cong (\otimes_{v|\infty} \pi_v, \otimes_{v|\infty} V_v)
\]

by Flath’s theorem (Theorem 5.7.1). The uniqueness statement follows from the corresponding uniqueness statements in each of the results cited in the proof. \(\square\)

6.4 Adelic automorphic forms over function fields

We assume throughout this section that \(F\) is a function field. The refined definition of an automorphic form in the function field case is fairly straightforward.

**Definition 6.7.** A function

\[
\varphi : G(A_F) \longrightarrow \mathbb{C}
\]

invariant on the left under \(G(F)\) and on the right under a compact open subgroup of \(G(A_F)\) is an **automorphic form** on \(G\) if the \(\mathbb{C}\)-vector space spanned by the functions

\[
\{ x \mapsto \varphi(xg) : g \in G(A_F) \}
\]

is an admissible representation of \(G(A_F)\).

As in the number field case, we denote by \(\mathcal{A}\) or \(\mathcal{A}(G)\) the \(\mathbb{C}\)-vector space of automorphic forms. It is naturally equipped with an action of \(G(A_F)\).

Let \(K = \prod_v K_v < G(A_F)\) be a compact open subgroup. As in the number field case, a function \(\varphi : G(A_F) \rightarrow \mathbb{C}\) is said to be \(K\)-**finite** if the \(\mathbb{C}\)-span of
the translates 
\[ \{ x \mapsto \varphi(xk) : k \in K \} \]
is finite dimensional. We observe that any element of \( \mathcal{A} \) is \( K \)-finite for any compact open subgroup \( K < G(\mathbb{A}_F) \). This is in sharp contrast to the number field case, in which \( \mathcal{A} \) depends on \( K_\infty \).

**Definition 6.8.** An **automorphic representation** of \( G(\mathbb{A}_F) \) is an irreducible admissible representation of \( G(\mathbb{A}_F) \) that is isomorphic to a subquotient of \( \mathcal{A} \).

In analogy with the number field case, two automorphic representations of \( G(\mathbb{A}_F) \) are isomorphic if they are isomorphic in the category of admissible representations of \( G(\mathbb{A}_F) \).

**Theorem 6.4.1** Every irreducible admissible \( G(\mathbb{A}_F) \)-module \( (\pi, V) \) admits a factorization
\[
(\pi, V) = (\otimes'_v \pi_v, \otimes'_v V_v)
\]
where \( (\pi_v, V_v) \) is an irreducible admissible representation of \( G(F_v) \) for each place \( v \). The restricted tensor product is taken with respect to a choice of nonzero vectors \( \varphi_v \in V_v^{K_v} \) for all but finitely many places \( v \). The factors \( (\pi_v, V_v) \) are uniquely determined by \( (\pi, V) \) up to isomorphism.

**Proof.** Combine Flath’s theorem (Theorem 5.7.1) with Theorem 5.7.2. \( \square \)

This is the analogue of Theorem 6.3.4 in the current context.

### 6.5 The cuspidal subspace

Recall that the definition of the space \( \mathcal{A} \) involves a choice of maximal compact subgroup \( K \leq G(\mathbb{A}_F) \). We fix such a choice for this section. We now isolate what is arguably the most important subspace of \( \mathcal{A} \):

**Definition 6.9.** An automorphic form \( \varphi \in \mathcal{A} \) is said to be **cuspidal** or a **cusp form** if for every proper parabolic subgroup \( P < G \) with unipotent radical \( N \), one has that 
\[
\int_{[N]} \varphi(ng)dn = 0
\]
for all \( g \in G(\mathbb{A}_F) \).

We recall that \([N]\) is compact (see Theorem 2.6.3, for instance) so the integral in the definition is well-defined. The subspace of \( \mathcal{A} \) consisting of cuspidal forms will be denoted \( \mathcal{A}_{\text{cusp}}(G) \) or \( \mathcal{A}_{\text{cusp}} \) if \( G \) is understood.

Before continuing, we motivate the definition of a cusp form, following the exposition in [God66]. It is natural to try and reduce the study of \( \mathcal{A}(G) \), or
subspaces of it, to spaces of functions on simpler groups. For example, we could consider Levi subgroups of $G$. Given a parabolic subgroup $P$ with Levi subgroup $M$ and unipotent radical $N$ and a $\varphi \in \mathcal{A}(G)$, we can define the constant term

$$\varphi_P(m) := \int_{[N]} \varphi(nm)dn.$$  

(6.13)

Here $m \in M(F) \setminus M(\mathbb{A}_F)$. This gives us a map

$$(\cdot)_P : \mathcal{A}(G) \rightarrow \mathcal{A}(M).$$

Then

$$\mathcal{A}_{\text{cusp}}(G) \leq \ker(\cdot)_P$$

for all proper parabolic subgroups $P < G$. Thus cusp forms cannot be related to automorphic forms on Levi subgroups via taking constant terms. This discussion and refinements of it are the motivation for the philosophy of cusp forms mentioned in §8.1 below.

The **cuspidal subspace**

$$L^2_{\text{cusp}}([G]) \leq L^2([G])$$  

(6.14)

is the space of $\varphi \in L^2([G])$ such that for all $f \in C_c^\infty(A_G \setminus G(\mathbb{A}_F))$ and every proper parabolic subgroup $P < G$ with unipotent radical $N$, one has

$$\int_{[N]} R(f)\varphi(ng)dn = 0$$  

(6.15)

for all $g \in A_G \setminus G(\mathbb{A}_F)$. An element of $L^2([G])$ is **cuspidal** if it is contained in $L^2_{\text{cusp}}([G])$. By Exercise 6.7, we could equivalently define a cuspidal function to be an element $\varphi \in L^2([G])$ such that the measurable function

$$g \mapsto \int_{[N]} \varphi(ng)dn$$  

(6.16)

vanishes for almost every $g$ with respect to a Haar measure on $G(\mathbb{A}_F)$. We will show in Lemma 9.2.1 that $L^2_{\text{cusp}}([G])$ is closed, and hence is a $G(\mathbb{A}_F)$-subrepresentation of $L^2([G])$.

**Definition 6.10.** A **cuspidal automorphic representation** of $G(\mathbb{A}_F)$ is an automorphic representation equivalent to a subquotient of $\mathcal{A}_{\text{cusp}}$.

Let us say that an irreducible subquotient of $L^2_{\text{cusp}}([G])$ is a **cuspidal automorphic representation in the $L^2$-sense**. We will show in Corollary 9.1.2 below that one has a discrete decomposition

$$L^2_{\text{cusp}}([G]) = \bigoplus_{\pi} L^2_{\text{cusp}}(\pi)$$
where $L^2_{\text{cusp}}(\pi)$ is the $\pi$-isotypic subspace of $L^2_{\text{cusp}}([G])$ and the Hilbert space direct sum is over all isomorphism classes of cuspidal automorphic representations $\pi$ of $A_G \backslash G(\mathbb{A}_F)$. In view of this fact, every irreducible subquotient of $L^2_{\text{cusp}}([G])$ is in fact a subrepresentation.

One justification for our claim that $\mathcal{A}_{\text{cusp}}$ is the most important subspace of $\mathcal{A}$ is that $L^2([G])$ admits a decomposition in terms of $\mathcal{A}_{\text{cusp}}(M)$ as $M$ runs over the Levi subgroups of parabolic subgroups of $G$. See §10.4 below.

Cuspidal automorphic representations in the $L^2$-sense and cuspidal automorphic representations (in the sense of Definition 6.10) are closely related. To explain the relationship, we observe that though $G(\mathbb{A}_F)$ does not act on $\mathcal{A}$ in the number field case, for any global field the subgroup $A_G$ does act on $\mathcal{A}$ and the action preserves $\mathcal{A}_{\text{cusp}}$. An automorphic representation of $A_G \backslash G(\mathbb{A}_F)$ is an automorphic representation of $G(\mathbb{A}_F)$ that is isomorphic to a subquotient of $A^{A_G}$. A cuspidal automorphic representation of $A_G \backslash G(\mathbb{A}_F)$ is an automorphic representation of $G(\mathbb{A}_F)$ isomorphic to a subquotient of $A^{A_G}_{\text{cusp}}$. For any cuspidal automorphic representation $\pi$, we denote by $A^{A_G}_{\text{cusp}}(\pi)$ the $\pi$-isotypic component of $A^{A_G}_{\text{cusp}}$.

To avoid repetition, in the following theorem we will refer to a cuspidal automorphic representation of $A_G \backslash G(\mathbb{A}_F)$ (resp. in the $L^2$-sense) simply as a cuspidal automorphic representation (resp. in the $L^2$-sense).

**Theorem 6.5.1** The space of cuspidal automorphic forms $A^{A_G}_{\text{cusp}}$ is a dense subspace of $L^2_{\text{cusp}}([G])$. If $(\pi, V)$ is a cuspidal automorphic representation in the $L^2$-sense, then the space of $K$-finite vectors $V_{\text{fin}}$ in $V$ is a cuspidal automorphic representation $(\pi, V_{\text{fin}})$, and

$$A^{A_G}_{\text{cusp}}(\pi) = L^2_{\text{cusp}}(\pi)_{\text{fin}},$$

the space of $K$-finite vectors in $L^2_{\text{cusp}}(\pi)$. The multiplicity of $(\pi, V_{\text{fin}})$ in $A^{A_G}_{\text{cusp}}(\pi)$ is finite and equal to the multiplicity of $(\pi, V)$ in $L^2_{\text{cusp}}(\pi)$. One has that

$$A^{A_G}_{\text{cusp}} = \bigoplus_{\pi} A^{A_G}_{\text{cusp}}(\pi)$$  \hspace{1cm} (6.17)

where the (algebraic) sum is over isomorphism classes of cuspidal automorphic representations. The association of $(\pi, V_{\text{fin}})$ to $(\pi, V)$ defines a bijection between isomorphism classes of cuspidal automorphic representations in the $L^2$-sense and isomorphism classes of cuspidal automorphic representations.

The decomposition (6.17) implies in particular that a cuspidal automorphic representation of $A_G \backslash G(\mathbb{A}_F)$ is in fact a submodule of $A^{A_G}_{\text{cusp}}$, though the original definition only required it to be a subquotient. We defer the proof of Theorem 6.5.1 to §9.7, where it is restated as Theorem 9.7.3. We observe that Exercise 6.4 implies that both $A^{A_G}_{\text{cusp}}$ and the space of $K$-finite vectors in $L^2_{\text{cusp}}([G])$ really depend only on $K_\infty$. In the function field case, essentially the same argument implies that they are additionally independent of $K_\infty$. 


6.6 The two definitions of an automorphic representation

We have given two definitions of an automorphic representation. First in Definition 3.3 we defined an automorphic representation of \(A_G \cap G(A_F)\) in the \(L^2\)-sense to be an irreducible subquotient of the unitary \(A_G \cap G(A_F)\)-representation \(L^2([G])\). In the number field case, we defined an automorphic representation in terms of irreducible admissible \((g, K_\infty) \times G(A_F)\)-modules in Definition 6.6. In the function field case, we defined an automorphic representation in terms of irreducible admissible \(G(A_F)\)-modules in Definition 6.8.

In this section, we prove Theorem 6.6.4, which describes the precise relationship between automorphic representations in the \(L^2\)-sense and automorphic representations.

We begin with considerations that do not require the space \([G]\). Thus let \((\pi, V)\) be an irreducible unitary representation of \(G(A_F)\).

**Theorem 6.6.1** One has that

\[
(\pi, V) \cong (\pi_\infty \otimes \pi_\infty, V_\infty \hat{\otimes} V_\infty)
\]

where \((\pi_\infty, V_\infty)\) is an irreducible unitary representation of \(G(F_\infty)\) and \((\pi_\infty, V_\infty)\) is an irreducible unitary representation of \(G(A_F)\). The representations \((\pi_\infty, V_\infty)\) and \((\pi_\infty, V_\infty)\) are determined up to unitary equivalence by \((\pi, V)\).

In the theorem, \(V_\infty \hat{\otimes} V_\infty\) is the Hilbert space tensor product of \(V_\infty\) and \(V_\infty\). The action of \(\pi_\infty \otimes \pi_\infty\) on the dense subspace \(V_\infty \otimes V_\infty < V_\infty \hat{\otimes} V_\infty\) extends by continuity.

**Proof.** The first assertion is [Fol95, Theorem 7.25] given Theorem 3.10.2. The second assertion is [Fol95, Corollary 7.22].

We let \(K = K_\infty K_\infty\) where \(K_\infty \leq G(F_\infty)\) is a maximal compact subgroup and \(K_\infty \leq G(A_F)\) is a compact open subgroup. Let

\[
V_{\text{fin}} < V, \quad V_{\infty} < V_\infty, \quad V_{\text{fin}} < V_\infty
\]

be the spaces of \(K\)-finite, \(K_\infty\)-finite, and \(K_\infty\)-finite vectors, respectively. We observe that these definitions are independent of the choice of \(K_\infty\), but in the number field case depend on the choice of \(K_\infty\).

**Theorem 6.6.2** The space \(V_{\text{fin}}\) is admissible and irreducible. One has that
$(\pi_\infty, V_{\text{fin}}) \cong (\pi_\infty \otimes \pi_\infty, V_{\text{fin}} \otimes V_{\text{fin}}^\infty)$.

If $(\pi_1, V_1)$ and $(\pi_2, V_2)$ are irreducible unitary representation of $G(\mathbb{A}_F)$ and $V_{1\text{fin}}$ is isomorphic to $V_{2\text{fin}}$ then $(\pi_1, V_1)$ and $(\pi_2, V_2)$ are unitarily equivalent.

Here when we say $V_{\text{fin}}$ is admissible (resp. irreducible), we mean as a $(g \hookrightarrow K_1) \lhd G(A_1 F)$-module in the number field case and as a smooth representation of $G(A_1 F)$ in the function field case. Similarly, when we say $V_{1\text{fin}}$ is isomorphic to $V_{2\text{fin}}$ we mean as $(g \hookrightarrow K_1) \lhd G(A_1 F)$-modules in the number field case and as smooth $G(A_1 F)$-representations in the function field case. We use the same convention for the remainder of this section.

**Proof.** Let $\tau$ be a $K_\infty$-type and let $K_\infty < G(\mathbb{A}_F^\infty)$ be a compact open subgroup. The isomorphism (6.18) induces an isomorphism

$$V_{\text{fin}}(\tau)^{K_\infty} \cong V_{\text{fin}}(\tau) \hat{\otimes} V^{\infty K_\infty}.$$  \hspace{1cm}(6.20)

The space $V_{\text{fin}}(\tau)$ is finite dimensional by Theorem 4.4.2 and the space $V^{\infty K_\infty}$ is finite dimensional by Theorem 5.3.7. Thus the tensor product in (6.20), which a priori is a completed tensor product in the sense of Hilbert spaces, is in fact algebraic, and $V_{\text{fin}}(\tau)^{K_\infty}$ is finite dimensional. This implies $V_{\text{fin}}$ is admissible. Irreducibility follows from Lemma 4.4.5 and Proposition 5.3.6. Taking the union over $K_\infty$-types $\tau$ and $K_\infty < G(\mathbb{A}_F^\infty)$, we deduce the isomorphism asserted in the theorem.

The last assertion follows from Theorem 4.4.6, Proposition 5.3.6, and Theorem 6.6.1.

In the case where $F$ is a number field, passing to $K_\infty$-finite vectors is often awkward because a general element of $G(F_\infty)$ will not preserve this subspace. With this in mind, it is often useful to work with smooth vectors. Let $(\pi, V)$ be an irreducible unitary representation of $G(\mathbb{A}_F)$. If $F$ is a function field then the space of smooth vectors $V_{\text{sm}}$ is just defined to be $V_{\text{fin}}$: it is stable under $G(\mathbb{A}_F)$. If $F$ is a number field, the space of smooth vectors $V_{\text{sm}}$ is the space of vectors $\varphi \in V$ that are smooth under the action of $G(F_\infty)$ in the sense of §4.2 and $K_\infty$-finite under the action of $G(\mathbb{A}_F^\infty)$. Unlike the subspace $V_{\text{fin}} < V$ of $K$-finite vectors, the space of smooth vectors $V_{\text{sm}} < V$ is preserved under $G(\mathbb{A}_F)$.

**Theorem 6.6.3** One has that

$$(\pi, V_{\text{sm}}) \cong (\pi_\infty \otimes \pi_\infty, V_{\text{sm}} \otimes V_{\text{fin}}^\infty),$$

where the product is algebraic.

**Proof.** For any compact open subgroup $K_\infty < G(\mathbb{A}_F^\infty)$, by restricting the isomorphism (6.18) we have

$$V^{K_\infty} \cong V_{\infty} \hat{\otimes} V^{\infty K_\infty} = V_{\infty} \otimes V^{\infty K_\infty}.$$
since $V^{\infty K^\infty}$ is finite dimensional by admissibility (see Theorem 5.3.7). Thus

$$V_{\text{sm}}^{K^\infty} \cong V_{\text{adm}} \otimes V^{\infty K^\infty}.$$  

Taking the union over $K^\infty$, we deduce the result. $\square$

The discussion above is valid for any unitary representation. In particular, it is valid for automorphic representations in the $L^2$-sense. This implies most of the following theorem:

**Theorem 6.6.4** The space of $K$-finite vectors in an automorphic representation in the $L^2$-sense is an automorphic representation of $A_G \backslash G(\mathbb{A}_F)$. Two automorphic representations in the $L^2$-sense are unitarily equivalent if and only if the associated automorphic representations are equivalent.

**Proof.** Suppose we are given an automorphic representation $(\pi, V)$ in the $L^2$-sense. Thus we have a $G(\mathbb{A}_F)$-equivariant continuous surjection of Hilbert spaces

$$L^2([G]) \twoheadrightarrow W$$

and a $G(\mathbb{A}_F)$-equivariant continuous injection $V \hookrightarrow W$. Since $W$ is a Hilbert space, we can choose an $G(\mathbb{A}_F)$-invariant closed orthogonal complement $V^\perp$ to $V$ in $W$, and we obtain a continuous $G(\mathbb{A}_F)$-equivariant surjection

$$L^2([G]) \twoheadrightarrow W \twoheadrightarrow W/V^\perp \cong V.$$  

By the Riesz representation theorem, this induces a continuous equivariant injection $V \hookrightarrow L^2([G])$. In other words we may identify $V$ with an irreducible subspace of $L^2([G])$. In particular, we can and do view elements of $V$ as functions on $[G]$.

The space $V_{\text{fin}}$ is admissible and irreducible by Theorem 6.6.2. Thus it suffices to show that $V_{\text{fin}}$ lies in $A^{4\infty}$. In the function field setting, this follows from admissibility. In the number field setting, we can apply Proposition 4.4.3 to deduce that the elements of $V_{\text{fin}}$ are smooth. They are $Z(\mathfrak{g})$-finite since any irreducible admissible $(\mathfrak{g}, K_\infty)$-module admits an infinitesimal character by Corollary 4.6.3. Finally it follows from [Wal92, §11.5.1] that they are of moderate growth. $\square$

In view of Theorem 6.6.4, for many purposes, we can work with either the definition of automorphic representation in terms of admissible representations presented in this chapter, or automorphic representations in the $L^2$-sense, whichever is convenient for the situation at hand. Both are often convenient for different reasons. However, in view of Lemma 3.8.1 and Langlands’ decomposition of $L^2([G])$ explained in §10.4 this definition has the disadvantage that the full decomposition of $L^2([G])$ cannot in general be described in terms of automorphic representations of $A_G \backslash G(\mathbb{A}_F)$ in the $L^2$-sense. Moreover this definition excludes many Eisenstein series (see Chapter 10 for an introduction to Eisenstein series). The definition of automorphic
representations in terms of admissible representations has the advantage that one can immediately reduce questions to finite dimensional settings, either by passing to \( K_\infty \)-types or vectors fixed under a compact open subgroup. It also has some drawbacks, mostly due to the fact that in the number field case automorphic representations are not representations of \( G(F_\infty) \). One way around this difficulty is through the use of Casselman-Wallach representations; we refer to [Wal92, Chapter 11].

Assume \( F \) is a number field. We warn the reader that later in this book we will often not be specific about whether we are viewing an automorphic representation as a unitary representation of \( G(\mathbb{A}_F) \) or as an admissible \( (\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty) \)-module. Similarly, if \( F \) is a function field we will often not be specific about whether we are viewing an automorphic representation as a unitary representation of \( G(\mathbb{A}_F) \) or an admissible representation of \( G(\mathbb{A}_F) \). Indeed, in practice one often has to switch back and forth between the two perspectives. The particular point of view must be deduced from the context.

### 6.7 From modular forms to automorphic forms

For readers familiar with modular forms, we make the connection between modular forms and automorphic forms precise in this section.

Let \( \Gamma \subseteq \text{GL}_2(\mathbb{Z}) \) be a congruence subgroup. Thus

\[
\Gamma = \text{GL}_2(\mathbb{Q}) \cap K^\infty
\]

where \( K^\infty \subseteq \text{GL}_2(\mathbb{A}_Q^\infty) \) is a compact open subgroup. In general \( \text{det} K^\infty \) is a compact open subgroup of \( \text{GL}_1(\mathbb{A}_Q^\infty) \). We assume that

\[
\text{det} K^\infty = \hat{\mathbb{Z}}^\times.
\]  

(6.21)

For example, we could set \( \Gamma \) equal to

\[
\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}) : N|c \right\}.
\]  

(6.22)

One has that \( \Gamma_0(N) = K_0(N) \cap \text{GL}_2(\mathbb{Q}) \), where

\[
K_0(N) := \hat{\Gamma_0(N)} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\hat{\mathbb{Z}}) : N|c \right\}.
\]

Here the hats denote profinite completions.

The key consequence of (6.21) is that it implies that the homeomorphism (2.20) is

\[
\Gamma \backslash \text{GL}_2(\mathbb{R}) \longrightarrow \text{GL}_2(\mathbb{Q}) \backslash \text{GL}_2(\mathbb{A}_Q)/K^\infty.
\]

(6.23)
In other words, there is only one component on the left in (2.20) in this case. In fact, one can do even better. Let
\[ GL_2(\mathbb{R})^+ := \{ g \in GL_2(\mathbb{R}) : \det g > 0 \}. \] (6.24)
Our assumption on \( \Gamma \) implies that there exists an element in \( \Gamma \) of negative determinant. Thus (6.23) implies that there is a homeomorphism
\[
(\Gamma \cap GL_2(\mathbb{R})^+) / GL_2(\mathbb{R})^+ \to GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_F) / K_{\infty} \\
(\Gamma \cap GL_2(\mathbb{R})^+) g \to GL_2(\mathbb{Q})(g, I_2) K_{\infty},
\] (6.25)
where \( I_2 \) is the identity of \( GL_2(\mathbb{A}_F) \).

We recall the definition of a classical modular form on \( \mathcal{H} \). For this definition \( \Gamma \) can be any congruence subgroup of \( GL_2(\mathbb{Q}) \); there is no need to assume (6.21).

**Definition 6.11.** Let \( k \in \mathbb{Z}_{>0} \) and let \( \mathcal{H} \) be the complex upper half plane. The space of **weight** \( k \) **modular forms** on \( \Gamma \) is the space \( M_k(\Gamma) \) of holomorphic functions \( f : \mathcal{H} \to \mathbb{C} \) satisfying the following conditions:

(a) \( f \left( \frac{az + b}{cz + d} \right) = (cz + d)^k f(z) \) for all \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma \cap SL_2(\mathbb{Z}) \) and all \( z \in \mathcal{H} \),

(b) \( f \) extends holomorphically to the cusps.

If \( f \) additionally vanishes at the cusps we say that \( f \) is a **cusp form**. The space of weight \( k \) cusp forms is denoted \( S_k(\Gamma) \).

There are many books on modular forms that the reader can consult for more details. We mention [DS05, GG12, Kob93, Lan95, Miy06, Ser73, Shi94]. A recent extension of the theory to larger classes of functions that arise naturally in applications is explained in [BFOR17].

We remark that if \( k \) is even then \( M_k(\Gamma) \) can be identified with a certain space of holomorphic differential forms. This explains the moniker “modular form.” This is explained in great detail in a slightly more general setting in [GG12, Chapter 6].

We now relate the space \( S_k(\Gamma) \) to a space of automorphic forms. One has an action
\[
GL_2(\mathbb{R})^+ \times \mathcal{H} \to \mathcal{H} \\
\left( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right), z \right) \mapsto \frac{az + b}{cz + d}.
\]

The stabilizer of \( i \) is \( A_{GL_2SO_2}(\mathbb{R}) \). We therefore obtain a quotient map
\[
A_{GL_2} \backslash GL_2(\mathbb{R})^+ \to A_{GL_2} \backslash GL_2(\mathbb{R})^+ / SO_2(\mathbb{R}) = \mathcal{H}.
\] (6.26)
In this setting
\[
A_{GL_2} := \{ (r, r) : r \in \mathbb{R}_{>0} \}.
\]
We set
\[ j(g, z) = (\det g)^{-1/2}(cz + d) \]
for
\[ g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{GL}_2(\mathbb{R})^+. \]
This is an example of an **automorphy factor** (see [GG12, §6.3] for details). Set
\[ \varphi_f(g) = j(g, i)^{-k}f(gi) : \text{GL}_2(\mathbb{R})^+ \to \mathbb{C}, \]
where \( g \) acts on \( i \) by fractional linear transformations. We extend the definition to \( \text{GL}_2(\mathbb{R}) \) by setting
\[ \varphi_f\left( g \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) := \varphi_f(g). \]
We also define
\[ \varphi'_f(g) := \varphi_f\left( g \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right). \]

In view of (6.25), these functions extend uniquely to functions on
\[ \text{GL}_2(\mathbb{Q}) \setminus \text{GL}_2(\mathbb{A}_\mathbb{Q})/K^\infty. \]
They will give us automorphic forms, but to specify the type, we require further notation.

Let \( \sigma_k \) be the induction of the representation
\[ \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \mapsto e^{ik\theta} \quad (6.27) \]
of \( \text{SO}_2(\mathbb{R}) \) to \( \text{O}_2(\mathbb{R}) \) and let
\[ \Delta = -\frac{1}{4}(H^2 + 2XY + 2YX) \]
be the Casimir element of \( \mathfrak{gl}_2^C \) (see §4.7). It and the element \( Z = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \) generate \( Z(\mathfrak{gl}_2) \).

Combining [Kud03, Proposition 1.4 and Remark 1.6], one has the following proposition:

**Proposition 6.7.1** For each integer \( k \geq 1 \) one has a \( \mathbb{C} \)-linear isomorphism of finite dimensional vector spaces
\[ S_k(\Gamma) \oplus S_k(\Gamma) \to \mathcal{A}_{\text{cusp}}(\Gamma, \langle \Delta - \frac{k}{2}(1 - \frac{k}{2}) \rangle, \sigma_k) \]
\[ \langle f_1, f_2 \rangle \mapsto \varphi_{f_1} + \varphi'_{f_2}. \]
One might ask if all cuspidal automorphic forms on $GL_2$ arise from elements of $S_k(\Gamma)$ in this fashion. In fact, we have missed many of them. The missing automorphic representations correspond to **Maass forms**. From the automorphic perspective it is very simple to define these objects; they are the automorphic representations of $GL_2(\mathbb{A}_\mathbb{Q})$ whose associated $(gl_2, O_2(\mathbb{R}))$-module is in the principal series in the sense of $\S 4.7$. Defining them classically involves at least as much work as required to formalize the notion of an element of $S_k(\Gamma)$. We refer the reader to [Bum97, §3.2] for more details. The ease with which we can talk about Maass forms and holomorphic cusp forms on equal footing illustrates the utility of the language of automorphic representation theory.

### 6.8 Ramanujan’s $\Delta$-function

We would be remiss not to give an example of a modular form from the classical perspective, and to an expert it should not be a surprise that we choose Ramanujan’s $\Delta$-function

$$\Delta(z) = e^{2\pi iz} \prod_{i=1}^{\infty} (1 - e^{2\pi inz})^{24}, \quad z \in \mathbb{H}.$$  

This $\Delta$ should not be confused with the Casimir operator from the previous section.

One has the following basic theorem:

**Theorem 6.8.1** The complex vector space $S_{12}(SL_2(\mathbb{Z}))$ is 1-dimensional and spanned by $\Delta(z)$.  

The proof of this fact can be found in any of the standard references [DS05, Lan95, Miy06, Kob93, Ser73, Shi94].

Define $\tau(n) \in \mathbb{C}$ to be the unique integers such that

$$\sum_{n=1}^{\infty} \tau(n)e^{2\pi inz} = \Delta(z).$$

In other words, the $\tau(n)$ can be viewed as the Fourier coefficients of $\Delta(z)$. Ramanujan conjectured that

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n^t} = \prod_{p} \frac{1}{1 - \tau(p)p^{-t} + p^{11-2t}} \quad (6.28)$$

for a sufficiently large real number $t$ [Ram00b, (101)]. Remarkably this is exactly his notation. He also conjectures that
\[ |\tau(p)| \leq 2p^{11/2} \quad (6.29) \]

[Ram00b, (104)]. For the uninitiated, we pause and explain how fantastically prescient these two assertions are. In (6.28) Ramanujan has written down the \( L \)-function of the automorphic representation attached to \( \Delta(z) \) over 20 years prior to Hecke’s general theory of \( L \)-functions for modular forms [Hec37b, Hec37a] (Ramanujan’s assertion had been proven some time earlier by Mordell [Mor]). The assertion (6.29) was proven by Deligne as a consequence of his proof of the Weil conjectures [Del73a], however, the natural generalization of (6.29) to automorphic representations of \( \text{GL}_n \) remains open even in the case \( n = 2 \); it will be discussed in Conjecture 10.6.4 later.

**Exercises**

**6.1.** Let \( \iota_1': G \to \text{GL}_{n_1}, \iota_2': G \to \text{GL}_{n_2} \) be a pair of faithful representations of a reductive group \( G \) over a global field \( F \) and let \( S \) be a set of places of \( F \) (finite or infinite). Let \( \|g\|_{i_1S}, \|g\|_{i_2S} \) be the corresponding norms defined in (6.8). Prove that there are constants \( c_1, c_2 \in \mathbb{R}_{>0} \) and \( r \in \mathbb{R}_{>1} \) such that

\[
c_1 \|g\|_{i_1S}^{1/r} \leq \|g\|_{i_2S} \leq c_2 \|g\|_{i_1S}^r.
\]

Deduce that the notion of moderate growth and uniform moderate growth is independent of the choice of faithful representation \( \iota': G \to \text{GL}_n \).

**6.2.** Let \( v \) be a place of the global field \( F \) and let \( \|\cdot\|_v \) be defined as in (6.7). Prove the following:

(a) There is a \( c \in \mathbb{R}_{>0} \) such that \( \|gh\|_v \leq c \|g\|_v \|h\|_v \) for all \( g, h \in G(F_v) \).
(b) One has that \( \|g^{-1}\|_v = \|g\|_v \) for all \( g \in G(F_v) \).
(c) For any compact subset \( \Omega \subset G(F_v) \) there are \( c_1(\Omega), c_2(\Omega) \in \mathbb{R}_{>0} \) such that

\[
c_1(\Omega) \leq \frac{\|gh\|}{\|g\|} < c_2(\Omega)
\]

for all \( (g, h) \in G(F_v) \times \Omega \).

**6.3.** Let \( F \) be an archimedean local field, \( G \) an affine algebraic group over \( F \) and let \( K \leq G(F) \) be a compact subgroup. For every irreducible finite dimensional representation \( \sigma \) of \( K \), let \( \chi_\sigma \) be the character of \( \sigma \) and let \( d(\sigma) \) be its degree. For any \( K \)-module \( (\pi, V) \), let

\[
V \to V
\]

\[
\varphi \mapsto \int_K \pi(k)\varphi \frac{\chi_\sigma(k)}{d(\sigma)}dk.
\]
Show that this linear map is an idempotent that projects $V$ onto the $\sigma$-isotypic subspace $V(\sigma)$.

6.4. Let $F$ be a global field and let $K = K_\infty K^\infty \leq G(\mathbb{A}_F)$ where $K_\infty \leq G(F_\infty)$ is a maximal compact subgroup and $K^\infty < G(\mathbb{A}_F)$ is a compact open subgroup. Show that a continuous function $\varphi : |G| \to \mathbb{C}$ is $K$-finite if and only if for all $x^\infty \in G(\mathbb{A}_F^\infty)$, $x^\infty \mapsto \varphi(x^\infty x^\infty)$ is $K_\infty$-finite and there is a compact open subgroup $K^{\infty \infty} \leq G(\mathbb{A}_F^\infty)$ such that $\varphi(xk) = \varphi(x)$ for all $x \in G(\mathbb{A}_F)$ and $k \in K^{\infty \infty}$.

6.5. Let $F$ be a local field and let $G$ be a reductive group over $F$. Let $K_1, K_2 \leq G(F)$ be maximal compact subgroups. If $F$ is nonarchimedean, show that $K_1 \cap K_2$ is a compact open subgroup of $G(F)$ and deduce that a function on $G(F)$ is $K_1$-finite if and only if it is $K_2$-finite. Construct an example to show that if $F$ is archimedean then a function that is $K_1$-finite need not be $K_2$-finite.

6.6. Prove Lemma 6.3.3.

6.7. Prove that a function $\varphi \in L^2(|G|)$ is cuspidal if and only if for all standard maximal parabolic subgroups $P \leq G$ with unipotent radical $N$, one has that
\[
\int_{|N|} \varphi(n g) d n = 0
\]
for almost every $g \in G(\mathbb{A}_F)$.

6.8. Let $\Gamma < \text{GL}_2(\mathbb{R})$ be an arithmetic subgroup. Using the classification of §4.7, identify the underlying $(\mathfrak{g}, K)$-module of $A_{cusp}(\Gamma, \langle \Delta - \frac{k}{2}(1 - \frac{k}{2}), Z \rangle)$.

6.9. Prove Ramanujan’s assertion that (6.28) is valid using Theorem 6.8.1.

6.10. Assume that $G$ is reductive and that $F$ is a number field. Define
\[
C_c^\infty(G(\mathbb{A}_F)\)^1 = \{ f|_{G(\mathbb{A}_F)\} } : f \in C_c^\infty(G(\mathbb{A}_F))\}.
\]
Prove that the canonical isomorphism $G(\mathbb{A}_F)\to A_G \setminus G(\mathbb{A}_F)$ induces an isomorphism
\[
C_c^\infty(A_G \setminus G(\mathbb{A}_F) \to C_c^\infty(G(\mathbb{A}_F)\)^1)
\]
via pullback of functions.
Chapter 7
Unramified Representations

Demazure nous indique que, derrière cette terminologie [épinglage], il y a l’image du papillon (que lui a fournie Grothendieck): le corps est un tore maximal \( T \), les ailes sont deux sous-groupes de Borel opposées par rapport à \( T \), on déploie le papillon en étalant les ailes, puis on fixe des éléments dans les groupes additifs (des épingles)...

Abstract
In this chapter, we describe the classification of unramified representations of reductive groups over nonarchimedean local fields. Along the way we discuss the Satake isomorphism and the Langlands dual group.

7.1 Unramified representations

Let \( K^\infty := \prod_{v \in \infty} K_v \leq G(\mathbb{A}_F^\infty) \) be a compact open subgroup. Then \( K_v \) is hyperspecial for all but finitely many \( v \) by Corollary 2.4.9. By Flath’s theorem (Theorem 5.7.1), if \( (\pi, V) \) is an automorphic representation of \( G(\mathbb{A}_F) \), then

\[
(\pi, V) \cong (\otimes'_v \pi_v, \otimes'_v V_v)
\]

where for all but finitely many \( v \), one has that \( V^K_v \neq 0 \) (see also theorems 6.3.4 and 6.4.1). In particular, for all but finitely many \( v \), the representation \( (\pi_v, V_v) \) is unramified in the sense of Definition 7.1 below. Thus unramified representations play an essential role in automorphic representation theory.
In this section we discuss the classification of unramified representations. It turns out that they can be explicitly parametrized in terms of conjugacy classes in the dual group of $G$ (see Theorem 7.2.1 and Theorem 7.5.1). This fundamental fact will be used in §7.7 to state a version of the Langlands functoriality conjecture.

Throughout this section we let $G$ be a reductive group over a nonarchimedean local field $F$. Recall that $G$ is unramified if $G$ is quasi-split and split over an unramified extension of $F$. In this case, there exists a hyperspecial subgroup $K \leq G(F)$. We recall Definition 5.7:

**Definition 7.1.** An irreducible smooth representation $(\pi, V)$ of $G(F)$ is called $K$-unramified if $G$ is unramified and $V^K \neq 0$.

As mentioned below Definition 5.7, unramified representations are admissible by Theorem 5.3.4.

Hyperspecial subgroups of $G(F)$ are not necessarily $G(F)$-conjugate; they are only conjugate under $(G \ltimes \mathbb{Z}/G)(F)$ (see §2.4). Thus the property of a representation being $K$-unramified in general depends on $K$. We will however follow tradition and usually suppress the dependence on $K$ (but see §12.5).

Let $K \leq G(F)$ be a hyperspecial subgroup. We recall from §5.5 that the subalgebra

$$C_c^\infty(G(F) \sslash K) \leq C_c^\infty(G(F))$$  \hspace{1cm} (7.1)

is known as the **unramified Hecke algebra** of $G(F)$ (with respect to $K$). It is commutative by Theorem 5.5.1. Let $f \in C_c^\infty(G(F) \sslash K)$ and let $\pi$ be unramified. By Corollary 5.5.2 we have $\dim_{\mathbb{C}} V^K = 1$. Thus $\pi(f)$ acts via a scalar on $V^K$. It is sensible to denote the scalar by $\text{tr} \pi(f)$ (see also §8.5). The map

$$C_c^\infty(G(F) \sslash K) \to \mathbb{C}$$

$$f \mapsto \text{tr} \pi(f)$$

is called the **Hecke character** of $\pi$.

**Proposition 7.1.1** Let $K \leq G(F)$ be a compact open subgroup (not necessarily hyperspecial). If $(\pi_1, V_1)$ and $(\pi_2, V_2)$ are irreducible representations of $G(F)$ such that $V_1^K$ and $V_2^K$ are nonzero and isomorphic as $C_c^\infty(G(F) \sslash K)$-modules, then $(\pi_1, V_1)$ is isomorphic to $(\pi_2, V_2)$ as a smooth representation. In particular, an unramified representation $\pi$ is determined up to isomorphism by its Hecke character.

For a refinement of this result, see [Lau96, Proposition D.1.8].

**Proof.** We follow an argument from [Cas73, p. 33]. Let $(\pi_1, V_1)$ and $(\pi_2, V_2)$ be as stated in the proposition, and let

$$I : V_1^K \to V_2^K$$
be a $C^\infty_c(G(F) \sslash K)$-intertwining map. We claim that for $f \in C^\infty_c(G(F))$ and $\varphi \in \mathcal{V}_1^F$ the map

$$I(\pi_1(f)\varphi) := \pi_2(f)I(\varphi)$$

is well-defined. Since $\mathcal{V}_1$ is irreducible, if $I$ is well-defined it is clearly an intertwining map of $C^\infty_c(G(F))$-modules from $\mathcal{V}_1$ to $\mathcal{V}_2$. By Lemma 5.3.2, $I$ therefore defines an intertwining map of smooth representations from $\mathcal{V}_1$ to $\mathcal{V}_2$. Since $\mathcal{V}_2$ is also irreducible, $I$ is an isomorphism.

To prove the claim, it suffices to check that if $\pi_1(f)\varphi = 0$ then $\pi_2(f)I(\varphi) = 0$. Suppose that $\pi_1(f)\varphi = 0$. Then, writing $e_K := \frac{1}{\text{meas}_K(K)}1_K$ as in (5.2), we have

$$\pi_1(e_K * f_1 * f * e_K)\varphi = 0$$

for all $f_1 \in C^\infty_c(G(F))$, which implies

$$\pi_2(e_K * f_1 * f * e_K)I(\varphi) = 0$$

for all $f_1 \in C^\infty_c(G(F))$ since $I$ is a $C^\infty_c(G(F) \sslash K)$-intertwining map. But then

$$\pi_2(e_K * f_1)\pi_2(f)I(\varphi) = 0$$

for all $f_1 \in C^\infty_c(G(F))$. Since $\mathcal{V}_2$ is irreducible and $\mathcal{V}_2^K$ is nonzero, we deduce that $\pi_2(f)I(\varphi) = 0$. \hfill \Box

### 7.2 The Satake isomorphism

Let $G$ be an unramified reductive group over the nonarchimedean local field $F$ and let $K \leq G(F)$ be a hyperspecial subgroup. If we did not know anything about $C^\infty_c(G(F) \sslash K)$, then we could hardly regard Proposition 7.1.1 as useful. However, it turns out that $C^\infty_c(G(F) \sslash K)$ has a simple description. Assume first that $G$ is split. Let $\widehat{G}$ be the complex reductive group with root datum dual to that of $G$ (see §1.8) and let $\widehat{T} \leq \widehat{G}$ be a maximal torus.

**Theorem 7.2.1 (Satake)** Assume that $G$ is split. There is an isomorphism of algebras

$$\mathcal{S} : C^\infty_c(G(F) \sslash K) \rightarrow \mathcal{O}(\widehat{T})^{W(\widehat{G}, \widehat{T})}.$$

We observe that $\widehat{T}$ is a $\mathbb{C}$-scheme, so the coordinate ring $\mathcal{O}(\widehat{T})$ is a $\mathbb{C}$-algebra. The group scheme $W(\widehat{G}, \widehat{T})$ acts on $\widehat{T}$ and hence we can define $\mathcal{O}(\widehat{T})^{W(\widehat{G}, \widehat{T})}$ as in (17.8). Informally, since $W(\widehat{G}, \widehat{T})$ acts on $\widehat{T}$, it acts on regular functions on $\widehat{T}$ and hence we can define $\mathcal{O}(\widehat{T})^{W(\widehat{G}, \widehat{T})} \leq \mathcal{O}(\widehat{T})$ to be the subalgebra fixed under this action. This can be made precise as explained below (17.8).

To gain intuition for the Satake isomorphism, let us consider the special case of $\text{GL}_n$. The Hecke algebra $C^\infty_c(\text{GL}_n(F) \sslash \text{GL}_n(\mathcal{O}_F))$, as a $\mathbb{C}$-module,
has a basis given by

$$\mathds{1}_\lambda := \mathds{1}_{\GL_n(\mathcal{O}_F)} \left( \begin{array}{c} \lambda_1 \\
 \vdots \\
 \lambda_n \end{array} \right)$$

with $\lambda := (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$, $\lambda_i \geq \lambda_{i+1}$ for all $1 \leq i \leq n-1$ (this follows from the theory of elementary divisors). As a $\mathbb{C}$-algebra it is generated by $\mathds{1}_\lambda$ with $\mathds{1}_\lambda := (\mathds{1} \mapsto \ldots \mapsto \mathds{1} \mapsto 0 \mapsto \ldots \mapsto 0)$ ($r$ ones and $n-r$ zeros) for $1 \leq r \leq n$ and $\lambda = ((-1)^n) := (-1, \ldots, -1)$.

Let $\hat{T} \leq \GL_n(\mathcal{O}_F) = \GL_n$ be the maximal torus of diagonal matrices. On the generating set above, the Satake isomorphism is given by

$$S(\mathds{1}_\lambda) = q^{r(n-r)/2} \text{tr}(\wedge^r t) \quad \text{and} \quad S(\mathds{1}_{(-1)^n}) = \det(t^{-1}). \quad (7.2)$$

Here $t \in \hat{T}(\mathbb{C})$ and $\text{tr}(\wedge^r t)$ is the polynomial function of $t$ given by the character of the representation $\wedge^r$ of $\GL_n$ (see [Gro98, (3.14)]).

It is instructive to indicate various ways of rephrasing this basic result. By the Chevalley restriction theorem, Theorem 17.2.3, the inclusion $\hat{T} \hookrightarrow \hat{G}$ induces an isomorphism

$$\text{Spec} \left( \mathcal{O}(\hat{T})^W(\hat{G}, \hat{T}) \right) = \hat{T}/W(\hat{G}, \hat{T}) \twoheadrightarrow \hat{G}/\text{conj} = \text{Spec} \left( \mathcal{O}(\hat{G})^{\hat{G}} \right) \quad (7.3)$$

where the quotient on the right is the quotient of $\hat{G}$ by the action of conjugation (see §17.1). In particular $(\hat{G}/\text{conj})(\mathbb{C})$ is just the set of closed conjugacy classes in $\hat{G}(\mathbb{C})$. The closed conjugacy classes are precisely the conjugacy classes of semisimple elements; a nice reference for this is [Ste65, Corollary 6.13]. Thus we have a sequence of isomorphisms

$$\text{Hom}(C^\infty_c(G(F) \parallel K), \mathbb{C}) \xrightarrow{(S^{-1})^*} \text{Hom}(\mathcal{O}(\hat{T})^W(\hat{G}, \hat{T}), \mathbb{C}) \rightarrow (\hat{G}/\text{conj})(\mathbb{C}) \quad (7.4)$$

where the first map is pullback along the inverse of the Satake isomorphism. Here homomorphism means homomorphism of $\mathbb{C}$-algebras. In Proposition 7.1.1 we saw that unramified representations were in bijection with Hecke characters, which are precisely elements of $\text{Hom}(C^\infty_c(G(F) \parallel K), \mathbb{C})$. We have therefore proven the following corollary of the Satake isomorphism:

**Corollary 7.2.2** Assume that $G$ is split. The composite isomorphism $(7.4)$ induces a bijection between semisimple conjugacy classes in $\hat{G}(\mathbb{C})$ and isomorphism classes of irreducible unramified representations of $G(F)$. \hfill \Box

From the point of view of automorphic representation theory, the fact that the Satake isomorphism is only valid for split groups is problematic. Indeed, suppose for the moment that $G$ is a reductive group over a global field $F$. Then, for all but finitely many places $v$ of $F$, the group $G_{F_v}$ is unramified by
Proposition 2.4.5 and hence in particular is quasi-split. However, the group $G_F$ can be nonsplit for infinitely many $v$ (see Exercise 7.2).

Langlands was able to extend the Satake isomorphism to the quasi-split case. Historically this was important because it gave crucial hints as to the structure of the Langlands dual group, which he introduced at the same time [Lan] (see also [Lan70]). We will define the Langlands dual group in the next section and use it to extend the Satake isomorphism in §7.5.

7.3 The Langlands dual group

For the moment, let $G$ be a reductive group over a field $F$ with separable closure $F^{sep}$ and let $T \leq G$ be a maximal torus. Then $T$ splits over $F^{sep}$. To $(G_{F^{sep}}, T_{F^{sep}})$, we associated in §1.8 a root datum $\Psi(G_{F^{sep}}, T_{F^{sep}}) = (X^*(T_{F^{sep}}), X_*(T_{F^{sep}}), \Phi, \Phi^\vee)$. We remind the reader that $X^*(T_{F^{sep}})$ and $X_*(T_{F^{sep}})$ are the character and cocharacter groups, $\Phi^\vee \subset X^*(T_{F^{sep}})$ is the set of roots of $T_{F^{sep}}$ in $g$ and $\Phi \subset X_*(T_{F^{sep}})$ is the set of coroots. We also remind the reader that the root datum characterizes $G_{F^{sep}}$. To ease notation, we let

$$ \text{Gal}_F := \text{Gal}(F^{sep}/F). $$

If $G$ is split, then we set

$$ L^G := \hat{G}(\mathbb{C}) \times \text{Gal}_F $$

where $\hat{G}$ is the complex dual group defined as in §1.8; it is the reductive group over $\mathbb{C}$ with root datum $(X_*(T_{F^{sep}}), X^*(T_{F^{sep}}), \Phi, \Phi^\vee)$.

If $G$ is not split then $L^G$ has a more complicated definition involving a Galois action on $\hat{G}(\mathbb{C})$ that records the fact that $G$ is a nonsplit group over $F$. Given a Galois action on $G_{F^{sep}}$ we obtain one on $\Psi(G_{F^{sep}}, T_{F^{sep}})$ and hence tautologically we obtain a Galois action on $\Psi(\hat{G}, \hat{T})$. This is not enough, as we really need a Galois action on $\hat{G}(\mathbb{C})$. We ask that this action is algebraic, and hence given in terms of elements of

$$ \text{Aut}(\hat{G}) $$

the group of automorphisms of the algebraic group $\hat{G}$. This group is actually the $\mathbb{C}$-points of another algebraic group, but we will not need this fact in this section. It would suffice to produce a splitting of the surjective map

$$ \text{Aut}(\hat{G}) \twoheadrightarrow \text{Aut}(\Psi(\hat{G}, \hat{T})) $$

described in Proposition 7.3.3 below. What we actually do is to produce a refinement $\Psi(\hat{G}, \hat{B}, \hat{T})$ of $\Psi(\hat{G}, \hat{T})$ called a based root datum and a surjective map
\[ \text{Aut}(\hat{G}) \to \text{Aut}(\Psi(\hat{G}, \hat{B}, \hat{T})). \]

We then provide an explicit description of splittings of this map via something known as a pinning, or épinglage, of \( \hat{G} \). This then suffices to construct our desired action of \( \text{Gal}_F \) on \( \hat{G}(\mathbb{C}) \).

We now begin this process. A useful reference is [Mil17, §23.d]. We assume until otherwise specified that \( G \) is a reductive group over a separably closed field \( k \). Thus \( G \) is split. Let \( \Delta \subseteq \Phi \) be a base for \( \Phi \) (see (1.18)). The set

\[ \Delta^\vee := \{ \alpha^\vee \in \Phi^\vee : \alpha \in \Delta \} \]

forms a base of \( \Phi^\vee \). With this in mind, a tuple

\[ (X, Y, \Delta, \Delta^\vee) \quad (7.5) \]

consisting of free abelian groups \( X \) and \( Y \) together with subsets \( \Delta \subseteq X \) and \( \Delta^\vee \subseteq Y \) is called a **based root datum** if there is a root datum \( (X, Y, \Phi, \Phi^\vee) \) such that \( \Delta \subseteq \Phi \) is a base for \( \Phi \) and \( \Delta^\vee \) is a dual base for \( \Phi^\vee \). We note that \( \Delta \) spans \( \Phi \) and \( \Delta^\vee \) spans \( \Phi^\vee \) as \( \mathbb{Z} \)-modules, so there is no ambiguity in the notation \( (7.5) \). There is an obvious notion of isomorphism of based root data; it is simply a pair of linear isomorphisms on the first two factors preserving the pairing and the sets of simple roots.

We let \( \Psi(G, B, T) := (X^*(T), X_*(T), \Delta, \Delta^\vee) \) be a choice of based root datum, and \( \Psi(G, T) \) the root datum it defines. The reason for the \( B \) in this notation is the following lemma:

**Lemma 7.3.1** The set of bases \( \Delta \subseteq \Phi \) is in natural bijection with the set of Borel subgroups \( B \leq G \) containing \( T \).

We have already mentioned a more general result in §1.9, but we did not prove it there. For each \( \alpha \in \Phi \) let \( N_\alpha \leq G \) be the corresponding root group defined as (1.20). For example if \( G = \text{GL}_n(R) \) and \( \alpha = e_{ij} \) in the notation in Example 1.12 then the corresponding root group is \( I_n + Re_{ij} \).

**Proof.** Given a Borel subgroup the set of roots of a maximal torus \( T \) in \( \text{Lie} B \) forms a set of positive roots [Spr09, Proposition 7.4.6] which provides us with a base.

Conversely, given a choice of a base \( \Delta \), let \( \Phi^+ \) be the associated set of positive roots. The group

\[ B = \langle T, \{ N_\alpha \}_{\alpha \in \Phi^+} \rangle \quad (7.6) \]

is a Borel subgroup by [Spr09, Proposition 8.2.4]. \( \square \)

To better describe \( \Psi(G, B, T) \), we introduce the notion of a pinning:

**Definition 7.2.** A pinning of \( G \) is a tuple

\[ (B, T, \{ X_\alpha \}_{\alpha \in \Delta}) \]
where $T$ is a maximal torus and $B$ is a Borel subgroup containing it, $\Delta$ is the set of simple roots attached to $B$ and $T$, and $X_\alpha \in \mathfrak{g}_\alpha - \{0\}$ for all $\alpha$.

We refer to [Mil17, §23.d] and [Con14, §1.5] for more information about pinnings; we reproduce some arguments from these references below.

We let $\text{Aut}(B \hookrightarrow T \hookrightarrow \{X_\alpha\}_{\alpha \in \Delta})$ be the group of automorphisms of $G$ that preserve $B$ and $T$ and the set $\{X_\alpha\}_{\alpha \in \Delta}$.

**Proposition 7.3.2** There is an isomorphism

$$\text{Aut}(B \hookrightarrow T \hookrightarrow \{X_\alpha\}_{\alpha \in \Delta}) \cong \text{Aut}(\Psi(G, B, T))$$

where we use $\Delta$ to define $\Psi(G, B, T)$.

**Proof.** It is clear that the map in the proposition is well-defined. The group $T(k)$ acts on $\{ (X_\alpha) \in \prod_{\alpha \in \Delta} (\mathfrak{g}_\alpha - \{0\}) \}$ by conjugation. Since the set $\Delta$ is linearly independent in $X^*(T)_\mathbb{Q}$, the action is surjective. Thus it follows from Theorem 1.8.3 that the map is surjective. For injectivity, Theorem 1.8.3 implies that anything in the kernel of the morphism in the proposition is an inner automorphism by an element $t \in T(k)$. Since this inner automorphism fixes the $X_\alpha$, it satisfies $\alpha(t) = 1$ for all $\alpha \in \Delta$. The element $t$ therefore centralizes every root group, hence the Bruhat cell $B_{op}TB \hookrightarrow$ where $B_{op}$ is the opposite Borel. The Bruhat cell $B_{op}TB$ is open in $G$ [Mil17, Theorem 21.84], so $t$ centralizes all of $G$. In particular, the inner automorphism defined by $t$ is trivial. \hfill \Box

**Proposition 7.3.3** There is a split exact sequence

$$1 \longrightarrow \text{Inn}(G) \xrightarrow{a} \text{Aut}(G) \xrightarrow{b} \text{Aut}(\Psi(G, B, T)) \longrightarrow 1. \quad (7.7)$$

The splittings are in bijection with pinnings $(B', T', \{X'_\alpha\}_{\alpha \in \Delta'})$ up to conjugation by $T'(k)$.

Here in the proposition $\text{Aut}(G)$ is the group (not group scheme) of automorphisms of the algebraic group $G$ and $\text{Inn}(G)$ is the subgroup of inner automorphisms, that is, automorphisms induced by conjugation by elements of $G(k)$.

**Proof.** The arrow $a$ is the obvious injection. To describe $b$, note that for any automorphism $\phi \in \text{Aut}(G)$ there is a $g \in G(k)$ such that $\phi \circ \text{Ad}(g)$ preserves $B$ and $T$. Indeed, all Borel subgroups are conjugate under $G(k)$ and all (split) maximal tori in $B$ are conjugate under $B(k)$. The fact that $\phi \circ \text{Ad}(g)$ preserves $B$ and $T$ implies that it preserves the set of simple roots $\Delta$ defined by $B$ as in the proof of Lemma 7.3.1. We claim this gives a well-defined homomorphism

$$b : \text{Aut}(G) \longrightarrow \text{Aut}(\Psi(G, B, T)). \quad (7.8)$$

To check that it is well-defined, assume that $\phi$ and $\phi \circ \text{Ad}(g)$ both preserve $B$ and $T$. Then
We therefore obtain an action of Gal $B$ Borel subgroup is again a based root datum with associated root datum. Now $Aut(G)$ we may assume $Aut(G)$ maps to the identity in $Aut(\Psi(G, B, T))$ under $b$. After replacing $\phi$ by $\phi \circ Ad(g)$ for some $g \in G(k)$, we may assume $\phi$ preserves $B$ and $T$. The fact that $b(\phi)$ is the identity implies that $\phi$ is an inner automorphism by an element of $T(k)$ by Theorem 1.8.3.

Since $Aut(B, T, \{X_\alpha\}_{\alpha \in \Delta})$ is defined to be a subset of $Aut(G)$, we see that a pinning defines a section of $Aut(G) \to Aut(\Psi(G, B, T))$. All sections are $G(k)$-conjugate. For $g \in G(k)$, the corresponding conjugate of this section may be identified with $Aut(gBg^{-1}, gTg^{-1}, \{gX_\alpha g^{-1}\}_{\alpha \in \Delta})$, and $(gBg^{-1}, gTg^{-1}, \{gX_\alpha g^{-1}\})$ is another pinning. To complete the proof, we claim that all pinnings are $G(k)$-conjugate. Indeed, all Borel subgroups are conjugate under $G(k)$, and all (split) maximal tori within a Borel subgroup $B$ are conjugate under $B(k)$, with normalizer $N_{B(k)}(T) = T$ (see the last equality of (7.9)). The choice of $B$ and $T$ determine $\Delta$ as explained in Lemma 7.3.1. Finally $T(k)$ acts transitively on $\{(X_\alpha) \in \prod_{\alpha \in \Delta}(\mathfrak{g}_\alpha - \{0\})\}$ as explained in the proof of Proposition 7.3.2. \hfill \square

The content of propositions 7.3.2 and 7.3.3 can be summarized informally as follows. The theorem of Chevalley and Demazure (Theorem 1.8.3) allows us to identify automorphisms of a reductive group over $k$ that fix a maximal torus $T$ with automorphisms of root data, but only up to an inner automorphism by an element of $T(k)$. Fixing a pinning and refining root data to based root data removes this ambiguity.

With this preparation complete, we can now define the Langlands dual group. Assume that $G$ is a reductive group over a local or global field $F$. Choose a Borel subgroup $B \leq G_{F^{\text{sep}}}$ and a maximal torus $T \leq B \leq G_{F^{\text{sep}}}$. We obtain a based root datum

$$\Psi(G_{F^{\text{sep}}}, B, T).$$

Using the exact sequence of Proposition 7.3.3, the homomorphism $Gal_F \to Aut(G_{F^{\text{sep}}})$ giving $G$ its $F$-structure yields an action of $Gal_F$ on the based root datum. Now

$$(X_\star(T), X^*(T), \Delta^\vee, \Delta)$$

is again a based root datum with associated root datum $\Psi(\widehat{G}, \widehat{T})$, so we have a Borel subgroup $\widehat{B} \leq \widehat{G}$ such that

$$(X_\star(T), X^*(T), \Delta^\vee, \Delta) = \Psi(\widehat{G}, \widehat{B}, \widehat{T}).$$

We therefore obtain an action of $Gal_F$ on $\Psi(\widehat{G}, \widehat{B}, \widehat{T})$.

Via a choice of pinning of $\widehat{G}$, we obtain a section of the map

$$g \in N_{G(k)}(B) \cap N_{G(k)}(T) = B(k) \cap N_{G(k)}(T) = T(k) \quad (7.9)$$

(see [Mil17, Theorem 17.48 and Proposition 17.53]) and hence $\phi \circ Ad(g)$ and $\phi$ induce the same automorphism of $\Psi(G, B, T)$. The map (7.8) is surjective by Theorem 1.8.3. We claim that the kernel $b$ is Inn($G$), and hence the sequence in the proposition is exact. Suppose $\phi \in Aut(G)$ maps to the identity in $Aut(\Psi(G, B, T))$ under $b$. After replacing $\phi$ by $\phi \circ Ad(g)$ for some $g \in G(k)$, we may assume $\phi$ preserves $B$ and $T$. The fact that $b(\phi)$ is the identity implies that $\phi$ is an inner automorphism by an element of $T(k)$ by Theorem 1.8.3.

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$$\Psi(G_{F^{\text{sep}}}, B, T).$$

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$$(X_\star(T), X^*(T), \Delta^\vee, \Delta)$$

is again a based root datum with associated root datum $\Psi(\widehat{G}, \widehat{T})$, so we have a Borel subgroup $\widehat{B} \leq \widehat{G}$ such that

$$(X_\star(T), X^*(T), \Delta^\vee, \Delta) = \Psi(\widehat{G}, \widehat{B}, \widehat{T}).$$

We therefore obtain an action of $Gal_F$ on $\Psi(\widehat{G}, \widehat{B}, \widehat{T})$.
\[ \text{Aut}(\tilde{G}) \to \text{Aut}(\Psi(\tilde{G}, \tilde{B}, \tilde{T})) \]

and hence a map \( \text{Gal}_F \to \text{Aut}(\tilde{G}) \). We define the **Langlands dual group** of \( G \) or the **L-group** of \( G \) to be the semidirect product

\[ L^G := \tilde{G}(\mathbb{C}) \rtimes \text{Gal}_F \]

with respect to this action. Notice that the action is canonical up to conjugation by \( \tilde{G}(\mathbb{C}) \) by Proposition 7.3.3.

We give \( \tilde{G}(\mathbb{C}) \) the Hausdorff topology and \( L^G \) the product topology. Then the neutral component \((L^G)^o\) of \( L^G \) is \( \tilde{G}(\mathbb{C}) \). Thus any continuous homomorphism \( L^H \to L^G \) maps \( \tilde{H}(\mathbb{C}) \) to \( \tilde{G}(\mathbb{C}) \). A morphism of \( L \)-groups

\[ L^H \to L^G \]

is simply a continuous homomorphism commuting with the projections to \( \text{Gal}_F \) such that its restriction to the neutral components is induced by a map of algebraic groups \( \tilde{H} \to \tilde{G} \).

Morphisms are also sometimes called **\( L \)-maps**. Stipulating that \( \tilde{H} \to \tilde{G} \) is algebraic is equivalent to requiring the map \( \tilde{H}(\mathbb{C}) \to \tilde{G}(\mathbb{C}) \) to be holomorphic [Con14, Proposition D.2.1].

Assume for the moment that \( F \) is a global field. For every place \( v \) upon choosing an embedding \( F^{\text{sep}} \to F_v^{\text{sep}} \), we obtain an injection

\[ \text{Gal}_{F_v} \to \text{Gal}_F \]

unique up to conjugacy. The embedding also induces an identification of the complex dual groups attached to \((G_{F^{\text{sep}}}, T_{F^{\text{sep}}})\) and \((G_{F_v^{\text{sep}}}, T_{F_v^{\text{sep}}})\) as explained after (1.26). This induces a morphism

\[ L^G_{F_v} \to L^G_F \]

which allows us to relate local and global \( L \)-groups.

Occasionally one works with modifications of the \( L \)-group. For example, we can replace \( \text{Gal}_F \) by its quotient \( \text{Gal}(E/F) \) where \( E/F \) is a finite degree Galois extension such that \( G_E \) splits. In some applications, one wants to modify or enlarge \( \text{Gal}_F \) (see [Mok15]). In general we can replace \( \text{Gal}_F \) by any topological group \( \Gamma \) admitting a continuous homomorphism \( \Gamma \to \text{Gal}_F \) that surjects onto \( \text{Gal}(E/F) \) for any field \( E \) such that \( G_E \) is split; all of the constructions above have obvious analogues in this level of generality.

It is useful to give some examples of \( L \)-groups for nonsplit reductive groups.

**Example 7.1.** Let \( G = \text{GL}_n \) over a field \( F \) and let \( E/F \) be a field extension (of finite degree). One has that

\[ L^{\text{Res}_{E/F}}\text{GL}_n \cong \text{GL}_n(\mathbb{C})^{\{\sigma \in E \to F^{\text{sep}}\}} \rtimes \text{Gal}_F \]
where \( \text{Gal}_F \) acts via permuting the factors.

**Example 7.2.** Let

\[
J_n := \begin{pmatrix}
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1
\end{pmatrix} \in \text{GL}_n(\mathbb{Z})
\]

be the antidiagonal matrix and let

\[
J'_{2n} := \begin{pmatrix}
-J_n & J_n \\
J_n & -J_n
\end{pmatrix},
J'_{2n+1} := \begin{pmatrix}
1 & J_n \\
-J_n & 1
\end{pmatrix}.
\]

Let \( M/F \) be a quadratic extension and let \( \sigma \in \text{Gal}(M/F) \) be the nontrivial element. For an \( F \)-algebra \( R \), consider the quasi-split unitary group attached to the extension \( M/F \):

\[
U_n(R) := \{ g \in \text{GL}_n(M \otimes_F R) : J'_{n-1} \sigma(g)^{-1} J'_n = g \}.
\]

One has that

\[
^L U_n \cong \text{GL}_n(\mathbb{C}) \rtimes \text{Gal}_F
\]

(7.14)

where \( \text{Gal}_F \) acts via its quotient \( \text{Gal}(M/F) \). The nontrivial element \( \sigma \in \text{Gal}(M/F) \) is sent to the isomorphism \( g \mapsto J'_{n-1} g^{-1} J'_{n} \) of \( \text{GL}_n(\mathbb{C}) \). There is an \( L \)-map

\[
^L U_n \to ^L \text{Res}_{M/F} \text{GL}_n
\]

\[
g \rtimes \sigma \mapsto (g, J'_{n-1} g^{-1} J'_{n}) \rtimes \sigma.
\]

### 7.4 Parabolic subgroups of \( L \)-groups

Let \( G \) be a reductive group over a local or global field \( F \). In Chapter 12, we will require the notion of a parabolic subgroup of an \( L \)-group. It is most natural to discuss this immediately after discussing the \( L \)-group, but the reader should feel free to skip this section and then refer back to it as needed.

We follow the discussion in [Bor79]. A **parabolic subgroup** of \( ^L G \) is a closed subgroup \( Q \) such that \( Q \cap \hat{G}(\mathbb{C}) \) is the complex points of a parabolic subgroup of \( \hat{G} \) and the restriction of the canonical map \( ^L G \to \text{Gal}_F \) to \( Q \) is surjective. Clearly

\[
^L B := \hat{B}(\mathbb{C}) \rtimes \text{Gal}_F
\]

(7.15)
is a parabolic subgroup, where $\hat{B}$ is the group of (7.10). A parabolic subgroup of $L^G$ is standard if it contains $L^B$. As in the case of algebraic groups, every parabolic subgroup of $L^G$ is conjugate under $\hat{G}(\mathbb{C})$ to one and only one standard parabolic subgroup.

We now isolate the subset of relevant parabolic subgroups of $L^G$. This will allow us to define a dual parabolic subgroup $L^P$ to any proper parabolic subgroup $P \leq G$. We will also explain how this duality behaves with respect to Levi subgroups.

For every field extension $F \leq E \leq F^{sep}$, we let $\mathcal{P}(E)$ denote the set of parabolic subgroups of $G_E$. Let $(\mathcal{P}/G)(F^{sep})$ denote the set of $G(F^{sep})$-conjugacy classes of parabolic subgroups in $G(F^{sep})$ and let

$$(\mathcal{P}/G)(E) = (\mathcal{P}/G)(F^{sep})^{Gal(F^{sep}/E)}.$$ 

This is the set of conjugacy classes that are fixed by $Gal(F^{sep}/E)$, or in other words are defined over $E$. There is a natural map

$$\mathcal{P}(E) \rightarrow (\mathcal{P}/G)(E).$$

Let $B \leq G_{F^{sep}}$ be a Borel subgroup and $T \leq B$ be a maximal torus. We then obtain a based root datum

$$\Psi(G_{F^{sep}}, B, T) = (X^*(T), X_*(T), \Delta, \Delta^\vee).$$

In §7.3 we explained how to endow the based root datum with an action of $Gal_{F^{sep}}$. One has a bijection

$$\{ J \subset \Delta \} \leftrightarrow (\mathcal{P}/G)(F^{sep})$$

by Theorem 1.9.2 and the discussion preceding it. The bijection is $Gal_{F^{sep}}$-equivariant.

**Lemma 7.4.1** If $G$ is quasi-split then the map $\mathcal{P}(F) \rightarrow (\mathcal{P}/G)(F)$ is surjective.

**Proof.** Let $B_0 \leq G$ be a Borel subgroup. We can and do assume $B = B_0_{F^{sep}}$, which implies in particular that $B$ is fixed by $Gal_F$. Let $P \leq G_{F^{sep}}$ be a parabolic subgroup whose conjugacy class is $Gal_F$-fixed. Upon replacing $P$ by a $G(F^{sep})$-conjugate, we can and do assume $P \geq B$. Now for all $\sigma \in Gal_F$ we have $P \cap \sigma(P) \geq B$ and $\sigma(P) = g^{-1}Pg$ for some $g \in G(F^{sep})$. Since every parabolic subgroup is conjugate to a unique standard parabolic subgroup, we conclude that $P = \sigma(P)$, and hence $P$ is defined over $F$, that is, $P = P'_{F^{sep}}$ for some parabolic subgroup $P' \leq G$.

Similarly, let $L^P$ denote the set of parabolic subgroups of $L^G$ and let $L^P/L^G$ denote the set of $L^G$-conjugacy classes of parabolic subgroups of $L^G$. Again by Theorem 1.9.2 and the discussion preceding it, we have a bijection
Here we are using the fact that a standard parabolic subgroup of \( L^\Delta \) (with respect to \( L B = \hat{B}(\mathbb{C}) \rtimes \text{Gal}_F \)) is of the form \( P(\mathbb{C}) \rtimes \text{Gal}_F \) where \( P \leq \hat{G} \) is a parabolic subgroup containing \( \hat{B} \) whose corresponding subset of \( \Delta^\vee \) is fixed by \( \text{Gal}_F \). Combining (7.16) with (7.17) yields a bijection

\[
(\mathcal{P}/G)(F) \leftrightarrow L^\mathcal{P}/L^G.
\]

We say that a parabolic subgroup of \( L^G \) is relevant if its \( L^G \)-conjugacy class is in the image of the composite

\[
\mathcal{P}(F) \rightarrow (\mathcal{P}/G)(F) \rightarrow L^\mathcal{P}/L^G.
\]

If \( P \in \mathcal{P}(F) \) we denote by \( L^P \) the standard parabolic subgroup in the class that is the image of \( P \) under the map (7.18).

Let \( Q \) be a parabolic subgroup of \( L^G \). The unipotent radical \( N \) of \( Q^\circ \) is normal in \( Q \) and will also be called the unipotent radical of \( Q \). Then \( Q \) is the semidirect product of \( N \) by the normalizer in \( Q \) of any Levi subgroup \( M^\circ \) of \( Q^\circ \). These normalizers will be called the Levi subgroups of \( Q \). A Levi subgroup of a parabolic subgroup \( L^P \) of \( L^G \) is relevant if \( L^P \) is.

For given parabolic subgroup \( P \leq G \), a unique \( G(\mathbb{F}^{\text{sep}}) \)-conjugate \( g_{P^{\text{sep}}} g^{-1} \) of \( P_{\mathbb{F}^{\text{sep}}} \) contains \( B \). Since the parabolic subgroups of \( G_{\mathbb{F}^{\text{sep}}} \) containing \( B \) are in bijection with subsets of \( \Delta \) as explained in Theorem 1.9.2, we can therefore associate a subset \( J(P) \subseteq \Delta \) to \( P \). Given a Levi subgroup \( M \leq P \), there is a \( g' \in G(\mathbb{F}^{\text{sep}}) \) such that \( g'Mg'^{-1} \) has based root datum

\[
(X^*(T), X_*(T), J(P), J(P)^\vee)
\]

and hence there is a Levi subgroup of \((L^P)^\circ \) with based root datum

\[
(X_*(T), X^*(T), J(P)^\vee, J(P)).
\]

The semidirect product of this Levi subgroup of \((L^P)^\circ \) with \( \text{Gal}(\mathbb{F}^{\text{sep}}/\mathbb{F}) \) is then a Levi subgroup of \( L^P \) denoted by \( L^M \). This is reasonable as \( L^M \) is the dual group of \( M \).

### 7.5 The Satake isomorphism for unramified groups

Let \( G \) be an unramified reductive group over a local field \( F \) and let \( K \leq G(F) \) be a hyperspecial subgroup. We wish to describe Langlands' extension of the Satake isomorphism to this setting. As in §12.1 below, we let \( \text{Fr} \in \text{Gal}(\mathbb{F}^{\text{sep}}/\mathbb{F}) \) be a choice of geometric Frobenius element. Since \( G \) is unramified we can take the \( L \)-group to be
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\[ \hat{G}(\mathbb{C}) \times \text{Fr}^Z. \]

In the split case, the Satake transform was phrased in terms of the quotient

\[ (\hat{G}/\text{conj})(\mathbb{C}). \]  \hfill (7.19)

In the nonsplit case, we replace \( \hat{G} \) by

\[ \hat{G} \times \text{Fr} \subset \hat{G} \times \text{Fr}^Z, \]

the coset of \( \hat{G} \times 1 \) in \( \hat{G} \times \text{Fr}^Z \) containing \( I \times \text{Fr} \) (\( I \) being the identity of \( \hat{G}(\mathbb{C}) \)). We replace the action of \( \hat{G} \) on itself by conjugation with the action given on points in a \( \mathbb{C} \)-algebra \( R \) by

\[
\begin{align*}
\hat{G}(R) \times (\hat{G}(R) \times \text{Fr}) & \longrightarrow \hat{G}(R) \times \text{Fr} \\
(g, (h \times \text{Fr})) & \longmapsto gh\text{Fr}(g)^{-1} \times \text{Fr}.
\end{align*}
\]  \hfill (7.20)

This action can also be described as the action of \( \hat{G} \) on itself via \( \text{Fr} \)-conjugacy, a notion that plays a key role in the twisted trace formula (see §18.5 below, where the role of \( \text{Fr} \) is played by \( \theta \)). Thus the appropriate analogue of (7.19) in the nonsplit case is

\[ (\hat{G} \times \text{Fr}/\sim)(\mathbb{C}) \]

where \( \sim \) denotes the action of (7.20).

Let \( T \leq G \) be a maximal torus contained in a Borel subgroup \( B \). Its dual \( \hat{T} \) is a maximal torus of \( \hat{G} \). We have a pair of homomorphisms

\[ N_{\hat{G}}(\hat{T})(\mathbb{C}) \longrightarrow W(\hat{G}, \hat{T})(\mathbb{C}) \longrightarrow W(G, T)(F_{\text{sep}}) \]  \hfill (7.21)

where the right isoismorphism is (1.25). We let

\[ F^N \]  \hfill (7.22)

be the constant group scheme over \( \mathbb{C} \) whose \( \mathbb{C} \)-points are the inverse image of \( W(G, T)(F) \leq W(G, T)(F_{\text{sep}}) \) in \( N_{\hat{G}}(\hat{T})(\mathbb{C}) \). One has an isomorphism of affine schemes

\[ \hat{T} \times \text{Fr}/F^N \sim \hat{G} \times \text{Fr}/\sim \]  \hfill (7.23)

(see [Bor79, Lemma 6.5]). This is the analogue of the Chevalley restriction theorem (Theorem 17.2.3) in this setting. With this isomorphism in mind the following theorem is natural:

**Theorem 7.5.1 (Langlands and Satake)** Assume that \( G \) is unramified over \( F \). There is an isomorphism of algebras

\[ S : C_c^\infty(G(F) \parallel K) \longrightarrow \mathcal{O}(\hat{T} \times \text{Fr})^{F^N}. \]
As in the case where \( G \) is split, we can rephrase this as an isomorphism

\[
\text{Hom}(C_1 \otimes (G(F) \backslash K), \mathbb{C}) \rightarrow (\hat{T} \rtimes \text{Fr}/F N)(\mathbb{C}) \rightarrow (\hat{G} \rtimes \text{Fr}/\sim)(\mathbb{C}). \tag{7.24}
\]

As explained in §17.1, one can identify points of a GIT quotient such as \( \hat{G} \rtimes \text{Fr}/\sim \) with closed orbits in \( \hat{G} \rtimes \text{Fr} \). An element of \( \hat{G}(\mathbb{C}) \rtimes \text{Fr} \) is said to be Fr-semisimple if its \( \hat{G} \)-orbit is closed. In view of (7.23), this is equivalent to the assertion that its orbit intersects \( \hat{T} \rtimes \text{Fr} \). Thus we obtain the following analogue of Corollary 7.2.2:

**Corollary 7.5.2** Assume that \( G \) is unramified over \( F \). The composite isomorphism (7.24) induces a bijection

\[
(\hat{G} \rtimes \text{Fr}/\sim)(\mathbb{C}) \leftrightarrow \left\{ \text{isomorphism classes of irreducible unramified representations of } G(F) \right\}.
\]

By way of terminology, the Fr-semisimple conjugacy class attached to an isomorphism class of unramified representations is called its Langlands class. In the split case, the eigenvalues of a representative of the conjugacy class are called its Satake parameters.

We will not prove either Theorem 7.2.1 or Theorem 7.5.1, though we will say more about the definition of the map \( S \) in a moment. The standard references are [Car79, Theorem 4.1], [Bor79, §7], and [Sat63] which we follow. There is a categorified version of the Satake correspondence due to Mirković and Vilonen that is known as the geometric Satake correspondence (see [MV07]). This result is an important tool in the geometric Langlands program. Moreover it has now been extended to mixed characteristic by work of Zhu [Zhu17] and Bhattacharyya-Scholze [BS17].

Let \( B \subseteq G \) be a Borel subgroup containing a maximal torus \( T \) of \( G \) and let \( K \) be a hyperspecial subgroup. We let \( N \) denote the unipotent radical of \( B \). We say that \( K \) is in good position with respect to \( (B, T) \) if the Iwasawa decomposition

\[
G(F) = B(F)K
\]

holds and

\[
B(F) \cap K = (T(F) \cap K)(N(F) \cap K).
\]

We also can and do assume that \( K_T := T(F) \cap K \) is a maximal compact subgroup of \( T \). In fact, given \( B \) and \( T \) we can always choose \( K \) so that these assumptions hold (see Theorem A.1.1). Conversely, if we are given a hyperspecial subgroup, we can always choose \( B \) and \( T \) so that \( K \) is in good position with respect to \( (B, T) \) (see Lemma A.5.2).
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Using the Iwasawa decomposition \( G(F) = B(F)K = T(F)N(F)K \), we have a decomposition of measures

\[
dg = ddbk = dtndk
\]

(see Proposition 3.2.1); we always assume that \( dk(K) = 1 \). Here we are using the fact that \( N(F) \) is unimodular (see Exercise 3.12). We define a \( \mathbb{C} \)-linear map

\[
C_c^\infty(G(F) \parallel K) \longrightarrow C_c^\infty(T(F)/K_T)
\]

\[
f \mapsto f^B
\]

where

\[
f^B(t) := \delta_B^{1/2}(t) \int_{N(F)} f(tn)dn.
\]

The function \( f^B \) is known as the constant term of \( f \) along \( B \). Note that \( C_c^\infty(T(F)/K_T) \) in (7.26) above has only a single slash since \( T(F) \) is commutative. Though we do not yet need this, we assume that measures are normalized so that (7.25) holds. This implies that the formula in Proposition 8.6.1 holds.

The following lemma is implied by Lemma 8.6.2 and Proposition 8.7.2 below:

**Lemma 7.5.3** The map \( f \mapsto f^B \) is an algebra homomorphism and has image in \( C_c^\infty(T(F)/K_T)^{W(G,T)(F)} \).

Let \( T_s \leq T \) be the maximal \( F \)-split torus in \( T \). Thus we can write \( T = T_sT_a \) where \( T_a \) is anisotropic and \( T_s \cap T_a \) is finite.

**Lemma 7.5.4** We have that

\[
W(G, T)(F) = W(G, T_s)(F) = W(G, T_s)(F^{\text{sep}}).
\]

**Proof.** The torus \( T_s \) is a maximal split torus of \( G \) [Bor91, Proposition 20.6(iii)]. Thus the right equality is a special case of Lemma 1.7.3.

Now \( C_G(T_s) = T = C_G(T) \) by [Bor91, Proposition 20.6(i)] and [Mil17, Proposition 17.61]. Thus Lemma 1.7.3 implies that

\[
W(G, T_s)(F) = N_G(T_s)(F)/T(F) = N_G(T_s)(F^{\text{sep}})/T(F^{\text{sep}})
\]

and by Proposition 17.1.8 we have that

\[
W(G, T)(F) = \{g \in N_G(T)(F^{\text{sep}})/T(F^{\text{sep}}) : g\xi(g^{-1}) \in T(F^{\text{sep}}) \text{ for all } \xi \in \text{Gal}_F \}.
\]
Any element \( g \in W(G, T)(F) \) normalizes \( T(F_{\text{sep}}) \) and induces an isomorphism \( T \to T \) (over \( F \)), hence necessarily normalizes \( T_s \). Thus we obtain an injection \( W(G, T)(F) \hookrightarrow W(G, T_s)(F) \). Suppose that \( g \in G(F) \) normalizes \( T_s \). Then \( g^{-1}Tg \) is a torus containing \( T_s \) which is therefore contained in \( C_G(T_s) = T \). In other words, any element of \( G(F) \) normalizing \( T_s \) normalizes \( T \). It follows that the map \( W(G, T)(F) \to W(G, T_s)(F) \) is surjective. \( \square \)

In view of Lemma 7.5.4, we have an injection
\[
W(G, T_s)(F) = W(G, T)(F) \hookrightarrow W(G, T)(F_{\text{sep}}) = W(\widehat{G}, \widehat{T})(\mathbb{C}).
\]
Let
\[
\widehat{W} \leq W(\widehat{G}, \widehat{T}) \tag{7.29}
\]
denote the constant subgroup scheme whose \( \mathbb{C} \)-points are the image of \( W(G, T_s)(F) \) under the injection above. Thus \( \widehat{W} \) acts on \( \widehat{T} \).

**Lemma 7.5.5** The action of \( \widehat{W} \) on \( \widehat{T} \) preserves \( \widehat{T}_s \). There is an isomorphism
\[
C^\infty_c(T(F)/K_T)^{W(G, T)(F)} \longrightarrow \mathcal{O}(\widehat{T}_s)^{\widehat{W}}.
\]

We refer to §17.1 for the precise meaning of the expression \( \mathcal{O}(\widehat{T}_s)^{\widehat{W}} \). The concrete meaning in the case at hand will be apparent from the proof of Lemma 7.5.5.

**Proof.** Let \( K_s = T_s(F) \cap K \). It is a maximal compact subgroup of \( T_s(F) \). We note that \( T_s(F) \) is compact, so via restriction we obtain an isomorphism
\[
C^\infty_c(T(F)/K_T) \longrightarrow C^\infty_c(T_s(F)/K_s).
\]
The group \( W(G, T)(F) \) preserves \( T_s \) as explained in the proof of Lemma 7.5.4 and it is clear that the isomorphism above is \( W(G, T)(F) \)-equivariant. Hence by Lemma 7.5.4 we obtain an isomorphism
\[
C^\infty_c(T(F)/K_T)^{W(G, T)(F)} \longrightarrow C^\infty_c(T_s(F)/K_s)^{W(G, T_s)(F)}.
\]
There is a \( W(G, T_s)(F) \)-equivariant isomorphism of \( \mathbb{C} \)-algebras
\[
C^\infty_c(T_s(F)/K_s) \longrightarrow \text{Hom}(X^*(T_s), \mathbb{C})
\]
\[
\mathbb{1}_{K_s} \mapsto (\chi \mapsto \text{ord}_v(\chi(t))).
\]
For a free \( \mathbb{Z} \)-module \( M \), let \( \mathbb{C}[M] \) denote the group algebra. As a \( \mathbb{C} \)-vector space, it is just \( \mathbb{C} \otimes_{\mathbb{Z}} M \). Thus we have a composite isomorphism of \( \mathbb{C} \)-vector spaces
\[
\text{Hom}(X^*(T_s), \mathbb{C}) \longrightarrow \mathbb{C}[X_s(T_s)] \longrightarrow \mathbb{C}[X^*(\widehat{T}_s)] \tag{7.30}
\]
where the first arrow is the canonical isomorphism and the second is given as part of the definition of $\hat{T}_s$. In fact, these are isomorphisms of $\mathbb{C}$-algebras. The first is $W(G,T_\kappa(T))(F)$-equivariant. Since $T_\kappa$ is split, we have $X_\kappa(T_\kappa) = X_\kappa(T_{\text{sep}})$. Thus the isomorphism $\mathbb{C}[X_\kappa(T_\kappa)] \cong \mathbb{C}[X_\kappa(T_{\text{sep}})]$ in (7.30) is the restriction of the $W(G,T)(F_{\text{sep}}) = W(\hat{G},\hat{T})(\mathbb{C})$-equivariant isomorphism

$$
\mathbb{C}[X_\kappa(T_{\text{sep}})] \longrightarrow \mathbb{C}[X_\kappa(T)].
$$

It follows that $\mathbb{C}[X_\kappa(T_{\text{sep}})]$ is preserved by $\mathcal{W}(\mathcal{C})$ and (7.30) is $\mathcal{W}(\mathbb{C})$-equivariant, where $\mathcal{W}(\mathbb{C})$ acts on $\mathbb{C}[X_\kappa(T_\kappa)]$ via the isomorphism $W(G,T)(F) \rightarrow \mathcal{W}(\mathbb{C})$.

By Theorem 1.7.1 we have a commutative diagram

$$
\begin{array}{ccc}
\mathbb{C}[X_\kappa(T_{\text{sep}})] & \cong & \mathcal{O}(\hat{T}_s) \\
\downarrow & & \downarrow \\
\mathbb{C}[X_\kappa(T)] & \cong & \mathcal{O}(\hat{T}).
\end{array}
$$

The bottom row is $W(\hat{G},\hat{T})(\mathbb{C})$-equivariant (where $W(\hat{G},\hat{T})(\mathbb{C})$ acts on $\mathcal{O}(\hat{T})$ as explained above (17.9)) and since $\mathbb{C}[X(T)]$ is preserved by the subgroup $\mathcal{W}(\mathbb{C})$, it follows that $\mathcal{O}(\hat{T}_s)$ is as well. Thus by (17.9) we obtain an isomorphism

$$
\mathbb{C}[X_\kappa(T_{\text{sep}})]^{\mathcal{W}(\mathbb{C})} \cong \mathcal{O}(\hat{T}_s)^{\mathcal{W}(\mathbb{C})}.
$$

\[\square\]

**Lemma 7.5.6** The inclusion $T_\kappa \rightarrow T$ induces an isomorphism

$$
\hat{T} \times \text{Fr}/ \sim \rightarrow \hat{T}_s/\mathcal{W}
$$

where on the left the quotient is modulo the conjugation action of $\mathcal{F}N$.

**Proof.** Let $\nu : \hat{T} \rightarrow \hat{T}_s$ be the map induced by the inclusion $T_\kappa \rightarrow T$ via duality. Let $R$ be a $\mathbb{C}$-algebra. One has that

$$
(\ker \nu)(R) = \{ u^{-1} \text{Fr}(u) : u \in \hat{T}(R) \}.
$$

(7.31)

We have a map

$$
\nu : (\hat{T} \times \text{Fr})(R) \longrightarrow \hat{T}_s(R)
$$

$$(t \times \text{Fr}) \longmapsto \nu(t).$$

It is surjective. By [Bor79, Lemma 6.2], every element of $\mathcal{F}N(R)$ is of the form $wu$ where $\text{Fr}(w) = w$ and $u \in \hat{T}(R)$. Thus

$$
(wu)^{-1}(t \times \text{Fr})wu = u^{-1} \text{Fr}(u) w^{-1} tw \times \text{Fr}
$$

Lemma 7.5.6
for all \( t \rtimes \text{Fr} \in \hat{T}(R) \rtimes \text{Fr} \). This implies \( \nu : \hat{T} \rtimes \text{Fr} \to \hat{T}_s \) is equivariant with respect to the action of \( fN\), where \( fN \) acts via its quotient \( \hat{W} \) on \( \hat{T}_s \). In particular \( \nu \) induces a morphism

\[
\hat{T} \rtimes \text{Fr} / \sim \to \hat{T}_s / \hat{W}
\]  
(7.32)

that is surjective on \( R \)-points for all \( \mathbb{C} \)-algebras \( R \). To prove injectivity on \( R \)-points for all \( \mathbb{C} \)-algebras \( R \) (and hence prove (7.32) is an isomorphism), assume that \( t \rtimes \text{Fr} \in \hat{T}(R) \) and

\[
\nu(t \rtimes \text{Fr}) = w^{-1}\nu(t' \rtimes \text{Fr})w
\]

for some \( w \in \hat{W}(R) \). Then \( \nu(t) = \nu(w^{-1}t'w) \) and \( t = xw^{-1}t'w \) for some \( x \in (\ker \nu)(R) \). By (7.31), all \( x \in (\ker \nu)(R) \) are of the form \( u^{-1}\text{Fr}(u) \), and we deduce that \( t \rtimes \text{Fr} = (wu)^{-1}(t' \rtimes \text{Fr})wu \). \( \square \)

Combining lemmas 7.5.3, 7.5.5 and the map of coordinate rings induced by Lemma 7.5.6, we see that we have constructed the Satake morphism

\[
S : C_c^\infty(G(F) \parallel K) \to C_c^\infty(T(F)/K_T)^W(G,T)(F) \to \mathcal{O}(\hat{T} \rtimes \text{Fr})^{fN}
\]  
(7.33)

though we have not proved that it is injective or surjective. The first map in this factorization of the Satake isomorphism is a special case of a parabolic descent map. In general, any construction that relates objects on the group \( G \) to objects on Levi subgroups of its parabolic subgroups (in this case the Levi subgroup \( T \) of the Borel subgroup \( B \)) is known as parabolic descent. We will use this idea in the next section to give a more explicit parametrization of unramified representations.

### 7.6 The principal series

We now explain how to explicitly realize unramified representations. Let \( G \) be an unramified reductive group over a nonarchimedean local field \( F \) with Borel subgroup \( B \subseteq G \) and maximal torus \( T \subseteq B \). Let \( K \subseteq G(F) \) be a hyperspecial subgroup. We assume that \( B \) and \( T \) are chosen so that the Iwasawa decomposition

\[
G(F) = B(F)K
\]

holds and that \( K_T := T(F) \cap K \) is a maximal compact open subgroup of \( T(F) \); it is always possible to arrange this by Lemma A.5.2. Let \( N \subseteq B \) be the unipotent radical of \( B \).

We recall from Proposition 3.6.1 that the modular quasi-character

\[
\delta_B := \delta_{B(F)} : B(F) \to \mathbb{R}_{>0}
\]  
(7.34)
is
\[ \delta_B(b) := |\det(\text{Ad}(b) : b \to b)|, \]
where \( b := \text{Lie } B \). The last ingredient we need to define unramified principal series representations is unramified quasi-characters.

**Definition 7.3.** A quasi-character \( \chi : T(F) \to \mathbb{C}^\times \) is **unramified** if it is trivial on \( K_T \).

Let us explicitly describe unramified quasi-characters. As in the previous section, we let \( T_s \leq T \) be the maximal split torus and choose an anisotropic torus \( T_a \leq T \) such that \( T_a T_s = T \) and \( T_a \cap T_s \) is finite. We remind the reader that \( X^*(T) \) is the abelian group of homomorphisms \( T \to \mathbb{G}_m \); this is in general a proper subgroup of the group \( X^*(T_{F_{\text{sep}}}) \) of homomorphisms \( T_{F_{\text{sep}}} \to \mathbb{G}_m_{F_{\text{sep}}} \).

In analogy with (4.18) we define a map
\[ H_T : T(F) \to \mathfrak{a}_T := \text{Hom}(X^*(T), \mathbb{R}) \]
via
\[ e^{(H_T(t), \lambda)} = |\chi(t)| \]
for \( \chi \in X^*(T) \). For each
\[ \lambda \in \mathfrak{a}_T^* := X^*(T) \otimes \mathbb{Z} \mathbb{C}, \]
we then obtain a quasi-character
\[ t \mapsto e^{(H_T(t), \lambda)}. \]
It is clearly unramified. We refine this description of unramified quasi-characters slightly as follows. The group \( \hat{W}(\mathbb{C}) \) of (7.29) acts on the set \( \hat{T}_s(\mathbb{C}) \) by Lemma 7.5.5. Lemma 7.5.4 provides an isomorphism
\[ W(G, T)(F) \cong \hat{W}(\mathbb{C}) \]
and hence \( \hat{W}(\mathbb{C}) \) also acts on the set \( \text{Hom}(T(F)/K_T, \mathbb{C}^\times) \).

**Lemma 7.6.1** One has a \( \hat{W}(\mathbb{C}) \)-equivariant bijection
\[ \hat{T}_s(\mathbb{C}) \to \text{Hom}(T(F)/K_T, \mathbb{C}^\times) \]
\[ q^{-\lambda} \mapsto (t \mapsto e^{(H_T(t), \lambda)}). \]
Here, as usual, \( q \) denotes the order of the residue field of \( \mathcal{O}_F \).

**Proof.** As in the proof of Lemma 7.5.5, the map \( T_s \to T \) induces a \( W(G, T)(F) \)-equivariant isomorphism of topological groups
Unramified Representations

\[ T_s(F)/K_s \rightarrow T(F)/K_T, \]

where \( K_s := T_s(F) \cap K \) is a maximal compact subgroup of \( T_s(F) \).

We now use the fact that \( W(G, T)(F) = W(G, T_s)(F) \) (see Lemma 7.5.4). The map

\[ \hat{T}_s(\mathbb{C}) \rightarrow \text{Hom}(T_s(F)/K_s, \mathbb{C}^\times) \]
\[ q^{-\lambda} \mapsto (t \mapsto e^{(H_T(t), \lambda)}) \]

is \( W(G, T_s)(F) \)-equivariant, so we need only check it is an isomorphism. By choosing an isomorphism \( T_s \cong \mathbb{G}_m^d \), we are reduced to observing that

\[ \mathbb{C}^\times \rightarrow \text{Hom}(F^\times / O_F^\times, \mathbb{C}^\times) \]
\[ q^{-\lambda} \mapsto (t \mapsto e^{\lambda \log |t|}) \]

is a bijection. Here we represent an element of \( \mathbb{C}^\times \) as \( q^{-\lambda} \) for some \( \lambda \in \mathbb{C} \). \( \square \)

For \( \lambda \in \mathfrak{a}_{T_C}^* \) we define the representation

\[ I(\lambda) := \text{Ind}_B^G(t \mapsto e^{(H_T(t), \lambda)}) \quad (7.35) \]

to be the (smooth) representation of \( G(F) \) on the space of functions

\[ \left\{ \varphi \in C^\infty(G(F)) : \varphi(tng) = \delta_B^{1/2}(t)e^{(H_T(t), \lambda)}\varphi(g) \right\} \quad (7.36) \]

for all \( t \in T(F), n \in N(F) \) and \( g \in G(F) \). Here \( G(F) \) acts via right translation:

\[ I(\lambda)(g)\varphi(x) := \varphi(xg). \quad (7.37) \]

This is an example of an induced representation (see §8.2 for the general construction). The factor of \( \delta_B^{1/2} \) is present so that if

\[ \lambda \in \mathfrak{a}_T^* = i(X^*(T) \otimes \mathbb{Z} \mathbb{R}) \]

then \( I(\lambda) \) is pre-unitarizable; this assertion is part of the following special case of propositions 8.2.2 and 8.2.3 below:

**Proposition 7.6.2** The representations \( I(\lambda) \) are admissible and satisfy

\[ I(\lambda)^\vee \cong I(-\lambda). \]

If \( \lambda \in \mathfrak{a}_T^* \) then \( I(\lambda) \) is pre-unitarizable. \( \square \)

We note that the condition \( \lambda \in \mathfrak{a}_T^* \) is equivalent to the statement that the quasi-character \( t \mapsto e^{(H_T(t), \lambda)} \) is unitary (i.e. it is a character).
Definition 7.4. An unramified principal series representation of $G(F)$ is a representation isomorphic to $I(\lambda)$ for some $\lambda \in \mathfrak{a}_T^\ast$.  

This definition is traditional, but it is a little inconsistent. We have assumed unramified representations are irreducible (see Definition 7.1), but unramified principal series representations need not be irreducible.

Lemma 7.6.3 There is a unique line in $I(\lambda)$ fixed by $K$.

Proof. Since $G(F) = B(F)K$, a function in $I(\lambda)$ is uniquely determined by its restriction to $K$. $\square$

Our next aim is to explain how the Weyl group acts on unramified principal series representations. First, we define semisimplifications of smooth representations. A smooth representation $(\pi, V)$ of $G(F)$ has finite length if there is a chain of $C_c(G(F))$-submodules

$$0 = V_0 < V_1 < \cdots < V_{n-1} < V_n$$

(7.38)

(with $n \in \mathbb{Z}_{>0}$) such that $V_i/V_{i-1}$ is irreducible for $0 < i \leq n$. As usual, the chain (7.38) is known as a composition series of $V$. If $\pi$ has finite length, its semisimplification is the representation

$$\bigoplus_{i=1}^n V_i/V_{i-1}.$$  

Up to isomorphism, the semisimplification depends only on the isomorphism type of $V$ by the Jordan-Hölder theorem. An irreducible smooth representation of $G(F)$ is a subquotient of $(\pi, V)$ if and only if it is isomorphic to $V_i/V_{i-1}$ for some $i$.

The Weyl group $W(G, T)(F)$ acts on $X^\ast(T)$, hence on $\mathfrak{a}_T^\ast$. It therefore makes sense to talk of $I(w(\lambda))$ for $\lambda \in \mathfrak{a}_T^\ast$ and $w \in W(G, T)(F)$. We say that $\lambda \in \mathfrak{a}_T^\ast$ is regular if $w(\lambda) = \lambda$ for $w \in W(G, T)(F)$ implies that $w$ is the identity. Thus $\lambda$ is regular if and only if $\lambda$ lies in an open Weyl chamber in $\mathfrak{a}_T^\ast$.

Theorem 7.6.4 Let $\lambda \in \mathfrak{a}_T^\ast$ and $w \in W(G, T)(F)$.

(a) The representation $I(\lambda)$ is of finite length.

(b) The semisimplifications of $I(\lambda)$ and $I(w(\lambda))$ are isomorphic.

(c) If $\lambda \in i\mathfrak{a}_T^\ast$ is regular then $I(\lambda)$ is irreducible. $\square$

For the proof of the theorem, one can refer to [Car79, Theorem 3.3]. These results are due to Casselman.

Lemma 7.6.5 The representation $I(\lambda)$ admits a unique (irreducible) unramified subquotient.
Proof. By Theorem 7.6.4 there exists a composition series \(0 = V_0 < \cdots < V_n = I(\lambda)\). There is a unique line of \(K\)-fixed vectors in \(I(\lambda)\) by Lemma 7.6.3. Let \(1 \leq i \leq n\) be the smallest index such that \(V_i\) contains this line. Then \(V_i/V_i^{-1}\) is unramified.

To prove that \(i\) is unique index such that \(V_i/V_i^{-1}\) is unramified, it suffices to observe that the map \(V_j^K \to (V_j/V_j^{-1})^K\) is surjective for all \(1 \leq j \leq n\) by Exercise 7.5.

We will also require the following result [Cas, Corollary 6.3.9]:

**Theorem 7.6.6** If \(\pi\) is an irreducible subquotient of \(I(\lambda)\) then there exists a \(w \in W(G,T)(F)\) such that \(\pi\) is a subrepresentation of \(I(w(\lambda))\).

Let \(J(\lambda)\) denote the unique (irreducible) unramified subquotient of \(I(\lambda)\). By Theorem 7.6.4, \(J(\lambda) \cong J(w(\lambda))\) for all \(w \in W(T,G)(F)\). This is consonant with (7.23), Corollary 7.5.2, and Lemma 7.5.6, which together yield a bijection

\[
\hat{T}_w(\mathbb{C})/\hat{W}(\mathbb{C}) \leftrightarrow \left\{ \text{isomorphism classes of irreducible unramified representations of } G(F) \right\}. \tag{7.39}
\]

**Theorem 7.6.7** The bijection (7.39) is given by \(q^{-\lambda} \mapsto J(\lambda)\). In particular, every unramified representation of \(G(F)\) is isomorphic to a \(J(\lambda)\) for some \(\lambda \in \mathfrak{a}_T^*\). Moreover \(J(\lambda) \cong J(\lambda')\) if and only if \(\lambda = \lambda'\).

This theorem implies in particular that the image of \(q^{-\lambda}\) in \((\hat{G} \times \text{Fr}/\sim)(\mathbb{C})\) is the Langlands class of \(J(\lambda)\).

The bijection (7.39) was constructed by identifying \(\hat{T}_w(\mathbb{C})/\hat{W}(\mathbb{C})\) with characters of \(C_\infty(C(G(F)//K)\), which in turn were identified with unramified representations via Proposition 7.1.1. Thus Theorem 7.6.7 is a consequence of the following proposition:

**Proposition 7.6.8** Let \(f \in C_\infty(C(G(F)//K)\). For \(\lambda \in \mathfrak{a}_T^*\) one has that

\[
\text{tr } J(\lambda)(f) = S(f)(q^{-\lambda}).
\]

**Proof.** One has that

\[
\text{tr } I(\lambda)(f) = \text{tr } e^{(H_T(\cdot),\lambda)}(f^B)
\]

by Proposition 8.6.1 and by unwinding the definition of the Satake isomorphism, one obtains that

\[
\text{tr } e^{(H_T(\cdot),\lambda)}(f^B) = S(f)(q^{-\lambda}).
\]

Thus to complete the proof, it suffices to show that \(\text{tr } I(\lambda)(f) = \text{tr } J(\lambda)(f)\). By Theorem 7.6.6 upon replacing \(\lambda\) by \(w\lambda\) for some \(w \in W(G,T)(F)\) we can assume that \(J(\lambda)\) is a subrepresentation of \(I(\lambda)\).
There is a unique line $C_{\varphi_0}$ in the space of $I(\lambda)$ fixed by $K$ by Lemma 7.6.3. It follows that $I(\lambda)(f)$ acts via the scalar $tr I(\lambda)(f)$ on $C_{\varphi_0}$. On the other hand, there is an equivariant map

$$J(\lambda) \rightarrow I(\lambda). \quad (7.40)$$

Since $J(\lambda)$, being unramified, has a unique spherical line, its image under (7.40) must be $C_{\varphi_0}$.

## 7.7 Weak global $L$-packets

Let $G$ be a reductive group over a global field $F$. When we say an admissible representation of $G(\mathbb{A}_F)$ we mean an admissible $(g, K_F) \times G(\mathbb{A}_F^\infty)$-module in the number field case and an admissible $G(\mathbb{A}_F)$-module in the function field case:

**Definition 7.5.** We say that two irreducible admissible representations $\pi_1$, $\pi_2$ of $G(\mathbb{A}_F)$ are **weakly globally L-indistinguishable** if $\pi_1^{\mathbb{T}} \cong \pi_2^{\mathbb{T}}$ for some finite set of places $S$ of $F$. An equivalence class of L-indistinguishable admissible representations is called an **weak global L-packet** (of $G$). If a weak global $L$-packet contains an automorphic representation, we say that it is a **weak automorphic L-packet** (of $G$).

In §12.5 we will define a notion of local $L$-packets that complements this definition. The definition of local $L$-packets will be used to give a refinement of the notion of a global $L$-packet; this is why the adjective weak is used above.

For the remainder of this section, we will omit the word global to save ink. It may seem odd that we include merely admissible, and not automorphic, representations in a weak automorphic $L$-packet, but this turns out to be convenient for technical reasons. In practice one describes the weak $L$-packet of an automorphic representation and then asks which representations in the weak global $L$-packet are automorphic.

To describe weak $L$-packets in more detail, we set some notational conventions. If $S$ is a finite set of places of $F$ including the infinite places such that $G_{F_v}$ is unramified for $v \notin S$, we say that $G$ is **unramified outside of $S$**. Now choose a maximal compact subgroup $K \leq G(\mathbb{A}_F)$. Let $S$ be a finite set of places of $F$ including the infinite places and assume that $K_v$ (the projection of $K$ to $G(F_v)$) is hyperspecial for $v \notin S$. We then say that $\pi$ is **unramified outside of $S$** if $\pi_v$ contains a nonzero vector fixed by $K_v$ for $v \notin S$. This depends on the choice of $K$, but if $K'$ is a different choice then $\pi_v$ will be unramified with respect to $K_v$ if and only if it is unramified with respect to $K'_v$ for all but finitely many $v$ by Corollary 2.4.8.

The Satake isomorphism provides a convenient way of thinking about weak $L$-packets. Assume that $G$ is unramified outside of $S$. Notice that if $\pi$ is an
admissible representation of $G(\mathbb{A}_F)$ unramified outside of $S$ then it defines a character
\[
\text{tr} \pi^S : C_{\infty}^G(G(\mathbb{A}^S_F) \backslash K^S) \longrightarrow \mathbb{C}
\]
where $K^S \leq G(\mathbb{A}_F^S)$ is a hyperspecial subgroup, i.e. $K_v \leq G(F_v)$ is hyperspecial for all $v \notin S$. By the Satake isomorphism, giving this character is equivalent to giving, for each $v \notin S$, a class $c_v \in (\hat{G} \times \text{Fr}_v / \sim)(\mathbb{C})$. Here $\text{Fr}_v$ is a choice of Frobenius element at $v$ and the quotient is with respect to the action of $\hat{G}$ given in (7.20). An element of $\hat{G}(\mathbb{C}) \times \text{Fr}_v$ is Fr-semisimple if its $\hat{G}$-orbit is (Zariski) closed. Then as explained in more detail in §17.1, $(\hat{G} \times \text{Fr}_v / \sim)(\mathbb{C})$ is the set of $\hat{G}(\mathbb{C})$-orbits of Fr-semisimple elements of $\hat{G}(\mathbb{C}) \times \text{Fr}_v$. When $G$ is split, this set may be identified with semisimple conjugacy classes in $\hat{G}(\mathbb{C})$. Let
\[
\Pi_w(G) := \{\text{weak global } L\text{-packets of } G\},
\]
\[
\Pi_{w,\text{aut}}(G) := \{\text{weak automorphic } L\text{-packets of } G\}
\]
and
\[
c(G) := \{(c_v) \in \prod_{v \notin S} (\hat{G} \times \text{Fr}_v / \sim)(\mathbb{C}) \}/ \sim
\]
where we say that $\prod_{v \notin S} c_v \sim \prod_{v \notin S} c'_v$ if and only if $c_v = c'_v$ for almost every $v \notin S$ (i.e. all $v$ outside of a finite set). There is no need to be specific about the set of places $S$ in the notation here because we can always enlarge $S$ by a finite set. Any weak $L$-packet has a representative that is unramified outside of $S$, so we have a bijection
\[
c : \Pi_w(G) \longrightarrow c(G)
\]
defined in the obvious manner.

Let
\[
r : {}^L H \longrightarrow {}^L G
\]
be an $L$-map. This gives rise to a map
\[
c(H) \longrightarrow c(G).
\]

We can now state a weak form of the Langlands functoriality conjecture:

**Conjecture 7.7.1 (Langlands)** Given a weak automorphic $L$-packet $\Pi$ of $H$, there is a weak automorphic $L$-packet $r(\Pi)$ of $G$ such that $c(r(\Pi)) = r(c(\Pi))$.

One can also phrase this as the existence of a top arrow making the following diagram commute:
The fact that the class \( c(P) \) is only defined up to a finite set of places is an irritant, but it is not as horrible as it first appears. To explain, we will require some results described in more detail in \( \S 10.6 \). Let \( n_1 \hookrightarrow \cdots \hookrightarrow n_d \) be a collection of positive integers with \( \sum_{i=1}^d n_i = n \). Let \( P = MN \leq GL_n \) be the parabolic subgroup of type \((n_1, \ldots, n_d)\) with its standard Levi decomposition (see Example 1.13). For each \( i \) let \( \pi_i \) be a cuspidal automorphic representation of \( A_{GL_{n_i}} \backslash GL_{n_i}(k_F) \) and let \( \lambda \in a_{MC}^* \). We can then form the induced representation
\[
I(\otimes_{i=1}^d \pi_i, \lambda)
\]
as in \( \S 10.3 \). In general it may have several irreducible subquotients, all of which turn out to be automorphic. However it always has a canonical subquotient that will be described in more detail in \( \S 10.6 \). Automorphic representations equivalent to this canonical subquotient are known as **isobaric representations**. In particular, if \( I(\otimes_{i=1}^d \pi_i, \lambda) \) is irreducible then it is isobaric; this holds in particular if \( n_1 = n \) which is to say that the representation is cuspidal.

We now record the following foundational fact from [JS81b, JS81a]:

**Theorem 7.7.2 (Jacquet and Shalika)** Any weak automorphic \( L \)-packet of \( GL_n \) contains a unique isobaric element.

Thus, at least for \( GL_n \), admissible representations that are automorphic and isobaric are very special; they are determined by their local factors at any cofinite set of places of \( F \) (see Exercise 7.7). In particular, forgetting about finite sets of places is not an issue for \( GL_n \). For other reductive groups \( H \), the automorphic elements in a weak \( L \)-packet are supposed to be the fibers of any transfer to a general linear group. The point is that the set of admissible representations in a weak \( L \)-packet is a large set of admissible representations that could possibly transfer to a given automorphic representation on a general linear group. The question then becomes which of these admissible representations is automorphic. This is subtle; in some circumstances the conjectural answer is explained in \( \S 12.6 \).

On the other hand, for a finite set of places \( S \) of \( F \), we could ask which irreducible admissible representations of \( G(F_S) \) are local components at \( S \) of automorphic representations. Assume that \( G \) is semisimple and simply connected over \( \mathbb{Q} \) and that \( S \) is a finite set of places of \( \mathbb{Q} \) such that \( G(\mathbb{Q}_S) \) is noncompact. Then any irreducible admissible tempered representation of \( G(\mathbb{Q}_S) \) is a limit, in a suitable sense, of local components at \( S \) of automorphic representations of \( G(k_{\mathbb{A}_F}) \). We refer to [Clo07, §3.3] for a precise statement. The method used in loc. cit. is due to Burger, Li, and Sarnak.
Exercises

7.1. Let $S_2$ act on $\mathbb{C}[t_1^\pm, t_2^\pm]$ by letting the nontrivial element act by $t_1 \mapsto t_2$. Define a linear map

$$S' : C_c^\infty(\text{GL}_2(F) \backslash \text{GL}_2(\mathcal{O}_F)) \rightarrow \mathbb{C}[t_1^\pm, t_2^\pm]_{S_2}$$

by setting, for each $n \in \mathbb{Z}$, $k \in \mathbb{Z}_{\geq 0}$,

$$S'\left(\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \mathbb{1}_{\text{GL}_2(\mathcal{O}_F)}(\omega^{-i+n} \text{Sym}^k(t_1, t_2))\right) = q^{k/2}(t_1 t_2)^n \text{Sym}^k(t_1, t_2).$$

Show that this is an algebra isomorphism and deduce that $S = S'$.

7.2. Let $E/F$ be a quadratic extension of number fields with nontrivial Galois automorphism $\sigma$. For any $F$-algebra $R$, let

$$U_n(R) := \{ g \in \text{GL}_2(E \otimes_F R) : (\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}) \sigma(g)^{-t} (\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}) = g \}.$$ 

This is a unitary group in two variables over $F$. Prove that $U_n$ is quasi-split over $F$, $U_{nF_v}$ is split at every place $v$ of $F$ where $E/F$ is split, and $U_{nF_v}$ is nonsplit at every place $v$ of $F$ where $E/F$ is nonsplit.

7.3. Prove that (7.12) is an isomorphism.

7.4. Prove that (7.14) is an isomorphism.

7.5. Let $G$ be a td-group and let $K \leq G$ be a compact open subgroup. Prove that the functor from smooth representations of $G$ to $C_c^\infty(G \backslash K)$-modules given on objects by

$$V \mapsto V^K$$

is exact. In other words, it sends exact sequences to exact sequences.

7.6. Let $F$ be a local field and let $T$ be a torus over $F$. Prove that if $T(F)$ is compact then $T$ is anisotropic.

7.7. Let $\pi$ be a cuspidal automorphic representation of $\text{GL}_n(\mathbb{A}_F)$ for some global field $F$. Show that if $S$ is a finite set of places of $F$ and $\pi_S \not\cong \pi_S$ is an admissible representation of $\text{GL}_n(F_S)$ then $\pi_S \otimes \pi_S$ is never an isobaric automorphic representation of $\text{GL}_n(\mathbb{A}_F)$.
Chapter 8
Nonarchimedean Representation Theory

Let us assume we know nothing, which is a reasonable approximation.

D. Kazhdan

Abstract In this chapter we explain how general admissible representations are built up out of supercuspidal representations via the process of parabolic induction.

8.1 Introduction

Let $G$ be a reductive group over a nonarchimedean local field $F$, let $P \leq G$ be a parabolic subgroup, and let $P = MN$ be a Levi decomposition with $M \leq P$ a Levi subgroup and $N$ the unipotent radical of $P$. This notation will be in force throughout the chapter.

We recall from Proposition 3.6.1 that the modular quasi-character

$$
\delta_P := \delta_{P(F)} : P(F) \to \mathbb{R}_{>0}
$$

(8.1)

is

$$
\delta_P(p) := |\det(\text{Ad}(p) : p \to p)|,
$$

where $p = \text{Lie } P$. Let $(\sigma, V)$ be a smooth irreducible representation of $M(F)$. Since $P/N \cong M$, it extends to a representation of $P(F)$. Using the modular quasi-character, we define the parabolically induced representation (or simply induced representation) $I(\sigma) := \text{Ind}_P^G(\sigma)$ to be the (smooth) representation of $G(F)$ on the space of functions

$$
\text{Ind}_P^G(V) = \left\{ \text{locally constant } \varphi : G(F) \to V : \varphi(mng) = \delta_P^{1/2}(m)\sigma(m)\varphi(g) \right\}
$$

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for all \((m, n, g) \in M(F) \times N(F) \times G(F)\). Note that a function \(\varphi : G(F) \to V\) is locally constant if and only if it is continuous with respect to the natural topology on \(G(F)\) and the usual topology (or even the discrete topology) on \(V\). The group \(G(F)\) acts via right translation:

\[
I(\sigma)(g)\varphi(x) = \varphi(xg).
\]  

(8.2)

The factor of \(\delta_p^{1/2}\) is present so that if \(\sigma\) is unitarizable then \(I(\sigma)\) is also unitarizable (see Proposition 8.2.2). This procedure yields a functor

\[
\text{Ind}^G_P : \text{Rep}_{\text{sm}}M(F) \to \text{Rep}_{\text{sm}}G(F) \tag{8.3}
\]

from smooth representations of \(M(F)\) to smooth representations of \(G(F)\) (in these categories, objects are smooth representations in the sense of §5.3 and morphisms are equivariant maps). This functor is called \textbf{parabolic induction}.

If \(V\) is a smooth representation of \(G(F)\) then its contragredient \(V^\vee\) is defined as in §5.4. Let

\[
\langle \cdot , \cdot \rangle : V \times V^\vee \to \mathbb{C}
\]

denote the canonical pairing. A function of the form

\[
m(g) := m_{\varphi, \varphi^\vee}(g) := \langle \pi(g)\varphi, \varphi^\vee \rangle
\]

for \((\varphi, \varphi^\vee) \in V \times V^\vee\) is called a \textbf{matrix coefficient} of \(\pi\) (for the relationship between this notion and Definition 4.2, see Exercise 8.1).

**Definition 8.1.** A \textbf{supercuspidal} (resp. \textbf{quasi-cuspidal}) representation of \(G(F)\) is an admissible (resp. smooth) representation all of whose matrix coefficients are compactly supported modulo the center of \(G(F)\).

We have taken the term “quasi-cuspidal” from [Ber]. The term “supercuspidal” is encountered more often the literature. We will show that a quasi-cuspidal irreducible representation is supercuspidal in Proposition 8.3.4 below.

The main result of this chapter is a theorem of Jacquet which states that every irreducible admissible representation is obtained as a subquotient of the parabolic induction of a supercuspidal representation (see Theorem 8.3.5 for the precise statement). One can profitably compare this result with the Langlands classification from §4.9 and the classification of automorphic representations explained in Chapter 10. These results can all be viewed as manifestations of what Harish-Chandra called the “philosophy of cusp forms” [HC70a]. Interpreted broadly, this is a slogan for the statement that all irreducible representations of reductive groups can be obtained via parabolic induction from cuspidal representations of parabolic subgroups. Of course, the correct notion of cuspidal representation varies based on the context.

We will not give a full proof of Theorem 8.3.5, but we will develop much of the machinery used in the proof because it appears in many contexts. In par-
8.2 Parabolic induction

Let us now prove some basic facts about parabolic induction. Let $(\sigma, V)$ be a smooth representation of $M(F)$.

We start with the following observation:

**Lemma 8.2.1** If $K \leq G(F)$ is a compact open subgroup and $\varphi \in \text{Ind}^G_P(V)^K$ then

$$\varphi(g) \in V^{M(F) \cap gKg^{-1}}.$$

**Proof.** Under the assumptions in the lemma if $k \in g^{-1}M(F)g \cap K$ we have

$$\varphi(g) = \varphi(gk) = \varphi(gkg^{-1}g) = \sigma(gkg^{-1})\varphi(g).$$

$\square$

**Proposition 8.2.2** One has the following:

(a) If $\sigma$ is admissible then $I(\sigma)$ is admissible.

(b) If $\sigma$ is unitary then $I(\sigma)$ is unitarizable.

**Proof.** Assume that $(\sigma, V)$ is admissible. Let $K \leq G(F)$ be a compact open subgroup. We are to show that $\text{Ind}^G_P(V)^K$ is finite dimensional. An element $\varphi \in \text{Ind}^G_P(V)^K$ is determined by its value on any set of representatives $X$ for the double cosets in $P(F)\backslash G(F)/K$. (8.5)
We can choose a maximal compact subgroup $K_{\text{max}} \leq G(F)$ so that
\[ P(F)K_{\text{max}} = G(F) \]  
and $M(F) \cap K_{\text{max}}$ is a maximal compact subgroup of $M(F)$ by Theorem A.1.1. Hence
\[ P(F)\backslash G(F)/(K \cap K_{\text{max}}) \cong (P(F) \cap K_{\text{max}})\backslash K_{\text{max}}/(K \cap K_{\text{max}}) \]
is finite, and it follows that (8.5) is finite. On the other hand, for any $x \in X$ one has $\varphi(x) \in V_{M(F) \cap Kx^{-1}}$ by Lemma 8.2.1. Since $M(F) \cap K_{\text{max}}$ is a maximal compact subgroup of $M(F)$, for any compact open subgroup $K' \leq G(F)$ the intersection $M(F) \cap K'$ is a compact open subgroup of $M(F)$. Hence $V_{M(F) \cap Kx^{-1}}$ is finite dimensional by admissibility. This proves (a).

For (b), let $(\cdot, \cdot)_V$ be an $M(F)$-invariant inner product on $V$. Choose a maximal compact open subgroup $K_{\text{max}} \leq G(F)$ so that the Iwasawa decomposition $G(F) = P(F)K_{\text{max}}$ holds (see Theorem A.1.1). Normalize the Haar measures $dg$ on $G(F)$, $dk$ on $K_{\text{max}}$, and $d_{\ell p}$ on $P(F)$ so that $dg = d_{\ell p}dk$ (see Exercise 3.5).

For $\varphi_1, \varphi_2 \in \text{Ind}_{P}^{G}(V)$, we define
\[ (\varphi_1, \varphi_2) := \int_{K_{\text{max}}} (\varphi_1(k), \varphi_2(k))_V dk. \]

Since every element of $\text{Ind}_{P}^{G}(V)$ is determined by its restriction to $K_{\text{max}}$, this is a positive definite inner product. We must check that it is $G(F)$-invariant.

Choose a nonnegative $f \in C_c^\infty(G(F))$ such that
\[ \int_{P(F)} f(pg)d_{\ell p} = 1 \]
for all $g \in G(F)$ (see Exercise 8.3). For $x \in G(F)$ one has that
\[
\begin{align*}
\int_{K_{\text{max}}} (\varphi_1(kx), \varphi_2(kx))_V dk &= \int_{K_{\text{max}}} (\varphi_1(kx), \varphi_2(kx))_V \int_{P(F)} f(pkx)d_{\ell p}dk \\
&= \int_{P(F) \times K_{\text{max}}} (\varphi_1(kx), \varphi_2(kx))_V f(pkx)\delta_{\ell p}(p)d_{\ell p}dk \quad \text{(see Exercise 3.3)} \\
&= \int_{P(F) \times K_{\text{max}}} (\varphi_1(pkx), \varphi_2(pkx))_V f(pkx)d_{\ell p}dk \\
&= \int_{G(F)} (\varphi_1(gx), \varphi_2(gx))_V f(gx)dg
\end{align*}
\]
\[ = \int_{G(F)} (\varphi_1(g), \varphi_2(g)) Vf(g) dg. \]

This expression is independent of \( x \), so we deduce the invariance of our inner product. \( \square \)

**Proposition 8.2.3** One has that \( I(\sigma^\vee) \cong I(\sigma)^\vee \).

**Proof.** Fix a maximal compact open subgroup \( K_{\text{max}} \trianglelefteq G(F) \) so that the Iwasawa decomposition \( G(F) = P(F) K_{\text{max}} \) holds by Theorem A.1.1.

Let \( \varphi \in \text{Ind}_P^G(V) \) and \( \varphi^\vee \in \text{Ind}_P^G(V^\vee) \). We then have a pairing

\[ (\varphi, \varphi^\vee) := \int_{K_{\text{max}}} \langle \varphi(k), \varphi^\vee(k) \rangle dk \]  \hspace{1cm} (8.7)

where the pairing on the right is the pairing between \( V \) and \( V^\vee \). Arguing as in the proof of part (b) of Proposition 8.2.2, we deduce that this pairing is \( G(F) \)-invariant.

Since \( \varphi^\vee \) is smooth, the linear form \( \langle \cdot, \varphi^\vee \rangle \) is smooth and we therefore obtain a \( \mathbb{C} \)-linear map

\[ \text{Ind}_P^G(V^\vee) \longrightarrow \text{Ind}_P^G(V)^\vee \]

\[ \varphi^\vee \longmapsto (\cdot, \varphi^\vee). \]  \hspace{1cm} (8.8)

It is an intertwining map because (8.7) is \( G(F) \)-equivariant. We must show that it is bijective. By smoothness it suffices to show that it induces an isomorphism

\[ \text{Ind}_P^G(V^\vee)^K \longrightarrow \text{Ind}_P^G(V)^\vee^K \]

for all compact open subgroups \( K \leq K_{\text{max}} \).

To do this, we first describe a basis of \( \text{Ind}_P^G(V)^K \). As in the proof of Proposition 8.2.2, the set \( P(F) \backslash G(F)/K \) is finite. Let \( X \subset K_{\text{max}} \) be a set of representatives for \( P(F) \backslash G(F)/K \). For each \( x \in X \), let \( B_x \) be a basis of \( V(M(F) \cap xKx^{-1}) \). By Lemma 8.2.1, \( \varphi(x) \in V^M(M(F) \cap xKx^{-1}) \). For each \( w \in B_x \) let \( \varphi_{x,w} \) be the function supported on \( P(F)xK \) such that

\[ \varphi_{x,w}(mnk) = \delta_{p}^{1/2}(m)\sigma(m)w \]

for \( (m, n, k) \in M(F) \times N(F) \times K \). Then

\[ \{ \varphi_{x,w} : x \in X, w \in B_x \} \]  \hspace{1cm} (8.9)

is a \( \mathbb{C} \)-basis of \( \text{Ind}_P^G(V)^K \). Now for each \( x \) let \( B_x^\vee \) be the basis of \( V^\vee M(F) \cap xKx^{-1} \) dual to \( B_x \). We define \( \varphi_{x,w^\vee} \) by replacing \( V \) with \( V^\vee \) in the construction above, and we obtain a basis

\[ \{ \varphi_{x,w^\vee} : x \in X, w^\vee \in B_x^\vee \} \]  \hspace{1cm} (8.10)
for \( \text{Ind}_{P}^{G}(V \nu)^{K} \). We are free to assume that the measure of \( K \) with respect to \( dk \) is 1, and if take this convention then (8.9) and (8.10) are a basis and dual basis with respect to the pairing (8.7), respectively.

It is often useful to introduce a variant of the parabolic induction functor. We define a map

\[
H_{M} : M(F) \longrightarrow a_{M} := \text{Hom}(X^{\times}(M), \mathbb{R})
\]

via

\[
\langle H_{M}(m), \lambda \rangle = \log |\lambda(m)|
\]

for \( \lambda \in X^{\times}(M) \). This is the same map as (4.18) in §4.9, except now we are considering the case where \( F \) is nonarchimedean. As in §4.9 we let

\[
H_{P} : P(M) \longrightarrow M(F) \longrightarrow a_{M}
\]

be the map that is the composite of the canonical map \( P(F) \rightarrow M(F) \) and \( H_{M} \). For each \( \lambda \in a_{MC}^{\times} := X^{\times}(M) \otimes_{\mathbb{Z}} \mathbb{C} \), we obtain a quasi-character

\[
m \mapsto e^{(H_{P}(m), \lambda)}.
\]

Define

\[
I(\sigma, \lambda) := I(\sigma \otimes e^{(H_{P}(\cdot), \lambda)}).
\]

The point of introducing this extra notation is that it makes clear that the induced representation \( I(\sigma, \lambda) \) is part of a continuous family of representations indexed by \( a_{MC}^{\times} \). In practice this can be very useful. We employed this construction in the special case where \( P \) is a Borel subgroup and \( \sigma \) is trivial in our construction of the unramified principal series in §7.6.

### 8.3 Jacquet modules

We now define left adjoints to the functors \( \text{Ind}_{P}^{G} \). Let \((\pi, V)\) be a smooth representation of \( G(F) \). Set

\[
V(N) := \langle \pi(n)\varphi - \varphi : \varphi \in V, n \in N(F) \rangle \quad \text{and} \quad V_{N} := V/V(N).
\]

The space \( V_{N} \) is referred to as the **space of coinvariants**. Since \( M(F) \) normalizes \( N(F) \), it preserves \( V(N) \) and hence \( M(F) \) acts on \( V_{N} \). We can therefore define the **Jacquet functor**

\[
(\cdot)_{N} : \text{Rep}_{\text{sm}} G(F) \longrightarrow \text{Rep}_{\text{sm}} M(F)
\]

\[
(\pi, V) \longmapsto (\pi_{N}, V_{N}),
\]
where
\[ \pi_N := \pi|_{M(F)} \otimes \delta_p^{-1/2}. \]
The representation \((\pi_N, V_N)\) is referred to as the **Jacquet module** of \((\pi, V)\) (with respect to \(N\)).

A version of Frobenius reciprocity is valid for this functor:

**Proposition 8.3.1** The Jacquet functor is left adjoint to induction; in other words for smooth representations \((\pi, V)\) of \(G(F)\) and \((\sigma, W)\) of \(M(F)\), one has a bijection
\[
\operatorname{Hom}_{G(F)}(V, \operatorname{Ind}_{P}^{G}(W)) \cong \operatorname{Hom}_{M(F)}(V_N, W)
\]
that is functorial in \((\pi, V)\) and \((\sigma, W)\). Here \(M(F)\) acts on \(V_N\) via \(\pi_N\).

**Proof.** There is an \(M(F)\)-equivariant map
\[
\operatorname{ev}_1 : \operatorname{Ind}_{P}^{G}(W) \to W
\]
\[ \varphi \mapsto \varphi(1) \]
where \(1 \in G(F)\) is the identity; it is clearly surjective. Thus we have a \(M(F)\)-equivariant map
\[
\operatorname{ev}_1 \circ (\cdot) : \operatorname{Hom}_{G(F)}(V, \operatorname{Ind}_{P}^{G}(W)) \to \operatorname{Hom}_{M(F)}(V_N, W)
\]
given by composition with \(\operatorname{ev}_1\). To construct the inverse, suppose we are given an \(M(F)\)-intertwining map \(\Phi : V_N \to W\). We define
\[
\Phi : V \to \operatorname{Ind}_{P}^{G}(W)
\]
\[ \varphi \mapsto (g \mapsto \Phi(g \cdot \varphi)). \]
The map \(\Phi \mapsto \tilde{\Phi}\) is inverse to \(\operatorname{ev}_1 \circ (\cdot)\). \(\Box\)

One would hope that the functor \((\cdot)_N\) preserves admissibility, and this is indeed the case:

**Theorem 8.3.2 (Jacquet)** The functor \((\cdot)_N\) takes admissible representations to admissible representations. \(\Box\)

Jacquet also proved the following elegant characterization of quasi-cuspidal representations using these functors:

**Theorem 8.3.3** A smooth irreducible representation \((\pi, V)\) of \(G(F)\) is quasi-cuspidal if and only if \(V_N = 0\) for all parabolic subgroups \(P \leq G\). \(\Box\)

It is because of these theorems that \(V_N\) called the Jacquet module. Strictly speaking Jacquet wrote up his results only for the case of \(\text{GL}_n\) [Jac71], but the proof carries over to the general case.
For the proof of Theorem 8.3.2 see [Cas, §3]. For the proof of Theorem 8.3.3 see [Ber, §I.5.1 and §II.2.2]. More details are given in [Ren10, §VI.2.1], but Renard assumes the field $F$ is of characteristic zero. In the remainder of this section, we use theorems 8.3.2 and 8.3.3 to prove a subrepresentation theorem for admissible representations over nonarchimedean local fields (see Theorem 8.3.5).

Before proving Theorem 8.3.5, we prove a warm up result. Recall that a supercuspidal representation is an admissible quasi-cuspidal representation.

**Proposition 8.3.4** A quasi-cuspidal irreducible representation is supercuspidal.

**Proof.** Let $(\pi, V)$ be a quasi-cuspidal irreducible representation of $G(F)$. Fix a nonzero $\varphi \in V$. For all compact open subgroups $K \leq G(F)$, one has that

$$V^K = \pi(e_K)V = \langle \pi(e_K)\pi(g)\varphi : g \in G(F) \rangle,$$

where

$$e_K := \frac{1}{\text{meas}(K)} \mathbb{1}_K$$

is the idempotent attached to $K$. We must show $V^K$ is finite dimensional. Since $\pi$ admits a central quasi-character (see Exercise 5.8), if $V^K$ is not finite dimensional then there exists $\{g_n : n \in \mathbb{Z}_{>0}\} \subseteq G(F)$, all inequivalent modulo $Z_G(F)$, such that $\pi(e_K)\pi(g_n)\varphi$ are linearly independent. Let $W \subseteq V^K$ be a $\mathbb{C}$-vector subspace such that

$$V^K = W \oplus \langle \pi(e_K)\pi(g_n)\varphi : g_n \in G(F), n \in \mathbb{Z}_{>0} \rangle.$$

As $V = V^K \oplus \ker \pi(e_K)$, we can define $\varphi^\vee \in \text{Hom}(V, \mathbb{C})$ such that

$$\langle \pi(e_K)\pi(g_n)\varphi, \varphi^\vee \rangle = n$$

for all $n$ and $\varphi^\vee|_{W \oplus \ker \pi(e_K)} = 0$. Then $\varphi^\vee$ is fixed by $K$, hence is smooth, and hence is an element of $V^{\vee}$. On the other hand, by construction, the support of the matrix coefficient $\langle \pi(g)\varphi, \varphi^\vee \rangle$ is not compact modulo the center $Z_G(F)$. This contradiction implies the proposition. \(\square\)

Combined with Proposition 8.3.4, Theorem 8.3.2 and Theorem 8.3.3 allow us to deduce the following concrete manifestation of the philosophy of cusp forms:

**Theorem 8.3.5** If $(\pi, V)$ is a smooth irreducible representation of $G(F)$ then the following are true:

(a) There exists a parabolic subgroup $P = MN \leq G$, a supercuspidal representation $(\sigma, W)$ of $M(F)$ and a nonzero intertwining map $V \to \text{Ind}^G_P(W)$.

(b) The representation $\pi$ is admissible.
Proof. The first assertion implies the second, as induction preserves admissibility by Proposition 8.2.2. For the first, we proceed by induction on the dimension of $G$ (as an $F$-algebraic group). If $G$ has dimension 1 then it is a torus (this follows from the proof of [Mil17, Corollary 17.27], for example), and so the result is trivial.

Assume that for all proper parabolic subgroups $P$ there is no nonzero intertwining map $V \to \text{Ind}^G_P(W)$ where $(\sigma, W)$ is a smooth irreducible representation of $M(F)$. Applying Frobenius reciprocity (Proposition 8.3.1) we see that $$\text{Hom}_{M(F)}(V_N, W) = 0$$ for all parabolic subgroups $P = MN$ and all smooth representations $(\sigma, W)$ of $M(F)$, which implies $\pi$ is supercuspidal by Theorem 8.3.3 and Proposition 8.3.4.

Now assume that there is a proper parabolic subgroup $P < G$, a Levi subgroup $M \leq P$, a smooth representation $(\sigma, W)$ of $M(F)$, and a nonzero (hence injective) intertwining map $V \to \text{Ind}^G_P(W)$. By Frobenius reciprocity, there is a nonzero intertwining map

$$V_N \to W$$

of representations of $M(F)$ so we can apply our inductive hypothesis to deduce that there is a parabolic subgroup $Q \leq M$ with Levi subgroup $M_Q$, a supercuspidal representation $(\rho, U)$ of $M_Q(F)$ and a nonzero intertwining map

$$V_N \to W \to \text{Ind}^M_Q(U).$$

Applying Frobenius reciprocity again we obtain a nonzero intertwining map

$$V \to \text{Ind}^G_P \circ \text{Ind}^M_Q(U) \cong \text{Ind}^G_{QN}(U)$$

(see Exercise 8.4 for the last isomorphism). Here we are implicitly using the fact that $QN$ is a parabolic subgroup of $G$ (see Exercise 1.15). \qed

A **cuspidal datum** is a tuple

$$(M, (\sigma, W))$$

consisting of a Levi subgroup $M \leq G$ and a supercuspidal representation $(\sigma, W)$ of $M(F)$. We say that two cuspidal data $(M, (\sigma, W))$ and $(M', (\sigma', W'))$ are **equivalent** if they are $G(F)$-conjugate, that is, there is a $g \in G(F)$ such that $gM(F)g^{-1} = M'(F)$ and $m' \mapsto \sigma(\sigma^{-1}m'g)$ is isomorphic to $\sigma'$.

Here is an amplification of the preceding theorem:

**Theorem 8.3.6** Assume that $F$ has characteristic zero. Suppose that $\pi$ is a smooth irreducible representation of $G(F)$ and $(M, (\sigma, W)), (M', (\sigma', W'))$ are cuspidal data. Assume that $P$ and $P'$ are parabolic subgroups of $G$ with Levi
subgroups $M$ and $M'$, respectively, and that there are nonzero intertwining maps
\[ V \rightarrow \text{Ind}_P^G(W) \quad \text{and} \quad V' \rightarrow \text{Ind}_{P'}^G(W'). \]
Then $(M, (\sigma, W))$ and $(M', (\sigma', W'))$ are equivalent. \qed

The only published reference for this theorem we know is [Ren10, Theorem VI.5.4] which assumes that $F$ is of characteristic zero.

The **cuspidal support** of an irreducible admissible representation $\pi$ is the equivalence class of the cuspidal datum $(M, (\sigma, W))$ such that there is a nonzero intertwining map $V \rightarrow \text{Ind}_P^G(W)$ for some parabolic subgroup $P$ containing $M$. This is well-defined by theorems 8.3.5 and 8.3.6.

### 8.4 The Bernstein-Zelevinsky classification

Theorem 8.3.5 provides the first step towards a classification of all irreducible admissible representations of $G(F)$ in terms of supercuspidal representations. The remaining task is to understand isomorphisms between subquotients of induced representations.

It is useful to start by stepping backwards. In §4.8 we defined, for any local field, the notions of square integrable (or discrete) representations, tempered representations, essentially square integrable representations and essentially tempered representations. These notions were employed to state the archimedean Langlands classification (Theorem 4.9.2). We will now state the Langlands classification in the nonarchimedean setting. It classifies irreducible admissible representations in terms of tempered representations.

Fix a minimal parabolic subgroup $P_0 \leq G$ and call a parabolic subgroup standard if it contains $P_0$. Let $(\sigma, V)$ be an irreducible admissible representation of $M(F)$. Given $\lambda \in \mathfrak{a}_{MC}$ we can form the induced representation $I(\sigma, \lambda)$ as in (8.12).

The following is the analogue of Theorem 4.9.1 in this setting:

**Theorem 8.4.1** If $\sigma$ is unitary and tempered and $\text{Re}(\lambda)$ is in the positive Weyl chamber then $I(\sigma, \lambda)$ admits a unique irreducible quotient $J(\sigma, \lambda)$. \qed

The representation $J(\sigma, \lambda)$ is known as the **Langlands quotient** of $I(\sigma, \lambda)$.

We now state the nonarchimedean analogue of Theorem 4.9.2:

**Theorem 8.4.2** Every irreducible admissible representation of $G(F)$ is isomorphic to some $J(\sigma, \lambda)$ with $\sigma$ unitary and tempered and $\text{Re}(\lambda)$ in the positive Weyl chamber. Moreover if we insist that the parabolic subgroup defining $J(\sigma, \lambda)$ is standard, fix a Levi decomposition $P = MN$ of each standard parabolic subgroup, insist that $\sigma$ is trivial on $A_M$, and stipulate that $\text{Re}(\lambda)$ is in the positive Weyl chamber then every irreducible admissible representation of $G(F)$ is isomorphic to a $J(\sigma, \lambda)$ that is unique up to replacing $\sigma$ by another representation of $M(F)$ equivalent to $\sigma$. \qed
In the degenerate case $P = G$ the assumption that $\text{Re}(\lambda)$ is in the positive Weyl chamber is automatically satisfied. For the proofs of these theorems, we refer to [BW00, §XI.2]. See also [Kont03, Theorem 8.4.2] allows us to attach, to every irreducible admissible representation $\pi$ of $G(F)$, parameters $(M, \sigma, \lambda)$ as in the theorem. These are called the **Langlands data** of $\pi$. They are unique in the sense explained in the theorem.

In view of these theorems, to classify the irreducible admissible representations of $G(F)$ it suffices to classify the irreducible tempered representations of $G(F)$. This work is largely complete in the case when $G$ is a classical group, see [Mg02, MgT02, Jan14] for example. The situation for general linear groups is arguably the simplest and we will discuss it in this section. The results are due to Bernstein and Zelevinsky [BZ77, Zel80] and the theory is known as the **Bernstein-Zelevinsky classification**.

We start with the classification of representations that are essentially square integrable. Let $n > 1$, let $a | n$, and let $\sigma$ be an irreducible supercuspidal representation of $\text{GL}_{n/a}(F)$. The external tensor product $\sigma^a := \sigma^a \otimes a$ can then be viewed as a (supercuspidal) representation of $\text{GL}_{n/a}$, which we view as a Levi subgroup of the standard parabolic subgroup of $\text{GL}_n$ of type $(n/a, \ldots, n/a)$. We refer to the reader Example 1.13 for our conventions regarding standard parabolic subgroups.

There is a unique irreducible quotient $Q(\sigma^a, \lambda_a)$ of $I(\sigma^a, \lambda_a)$ where

$$\lambda_a := \left( -\frac{a-1}{2}, \frac{a-3}{2}, \ldots, -\frac{a-3}{2}, -\frac{a-1}{2} \right) \in X^*(M) \otimes_{\mathbb{Z}} \mathbb{C}.$$  

This quotient is essentially square integrable. Conversely, all irreducible essentially square integrable representations arise in this manner:

**Theorem 8.4.3 ( Bernstein)** Every irreducible admissible essentially square integrable representation $\pi$ of $\text{GL}_n(F)$ is isomorphic to $Q(\sigma^a, \lambda_a)$ for a pair $(\sigma^a, \lambda_a)$ where $a | n$ and $\sigma$ is an irreducible supercuspidal representation of $\text{GL}_{n/a}(F)$.

The theorem is stated in [Zel80, §9.3]. There is a sketch of the proof in [JS83, §1.2]. We observe that unlike in the setting of Theorem 8.4.2, the parameter $\lambda_a$ does not lie in the positive Weyl chamber. If $\sigma$ is a unitary supercuspidal representation, then the Langlands data of $Q(\sigma^a, \lambda_a)$ are $(G, Q(\sigma^a, \lambda_a), 0)$ and a similar statement holds if we merely assume $\sigma$ is supercuspidal.

To proceed we need the notion of linked representations. Following [BZ77] one calls any set of supercuspidal representations of $\text{GL}_r(F)$ for some $r$ of the form

$$\left\{ \sigma, \sigma \otimes e^{(H_{GL_r}(-), 1)}, \ldots, \sigma \otimes e^{(H_{GL_r}(-), d)} \right\}$$

for some integer $d$ a **segment**. Two segments are **linked** if neither is included in the other and their union is a segment.
To each $Q(\sigma^a, \lambda_a)$ we associate the segment
\[ \left\{ \sigma \otimes e^{(H_{GL_n/a}(\cdot), \frac{a_i}{n_i})}, \ldots, \sigma \otimes e^{(H_{GL_n/a}(\cdot), \frac{a_k}{n_k})} \right\}. \]

We say that
\[ Q(\sigma^a, \lambda_a) \quad \text{and} \quad Q(\sigma'^{a'}, \lambda_{a'}) \] are **linked** if their associated segments are linked.

Now assume $\sum_{i=1}^{k} n_i = n$ and $a_i/n_i$ for all $i$. If $P$ is the standard parabolic of type $(n_1, \ldots, n_k)$ then, for any collection of square integrable representations $Q(\sigma_i^{a_i}, \lambda_{a_i})$ of $GL_n(F)$ as above, we can form the induced representation
\[ I(Q(\sigma_1^{a_1}, \lambda_{a_1}) \otimes \cdots \otimes Q(\sigma_k^{a_k}, \lambda_{a_k}), 0). \]

In the case where $Q(\sigma_i^{a_i}, \lambda_{a_i})$ is linked with $Q(\sigma_j^{a_j}, \lambda_{a_j})$ if and only if $i = j$ we say that none of the $Q(\sigma_i^{a_i}, \lambda_{a_i})$ are linked.

We require the notion of a **generic representation** to state the next theorem. Generic representations are defined in §11.3. The reader can treat the definition as a black box for the time being. To aid the reader, we point out that irreducible admissible representations are nondegenerate in the terminology of [BZ77, Zel80] if and only if they are generic (see [BZ76]).

The following theorem is [Zel80, Theorem 9.7]:

**Theorem 8.4.4** If the representation (8.17) is irreducible then it is generic. The representation (8.17) is irreducible if and only if none of the $Q(\sigma_i^{a_i}, \lambda_{a_i})$ are linked. Any irreducible admissible generic representation $\pi$ is isomorphic to (8.17) for some $Q(\sigma_i^{a_i}, \lambda_{a_i})$, $1 \leq i \leq k$, uniquely determined up to reordering indices. \hfill \Box

**Theorem 8.4.5** An admissible representation $(\pi, V)$ of $GL_n(F)$ is tempered and irreducible if and only if it is of the form (8.17) where all of the $Q(\sigma_i^{a_i}, \lambda_{a_i})$ are square integrable.

To clarify the assumptions in the theorem, we note that the $Q(\sigma_i^{a_i}, \lambda_{a_i})$ are always essentially square integrable, so the assumption of square integrability in the theorem amounts to assuming that the central character of $Q(\sigma_i^{a_i}, \lambda_{a_i})$ is unitary for each $i$.

**Proof.** Assume $(\pi, V)$ is tempered. Then by [Wal03, Proposition III.4.1] and Theorem 8.4.3, $\pi$ is a subquotient of a representation $\pi'$ of the form (8.17) for some square integrable $Q(\sigma_i^{a_i}, \lambda_{a_i})$. Since the $Q(\sigma_i^{a_i}, \lambda_{a_i})$ are all square integrable, none of them are linked by Exercise 8.15. Thus $\pi'$ is irreducible by Theorem 8.4.4 and we deduce that $\pi' \cong \pi$.

Conversely suppose that an admissible representation $\pi$ is of the form (8.17) where all of the $Q(\sigma_i^{a_i}, \lambda_{a_i})$ are square integrable. Then none of the
$Q_i(\sigma_i, \lambda_n)$ are linked (see Exercise 8.15), so $\pi$ is irreducible by Theorem 8.4.4. Since the (normalized) induction of a tempered representation is tempered [Wal03, Lemme III.2.3], we conclude that $\pi$ is tempered.

Combining the Langlands classification (Theorem 8.4.2) and the subsequent description of tempered representations given by Theorem 8.4.5, we can give an analogue of the Langlands classification in terms of square integrable representations. We defer a discussion of this to §10.5.

8.5 Traces, characters, coefficients

Let $(\pi, V)$ be an admissible representation of $G(F)$. Then for all $f \in C_c^\infty(G(F))$ one has an operator

$$\pi(f) : V \to V.$$ 

There is a compact open subgroup $K \leq G(F)$ such that $f \in C_c^\infty(G(F)/K)$ and hence $\pi(f)$ induces an operator

$$\pi(f) : W \to W$$

for any finite dimensional subspace $V^K \leq W \leq V$. We define the trace of $\pi(f)$ by

$$\text{tr} \pi(f) := \text{tr} \pi(f)|_W \quad (8.18)$$

for any such $W$. The notion of the trace of a representation will play a crucial role in the trace formula in later chapters.

In this nonarchimedean setting, a distribution on $G(F)$ is simply a complex linear functional on $C_c^\infty(G(F))$. Thus the trace map

$$\text{tr} \pi : C_c^\infty(G(F)) \to \mathbb{C}$$

is a distribution on $G(F)$, called the character of $\pi$. Of course this distribution depends on a choice of Haar measure. Let $G_{\text{reg}} \leq G$ denote the (open) subscheme consisting of regular semisimple elements. The notion of a regular semisimple element will be discussed in more detail in §17.1 below. For the moment we point out that, for fields $E/F$,

$$G_{\text{reg}}(E) := \{ \gamma \in G(E) : C_{\gamma, G_E} \leq G_E \text{ is a maximal torus} \} . \quad (8.19)$$

Here $C_{\gamma, G_E}$ is the centralizer of $\gamma$ in $G_E$. The following is a fundamental and deep result:
Theorem 8.5.1 (Harish-Chandra) Assume that $F$ has characteristic zero. The distribution $\text{tr} \pi$ is represented by a locally constant function $\Theta_\pi$ with support in $G^{\text{reg}}(F)$.

See [HC99] for the proof and [Kot05] for a detailed exposition.

In other words, if $F$ has characteristic zero there is a locally constant function $\Theta_\pi$ on $G^{\text{reg}}(F)$ such that

$$\text{tr} \pi(f) = \int_{G(F)} \Theta_\pi(g) f(g) \, dg$$

for all $f \in C^\infty_c(G(F))$. This result tells us that we can almost regard $\text{tr} \pi$ as a function.

The following is a version of linear independence of characters adapted to this setting:

Proposition 8.5.2 (Linear independence of characters) If $\pi_1, \ldots, \pi_n$ is a finite set of irreducible admissible representations such that $\pi_i \cong \pi_j$ implies $i = j$, then the distributions $\text{tr} \pi_i$ are linearly independent.

Proof. We use admissibility to reduce the assertion to a finite dimensional setting. Fix a compact open subgroup $K \leq G(F)$ such that $V^K_i \neq 0$ for all $i$. This implies that $\{V^K_i\}$ is a finite family of finite dimensional $\mathbb{C}$-vector spaces with an action of $C^\infty_c(G(F) \backslash K)$. They are all simple (that is, irreducible) for this action. Moreover, they are pairwise nonisomorphic as $C^\infty_c(G(F) \backslash K)$-modules by Proposition 7.1.1. Let

$$A := \text{Im} \left( C^\infty_c(G(F) \backslash K) \rightarrow \prod_i \text{End}_\mathbb{C}(V^K_i) \right).$$

Then $A$ is a finite dimensional $\mathbb{C}$-algebra and the $\{V^K_i\}$ are a finite family of finite dimensional nonisomorphic simple $A$-modules. Hence the traces $\text{tr} \pi_i|_{C^\infty_c(G(F) \backslash K)}$ are linearly independent. As a reference for this last statement we give [GW09, Lemma 4.1.18].

Thus traces can be used to distinguish between a finite set of representations. In particular, if $\{\pi_1, \ldots, \pi_n\}$ is a finite set of pairwise nonisomorphic irreducible representations then linear independence of characters implies that we can find an $f \in C^\infty_c(G(F))$ such that

$$\text{tr} \pi_i(f) = 0 \text{ if and only if } i \neq 1.$$  

For a refinement of this result at the level of operators, see Exercise 8.9.

One can ask for more. Let $\pi$ be an irreducible admissible representation. A coefficient of $\pi$ is a smooth function $f_\pi \in C^\infty_c(G(F))$ such that $\text{tr} \pi(f_\pi) \neq 0$ and $\text{tr} \pi'(f_\pi) = 0$ for $\pi' \ncong \pi$. Thus if a coefficient for $\pi$ exists, we can use it to isolate $\pi$ among any set of irreducible admissible representations, finite or
not. We observe that if a coefficient exists it is not unique. For example, any $G(F)$ conjugate of a coefficient is another coefficient. For general $\pi$, it is not necessarily true that such functions $f_{\pi}$ exist (see Exercise 8.12). However, in certain circumstances we can construct coefficients:

**Proposition 8.5.3** Assume that $Z_G(F)$ is compact and let $(\pi, V)$ be an irreducible supercuspidal representation of $G(F)$. Then for all $f \in C^\infty_c(G(F))$ there exists a unique $f_\pi \in C^\infty_c(G(F))$ such that $\pi(f_\pi) = \pi(f)$ and if $\pi'$ is an irreducible admissible representations of $G(F)$ with $\pi' \not\cong \pi$ then $\pi'(f_\pi) = 0$.

As an immediate consequence, we see that coefficients exist for supercuspids.

To prove Proposition 8.5.3, we collect some observations on the space of endomorphisms of a representation. For any smooth representation $(\pi \hookrightarrow V)$ of $G(F)$, the $\mathbb{C}$-vector space

$$\text{End}(V) := \text{Hom}_\mathbb{C}(V, V)$$

is naturally a $G(F) \times G(F)$-module, where one copy of $G$ acts via precomposition and the other via postcomposition. The action is given explicitly by

$$G(F) \times G(F) \times \text{End}(V) \rightarrow \text{End}(V)
\quad ((g_1, g_2), A) \mapsto \pi(g_1) \circ A \circ \pi(g_2^{-1}).$$

We let

$$\text{End}_{\text{sm}}(V) \leq \text{End}(V)$$

denote the subspace consisting of smooth endomorphisms, that is, endomorphisms that are left and right invariant by a compact open subgroup $K \leq G(F)$. This is a smooth representation of $G(F) \times G(F)$.

The usual morphism

$$V \otimes \text{Hom}_\mathbb{C}(V, \mathbb{C}) \rightarrow \text{End}(V)$$

given on pure tensors by

$$\varphi \otimes \varphi^\vee \mapsto (\varphi_0 \mapsto (\varphi_0, \varphi^\vee) \varphi)$$

is $G(F) \times G(F)$-equivariant and upon restriction induces an isomorphism $V \otimes V^\vee \rightarrow \text{End}_{\text{sm}}(V)$. Here all of the tensor products are over $\mathbb{C}$. Thus the action of $G(F) \times G(F)$ on $\text{End}_{\text{sm}}(V)$ can be reasonably denoted by $\pi \otimes \pi^\vee$ (for more details on product representations see Theorem 5.7.2 above). If $(\pi, V)$ is admissible, then $\text{End}_{\text{sm}}(V)$ is also admissible (see Exercise 8.13).

One also has an intertwining map

$$\beta : (\pi \otimes \pi^\vee, \text{End}_{\text{sm}}(V)) \rightarrow (\rho, C^\infty_c(G(F)))$$
given by
\[ \beta(A)(g) = \text{tr}(\pi(g^{-1}) \circ A). \]
Here \( \rho \) acts via \( \rho(g_1, g_2)(f)(h) = f(g_1 h g_2) \). We note that for each \( A \in \text{End}_{\text{sm}}(V) \), the function \( \beta(A) \) is a sum of matrix coefficients of \( \pi^\vee \). Indeed, if we choose a compact open subgroup \( K \leq G(F) \) such that \( A \) is fixed on the left and right under \( K \), choose a basis \( \varphi_1, \ldots, \varphi_n \) of \( V^K \) and a dual basis \( \varphi_1^\vee, \ldots, \varphi_n^\vee \) of \( V^\vee^K \) then
\[ \beta(A)(g) = \sum_{i=1}^n \langle \pi(g^{-1}) \circ A \varphi_i, \varphi_i^\vee \rangle = \sum_{i=1}^n (A \varphi_i, \pi^\vee(g) \varphi_i^\vee). \tag{8.20} \]

**Proof of Proposition 8.5.3:** Since \( Z_G(F) \) is compact by assumption we deduce that \( \beta(\text{End}_{\text{sm}}(V)) \leq C_c^\infty(G(F)) \). Since there are \( \varphi \in V \) and \( \varphi^\vee \in V^\vee \) such that
\[ \langle \varphi, \varphi^\vee \rangle = \langle \pi(1) \varphi, \varphi^\vee \rangle \neq 0, \]
we have that \( \beta \) is not identically zero. Thus since the exterior tensor product \( \pi \otimes \pi^\vee \) is irreducible, \( \beta \) is an embedding.

Consider
\[ \beta' : (\rho, C_c^\infty(G(F))) \to (\pi \otimes \pi^\vee, \text{End}_{\text{sm}}(V)) \]
\[ f \mapsto \pi(f). \tag{8.21} \]
Then \( \beta' \circ \beta \) is an endomorphism of the irreducible representation \( \text{End}_{\text{sm}}(V) \) of \( G(F) \times G(F) \). Hence \( \beta' \circ \beta \) is multiplication by a scalar by Schur’s lemma (see Exercise 5.6), say \( \beta' \circ \beta = \lambda I \) for some \( \lambda \in \mathbb{C} \) where \( I : V \to V \) is the identity. We will now show that \( \lambda \neq 0 \) and that we can take
\[ f_\pi := \lambda^{-1} \beta \circ \beta'(f). \]
To show that \( \lambda \neq 0 \), note that we can find \( f \in C_c^\infty(G(F)) \) such that \( \pi(f) \neq 0 \) (take, for example, \( f \) to be the characteristic function of a sufficiently small compact open subgroup). Thus \( \beta'(f) = \pi(f) \neq 0 \) and since \( \beta \) is an embedding, we deduce that \( \beta \circ \beta'(f) \neq 0 \).

To show that we can take \( f_\pi \) as claimed, we first observe that
\[ \pi(\beta \circ \beta'(f)) = \beta' \circ \beta \circ \beta'(f) = \lambda \beta'(f) = \lambda \pi(f). \]
Second, let \( (\pi', V') \) be a smooth irreducible representation of \( G(F) \), let \( \varphi' \in V' \) be a nonzero vector, and let
\[ \gamma : (\rho|_{G(F) \times 1}, C_c^\infty(G(F))) \to (\pi', V') \]
\[ f \mapsto \pi'(f) \varphi'. \tag{8.22} \]
As a representation of $G(F)$, one has that $\text{End}_{sm}(V)|_{G(F)\times 1}$ is isomorphic to a direct sum of copies of $\pi$, and thus the same is true of $\gamma(\beta(\text{End}_{sm}(V)))$. Thus $\gamma(\beta(\text{End}_{sm}(V))) = 0$ unless $\pi' \cong \pi$ (since whenever the former is nonzero we obtain an intertwining operator between $\pi'$ and $\pi$). It follows that $\pi'(f_\pi) = 0$ if $\pi' \not\cong \pi$. This completes the proof of the proposition. \hfill \Box

8.6 Parabolic descent of representations

Parabolic descent is a term for the process by which one passes from objects (say, representations) on a group $G$ to objects on a Levi subgroup of a parabolic subgroup of $G$. In this chapter we give two examples of this phenomenon. First we show that the character of a parabolically induced representation is easily related to the character of the inducing representation (see Proposition 8.6.1). Then in §8.7 we prove an analogous result (Proposition 8.7.1) for orbital integrals.

Let $P \leq G$ be a parabolic subgroup with Levi decomposition $P(F) = M(F)N(F)$ and let $K \leq G(F)$ be a maximal compact subgroup.

**Definition 8.2.** A maximal compact subgroup $K \leq G(F)$ is said to be in good position with respect to $(P,M)$ if $G(F) = P(F)K$ and $P(F) \cap K = (M(F) \cap K)(N(F) \cap K)$.

Given a parabolic subgroup $P$ with Levi decomposition $P = MN$, we can always find a maximal compact subgroup in good position with respect to $(P,M)$. We can even assume that

$$K_M := M(F) \cap K$$

is a maximal compact subgroup of $M(F)$ (see Theorem A.4.2); we will also assume this in what follows.

We assume throughout this section that Haar measures are chosen so that with respect to the decomposition $G(F) = KM(F)N(F)$ one has

$$dg = \delta_p(m)dkdmndn \quad (8.23)$$

where

$$\text{meas}_d(K) = \text{meas}_d(N(F) \cap K) = \text{meas}_d(K_M) = 1$$

(see Exercise 8.2). This is equivalent to stipulating that with respect to the decomposition $G(F) = M(F)N(F)K$ one has

$$dg = dmndk. \quad (8.24)$$

The constant term of $f \in C_c^\infty(G(F))$ along $P$ is the function
\[ f^P(m) := \delta_{\mathcal{P}}^{1/2}(m) \int_{N(F)} f(mn)dn. \quad (8.25) \]

This is equal to (7.27) in the special case considered in §7.5.

Let \( T \leq M \) be a maximal split torus and let

\[ C_c^\infty(G(F); K) := \{ f \in C_c^\infty(G(F)) : f(k^{-1}xk) = f(x) \text{ for all } k \in K \}. \]

We note that \( C_c^\infty(G(F); K) \) is precisely the image of the map

\[ C_c^\infty(G(F)) \to C_c^\infty(G(F); K) \]
\[ f \mapsto f_K \quad (8.26) \]

where \( f_K(x) := \int_K f(k^{-1}xk)dk \). The space \( C_c^\infty(G(F); K) \subseteq C_c^\infty(G(F)) \) is a subalgebra under convolution and one trivially has that

\[ \text{tr } \pi(f) = \text{tr } \pi(f_K). \quad (8.27) \]

**Proposition 8.6.1** The constant term gives a map

\[ C_c^\infty(G(F); K) \to C_c^\infty(M(F); K_M) \]
\[ f \mapsto f^P. \]

If \((\sigma, V)\) is an admissible representation of \( M(F) \), then

\[ \text{tr } I(\sigma)(f) = \text{tr } \sigma(f^P). \]

**Remark 8.1.** The archimedean analogue of this proposition is valid as well. One reference is [Kna86, (10.23)].

**Proof.** We follow the proof of [Lau96, Lemma 7.5.7]. Since \( K \) is compact, \( \delta_P(k) = 1 \) for all \( k \in K_M \). It follows that \( f^P \in C_c^\infty(M(F); K_M) \) as claimed.

We now prove the equality of traces. Consider the space \( \text{Ind}^G_P(V)^\circ \) of locally constant functions

\[ \varphi : K \to V \]

such that \( \varphi(mnk) = \sigma(m)\varphi(k) \) for \((m, n, k) \in K_M \times (N(F) \cap K) \times K\). There is a representation \( I(\sigma)^\circ \) of \( K \) on this space given by

\[ I(\sigma)^\circ(k)\varphi(g) = \varphi(gk). \quad (8.28) \]

By the Iwasawa decomposition, restriction of functions to \( K \) induces an isomorphism

\[ \text{Ind}^G_P(V) \to \text{Ind}^G_P(V)^\circ. \quad (8.29) \]

Thus for all \( f \in C_c^\infty(G(F)) \), \( I(\sigma)(f) \) induces an endomorphism of \( \text{Ind}^G_P(V)^\circ \).

Let us compute this endomorphism. Define
8.6 Parabolic descent of representations

\[ \Phi_{k,k'}(m) := \delta^{1/2}_P(m) \int_{N(F)} f(k^{-1}mnk')dn \]

for \( k, k' \in K \) and \( m \in M(F) \). Then if \( \varphi \in \text{Ind}^G_P(V) \) one has that

\[
I(\sigma)(f) \varphi(k) = \int_{G(F)} f(g) \varphi(kg) dg
\]

\[
= \int_{G(F)} f(k^{-1}g) \varphi(g) dg
\]

\[
= \int_{M(F)} \int_{N(F)} \int_K f(k^{-1}mnk') \varphi(mnk')dk' d\eta dm
\]

\[
= \int_{M(F)} \int_{N(F)} \int_K f(k^{-1}mnk') \delta^{1/2}_P(m) \sigma(m) \varphi(k') dk' d\eta dm
\]

\[
= \int_K \sigma(\Phi_{k,k'}) \varphi(k') dk'.
\]

Thus the endomorphism on \( \text{Ind}^G_P(V) \) induced by \( I(\sigma)(f) \) is

\[
\varphi \mapsto \left( k \mapsto \int_K \sigma(\Phi_{k,k'}) \varphi(k') dk' \right).
\]

(8.30)

In particular \( \text{tr} I(\sigma)(f) \) is the trace of the endomorphism (8.30). This endomorphism has trace

\[
\int_K \text{tr} \sigma(\Phi_{k,k'}) dk.
\]

On the other hand

\[
f^P = \int_K \Phi_{k,k} dk.
\]

The lemma follows. \( \square \)

As an addendum, we prove the following lemma:

**Lemma 8.6.2** The constant term provides an algebra homomorphism

\[
C_c^\infty(G(F) \parallel K) \longrightarrow C_c^\infty(M(F) \parallel K_M).
\]

We used this lemma in the special case where \( G \) is unramified and \( P = B \) in Lemma 7.5.3:

**Proof.** Let \( f_1, f_2 \in C_c^\infty(G(F) \parallel K) \). One has that

\[
(f_1 * f_2)^P(m) = \delta^{1/2}_P(m) \int_{N(F)} (f_1 * f_2)(mn)dn
\]

\[
= \delta^{1/2}_P(m) \int_{N(F)} \left( \int_{G(F)} f_1(mng^{-1})f_2(g)dg \right) dn.
\]
Using Exercise 8.2, we rewrite this as
\begin{align*}
\delta_{P}^{1/2}(m) \int_{N(F) \times K \times M(F) \times N(F)} \delta_{P}(m') f_1(mn(k'm'n')^{-1}) f_2(k'm'n') dn' dk' dm' \ \\
= \delta_{P}^{1/2}(m) \int_{N(F) \times M(F) \times N(F)} \delta_{P}(m') f_1(mm'n^{-1}m'^{-1}) f_2(m'n') dm' dn'.
\end{align*}
Changing variable \( n \mapsto mn' \) this is
\begin{align*}
\delta_{P}^{1/2}(m) \int_{N(F) \times M(F) \times N(F)} \delta_{P}(m') f_1(mm'n^{-1}) f_2(m'n') dm' dn'
\end{align*}
and then changing variables \( n \mapsto m'^{-1}nm' \) this is
\begin{align*}
\delta_{P}^{1/2}(m) \int_{N(F) \times M(F) \times N(F)} f_1(mm'n^{-1}) f_2(m'n') dm' dn' \ \\
= \int_{M(F)} \left( \delta_{P}^{1/2}(mm'n^{-1}) \int_{N(F)} f_1(mm'n^{-1}) dn' \delta_{P}^{1/2}(m') \int_{N(F)} f_2(m'n') dm' \right) dm' \\
= (f_{1}^{P} \ast f_{2}^{P})(m).
\end{align*}
This proves our assertion that the constant term is an algebra morphism. \( \square \)

### 8.7 Parabolic descent of orbital integrals

In our discussion of the Satake isomorphism in §7.5, we claimed that the constant term map has image in Weyl invariant functions on the maximal torus. We prove this in the current section and use it as an opportunity to discuss descent of orbital integrals.

We keep the notation of the previous section; thus \( P = MN \) is a parabolic subgroup of \( G \), \( K \leq G(F) \) is a maximal compact subgroup in good position with respect to \( (P, M) \), and \( K_M := M(F) \cap K \) is maximal compact subgroup of \( M(F) \).

Let \( \gamma \in G(F) \) and let \( C_{\gamma} \) be the centralizer of \( \gamma \) defined as
\begin{equation}
C_{\gamma}(R) = \{ g \in G(R) : g^{-1}\gamma g = \gamma \}
\end{equation}
for an \( F \)-algebra \( R \). Earlier in Definition 1.13, we only defined semisimple elements for perfect fields. If \( F \) is not perfect then we say that \( \gamma \in G(F) \) is semisimple if the image of \( \gamma \) in \( G(\overline{F}) \) is semisimple. If \( \gamma \in G(F) \) is semisimple then the orbit \( O(\gamma) \subset G \) of \( \gamma \) is closed [Ste65, Corollary 6.13, Proposition 6.14]. For a semisimple element \( \gamma \), the neutral component of the centralizer \( C_{\gamma} \) is reductive (see Theorem 17.1.5). For more details on orbits and stabilizers, we refer to §17.1.
Assume that $\gamma \in G(F)$ is semisimple. Then $C_\gamma(F)$ is unimodular, and upon choosing a Haar measure $dg_\gamma$ on $C_\gamma^0(F)$ we can form a right $G(F)$-invariant Radon measure $\frac{dg}{dg_\gamma}$ on the quotient $C_\gamma^0(F)\backslash G(F)$ (see Theorem 3.2.2). We then define the orbital integral

$$O_\gamma^0(f) := \int_{C_\gamma^0(F)\backslash G(F)} f(g^{-1}\gamma g) \frac{dg}{dg_\gamma}. \quad (8.32)$$

Note that this is not the same as the local version of the orbital integral appearing in (8.21). It differs in that in (8.21) we use $C_\gamma$ instead of $C_\gamma^0$. This is why there is a superscript $\circ$ in the notation. The quotient map

$$C_\gamma^0(F)\backslash G(F) \longrightarrow C_\gamma(F)\backslash G(F)$$

is proper as is the map

$$C_\gamma(F)\backslash G(F) \longrightarrow G(F)$$

$$C_\gamma(F)g \mapsto g^{-1}\gamma g$$

by the argument in the proof of Theorem 17.4.1. It follows as in the proof of Theorem 17.4.1 that the integral (8.32) is absolutely convergent.

Let $H \leq G$ be a connected subgroup. For $F$-algebras $R$ and $h \in H(R)$, it is convenient to define

$$D_{H\backslash G}(h) := \det \left( I - \text{Ad}(h^{-1}) : \mathfrak{h}\backslash \mathfrak{g} \longrightarrow \mathfrak{h}\backslash \mathfrak{g} \right) \quad (8.33)$$

where $\mathfrak{h} := \text{Lie } H$, and $\mathfrak{g} := \text{Lie } G$, and $I$ is the identity map.

There is a dual relationship between orbital integrals and characters that can be made precise in many ways; one of the most profound is the absolute trace formula (see §18.4). In view of Proposition 8.6.1 the following proposition is consonant with this duality:

**Proposition 8.7.1** Assume that $\gamma \in M(F)$ is semisimple and $C^0_\gamma \leq M$. If $D_{M\backslash G}(\gamma) \neq 0$ then one has that

$$O_\gamma^0(f) = |D_{M\backslash G}(\gamma)|^{-1/2}O_\gamma^0(f^P).$$

Here we normalize the Haar measure $dg$ on $G(F)$ as in (8.23) and we take the Haar measures on $C_\gamma(F)$ to be equal on either side of the equation. If we replace the assumption that $C^0_\gamma \leq M$ with the stronger assumption that $C_\gamma \leq M$ and define $O_\gamma^0(f)$ with $C^0_\gamma(F)$ replaced by $C_\gamma(F)$ then the proposition remains valid, with an identical proof.

The following proof is taken from [Lau96, Proposition 4.3.11]:

**Proof.** One has that
\[
O^\gamma_1(f) = \int_{C^\gamma_1(F) \setminus M(F)} \int_{N(F)} \int_K f(k^{-1}m^{-1}\gamma mnk)dkdn \frac{dm}{dm_\gamma}.
\]

For each \( m \in M(F) \) there is a morphism of affine \( F \)-schemes \( N \to N \) given on points in an \( F \)-algebra \( R \) by

\[
N(R) \longrightarrow N(R) \\
n \mapsto (m^{-1}\gamma m)^{-1}n^{-1}(m^{-1}\gamma m)n.
\]

(8.34)

We claim that this morphism is an isomorphism. The image is the orbit of the identity under an action of the unipotent group \( N \hookrightarrow \) and is therefore closed in \( N \) by [Mil17, Theorem 17.64]. On the other hand, the morphism is injective at the level of points. Indeed if \( \gamma^{-1}n^{-1}\gamma n = \gamma^{-1}n'^{-1}\gamma n' \) for \( n, n' \in N(R) \) then \( n'n^{-1} \in C_\gamma(R) \). On the other hand \( C_\gamma/C_\gamma^\circ \) is abelian and consists of semisimple elements by [Ste68, Corollary 9.4], which implies \( n'n^{-1} = I \) by our assumption that \( C_\gamma^\circ \leq M \). The same is true if we replace \( \gamma \) by \( m^{-1}\gamma m \) and injectivity follows. In particular the stabilizer of the identity element is trivial. We concluded using Proposition 17.1.2 that (8.34) is an isomorphism onto its image, a closed subscheme of \( N \). By considering dimensions we deduce that (8.34) is an isomorphism.

The Jacobian depends only on \( \gamma \) and is equal to

\[
J(\gamma) = |\det (I - \text{Ad}(\gamma^{-1}) : n \mapsto n)|.
\]

Thus we can change variables and deduce that the integral above is

\[
\int_{C^\gamma_1(F) \setminus M(F)} \int_{N(F)} \int_K f(k^{-1}m^{-1}\gamma mnk)dkdn \frac{dm}{J(\gamma) \frac{dm}{dm_\gamma}}.
\]

Observing that

\[
|D_{M\setminus G}(\gamma)| = \delta_P(\gamma)J(\gamma)^2
\]

we deduce the proposition. □

This is the first time we have used the usual change of variables formula in the nonarchimedean setting. A general discussion of integration on manifolds over nonarchimedean fields, including the change of variables formula, is in [Igu00, Chapter 7]. We refer especially to [Igu00, Proposition 7.4.1].

We now prove the following proposition, used above to prove Lemma 7.5.3:

**Proposition 8.7.2** Assume that \( G \) is quasi-split with Borel subgroup \( B \), that \( M \leq B \) is a maximal torus, and that \( T \leq M \) is its maximal split subtorus. The constant term induces a homomorphism

\[
C_\infty^c(G(F) \setminus K) \longrightarrow C_\infty^c(T(F)/K_T)^W(G,T)(F),
\]

where \( K_T = T(F) \cap K \).
To prove Proposition 8.7.2 we require the following proposition:

**Proposition 8.7.3** With assumptions as in Proposition 8.7.2, for any $\gamma \in M(F)$

$$|D_{M \setminus G}(\gamma)| \neq 0$$

if and only if $C_\gamma^o = M$. The set

$$\{ \gamma \in M(F) : C_\gamma^o = M \}$$

is dense in $M(F)$ in the natural topology.

**Proof.** Clearly $M \subseteq C_\gamma^o$ because $M$ is connected and commutative. Let $\mathfrak{g}_\gamma$ be the Lie algebra of $C_\gamma$ and $\mathfrak{m}$ be the Lie algebra of $M$. Since $I - \text{Ad}(\gamma^{-1})$ acts by zero on the subspace

$$\mathfrak{m} \setminus \mathfrak{g}_\gamma \leq \mathfrak{m} \setminus \mathfrak{g}$$

we deduce that $|D_{M \setminus G}(\gamma)| \neq 0$ if and only if $\mathfrak{m} = \mathfrak{g}_\gamma$, which is to say that $C_\gamma^o = M$.

To deduce that the set of elements of $M(F)$ where $C_\gamma^o = M$ is dense in $M(F)$ in the natural topology, we remark that the set of points where a nonzero polynomial on a (nonarchimedean) analytic manifold does not vanish is necessarily dense. \(\square\)

There are surprisingly few textbook treatments of analytic manifolds in the nonarchimedean setting; two are [Ser06, Sch11] and some additional topics are in [Igu00]. Perhaps the reason is that more sophisticated treatments involving rigid analytic spaces, Berkovich spaces, etc. are often needed and have drawn more attention.

**Proof of Proposition 8.7.2:** By Lemma 1.7.3, every element of $W(G, T)(F)$ has a representative $w \in N_G(T)(F)$, the normalizer of $T(F)$ in $G(F)$. Thus it suffices to show that for $f \in C_\gamma^o(G(F)/K)$ one has that

$$f^B(w^{-1}tw) = f^B(t)$$

for $t \in T(F)$. We will in fact prove the stronger statement that

$$f^B(w^{-1}\gamma w) = f^B(\gamma)$$

(8.36)

for all $\gamma \in M(F)$.

Assume that $\gamma \in M(F)$ satisfies $C_\gamma^o = M$. By Proposition 8.7.3 this implies $|D_{M \setminus G}(\gamma)| \neq 0$. We then have that

$$O^o_\gamma(f) = |D_{M \setminus G}(\gamma)|^{-1/2}O^o_\gamma(f^B)$$

$$= |D_{M \setminus G}(\gamma)|^{-1/2}f^B(\gamma)$$
by Proposition 8.7.1. Here the last equality follows since \( M(F) \) is commutative. It is clear that \( O_{w^{-1},w}^\circ(f) = O_w^\circ(f) \), so at least for \( \gamma \in M(F) \) such that \( C_\gamma = M \), we deduce (8.36). This set is dense in \( M(F) \) by Proposition 8.7.3. It is clear that \( f^B \) is continuous, so we deduce (8.36) for all \( \gamma \in M(F) \). □

**Exercises**

In all of these exercises \( G \) is a reductive group over a nonarchimedean local field \( F \) and \( P \leq G \) is a parabolic subgroup with a fixed Levi subgroup \( M \leq P \) and unipotent radical \( N \).

8.1. Let \((\pi, V)\) be a unitary representation of \( G(F) \) on a Hilbert space \( V \). Let \( V_{sm} \leq V \) be the subspace of smooth vectors (see Exercise 5.3). Construct an isomorphism \( V_{sm} \cong (V_{sm})^V \). Using this isomorphism, show that a matrix coefficient in the sense of (8.4) is a matrix coefficient in the sense of Definition 4.2.

8.2. Assume that \( K \leq G(F) \) is a maximal compact subgroup in good position with respect to \((P, M)\). Show that we can normalize the Haar measures \( dg, dk, dm, dn \) on \( G(F), K, M(F) \) and \( N(F) \), respectively, so that with respect to the decomposition \( G(F) = KM(F)N(F) \) one has

\[
dg = \delta_P(m)dkdmn
\]

where

\[
\text{meas}_{dk}(K) = \text{meas}_{dn}(N(F) \cap K) = \text{meas}_{dm}(M(F) \cap K) = 1.
\]

8.3. Show that we can choose a function \( f \in C_c^\infty(G(F)) \) such that

\[
\int_{P(F)} f(px)dx = 1
\]

for all \( x \in G(F) \).

8.4. Let \( Q \leq M \) be a parabolic subgroup. Prove there is a natural transformation of functors

\[
\text{Ind}_P^G \circ \text{Ind}_Q^M \cong \text{Ind}_Q^N.
\]

State and prove the analogous statement for Jacquet modules.

8.5. Prove that the functor \( \text{Ind}_P^G \) is exact (that is, it preserves exact sequences) and additive (that is, takes direct sums to direct sums).

8.6. Prove that the functor \( \pi \mapsto \pi_N \) is exact (sends exact sequences to exact sequences) and additive (that is, takes direct sums to direct sums).
8.7. Let \((\pi, V)\) be a smooth representation of \(G(F)\). Prove that an element \(\varphi \in V\) is in \(V(N)\) if and only if \(\int_{K_N} \pi(n) \varphi \, dn = 0\) for some compact subgroup \(K_N \leq N(F)\).

8.8. Prove that \(G^{\text{reg}}\), defined as in (8.19), is an open subscheme of \(G\).

8.9. Let \((\pi_1, V_1), \ldots, (\pi_n, V_n)\) be a finite set of irreducible admissible representations of \(G(F)\) such that \(\pi_i \cong \pi_j\) if and only if \(i = j\). Let \(K \leq G(F)\) be a compact open subgroup such that \(V_1^K \neq 0\). Show that there exists an \(f \in C_c^\infty(G(F) \sslash K)\) such that \(\pi_1(f)|_{V_1^K}\) is the identity and \(\pi_i(f) = 0\) if \(1 < i \leq n\).

8.10. Prove that an unramified irreducible admissible representation of \(GL_n(F)\) is tempered if and only if its Satake parameters have complex norm 1.

8.11. Suppose that \(\pi\) is an irreducible supercuspidal representation of \(G(F)\). Show that there exists a function \(f_\pi \in C_c^\infty(G(F))\) such that \(\text{tr} \, \pi(f_\pi) = 1\) and if \(\text{tr} \, \pi'(f_\pi) \neq 0\) for some irreducible admissible representation \(\pi'\) of \(G(F)\) then \(\pi \cong \pi' \otimes \chi\) for some quasi-character \(\chi : Z_G(F) \to \mathbb{C}^\times\).

8.12. Let \(\pi\) be an irreducible principal series representation of \(G(F)\). Prove that there are no coefficients of \(\pi\) in the sense of §8.5.

8.13. Let \((\pi, V)\) be an admissible representation of a td-type group \(H\). Prove that the space of smooth endomorphisms \(\text{End}_{\text{sm}}(V)\) is an admissible representation of \(H \times H\).


8.15. In the notation of §8.4, show that if \(Q(\sigma^a, \lambda_\alpha)\) and \(Q(\sigma'^a, \lambda_\alpha')\) are both square integrable (not just essentially square integrable) then they are not linked.
Chapter 9
The Cuspidal Spectrum

God exists since mathematics is consistent, and the Devil exists since we cannot prove it.

André Weil

Abstract The cuspidal spectrum of $L^2([G])$ decomposes discretely into a Hilbert space direct sum with finite multiplicities. We give a complete proof of this fact in this chapter. As consequences of the techniques used in the proof, we prove that cuspidal automorphic forms are rapidly decreasing in the number field case and are compactly supported in the function field case.

9.1 Introduction

Let $G$ be a reductive group over a global field $F$. For $x \in \mathbb{A}_G \backslash G(\mathbb{A}_F)$, one has the operator

$$ R(x) : L^2([G]) \rightarrow L^2([G]) $$

$$ \varphi \mapsto (g \mapsto \varphi(gx)). $$

(9.1)

This defines the regular representation of $\mathbb{A}_G \backslash G(\mathbb{A}_F)$ on $L^2([G])$. Here as we recall $[G] = \mathbb{A}_G G(F) \backslash G(\mathbb{A}_F)$. As we have seen in various circumstances, it is often convenient to work instead with the operators

$$ R(f) : L^2([G]) \rightarrow L^2([G]) $$

$$ \varphi \mapsto \left( g \mapsto \int_{\mathbb{A}_G \backslash G(\mathbb{A}_F)} f(x) \varphi(gx) dx \right) $$

(9.2)

for $f \in C^\infty_c(\mathbb{A}_G \backslash G(\mathbb{A}_F))$. Here, as in (6.1),
\[ C^\infty_c(A_G \backslash G(\A_F)) = C^\infty_c(A_G \backslash G(F_\infty)) \otimes C^\infty_c(G(\A_F^\infty)) \]

where the tensor product is algebraic. Later in this book we will be interested in various notions of the trace for operators of this type. There is a problem, however, in that \( R(f) \) does not in general have a well-defined trace. Instead, one restricts the operator to the cuspidal subspace of (6.14). The main goal of the current chapter is to prove that this restriction has a well-defined trace (see Theorem 9.1.1 below). One is then left with understanding the complement of the cuspidal subspace; this is possible thanks to the theory of Eisenstein series which will be discussed in Chapter 10.

We now explain what is meant by a trace in this infinite dimensional setting. Let \( V \) be a Hilbert space with inner product \((\cdot,\cdot)\) whose induced norm is \( \|\cdot\|_2 \). We recall for all bounded operators \( A : V \to V \) there exists a unique bounded operator \( |A| \) characterized by the property that \( |A| \circ |A| = A \circ A^* \) (where the \( * \) denotes the adjoint) \cite[Theorem 5.1.3]{DE09}. As the notation suggests, the operator \( |A| \) is positive in the sense that \((|A|\varphi,\varphi) \geq 0\) for all \( \varphi \in V \).

**Definition 9.1.** A bounded linear operator \( A : V \to V \) is **Hilbert-Schmidt** if there is an orthonormal basis \( \{\varphi_i\}_{i=1}^\infty \) of \( V \) such that

\[
\sum_{i=1}^\infty \|A\varphi_i\|_2^2 < \infty.
\]

A bounded linear operator \( A : V \to V \) is of **trace class** if there is an orthonormal basis \( \{\varphi_i\}_{i=1}^\infty \) of \( V \) such that

\[
\sum_{i=1}^\infty (|A|\varphi_i,\varphi_i) < \infty.
\]

If \( A \) is Hilbert-Schmidt (resp. of trace class) we let

\[
\|A\|_{\text{HS}} = \left( \sum_{i=1}^\infty \|A\varphi_i\|_2^2 \right)^{1/2} \quad \text{and} \quad \|A\|_{\text{tr}} = \sum_{i=1}^\infty (|A|\varphi_i,\varphi_i). \tag{9.3}
\]

This is known as the **Hilbert-Schmidt norm** (resp. **trace norm**) of \( A \). If \( A \) is of trace class we define its **trace** to be

\[
\text{tr } A := \sum_{i=1}^\infty (A\varphi_i,\varphi_i).
\]

The Hilbert-Schmidt norm, trace norm, and trace, are independent of the choice of basis \( \{\varphi_i\}_{i=1}^\infty \) (see Exercise 9.1).

We recall from (6.14) that the subspace \( L^2_{\text{cusp}}([G]) \leq L^2([G]) \) is **cuspidal** if for all \( f \in C^\infty_c(A_G \backslash G(\A_F)) \) and every parabolic subgroup \( P \leq G \) with unipotent radical \( N \), one has that
\[ \int_{[N]} R(f)\varphi(ng)dn = 0. \]

As mentioned above, our aim in this chapter is to prove the following fundamental theorem:

**Theorem 9.1.1 (Gelfand and Piatetski-Shapiro)** The subspace

\[ L^2_{\text{cusp}}([G]) \leq L^2([G]) \]

is closed and \( R_{\text{cusp}}(f) \) is of trace class for all \( f \in C^\infty_c(A_G \backslash G(A_F)) \).

Here \( R_{\text{cusp}}(f) \) is the restriction of \( R(f) \) to the cuspidal subspace. Most of this chapter is devoted to the proof of Theorem 9.1.1. We complete the proof in §9.5. We will follow [God66], filling in some details using results from [Bor97]. These two references proceed classically whereas we will proceed adelicly. This leads to some (mostly notational) simplifications.

We will show Theorem 9.1.1 implies the following corollary in §9.3 (see Corollary 9.3.2):

**Corollary 9.1.2** The subspace \( L^2_{\text{cusp}}([G]) \) decomposes into a Hilbert space direct sum of irreducible representations of \( A_G \backslash G(A_F) \), each occurring with finite multiplicity.

In other words, we can write

\[ L^2_{\text{cusp}}([G]) = \bigoplus_{\pi} L^2_{\text{cusp}}(\pi), \] (9.4)

where the Hilbert space direct sum is over isomorphism classes of cuspidal automorphic representations \( \pi \) of \( A_G \backslash G(A_F) \) and the multiplicity of \( \pi \) in \( L^2_{\text{cusp}}(\pi) \) is finite. It does not matter if we take this to mean automorphic representations in the \( L^2 \)-sense or with the refined definition of Chapter 6 by Theorem 6.6.4.

The techniques used to prove Theorem 9.1.1 will also allow us to prove that cuspforms are rapidly decreasing in the sense of Definition 9.2 below in the number field case (see Corollary 9.6.2). They are even compactly supported in the function field case (see Theorem 9.5.1). These facts are used constantly when working with cuspidal automorphic forms. The techniques also allow us to prove Theorem 6.5.1 above in §9.7.

We close this section by discussing a generalization of Theorem 9.1.1. Let

\[ L^2_{\text{disc}}([G]) \leq L^2([G]) \] (9.5)

denote the **discrete spectrum**. It is the largest closed subspace of \( L^2[G] \) that decomposes as a Hilbert space direct sum of irreducible subrepresentations of \( L^2([G]) \). An automorphic representation of \( A_G \backslash G(A_F) \) is **discrete** if it can be realized as subrepresentation of \( L^2_{\text{disc}}([G]) \). Thus cuspidal representations...
are discrete by Corollary 9.1.2, but the converse is false. A far more difficult theorem than Corollary 9.1.2 is the following:

**Theorem 9.1.3** Assume $F$ is a number field. The restriction of $R(f)$ to the full discrete spectrum of $L^2([G])$ is of trace class.

Müller [Mül89] first proved the theorem in the special case where $f$ is finite under a maximal compact subgroup $K_\infty \leq G(F_\infty)$. Ji [Ji98] and Müller [Mül98] then independently removed this assumption.

### 9.2 The cuspidal subspace

We begin with the easiest part of Theorem 9.1.1:

**Lemma 9.2.1** The space $L^2_{\text{cusp}}([G])$ is closed in $L^2([G])$.

We will use the proof of this lemma as an opportunity to introduce Poincaré series. To keep straight the techniques used in the proof, we do not assume that $G$ is reductive or even connected; any affine algebraic group over a global field will do. We recall the adelic quotient

$$[G] = A_G G(F) \backslash G(\mathbb{A}_F)$$

defined as in (2.18). Let $H \leq G$ be a subgroup and let

$$\tilde{H}(\mathbb{A}_F) := A_G \backslash A_G H(\mathbb{A}_F) \leq A_G \backslash G(\mathbb{A}_F).$$

Despite the notation, this is not necessarily the $\mathbb{A}_F$-points of an algebraic group over $F$. Assume that $H(F) \backslash \tilde{H}(\mathbb{A}_F)$ is compact and let $\chi : \tilde{H}(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$ be a character trivial on $H(F)$. It defines a function on $H(F) \backslash \tilde{H}(\mathbb{A}_F)$ that we again denote by $\chi$. The fact that $H(F) \backslash \tilde{H}(\mathbb{A}_F)$ is compact implies that $\tilde{H}(\mathbb{A}_F)$ is unimodular (see Exercise 9.6); we let $dh$ be a choice of Haar measure on it. When $N \leq G$ is a unipotent group then the natural map $N(\mathbb{A}_F) \rightarrow \tilde{N}(\mathbb{A}_F)$ is an isomorphism. This is the situation of interest in Lemma 9.2.1.

For continuous functions $\varphi : [G] \rightarrow \mathbb{C}$, we let

$$\mathcal{P}_\chi(\varphi) := \int_{H(F) \backslash \tilde{H}(\mathbb{A}_F)} \varphi(h)\overline{\chi(h)}dh.$$  \hspace{1cm} (9.6)

This is an example of a **period integral.** It is absolutely convergent since $H(F) \backslash \tilde{H}(\mathbb{A}_F)$ is compact.

We consider the subspace

$$V := \{ \varphi \in L^2([G]) : \mathcal{P}_\chi(R(g)\varphi) = 0 \text{ for almost every } g \in A_G \backslash G(\mathbb{A}_F) \},$$

9.2 The cuspidal subspace

where \( R \) is defined as in (9.1). We will show that \( V \) is closed:

**Proposition 9.2.2** The subspace \( V \subseteq L^2([G]) \) is closed.

The proof of Proposition 9.2.2 will be given at the end of this section. As a corollary of Proposition 9.2.2, it is easy to obtain Lemma 9.2.1:

**Proof of Lemma 9.2.1:** For parabolic subgroups \( P \leq G \) with unipotent radical \( N_P \), let \( V_P \) be the space \( V \) of Proposition 9.2.2 in the special case where \( H = N_P \) and \( \chi \) is the trivial character. Then

\[
L_{\text{cusp}}^2([G]) = \bigcap_P V_P.
\]

Thus \( L_{\text{cusp}}^2([G]) \), being the intersection of closed subspaces, is closed. \( \square \)

For \( f \in C_c^\infty(A_G \backslash G(A_F)) \), we consider the **kernel function**

\[
K_f(x, y) = \sum_{\gamma \in G(F)} f(x^{-1} \gamma y).
\]

(9.7)

It is easy to see that the sum is absolutely convergent (in fact finite for \( x \) and \( y \) in a fixed compact set) and that \( K_f(x, y) \) defines a smooth function on \( A_G \backslash G(A_F) \times A_G \backslash G(A_F) \) (see §16.1). We define the **Poincaré series**

\[
P\hat{\chi}_f(g) := P\hat{\chi}_{f,H}(g) := \int_{H(F) \backslash H(A_F)} K_f(h, g) \overline{\chi}(h) dh.
\]

(9.8)

Since \( H(F) \backslash H(A_F) \) is compact, the integral converges absolutely. If \( G \) is reductive, \( H = N \) is the unipotent radical of a Borel subgroup and \( \chi \) is a generic character of \( N(A_F) \) in the sense of §11.2 below, then this coincides with the Poincaré series studied in many places, including [Fri87] and [Ste87].

**Lemma 9.2.3** One has that \( P\hat{\chi}_f \in L^2([G]) \).

**Proof.** One has that

\[
P\hat{\chi}_f(g) P\hat{\chi}_f(g) = \left( \int_{H(F) \backslash H(A_F)} \sum_{\gamma_1 \in G(F)} f(h_1^{-1} \gamma_1 g) \overline{\chi}(h_1) dh_1 \right) \times \left( \int_{H(F) \backslash H(A_F)} \sum_{\gamma_2 \in G(F)} f(h_2^{-1} \gamma_2 g) \chi(h_2) dh_2 \right).
\]

(9.9)

Choose a measurable fundamental domain \( F \) for \( H(F) \backslash H(A_F) \). Then

\[
\tilde{f}(g) := \int_F f(h^{-1} g) \overline{\chi}(h) dh
\]

(9.10)
is in $C_c^\infty(A_G\setminus G(\mathbb{A}_F))$. Thus from (9.9) and (9.10), we obtain that
\[
\int_{[G]} \hat{P}_f(g)\overline{\hat{P}_f(g)} dg = \int_{[G]} \left( \sum_{\gamma_1 \in G(F)} \overline{\hat{f}(\gamma_1 g)} \sum_{\gamma_2 \in G(F)} \hat{f}(\gamma_2 g) \right) dg.
\]
Since $\sum_{\gamma \in G(F)} \overline{\hat{f}(\gamma)}$ is a compactly supported continuous function on $[G]$, this integral converges. \hfill \Box

In the proof of Lemma 9.2.5 below, we will encounter for the first time the trivial, but useful, technique known as unfolding. We formalize this in the following lemma, the proof of which is a special case of Theorem 3.2.2:

**Lemma 9.2.4** Suppose that $G$ is a Hausdorff, locally compact, second countable topological group with right Haar measure $dg$. If $f \in L^1(G)$ and $\Gamma \leq G$ is a discrete subgroup such that the modular character of $G$ is trivial on $\Gamma$ then
\[
\int_{\Gamma \setminus G} \sum_{\gamma \in \Gamma} f(\gamma g) dg = \int_{G} f(g) dg.
\]
Here in the lemma we have denoted also by $dg$ the right $G$-invariant measure on $\Gamma \setminus G$ induced by $dg$. Use this technique, we prove the following lemma.

**Lemma 9.2.5** One has that
\[
\int_{[G]} \hat{P}_f(g)\varphi(g) dg = \mathcal{P}_\lambda(R(f)\varphi).
\]

**Proof.** Let $\varphi \in L^2([G])$. Then by Lemma 9.2.3 the integral
\[
\int_{[G]} \hat{P}_f(g)\varphi(g) dg
\]
is absolutely convergent. It is equal to
\[
\int_{[G]} \int_{H(F) \setminus \overline{H(\mathbb{A}_F)}} K_f(h, g)\overline{\chi(h)} dh \varphi(g) dg
\]
for $f \in C_c^\infty(A_G\setminus G(\mathbb{A}_F))$. Manipulating formally, we have
\[
\int_{[G]} \int_{H(F) \setminus \overline{H(\mathbb{A}_F)}} K_f(h, g)\overline{\chi(h)} dh \varphi(g) dg
\]
\[
= \int_{H(F) \setminus \overline{H(\mathbb{A}_F)}} \left( \int_{[G]} K_f(h, g)\varphi(g) dg \right) \overline{\chi(h)} dh
\]
\[ = \int_{H(F) \backslash \tilde{H}(\mathcal{A}_F)} \left( \int_{A_G \backslash G(\mathcal{A}_F)} f(g)\varphi(hg)dg \right) \chi(h)dh. \]

Here, in the last equality, we have unfolded as in Lemma 9.2.4 to deduce that

\[ \int_{[G]} K_f(h, g)\varphi(g)dg = \int_{A_G \backslash G(\mathcal{A}_F)} f(g)\varphi(hg)dg. \quad (9.11) \]

The integral on the right of (9.11) converges absolutely and defines a continuous function of \( h \). Combining this with the compactness of \( H(F) \backslash \tilde{H}(\mathcal{A}_F) \) and dominated convergence, we deduce that our formal manipulations above are justified. \( \square \)

Given our preparation, the proof of Proposition 9.2.2 is now straightforward:

**Proof of Proposition 9.2.2:** Each \( f \in C^\infty_c(A_G \backslash G(\mathcal{A}_F)) \) gives rise to a linear form \((\cdot, \Phi_f)\) on \( L^2([G]) \). It is continuous by Lemma 9.2.3. In view of Lemma 9.2.5 the space \( V \) is the intersection over all \( f \in C^\infty_c(A_G \backslash G(\mathcal{A}_F)) \) of the kernels of these linear forms. \( \square \)

### 9.3 Deduction of the discreteness of the spectrum

We now explain how to deduce the discreteness of the spectrum in Corollary 9.1.2 from the assertion that certain operators are of trace class in Theorem 9.1.1. As in the previous section, we do not yet assume that \( G \) is reductive.

A **Dirac sequence** on \( A_G \backslash G(\mathcal{A}_F) \) is a sequence of nonnegative real valued functions \( f_n \in C^\infty_c(A_G \backslash G(\mathcal{A}_F)) \) indexed by \( n \in \mathbb{Z}_{>0} \) such that

(a) For any open neighborhood \( U \ni 1 \) in \( A_G \backslash G(\mathcal{A}_F) \) with compact closure, the support of \( f_n \) is contained in \( U \) for \( n \) large enough.

(b) One has that \( f_n(x^{-1}) = f_n(x) \).

(c) One has that

\[ \int_{A_G \backslash G(\mathcal{A}_F)} f_n(x)dx = 1 \]

for all \( n \).

It is not difficult to see that a Dirac sequence on \( A_G \backslash G(\mathcal{A}_F) \) exists (see Exercise 9.7). They are known as Dirac sequences because for representations \((R, V)\) of \( A_G \backslash G(\mathcal{A}_F) \) and \( \varphi \in V \) one has \( R(f_n)\varphi \to \varphi \) as \( n \to \infty \) (see Exercise 9.8). We also note that if \( \{f_n\} \) is a Dirac sequence and \((R, V)\) is a unitary representation of \( A_G \backslash G(\mathcal{A}_F) \) then \( R(f_n) \) is self-adjoint for every \( n \) by condition (b).
We recall that a compact operator $T$ on a Hilbert space $V$ with an inner product $(\cdot, \cdot)$ can be written as

$$T = \sum_{n=1}^{\infty} \lambda_n (\varphi_n, \cdot) \varphi'_n,$$

where $\{\varphi_n\}_{n>0}$ and $\{\varphi'_n\}_{n>0}$ are orthonormal sequences of vectors in $V$ and $\{\lambda_n\}_{n>0} \subset \mathbb{C}$ is a sequence of numbers such that $\lim_{n \to \infty} \lambda_n = 0$. If $T$ is self-adjoint, then we can take $\varphi_n = \varphi'_n$, and in particular in this case $T$ has an orthonormal basis of eigenvectors (see [Zhu07, §1.3], for example).

We observe that if $T$ is a compact operator on $V$ with nonzero eigenvalue $\lambda$ then the $\lambda$-eigenspace is finite dimensional. This follows from the fact that $\lim_{n \to \infty} \lambda_n = 0$ in the decomposition (9.12).

**Lemma 9.3.1** Let $(R, V)$ be a unitary representation of $A_G \backslash G(\mathbb{A}_F)$. Assume that there is a Dirac sequence $\{f_n\}$ on $A_G \backslash G(\mathbb{A}_F)$ such that $R(f_n)$ is compact for all $n$. Then $(R, V)$ is the Hilbert space direct sum of its irreducible subrepresentations. The multiplicity of each subrepresentation is finite.

**Proof.** Our proof is an adaptation of the proof of [Bor97, Lemma 16.1]. We first show that a closed $A_G \backslash G(\mathbb{A}_F)$-invariant nonzero subspace $W \leq V$ contains a closed $A_G \backslash G(\mathbb{A}_F)$-invariant subspace that is minimal among nonzero closed $A_G \backslash G(\mathbb{A}_F)$-invariant subspaces. Let $W$ be a closed $A_G \backslash G(\mathbb{A}_F)$-invariant subspace. Clearly there exists a $j$ such that $R(f_j)|_W \neq 0$. Let $\lambda$ be a nonzero eigenvalue of $R(f_j)$ in $W$ and let $E_\lambda \neq 0$ be the corresponding finite dimensional eigenspace in $W$. Let $M$ be a minimal space among the nonzero intersections of $E_\lambda$ with the closed $A_G \backslash G(\mathbb{A}_F)$-invariant subspaces of $W$; such a space exists because $E_\lambda$ is finite dimensional. Let $\varphi$ be a nonzero element in $M$ and let $P$ be the smallest closed $A_G \backslash G(\mathbb{A}_F)$-invariant subspace containing $\varphi$. By construction, $P \cap E_\lambda = M$. We claim that $P$ is a minimal closed $A_G \backslash G(\mathbb{A}_F)$-invariant subspace of $P$ and $Q^\perp$ be its orthogonal complement in $P$. Then $P = Q \oplus Q^\perp$. The spaces $P$, $Q$, $Q^\perp$ and $E_\lambda$ are invariant under $R(f_j)$, hence $M = P \cap E_\lambda = (Q \cap E_\lambda) \oplus (Q^\perp \cap E_\lambda)$. Therefore, either $M = Q \cap E_\lambda$ with $P = Q$ and $Q^\perp = 0$ or $M = Q^\perp \cap E_\lambda$ with $P = Q^\perp$ and $Q = 0$.

Next, we prove that $(R, V)$ has a discrete decomposition into closed irreducible $A_G \backslash G(\mathbb{A}_F)$-invariant subspaces. Let $S$ be the set consisting of the nonzero closed $A_G \backslash G(\mathbb{A}_F)$-invariant subspaces of $V$ admitting a discrete decomposition. If we take a minimal closed $A_G \backslash G(\mathbb{A}_F)$-invariant subspace $M$ of $V$ (which exists by the previous paragraph) then $M$ is irreducible, hence $S$ is not empty. The set $S$ is partially ordered by the relation $X \leq Y$ if the space of $X$ is contained in that of $Y$. Let $W$ be the space given by a maximal element. If $W \neq V$, then we could add to $W$ a minimal closed nonzero $A_G \backslash G(\mathbb{A}_F)$-invariant subspace in the orthogonal complement of $W$, which exists by the previous claim. This contradiction implies that $W = V$. 

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Finally, we will show that the multiplicities are finite. Let \( \pi \) be an irreducible unitary representation occurring in \( V \) and let \( m_\pi \) be its multiplicity. Let \( j \) be such that \( \pi(f_j) \) is not zero on the space of \( \pi \), let \( \lambda \) be a nonzero eigenvalue of \( \pi(f_j) \) and \( m_\lambda \) be its multiplicity. Then the dimension of the eigenspace of \( R(f_j) \) in \( V \) with eigenvalue \( \lambda \) is finite and is at least \( m_\lambda m_\pi \). Therefore \( m_\pi \) is finite.

\[ \square \]

**Corollary 9.3.2** Theorem 9.1.1 implies Corollary 9.1.2.

**Proof.** Trace class operators are Hilbert-Schmidt and Hilbert-Schmidt operators are compact (see Exercise 9.3).

In view of Lemma 9.2.1 and Lemma 9.3.1, to prove Theorem 9.1.1 it suffices to show that \( R_{\text{cusp}}(f) \) is compact for all \( f \in C_0^\infty(A_G \backslash G(\mathbb{A}_F)) \). The following lemma gives us a criterion:

**Lemma 9.3.3** Let \( V \subseteq L^2([G]) \) be a closed \( A_G \backslash G(\mathbb{A}_F) \)-invariant subspace. For \( f \in C_0^\infty(A_G \backslash G(\mathbb{A}_F)) \), assume that one has an estimate

\[ \|R(f)\varphi\|_\infty \ll_f \|\varphi\|_2 \]

for all \( \varphi \in V \). Then the operator \( R(f)|_V \) is of trace class.

In the lemma and below \( \|\cdot\|_\infty \) denotes the uniform norm.

**Proof.** This follows from the argument of the proof of [Bor97, Theorem 9.5] which we will recall here: For given \( x \in A_G \backslash G(\mathbb{A}_F) \), the assumption implies that the map \( \varphi \mapsto R(f)\varphi(x) \) is a continuous linear form on \( V \). Hence by the Riesz representation theorem, there exists an element \( K_{R(f),x} \in V \) such that

\[ R(f)\varphi(x) = \int_{[G]} K_{R(f),x}(y)\varphi(y)dy \]

for all \( \varphi \in V \). Taking \( \varphi = K_{R(f),x} \) we obtain

\[ \|K_{R(f),x}\|_2^2 \leq \|R(f)K_{R(f),x}\|_\infty \ll_f \|K_{R(f),x}\|_2 \]

so it follows that \( \|K_{R(f),x}\|_2 \leq C \) for some constant \( C > 0 \). Write

\[ K_{R(f)}(x,y) := K_{R(f),x}(y) \]

for \( x, y \in [G] \). Then

\[ \int_{[G]} \int_{[G]} |K_{R(f)}(x,y)|^2dxdy = \int_{[G]} \left( \int_{[G]} K_{R(f)}(y)K_{R(f),x}(y)dy \right)dx \]

\[ \leq C^2 \int_{[G]} dx \]
is finite and
\[ R(f)\varphi(x) = \int_{[G]} K_{R(f)}(x, y)\varphi(y)dy \]
for \( \varphi \in V \). Therefore the operator \( R(f) \) on \( V \) is represented by an \( L^2 \)-kernel. This implies that it is Hilbert-Schmidt (see Exercise 9.4).

It follows from the Dixmier Malliavin lemma (Theorem 4.2.7) that \( f \) is a finite linear combination of convolutions \( f_1 * f_2 \) with \( f_1, f_2 \in C_c^\infty(A_G \setminus G(\mathbb{A}_F)) \) (see Exercise 9.9). Therefore \( R(f) \) is a finite linear combination of convolutions of two Hilbert-Schmidt operators and hence is of trace class (see Exercise 9.2).

In view of lemmas 9.2.1 and 9.3.3, to prove Theorem 9.1.1, it suffices to prove that for \( f \in C_c^\infty(A_G \setminus G(\mathbb{A}_F)) \) one has an estimate
\[ \|R(f)\varphi\|_\infty \ll_f \|\varphi\|_2 \quad (9.13) \]
for \( \varphi \in L^2_\text{cusp}([G]) \). This estimate is really the heart of the proof. In the function field case we will be able to deduce something much stronger, and in the number field case we derive it using the argument of [God66]. It is only in this part of the proof where we use the assumption that \( G \) is reductive. We have been careful in isolating when this assumption is used for two reasons. First, it is helpful to see at what point the argument passes from abstract representation theory to something that uses the specific structure of the adelic quotient in the reductive case. Second, there ought to be a notion of a cuspidal representation on more general groups than those that are reductive, but we do not know the natural level of generality. We feel that this is an interesting research problem. One example that has been worked out reasonably thoroughly is the case where \( G \) is the Jacobi group, a semidirect product of \( \text{SL}_2 \) and the 3-dimensional Heisenberg group [BS98, §4.2]. The work in [Sli84] is probably relevant here as well.

### 9.4 The basic estimate

We now assume that \( G \) is reductive and that \( F \) is a number field. Our aim is prove (9.13). As explained in the previous section, this will complete the proof of Theorem 9.1.1 in the number field case. To motivate the approach, let us consider the case \( G = \text{GL}_2 \) over \( \mathbb{Q} \). To make things as concrete as possible, we assume that
\[ \varphi : [\text{GL}_2] \to \mathbb{C} \]
is a smooth cuspidal function invariant under \( O_2(\mathbb{R})\text{GL}_2(\mathbb{Z}) \); in this case we can identify \( \varphi \) with a function
\[ \varphi : \text{SL}_2(\mathbb{Z}) \setminus \mathfrak{H} \to \mathbb{C} \]
9.4 The basic estimate

(compare (2.20), (6.26)). We can choose a fundamental domain for $\text{SL}_2(\mathbb{Z})$ contained in the Siegel set

$$\{ z \in \mathcal{H} : \text{Im}(z) > \frac{1}{2}, 0 \leq \text{Re}(z) \leq 1 \}$$

(9.14)

by Exercise 2.18. Thus it suffices to bound $\varphi$ in this region. Let $B \leq \text{GL}_2$ be the Borel subgroup of upper triangular matrices and $N_B \leq B$ the unipotent radical of $B$. Then $\varphi$ is invariant under $N_B(\mathbb{Z})$ on the left, which is to say that it is invariant under $z \mapsto z + 1$. Thus it can be expanded in a Fourier series

$$\varphi(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i nz}.$$ 

The key point here is that cuspidality implies that only positive $n$ contribute. Since $\varphi$ is continuous, the $a_n$ are bounded and this implies that $\varphi(z)$ is rapidly decreasing as $\text{Im}(z) \to \infty$. Though this is not yet enough to immediately deduce (9.13), it is a strong indication that the growth of $\varphi(z)$ is reasonable on a fundamental domain. In the discussion above, we have used Fourier analysis on

$$\mathbb{Z} \setminus \mathbb{R} \cong N_B(\mathbb{Z}) \setminus N_B(\mathbb{R}).$$

In the general case, we will again use Fourier analysis on the unipotent radical of a minimal parabolic subgroup.

We now begin our discussion of the general case, so we let $G$ be a reductive group over a number field $F$. By applying Weil restriction of scalars, we can and do assume $F = \mathbb{Q}$. Let $P$ be a fixed minimal parabolic subgroup of $G$ with unipotent radical $N$ and let $T \leq P$ be a maximal split torus. Then $M := C_G(T)$ is a Levi subgroup of $P$ [Mil17, Theorem 25.6]. Let $T_G \leq Z_G$ be a maximal split torus, and let $T_0 \leq G^{\text{der}}$ be a maximal split torus such that $T_0 T_G = T$. The exponential map induces an isomorphism of affine schemes over $\mathbb{Q}$:

$$\exp : n \longrightarrow N$$

[Mil17, Proposition 14.32]. Here we are slightly abusing notation in that we are denoting by $n$ the commutative algebraic group over $\mathbb{Q}$ whose points in a $\mathbb{Q}$-algebra $R$ are

$$n(R) := n \otimes_\mathbb{Q} R.$$ 

This group scheme is isomorphic to $G^{\text{dim} n}$. Note that we are not claiming that the exponential map is an isomorphism with respect to any sort of group structure.

Any element $\varphi \in L^2([G])$ is left-invariant under $G(\mathbb{Q})$. Thus for $f \in C_c^\infty(A_G \setminus G(\mathbb{A}_\mathbb{Q}))$ we can write
\[ R(f)\varphi(x) = \int_{A_0 \backslash G(A_\mathbb{Q})} f(x^{-1}y)\varphi(y)dy \]
\[ = \int_{A_0 N(\mathbb{Q}) \backslash G(A_\mathbb{Q})} \sum_{\gamma \in N(\mathbb{Q})} f(x^{-1}\gamma y)\varphi(y)dy \quad (9.15) \]
\[ = \int_{A_0 N(\mathbb{Q}) \backslash G(A_\mathbb{Q})} K_{f,P}(x,y)\varphi(y)dy, \]

where for \( x, y \in G(A_\mathbb{Q}) \) we set

\[ K_{f,P}(x,y) = \sum_{\gamma \in N(\mathbb{Q})} f(x^{-1}\gamma y) = \sum_{\eta \in n(\mathbb{Q})} f(x^{-1}\exp(\eta)y). \quad (9.16) \]

We view \( n \) as an affine abelian algebraic group over \( \mathbb{Q} \) with addition as the law of composition. In particular,

\[ [n] := n(\mathbb{Q}) \backslash n(A_\mathbb{Q}) \]

is an abelian group. We note that for fixed \( x, y \in G(A_\mathbb{Q}) \) the function

\[ n \mapsto f(x^{-1}\exp(n)y) \]

is smooth on \( n(A_\mathbb{Q}) \). Let \( \psi : \mathbb{Q} \backslash A_\mathbb{Q} \to \mathbb{C}^\times \) be a nontrivial character and let

\[ (, ) : n \times n \to \mathbb{G}_a \]

be a perfect pairing (over \( \mathbb{Q} \)). Then by Poisson summation one has

\[ K_{f,P}(x,y) = \sum_{\eta \in n(\mathbb{Q})} \int_{n(A_\mathbb{Q})} f(x^{-1}\exp(n)y)\psi((\eta,n))dn \quad (9.17) \]

for an appropriate choice of (additive) Haar measure \( n(A_\mathbb{Q}) \). For a primer on Poisson summation over global fields, see Appendix B. Using (9.15) and (9.17) we obtain that

\[ R(f)\varphi(x) = \int_{A_0 N(\mathbb{Q}) \backslash G(A_\mathbb{Q})} K_{f,P}(x,y)\varphi(y)dy \]
\[ = \int_{A_0 N(\mathbb{Q}) \backslash G(A_\mathbb{Q})} \left( \sum_{\eta \in n(\mathbb{Q})} \int_{n(A_\mathbb{Q})} f(x^{-1}\exp(n)y)\psi((\eta,n))dn \right)\varphi(y)dy. \quad (9.18) \]

We will estimate \( R(f)\varphi(x) \) in Proposition 9.4.5 below using this expansion.

In the following we will freely use notation from our discussion of Siegel sets in §2.7. By the Iwasawa decomposition (2.21) we can choose a maximal compact subgroup \( K \leq G(A_\mathbb{Q}) \) such that any element \( y \in G(A_\mathbb{Q}) \) can be written as
y = u_y m_y s_y k_y \quad (9.19)

with \((u_y, m_y, s_y, k_y) \in N(\mathbb{A}_Q) \times M(\mathbb{A}_Q)^1 \times A_{T_0}(t) \times K\) for some \(t > 0\). The decomposition is not unique, but \(s_y\) is uniquely determined by \(y\), and can be recovered from \(y\) using the Harish-Chandra map \(H_{MN}\) defined below (4.18).

We shall denote

\[ \mathcal{S}(t) = \Omega_N \Omega_M A_{T_0}(t) K \quad (9.20) \]

a fixed Siegel domain in \(G(\mathbb{A}_Q)^1\). Here \(\Omega_N\) and \(\Omega_M\) are fixed compact subsets of \(N(\mathbb{A}_Q)\) and \(M(\mathbb{A}_Q)^1\), respectively.

**Lemma 9.4.1** Assume \(x \in \mathcal{S}(t)\). Then \(s_x^{-1} x\) lies in a compact subset of \(G(\mathbb{A}_Q)^1\) depending only on \(\mathcal{S}(t)\).

**Proof.** If \(x \in \mathcal{S}(t)\) we have

\[ x \in \Omega_N \Omega_M s_x K = \Omega_N s_x \Omega_M K \]

for some compact subsets \(\Omega_N \subset N(\mathbb{A}_Q)\) and \(\Omega_M \subset M(\mathbb{A}_Q)^1\). Here we are using the fact that \(s_x\) commutes with \(M(\mathbb{A}_Q) = C_G(T)(\mathbb{A}_Q)\).

As \(x\) varies over \(\mathcal{S}(t)\), one has that \(s_x^{-1} \Omega_N s_x\) lies in a compact subset of \(N(\mathbb{A}_Q)\) (see Exercise 9.10). Thus \(x \in s_x \Omega_G\) for some compact subset \(\Omega_G\) of \(G(\mathbb{A}_Q)^1\).

**Lemma 9.4.2** Fix a compact subset \(\Omega_N \subset N(\mathbb{A}_Q)\). Assume that \(x \in \mathcal{S}(t)\) and \(y \in \Omega_N M(\mathbb{A}_Q) K\). Let \(\Omega \subset G(\mathbb{A}_F)^1\) be a compact set. If

\[ x^{-1} \gamma y \in \Omega \]

for some \(\gamma \in N(F)\), then \(x^{-1} s_x\) and \(s^{-1}_x y\) lie in compact subsets of \(G(\mathbb{A}_Q)^1\) depending only on the support of \(f\), on \(\mathcal{S}(t)\) and on \(\Omega_N\). Moreover \(m_y\) lies in a compact subset of \(M(\mathbb{A}_Q)^1\) that depends only on the support of \(f\) and on \(\mathcal{S}(t)\).

**Proof.** Since \(x \in \mathcal{S}(t)\), one has by Lemma 9.4.1 that \(x \in s_x \Omega_G\) for some compact subset \(\Omega_G\) of \(G(\mathbb{A}_Q)^1\) depending only on the Siegel set \(\mathcal{S}(t)\). We can take \(\Omega_G\) to be invariant under \(g \mapsto g^{-1}\) and then it follows that \(x^{-1} \in \Omega_G s_x^{-1}\).

Now if \(f(x^{-1}\gamma y) \neq 0\) then \(x^{-1} \gamma y\) lies in a compact subset \(\Omega_G\) of \(G(\mathbb{A}_Q)^1\) depending only on \(f\). By our earlier considerations we see that upon enlarging \(\Omega_G\) if necessary

\[ s_x^{-1} \gamma y = s_x^{-1} \gamma u_y s_x s^{-1}_x m_y s_y k_y \in \Omega_G \quad (9.21) \]

and hence (again enlarging \(\Omega_G\) if necessary) \(s_x^{-1} \gamma u_y s_x m_y s_y^{-1} s_y \in \Omega_G\).

We deduce that \(m_y\) lies in a compact subset \(\Omega_M \subset M(\mathbb{A}_Q)^1\) and \(s_x^{-1} s_y\) lies in a compact subset of \(A_{T_0}\) depending only on \(f\) and \(\mathcal{S}(t)\). Using Lemma 9.4.1 we deduce that \(s_y y\) lies in a compact subset of \(G(\mathbb{A}_Q)^1\) depending only
on \( f, \Omega_N, \) and \( \mathcal{S}(t) \). Since \( s_x^{-1}s_y \) lies in a compact subset of \( A_Tn \), depending only on \( f \) and \( \mathcal{S}(t) \) it follows that \( s_x^{-1}y \) lies in a compact subset of \( G(A_Q)^1 \) depending only on \( f, \Omega_N, \) and \( \mathcal{S}(t) \).

In the previous two lemmas we worked with subsets of \( G(A_Q)^1 \) instead of \( A_G \setminus G(A_Q) \). This was merely for notational convenience. Indeed, Lemma 2.6.2 implies the natural map

\[
G(A_Q)^1 \rightarrow A_G \setminus G(A_Q)
\]  

(9.22)
is an isomorphism of topological groups. Using this isomorphism we will identify elements of \( G(A_Q)^1 \) with their images in \( A_G \setminus G(A_Q) \) when convenient.

Let \( \Phi \) be the set of roots of \( T \) in \( G \). The subset \( \Phi^+ \subset \Phi \) of roots appearing in \( n \) is a set of positive roots. Choose a norm

\[
\| \cdot \|_n
\]
on the real vector space \( n(\mathbb{R}) \). The next proposition, Proposition 9.4.3, gives us good analytic control over many of the terms in the sum over \( \eta \) in (9.15).

It will turn out that the terms that cannot be estimated using Proposition 9.4.3 vanish since \( \varphi \) is cuspidal (see the proof of Proposition 9.4.5 below).

Let \( n_\lambda < n \) be the root space of a root \( \lambda \in \Phi^+ \).

**Proposition 9.4.3** Assume that \( y \in \Omega_NM(A_Q)K \) for some fixed compact subset \( \Omega_N \subset N(A_Q) \). Let \( B > 0 \) and assume \( \eta \in n(\mathbb{Q}) \) is not zero. There is a compact subset \( \Omega_n^\infty \subset n(A_Q^\infty) \) such that for any \( x \in \mathcal{S}(t) \) one has that

\[
\int_{n(A_Q)} f(x^{-1}\exp(n)y)\psi((\eta, n)) \, dn
\]

\[
\ll_{f, B, \Omega_N, \mathcal{S}(t)} \|\eta\|_n^{-B} \mathbb{1}_{\Omega_n^\infty}(\eta) \delta_P(s_x)\lambda(s_x)^{-B}
\]

for any \( \lambda \in \Phi^+ \) such that \( \langle \eta, \cdot \rangle|_{n_{\lambda}} \neq 0 \).

**Proof.** One has that

\[
\int_{n(A_Q)} f(x^{-1}\exp(n)y)\psi((\eta, n)) \, dn
\]

\[
= \int_{n(A_Q)} f(x^{-1}s_x \exp(\text{Ad}(s_x^{-1})n)s_x^{-1}y)\psi((\eta, n)) \, dn
\]

\[
= \delta_P(s_x) \int_{n(A_Q)} f(x^{-1}s_x \exp(n)s_x^{-1}y)\psi((\eta, \text{Ad}(s_x)n)) \, dn.
\]

Using Lemma 9.4.2 and integration by parts at the infinite place we deduce that for all \( B > 0 \)

\[
\int_{n(A_Q)} f(x^{-1}\exp(n)y)\psi((\eta, n)) \, dn \ll \delta_P(s_x)\|\text{Ad}(s_x^{-1})\eta\|_n^{-2B} \mathbb{1}_{\Omega_n^\infty}(\eta),
\]
where the implied constant depends on $f$, $\mathcal{G}(t)$ and $\Omega_N$. Now for all $\lambda \in \Phi^+$ we have that $\lambda(s_x)$ is bounded below by a nonzero constant depending on $t$. Moreover, if $\eta \in \Omega^\infty_n \cap n(\mathbb{Q})$ then $(\eta, \cdot)|_{n_\lambda}$ lies in a lattice in the $\mathbb{R}$-linear dual $n^\vee_\lambda(\mathbb{R})$ of $n_\lambda(\mathbb{R})$ depending only on $\Omega^\infty_n$. In particular, regarded as an element of the real vector space $n^\vee_\lambda(\mathbb{R})$, the linear form $(\eta, \cdot)|_{n_\lambda}$ is either zero or lies in the complement of an open neighborhood of zero depending only on $\Omega^\infty_n$. Thus if $(\eta, \cdot)|_{n_\lambda} \neq 0$, then

$$\mathbb{1}_{\Omega^\infty_n}(\eta)\|\text{Ad}^\vee(s_x^{-1})\eta\|_n \gtrsim \mathbb{1}_{\Omega^\infty_n}(\eta)\max(\|\eta\|, \lambda(s_x)) \geq \mathbb{1}_{\Omega^\infty_n}(\eta)\|\eta\|^{1/2}\lambda(s_x)^{1/2}.$$

Here the implied constant depends on $t$ and $\Omega^\infty_n$. The proposition follows. $\square$

Let $\Delta$ be the set of simple roots attached to the set of positive roots $\Phi^+$. For each simple root $\alpha \in \Delta$, let

$$\Phi_\alpha^+ \subset \Phi^+$$

be the set of roots of $T$ in $G$ of the form $\lambda = \sum_{\beta \in \Delta} m_\beta \beta$ with $m_\beta \geq 0$ for all $\beta \neq \alpha$ and $m_\alpha > 0$. Let

$$n(\Phi_\alpha^+) := \oplus_{\lambda \in \Phi^+_\alpha} n_\lambda.$$

(9.23)

It is clear that this is an ideal in $n$. Let $N(\Phi_\alpha^+) \triangleleft N$ be the connected subgroup of $N$ with Lie algebra $n(\Phi_\alpha^+)$. It is the unipotent radical of a maximal parabolic subgroup of $G$ containing $P$. In fact, if we take $P_0 = P$ in the notation of §1.9, then in the notation of Theorem 1.9.2 the maximal parabolic with unipotent radical $N(\Phi_\alpha^+)$ is $P_{\Delta \setminus \{\alpha\}}$.

**Lemma 9.4.4** Let $\alpha \in \Delta$ and assume that $(\eta, \cdot)$ is trivial on $n(\Phi_\alpha^+)$. Then

$$y \mapsto \int_{n(\mathbb{A}_Q)} f(x^{-1}\exp(n)y)\psi((\eta, n))dn$$

is invariant under multiplication by $N(\Phi_\alpha^+)$. $\mathbb{A}_Q$ on the left.

**Proof.** Let $n \in n(\mathbb{A}_Q)$ and $u \in N(\Phi_\alpha^+)$. Choose $n' \in n(\mathbb{A}_Q)$ such that $\exp(n') = u$. One has

$$\exp(n)u = \exp(n + w(n, n'))$$

for some $w(n, n') \in n(\mathbb{A}_Q)$ by the Baker-Campbell-Hausdorff formula. As mentioned above, $n(\Phi_\alpha^+)$ is an ideal in $n$. Combining this with the fact that $w(n, n')$ is given in terms of commutators of $n$ and $n'$ we deduce that $w(n, n') \in n(\Phi_\alpha^+)$. Thus

$$\int_{n(\mathbb{A}_Q)} f(x^{-1}\exp(n)uy)\psi((\eta, n))dn$$


The main result of this section is the following proposition:

**Proposition 9.4.5** Let \( \varphi \in L_c^{2}(\mathcal{G}) \). Then for any \( f \in C_{c}^{\infty}(A_{G}\backslash G(A_{\mathfrak{q}})) \), any \( B \in \mathbb{R}_{>0} \), and any \( \alpha \in \Delta \) one has that

\[
|R_{cusp}(f)\varphi(x)| \ll_{f,B,\mathcal{S}(t)} \alpha(s_{x})^{-B} \|\varphi\|_{2}
\]

for \( x \in \mathcal{S}(t) \).

Combined with Lemma 9.2.1 and Lemma 9.3.3 this completes the proof of Theorem 9.1.1 in the number field case. We will complete the proof in the function field case in Section 9.5 below.

**Proof.** In view of (9.18) to prove the proposition it suffices to bound

\[
\int_{A_{G}N(\mathbb{Q})\backslash G(A_{\mathfrak{q}})} \left( \sum_{\eta \in \mathfrak{n}(\mathbb{Q})} \int_{\mathfrak{n}(A_{\mathfrak{q}})} f(x^{-1}\exp(n)y)\psi((\eta, n))dn \right) \varphi(y)dy \quad (9.24)
\]

by a constant times \( \delta_{P}(s_{x})\alpha(s_{x})^{-B} \|\varphi\|_{2} \). Here and throughout the proof all constants are allowed to depend on \( f, B, \) and \( \mathcal{S}(t) \).

If \( \eta \in \mathfrak{n}(\mathbb{Q}) \) has the property that \( \langle \eta, \cdot \rangle \) is identically zero on \( \mathfrak{n}(\mathfrak{F}_{\alpha}^{+})(A_{\mathfrak{q}}) \) then

\[
y \mapsto \int_{\mathfrak{n}(A_{\mathfrak{q}})} f(x^{-1}\exp(n)y)\psi((\eta, n))dn
\]

is left-invariant under \( N(\mathfrak{F}_{\alpha}^{+})(A_{\mathfrak{q}}) \) by Lemma 9.4.4. Thus for such an \( \eta \)

\[
\text{meas}_{du}(\mathfrak{n}(\mathfrak{F}_{\alpha}^{+})) \int_{A_{G}N(\mathbb{Q})\backslash G(A_{\mathfrak{q}})} \left( \int_{\mathfrak{n}(A_{\mathfrak{q}})} f(x^{-1}\exp(n)y)\psi((\eta, n))dn \right) \varphi(y)dy
\]

\[
= \int_{A_{G}N(\mathbb{Q})\backslash G(A_{\mathfrak{q}})} \left( \int_{(N(\mathfrak{F}_{\alpha}^{+}))} \left( \int_{\mathfrak{n}(A_{\mathfrak{q}})} f(x^{-1}\exp(n)y)\psi((\eta, n))dn \right) du \varphi(y)dy \right)
\]

\[
= \int_{A_{G}N(\mathbb{Q})\backslash G(A_{\mathfrak{q}})} \left( \int_{\mathfrak{n}(A_{\mathfrak{q}})} f(x^{-1}\exp(n)y)\psi((\eta, n))dn \int_{(N(\mathfrak{F}_{\alpha}^{+}))} \varphi(u^{-1}y)du \right) dy
\]

\[
= 0
\]
since \( \varphi \) is cuspidal. Here we are using the fact that \( \mathcal{N}(\Phi^+_\alpha) \) is compact to justifying manipulating the integrals. Thus in (9.24) one can omit the summands indexed by those \( \eta \) for which \( \langle \eta, \cdot \rangle \) is identically zero on \( n(\Phi^+_\alpha)(\mathbb{A}_Q) \).

Suppose that \( \eta \in n(Q) \) has the property that \( \langle \eta, \cdot \rangle \) is not identically zero on \( n(\Phi^+_\alpha)(\mathbb{A}_Q) \). Then \( \langle \eta, \cdot \rangle \big|_{n_x} \neq 0 \) for some \( \lambda \in \Phi^+_\alpha \). Since have assumed \( x \in \mathcal{G}(t) \), \( \alpha'(s_x) \) is bounded below by a constant depending on \( t \) for all \( \alpha' \in \Delta \). Given this and the definition of \( \Phi^+_\alpha \), we see that Proposition 9.4.3 implies there is a compact subset \( \Omega^\infty_n \subset n(\mathbb{A}_Q) \) such that for any \( B > 0 \) one has that

\[
\int_{n(\mathbb{A}_Q)} f(x^{-1} \exp(n)y)\varphi(\langle \eta, n \rangle) dn \ll \|\eta\|^{-B}_n \mathbb{1}_{\Omega^\infty_n}(\eta) \delta_P(s_x) \alpha(s_x)^{-B}. \tag{9.25}
\]

Note that \( \sum_{\eta \in n(Q)} \|\eta\|^{-B}_n \mathbb{1}_{\Omega^\infty_n}(\eta) \) is bounded for \( B \) sufficiently large. Moreover we may choose a fundamental domain for \( N(Q) \) acting on \( A_G \backslash G(\mathbb{A}_Q) \) in (9.24) and thereby assume that the integral over \( y \) is supported in \( N(\mathbb{A}_Q) \) for some compact subset \( \Omega_n \subset N(\mathbb{A}_Q) \) and some maximal compact subgroup \( K \leq A_G \backslash G(\mathbb{A}_Q) \). Thus for \( B \) sufficiently large, Lemma 9.4.2 and (9.25) imply that (9.24) is bounded by

\[
R(f)\varphi(x) \ll \delta_P(s_x) \alpha(s_x)^{-B} \int_{s_x \Omega_G} \varphi(y) dy
\]

where \( \Omega_G \subset A_G \backslash G(\mathbb{A}_Q) \) is a \( dy \)-measurable compact set. By the Hölder inequality this is bounded by

\[
\delta_P(s_x) \alpha(s_x)^{-B} \text{meas}_{dy}(s_x \Omega_G)^{1/2} \left( \int_{s_x \Omega_G} |\varphi(y)|^2 dy \right)^{1/2} \leq \delta_P(s_x) \alpha(s_x)^{-B} \text{meas}_{dy}(\Omega_G)^{1/2} \|\varphi\|_2.
\]

Upon increasing \( B \) and varying \( \alpha \) if necessary we see that this estimate implies the estimate in the proposition. \( \square \)

### 9.5 The function field case

We still owe the reader a proof of Theorem 9.1.1 when \( F \) is a function field. Thus we continue to assume that \( G \) is reductive, but now assume that \( F \) is a function field. Rather than prove (9.13) we will argue more directly. What makes this possible is that the asymptotic behavior of cuspidal functions is much simpler in the function field case than their behavior in the number field case:
Theorem 9.5.1 (Harder) Let $\varphi \in L^2_{\text{cusp}}([G])$ be a function that is invariant under the compact open subgroup $K' \leq A_G \backslash G(\mathfrak{A}_F)$. Then $\varphi(x) = 0$ for $x$ outside a compact subset of $[G]$ that depends only on $K'$.

We follow Harder’s original proof [Har74], but generalize the argument from split to arbitrary reductive groups. Let $P$ be a fixed minimal parabolic subgroup of $G$ with unipotent radical $N$ and let $T \leq P$ be a maximal split torus. Then $M := C_G(T)$ is a Levi subgroup of $P$ [Mil17, Theorem 25.6]. Let $T_G \leq Z_G$ be a maximal split torus, and let $T_0 \leq G^{\text{der}}$ be a maximal split torus such that $T_0T_G = T$. Let $K \leq G(\mathfrak{A}_F)$ be a maximal compact subgroup such that the Iwasawa decomposition $N(\mathfrak{A}_F)M(\mathfrak{A}_F)K = G(\mathfrak{A}_F)$ holds. Let $K'$ be as in the statement of Theorem 9.5.1. By replacing $K'$ with $K' \cap K$ we can and do assume $K' \leq K$. Fix a compact subset $\Omega \subseteq N(\mathfrak{A}_F)M(\mathfrak{A}_F)^1$, $t \in \mathbb{R}_{>0}$ and let $\mathcal{S}(t)$ be the corresponding Siegel set as in (2.22). We assume that $\Omega$ and $K$ are chosen so that, for $t$ sufficiently small, one has that

$$G(F)\mathcal{S}(t) = G(\mathfrak{A}_F)^1;$$

this is possible by Theorem 2.7.2.

Let $\Phi^+$ be the set of positive roots of the split maximal torus $T \leq M$ acting on $N$ and $\Delta$ the corresponding base. For each simple root $\alpha \in \Delta$, let

$$\Phi_\alpha^+ \subset \Phi^+$$

be the set of positive roots of $T$ in $G$ of the form $\lambda = \sum_{\beta \in \Delta} m_\beta \beta$ with $m_\beta \geq 0$ for all $\beta \neq \alpha$ and $m_\alpha > 0$. Let $N(\Phi_\alpha^+) \triangleleft N$ be the closed connected subgroup of $N$ with Lie algebra $\mathfrak{n}(\Phi_\alpha^+)$ defined as in (9.23).

Lemma 9.5.2 If $t > 0$ is chosen sufficiently large, then for any $h \in \mathcal{S}(t)$ one has that

$$N(\Phi_\alpha^+)(F)(N(\Phi_\alpha^+) \cap hK'h^{-1}) = N(\Phi_\alpha^+)(\mathfrak{A}_F).$$

Proof. Observe that for any compact open subgroup $K_1 \leq \mathfrak{A}_F$ there is a constant $c_{K_1} > 1$ such that if $a \in \mathfrak{A}_{G_{m}}$ satisfies $|a| \geq c_{K_1}$ then $F + aK_1 = \mathfrak{A}_F$. Write

$$N(\Phi_\alpha^+) = \prod_{\lambda \in \Phi_\alpha^+} N_\lambda$$

where $N_\lambda$ is the root group of $\lambda$ as in (1.20). Then $N_\lambda(\mathfrak{A}_F) \cap hK'h^{-1} \subset N_\lambda(\mathfrak{A}_F)$ is a compact open subgroup for any $h \in G(\mathfrak{A}_F)$. Thus by our observation one has $N_\lambda(F)(N_\lambda(\mathfrak{A}_F) \cap hK'h^{-1}) = N_\lambda(\mathfrak{A}_F)$ for $h \in \mathcal{S}(t)$ if $t$ is sufficiently large.

We now complete the proof using an inductive argument. The set of positive roots $\Phi^+$ is partially ordered in the usual manner. Extend this partial order to a total order. For $\lambda' \in \Phi^+$, let
\[ N(\alpha, \lambda') := \prod_{\lambda' \geq \lambda} N_{\lambda}. \]

We have that \( N(\Phi^+_\alpha) := \bigcup_{\lambda' \in \Phi^+} N(\alpha, \lambda), \) i.e. \( N(\Phi^+_\alpha) \) is an increasing chain of products of the \( N_{\lambda}. \) We can now apply induction to deduce the lemma. \( \Box \)

**Proof of Theorem 9.5.1:** Let \( \varphi \in L^2_{\text{cusp}}(\Gamma) \)^{K'}. By Lemma 2.6.2, \( \Gamma \) is a finite index subgroup of \( G(\mathbb{A}_\mathbb{F}). \) Choose a set of representatives \( g_1, \ldots, g_n \) for \( \Gamma \). Upon replacing \( \Gamma \) by a finite index subgroup if necessary, we can and do assume that the functions

\[ x \mapsto \varphi(xg_i) \]

all lie in \( L^2_{\text{cusp}}(\Gamma) \)^{K'}. Choose \( t_0 > 0 \) and a Siegel set \( \mathcal{S}(t_0) := \Omega A_{F_0}(t_0)K \) so that \( G(F)\mathcal{S}(t_0) = G(\mathbb{A}_\mathbb{F}) \). Moreover choose \( t > t_0 \) so that the conclusion of Lemma 9.5.2 holds. Then for \( h \in \mathcal{S}(t) \) the functions

\[ N(\Phi^+_\alpha)(\mathbb{A}_\mathbb{F}) \rightarrow \mathbb{C} \]

\[ n \mapsto \varphi(nhg_i) \]

are left invariant under \( N(\Phi^+_\alpha)(F) \) and right invariant under \( N(\Phi^+_\alpha)(\mathbb{A}_\mathbb{F}) \cap hK'h^{-1}. \) Thus by Lemma 9.5.2 they are constant. On the other hand \( N(\Phi^+_\alpha) \) is the unipotent radical of a proper parabolic subgroup of \( G \) and we obtain

\[ 0 = \int_{[N(\Phi^+_\alpha)]} \varphi(nh)dn = \text{meas}_{dn}([N(\Phi^+_\alpha)])\varphi(hg_i) \]

and hence \( \varphi(hg_i) = 0. \) Thus \( \varphi \) vanishes on \( \mathcal{S}(t)g_i \) for sufficiently large \( t \) for each \( i. \) \( \Box \)

Using Theorem 9.5.1 we prove Theorem 9.1.1 in the function field case:

**Proof of Theorem 9.1.1 (for function fields):** By Lemma 9.2.1 the cuspidal subspace \( L^2_{\text{cusp}}(\Gamma) \) is closed in \( L^2(\Gamma). \) Let \( f \in C^\infty(\Gamma \setminus \mathbb{A}_\mathbb{F}). \) We must show \( R_{\text{cusp}}(f) \) is of trace class. As explained in more detail in §16.1, the operator

\[ R(f) : L^2(\Gamma) \rightarrow L^2(\Gamma) \]

is represented by the kernel

\[ K_f(x, y) := \sum_{\gamma \in G(F)} f(x^{-1}\gamma y). \]

It is not hard to see that this function is continuous as a function of \( (x, y) \in [\Gamma] \times [\Gamma] \) for more details see §16.1. Thus for any compact subset \( \Omega \subseteq [\Gamma] \)
the restriction of $K_f(x, y)$ to $\Omega \times \Omega$ is in $L^2(\Omega \times \Omega)$, and hence is a Hilbert-Schmidt operator.

Now there is a compact open subgroup $K \leq A_G \backslash G(A_F)$ such that $f \in C_c^{\infty}(A_G \backslash G(A_F) \sslash K)$. Thus for any $\varphi \in L^2_{\text{cusp}}([G])$ such that $R(f)\varphi \neq 0$ one has $\varphi \in L^2_{\text{cusp}}([G])^K$. Using Theorem 9.5.1 choose a compact subset $\Omega \subset [G]$ such that any element of $L^2_{\text{cusp}}([G])^K$ is supported in $\Omega$. In other words,

$$R_{\text{cusp}}(f) = R(f)|_{L^2(\Omega) \cap L^2_{\text{cusp}}([G])}.$$ 

Now $R(f)|_{L^2(\Omega)}$ is Hilbert-Schmidt, so its restriction to the closed subspace $L^2(\Omega) \cap L^2_{\text{cusp}}([G])$ of $L^2(\Omega)$ is also Hilbert-Schmidt (see Exercise 9.5). Thus we have proven that

$$R_{\text{cusp}}(f) : L^2_{\text{cusp}}([G]) \rightarrow L^2_{\text{cusp}}([G])$$

is Hilbert-Schmidt. On the other hand, every $f \in C_c^{\infty}(A_G \backslash G(A_F))$ is a finite linear combination of convolutions: $f = \sum_{i=1}^k h_{i1} \ast h_{i2}$ where $h_{ij} \in C_c^{\infty}(A_G \backslash G(A_F))$ (see Exercise 9.9). We deduce that

$$R_{\text{cusp}}(f) = \sum_{i=1}^k R(h_{i1}) \ast R(h_{i2}).$$

This says that $R_{\text{cusp}}(f)$ is a finite sum of convolutions of two Hilbert-Schmidt operators, and hence is of trace class (see Exercise 9.2).

## 9.6 Rapidly decreasing functions

We now assume that $F$ is a number field. We fix a Siegel set $\mathcal{S}(t)$ as in (9.20). For $x \in \mathcal{S}(t)$ we define $s_x \in A_{T_0}(t)$ as in (9.19). The estimate of Proposition 9.4.5 for cuspidal functions motivates the following definition:

**Definition 9.2.** A function $\varphi : [G] \rightarrow \mathbb{C}$

is **rapidly decreasing** if it is smooth and for all Siegel sets as above and all $r \in \mathbb{R}_{>0}$ there is a constant $c \in \mathbb{R}_{>0}$ (depending on $\mathcal{S}(t)$ and $r$) such that one has

$$|\varphi(x)| \leq c\alpha(s_x)^{-r}$$

for all $x \in \mathcal{S}(t)$ and $\alpha \in \Delta$.

Observe that in Definition 9.2 the exponent of $\alpha(s_x)$ can be taken to be arbitrarily small, whereas for a moderate growth function (using the definition of
moderate growth provided by Lemma 6.3.1) the exponent of $\alpha(s_x)$ is assumed to be some fixed positive number. Rapidly decreasing functions go to zero faster than the inverse of any polynomial as one goes to infinity in a Siegel set whereas functions of moderate growth have at most polynomial growth in a Siegel set.

Using this terminology we see that Proposition 9.4.5 implies the following theorem:

**Theorem 9.6.1** If $f \in C_c^\infty(A_G \setminus G(\mathbb{A}_F))$ and $\varphi \in L^2_{\text{cusp}}([G])$ then $R(f)\varphi$ is rapidly decreasing.

**Corollary 9.6.2** Any smooth $\varphi \in L^2_{\text{cusp}}([G])$ is rapidly decreasing.

Here when we say that $\varphi$ is smooth we mean that there is a compact open subgroup $K \leq G(\mathbb{A}_F^\infty)$ such that $\varphi \in L^2_{\text{cusp}}([G])^K$, and $\varphi$ lies in the set of vectors in $L^2_{\text{cusp}}([G])^A$ that are smooth with respect to the action of $G(F_\infty)$ in the sense of §4.2.

**Proof.** It suffices to show that such a $\varphi$ can be written as a finite sum

$$\varphi = \sum_i R(f_i)\varphi_i$$

for some $f_i \in C_c^\infty(A_G \setminus G(\mathbb{A}_F))$ and $\varphi_i \in L^2_{\text{cusp}}([G])$. This follows from Theorem 4.2.6.

**9.7 Cuspidal automorphic forms**

We assume again that $G$ is a reductive group over a global field $F$ and let $K = K_\infty K^\infty \leq G(\mathbb{A}_F)$ be a maximal compact subgroup. We use it to define the space of automorphic forms $A = A(G)$ as in definitions 6.5 and 6.7. Recall that the definition only depends on $K_\infty$ in the number field case, and is independent of the choice of $K$ in the function field case. In this section we prove Theorem 6.5.1, which constructs a natural bijection between isomorphism classes of cuspidal automorphic representations in the $L^2$-sense and cuspidal automorphic representations.

To prove this we first state the following result:

**Theorem 9.7.1** An element of $A_{\text{cusp}}^A$ is compactly supported in the function field case and rapidly decreasing in the number field case.

We refer to [MW95, §1.2.12] for the proof.

We also require the following lemma:

**Lemma 9.7.2** Suppose that $(\pi, V)$ is an irreducible unitary representation of $A_G \setminus G(\mathbb{A}_F)$ on a Hilbert space $V$. The subspace $V_{\text{fin}} < V$ of $K$-finite vectors...
is dense. For every \( \varphi \in V_\mathrm{fin} \), there is an \( f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F)) \) such that \( \pi(f) \varphi = \varphi \).

**Proof.** If \( F \) is a function field, then the density of \( V_\mathrm{fin} \) in \( V \) follows from Lemma 5.3.5. In this case every vector \( \varphi \in V_\mathrm{fin} \) is fixed by a compact open subgroup \( K' \leq K \), so \( \pi(1_{K'}) \varphi = \varphi \).

If \( F \) is a number field then by Theorem 6.6.1 we can write \( V \subset V_\infty \otimes V_\infty' \) where the \( V_\infty \) is an irreducible unitary representation of \( A_G \backslash G(F_\infty) \) and \( V_\infty' \) is an irreducible unitary representation of \( G(\mathbb{A}_F^\infty) \). By Theorem 6.6.2 we can then write \( V_\mathrm{fin} = V_\infty \otimes V_\mathrm{fin}' \) where \( V_\infty \) and \( V_\mathrm{fin}' \) are the spaces of \( K_\infty \)- and \( K_\mathrm{fin} \)-finite vectors, respectively. Thus the density assertion is implied by Proposition 4.4.3 and Lemma 5.3.5. The fact that every element in \( V_\mathrm{fin} \) is of the form \( \pi(f) \varphi \) for some \( \varphi \in V \) is implied by the corresponding statements for \( V_\infty \) and \( V_\infty' \). For \( V_\infty' \) one proceeds as in the function field case. For \( V_\infty \) we use Proposition 4.5.5. 

We restate Theorem 6.5.1 for the convenience of the reader:

**Theorem 9.7.3** The space of cuspidal automorphic forms \( \mathcal{A}_{\mathrm{cusp}}^{A_G} \) is a dense subspace of \( L^2_{\mathrm{cusp}}([G]) \). If \( (\pi, V) \) is a cuspidal automorphic representation in the \( L^2 \)-sense, then the space of \( K \)-finite vectors \( V_\mathrm{fin} \) in \( V \) is a cuspidal automorphic representation \( (\pi, V_\mathrm{fin}) \), and

\[
\mathcal{A}_{\mathrm{cusp}}^{A_G}(\pi) = L^2_{\mathrm{cusp}}(\pi)_\mathrm{fin},
\]

(9.26)

the space of \( K \)-finite vectors in \( L^2_{\mathrm{cusp}}(\pi) \). The multiplicity of \( (\pi, V_\mathrm{fin}) \) in \( \mathcal{A}_{\mathrm{cusp}}^{A_G}(\pi) \) is finite and equal to the multiplicity of \( (\pi, V) \) in \( L^2_{\mathrm{cusp}}(\pi) \). One has

\[
\mathcal{A}_{\mathrm{cusp}}^{A_G} = \bigoplus_{\pi} \mathcal{A}_{\mathrm{cusp}}^{A_G}(\pi)
\]

(9.27)

where the (algebraic) sum is over isomorphism classes of cuspidal automorphic representations. The association of \( (\pi, V_\mathrm{fin}) \) to \( (\pi, V) \) defines a bijection between isomorphism classes of cuspidal automorphic representations in the \( L^2 \)-sense and isomorphism classes of cuspidal automorphic representations.

In the theorem we have referred to a cuspidal automorphic representation of \( A_G \backslash G(A_F) \) (resp. in the \( L^2 \)-sense) simply as a cuspidal automorphic representation (resp. in the \( L^2 \)-sense). We will continue this practice in the proof.

**Proof.** By Theorem 9.7.1 an element of \( \mathcal{A}_{\mathrm{cusp}}^{A_G} \) is bounded. Since \( [G] \) has finite measure we deduce that \( \mathcal{A}_{\mathrm{cusp}}^{A_G} \leq L^2_{\mathrm{cusp}}([G]) \).

Let \( (\pi, V) \) be a cuspidal automorphic representation in the \( L^2 \)-sense. The admissible representation \( (\pi, V_\mathrm{fin}) \) is an automorphic representation and the isomorphism types of \( (\pi, V) \) and \( (\pi, V_\mathrm{fin}) \) determine each other by Theorem 6.6.4. Since \( \mathcal{A}_{\mathrm{cusp}}^{A_G} < L^2_{\mathrm{cusp}}([G]) \) we have that
9.7 Cuspidal automorphic forms

\[ A_{\text{cusp}}^G(\pi) = A_{\text{cusp}}^G \cap L^2_{\text{cusp}}(\pi). \]

Let us now prove (9.26). By Corollary 9.1.2, \( L^2_{\text{cusp}}(\pi) \) is a finite direct sum of irreducible representations, all isomorphic to \((\pi, V)\). Thus by the proof of Theorem 6.6.4, \( L^2_{\text{cusp}}(\pi)_{\text{fin}} \) is contained in \( A_{\text{cusp}}^G(\pi) \). For every \( f \in \mathcal{C}^\infty_c(AG \backslash G(\mathbb{A}_F)) \) such that \( \pi(f) \varphi = \varphi \) by Lemma 9.7.2, so any function \( \varphi \in L^2_{\text{cusp}}(\pi)_{\text{fin}} \) is cuspidal in the sense of Definition 6.9. Thus (9.26) is valid.

The underlying admissible representation of an automorphic representation in the \( L^2 \)-sense is irreducible (this is part of Theorem 6.6.4), so we deduce that the (finite) multiplicity of \( \pi \) in \( L^2_{\text{cusp}}(\pi) \) is equal to the multiplicity of \( \pi \) in \( A_{\text{cusp}}^G(\pi) \). Since \( A_{\text{cusp}}^G < L^2_{\text{cusp}}([G]) \) the decomposition (9.27) follows from (9.26) and Corollary 9.1.2. Using Corollary 9.1.2, (9.26), (9.27) and the assertion on multiplicities we deduce the bijection asserted in the theorem. Finally, Lemma 9.7.2 and (9.26) imply that \( A_{\text{cusp}}^G(\pi) \) is dense in \( L^2_{\text{cusp}}(\pi) \). Combining this with (9.27) and Corollary 9.1.2 we deduce that \( A_{\text{cusp}}^G \) is dense in \( L^2_{\text{cusp}}([G]) \).

**Exercises**

9.1. Prove that the Hilbert-Schmidt norm of a Hilbert-Schmidt operator is independent of the choice of basis and the trace norm and trace of a trace-class operator is independent of the choice of basis.

9.2. Prove that if \( A \) and \( B \) are Hilbert-Schmidt operators then \( A \circ B \) is a trace class operator.

9.3. Prove that a trace class operator is Hilbert-Schmidt and a Hilbert-Schmidt operator is compact.

9.4. Let \((Y, \mu)\) be a \( \sigma \)-finite measurable space. Then \( L^2(Y, d\mu) \) is a separable Hilbert space. Let \( K(x, y) \in L^2(Y \times Y, d\mu \times d\mu) \). Prove that the operator

\[ L^2(Y, d\mu) \rightarrow L^2(Y, d\mu) \]

\[ \varphi \mapsto \left( x \mapsto \int_Y K(x, y)\varphi(y)d\mu(y) \right) \]

is Hilbert-Schmidt.

9.5. Let \( V \) be a Hilbert space and let \( A : V \rightarrow V \) be a Hilbert-Schmidt operator. Prove that the restriction of \( A \) to a closed subspace \( W \leq V \) is also a Hilbert-Schmidt operator.

9.6. Let \( G \) be a reductive group over a global field \( F \) and let \( H \leq G \) be a subgroup such that \((A_G \cap H(\mathbb{A}_F))H(F) \backslash H(\mathbb{A}_F)\) is compact. Prove that \((A_G \cap H(\mathbb{A}_F)) \backslash H(\mathbb{A}_F)\) is unimodular.
9.7. Prove that a Dirac sequence on \( A \setminus \mathcal{G}(\mathbb{A}_F) \) exists.

9.8. If \( \{f_n\} \subset C_c^\infty(\mathcal{A}_\mathcal{G} \setminus \mathcal{G}(\mathbb{A}_F)) \) is a Dirac sequence, \((R, V)\) is a representation of \( A \setminus \mathcal{G}(\mathbb{A}_F) \) and \( \varphi \in V \) then

\[
R(f_n)\varphi \rightarrow \varphi
\]
as \( n \to \infty \).

9.9. Using Theorem 4.2.7 prove that every \( f \in C_c^\infty(\mathcal{A}_\mathcal{G} \setminus \mathcal{G}(\mathbb{A}_F)) \) can be written as

\[
f = \sum_{j=1}^r f_{1j} * f_{2j}
\]
for some integer \( r \) and some \( f_{ij} \in C_c^\infty(\mathcal{A}_\mathcal{G} \setminus \mathcal{G}(\mathbb{A}_F)) \).

9.10. With notation as in Lemma 9.4.1 prove that when \( x \) varies over \( \mathcal{S}(t) \) one has that \( s_x^{-1} \Omega_N s_x \) lies in a compact subset of \( \mathcal{N}(\mathbb{A}_Q) \).

9.11. For \( n \geq 1 \), let \( J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \). For \( \mathbb{Q} \)-algebras \( R \), let

\[
\text{Sp}_{2n}(R) = \{g \in \text{GL}_{2n}(R) : Jg^{-t}J^{-1} = g\}.
\]

Let \( \mathbb{H}_n \) be the set of \( n \times n \) symmetric complex matrices \( Z = X + iY \) whose imaginary part \( Y \) is positive definite (this is called \textbf{Siegel’s upper-half space} of degree \( n \)). The group \( \text{Sp}_{2n}(\mathbb{R}) \) acts on \( \mathbb{H}_n \) via

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}.
\]

Here \( A, B, C, D \) are \( n \times n \) matrices.

(a) Show that \( \text{Sp}_{2n}(\mathbb{R}) \) acts transitively on \( \mathbb{H}_n \).

(b) Choose a Borel subgroup \( B < \text{Sp}_{2n} \) and compute the image of the associated Siegel domain \( \mathcal{S}(t) \) under

\[
\text{Sp}_{2n}(\mathbb{A}_Q) \longrightarrow \text{Sp}_{2n}(\mathbb{R}) \longrightarrow \mathbb{H}_n
\]
for \( n = 1 \) and \( n = 2 \).
Chapter 10
Eisenstein Series

Abstract In this chapter we survey the theory of Eisenstein series. Our main goal is to state Langlands' decomposition of $L^2([G])$.

10.1 Induced representations

In §8.1 we mentioned Harish-Chandra's philosophy of cusp forms [HC70a]. It is a slogan for the assertion that the irreducible representations of a reductive group can all be obtained as subquotients of the parabolic inductions of cuspidal representations on Levi subgroups. This philosophy is in part based on the foundational results of Langlands [Lan76] which we survey in this chapter. We largely follow [Art05, §7] in our exposition. It is beyond the scope of this book to provide proofs. Indeed, whole books have been devoted to solely to the theorems we state here [MW95].

We begin by setting notation for the induction of automorphic representations from Levi subgroups of $G$ to $G$ itself. Intertwining operators involving such representations are discussed in §10.2. Eisenstein series, defined in §10.3, allow one to prove that these induced representations are again automorphic, and in §10.4 we use Eisenstein series to give Langlands’ decomposition of $L^2([G])$. In the case of $GL_n$ all of this theory simplifies significantly. In particular, one can use it to define an analogue of direct sums in the category of automorphic representations of general linear groups. This is discussed in §10.5 and 10.6. We end the chapter with a statement, in §10.7, of Moeglin
and Waldspurger’s foundational result which precisely describes the discrete spectrum of $L^2([GL_n])$ in terms of cuspidal representations of smaller general linear groups.

Let $G$ be a reductive group over a global field $F$ with minimal parabolic subgroup $P_0 \leq G$. As usual, parabolic subgroups containing $P_0$ are said to be standard. We fix a Levi subgroup $M_0 \leq P_0$. Then $P_0 = M_0N_0$ where $N_0 \leq P_0$ is the unipotent radical. By Proposition 1.9.5 there is then a unique Levi decomposition

$$P = MN$$

for each standard parabolic subgroup $P$ such that $M$ contains $M_0$; we always use this Levi decomposition. We can and do choose a maximal compact subgroup $K \leq G(A_F)$ in good position with respect to $(P_0 \cap M_0)$ and hence all $(P,M)$ in the sense of §A.1. Indeed, this follows upon combining Proposition A.3.1, Theorem A.4.1 and Theorem A.4.2. We normalize the Haar measures $dg$ on $G(A_F)$, $dk$ on $K$ and $dp$ on $P(A_F)$ so that

$$dg = dpdk$$

(see Exercise 3.5). Let $a_M := Hom(X^*(M), \mathbb{R})$. We have a map

$$H_M : M(\mathbb{A}_F) \longrightarrow a_M$$

defined by

$$\langle H_M(m), \lambda \rangle = \log |\lambda(m)|$$

for $\lambda \in X^*(M)$. We observe that $H_M$ is trivial on $K \cap M(\mathbb{A}_F)$. Thus using the Iwasawa decomposition $G(\mathbb{A}_F) = N(\mathbb{A}_F)M(\mathbb{A}_F)K$, we can define a morphism

$$H_P : G(\mathbb{A}_F) \longrightarrow a_M$$

$$nmk \longmapsto H_M(m)$$

for $(n, m, k) \in N(\mathbb{A}_F) \times M(\mathbb{A}_F) \times K$.

The discrete spectrum $L^2_{\text{disc}}([M]) \leq L^2([M])$ is the largest closed subspace of $L^2([M])$ that decomposes as a Hilbert space direct sum of irreducible subrepresentations. An irreducible subrepresentation of $L^2_{\text{disc}}([M])$ is called a discrete series representation. We denote by $R_{\text{disc}}$ the restriction of the regular representation $R$ of $L^2([M])$ to this subspace. Let $(\sigma, V)$ be an $A_M \setminus M(\mathbb{A}_F)$-subrepresentation of $L^2_{\text{disc}}([M]) \leq L^2([M])$.

For

$$\lambda \in a^\ast_{MC} := Hom(a_M, \mathbb{C}) = X^*(M) \otimes_{\mathbb{Z}} \mathbb{C},$$

we then form the global induced representation

$$(I(\sigma, \lambda), \text{Ind}^G_P(V)).$$

Here $\text{Ind}^G_P(V)$ is the space of measurable functions
10.2 Intertwining operators

\[ \varphi : N(\mathbb{A}_F)M(F)A_M \backslash G(\mathbb{A}_F) \rightarrow \mathbb{C} \]

such that for all \( x \in G(\mathbb{A}_F) \) the function

\[ m \mapsto \varphi(mx) \]

lies in \( V \subseteq L^2_{\text{disc}}([M]) \) and such that

\[ \|\varphi\|^2 = \int_K \int_{A_M M(F) \backslash M(\mathbb{A}_F)} |\varphi(mk)|^2 dmdk < \infty. \]

Thus \( \text{Ind}^G_\mathcal{P}(V) \) is a Hilbert space. The action is given by

\[ I(\sigma, \lambda)(g)\varphi(x) = \varphi(xg) e^{(H_P(xg), \lambda + \rho_P)} e^{- (H_P(x), \lambda + \rho_P)}, \]

where \( \rho_P \in a_\mathfrak{M}^+ \) is the half-sum of positive roots of a maximal split torus of \( G \) contained in \( P \).

Suppose now that \( \sigma \) is irreducible, so it is an automorphic representation of \( A_M \backslash M(\mathbb{A}_F) \). Since

\[ e^{(H_M(m), \lambda)} = \prod_v e^{(H_M(m_v), \lambda)}, \]

for \( m \in M(\mathbb{A}_F) \), we have a factorization

\[ I(\sigma, \lambda) \cong \otimes'_v I(\sigma_v, \lambda) \]

where the local factors \( I(\sigma_v, \lambda) \) are defined as in §4.9 in the archimedean setting and §8.2 in the nonarchimedean setting. There is a slight difference between our conventions in the global setting and in the local setting in terms of the spaces on which \( I(\sigma, \lambda) \) acts. In the global setting, we are normalizing things so that the space on which \( I(\sigma, \lambda) \) acts is independent of \( \lambda \). In the local setting, we incorporated \( \lambda \) into the definition of the functions themselves. The two definitions only differ by the character \( e^{-(H_P(x_v), \lambda + \rho_P)} \) at a place \( v \).

10.2 Intertwining operators

We now need to discuss the relationships between various parabolic subgroups. As in the previous section, we write \( P_0 = M_0 N_0 \) for a minimal parabolic subgroup of \( G \) with Levi subgroup \( M_0 \) and unipotent radical \( N_0 \). We let \( P \) and \( P' \) be standard, and let

\[ P = MN \quad \text{and} \quad P' = M'N' \]
be the unique Levi decompositions such that $M$ and $M'$ contain $M_0$. Assume for the moment that 

$$P \preceq P'$$

so that $M \geq M'$. The inclusions $Z_M \leq Z_{M'} \leq M' \leq M$ induce restriction morphisms

$$X^*(M) \to X^*(M') \text{ and } X^*(Z_{M'}) \to X^*(Z_M).$$

(10.2)

For any reductive group $H$, we have an isomorphism $a_H^* \to a_{Z_H}^*$ by Exercise 10.1. Thus (10.2) induces an injection and a surjection

$$a_M^* \inj a_{M'}^* \longrightarrow a_M^*$$

such that the composite is an isomorphism. Similarly, by duality we obtain

$$a_M \inj a_{M'} \longrightarrow a_M,$$

(10.3)

where again the composite is an isomorphism. We often identify $a_M^*$ and $a_M$ with their images in $a_{M'}^*$ and $a_{M'}$, respectively. We write

$$a_{M'}^M := \ker (a_{M'} \longrightarrow a_M)$$

(10.4)

and $(a_{M'}^M)^*$ for its dual in $a_M^*$. Then the considerations above imply

$$a_{M'}^* = a_M^* \oplus (a_{M'}^M)^* \text{ and } a_{M'} = a_M \oplus a_{M'}^M.$$  

(10.5)

Let $T_0 \leq M_0$ be a maximal split torus and let

$$W_0 := W(G, T_0)(F) = W(G, T_0)(\overline{F})$$

be the Weyl group of $T_0$ in $G$ (see Lemma 1.7.3 for the second equality). The Weyl group $W_0$ acts on $a_{M_0}$. For a pair of standard parabolic subgroups $P$ and $P'$, we have injections $a_M \inj a_{M_0}$ and $a_M' \inj a_{M_0}'$ as discussed above and we identify $a_M$ and $a_{M'}$ with their images as mentioned above. Thus we can define

$$W(a_M, a_{M'})$$

be the set of linear isomorphisms from $a_M$ onto $a_{M'}$ obtained by restricting elements in the Weyl group $W_0$. If $W(a_M, a_{M'})$ is nonempty then $P$ and $P'$ are said to be **associate**. An **association class** (with respect to a given minimal parabolic subgroup) is a set of standard parabolic subgroups that are all associate to each other.

**Lemma 10.2.1** If $P$ and $P'$ are associate if and only if their Levi subgroups $M$ and $M'$ are $G(F)$-conjugate.

**Proof.** This is essentially [Sol20, Lemma 1]; we reproduce the proof here.

Let $\Delta$ be the set of simple roots of $T_0$ in $G$ corresponding to $P_0$. We recall
that standard parabolic subgroups are in bijection with subsets \( J \subseteq \Delta \) by Theorem \ref{thm:parabolic-subgroups}. We write \( P_J \) for the parabolic subgroup corresponding to \( J \) and \( \Phi(J) := \mathbb{Z} J \cap \Phi(G, T_0) \). As in (1.33), let \( M_J \) be the group generated by \( Z_G(T_0) \) and \( G_\alpha \) for \( \alpha \in \Phi(J) \). Then by Proposition 1.9.5 (or more precisely its proof), \( M_0 = M_0' \) and \( M_J \) is the unique Levi subgroup of \( P_J \) containing \( M_0 \).

Now assume that \( P_J \) and \( P_{J'} \) are associate. We observe that \( \mathfrak{a}_{M_J}' \) is just the \( \mathbb{R} \)-span of \( \mathfrak{a}_G' \) and \( \Delta - J \), and similarly for \( \mathfrak{a}_{M_{J'}}' \). Thus by Lemma 1.7.3 there exists an element \( g \in N_G(T_0)(F) \) such that \( g\Phi(J)g^{-1} = \Phi(J') \). We deduce that \( gM_Jg^{-1} = M_{J'} \). Conversely if \( gM_Jg^{-1} = M_{J'} \) then \( gT_0 \) is a maximal split torus of \( M_{J'} \), hence there is an \( m \in M_{J'}(F) \) such that \( mgT_0m^{-1} = T_0 \) by Theorem 1.7.2. It follows that \( mg \in N_G(T_0)(F) \) provides an element of \( W(\mathfrak{a}_{M_J}, \mathfrak{a}_{M_{J'}}) \).

In view of the lemma, it is natural to expect a relationship between representations of \( G \) induced from \( P \) and those induced from \( P' \) when \( P \) and \( P' \) are associate.

As above let \( V \leq L^2_{\text{disc}}([M]) \) be a subrepresentation. The relationship mentioned above is obtained by introducing intertwining operators

\[ M(w, \lambda) : \text{Ind}^{G}_{P'(V)} \longrightarrow \text{Ind}^{G}_{P}(V) \]

defined by

\[ M(w, \lambda)\varphi(x) := \int \varphi(\bar{w}^{-1}nx)e^{(H_P(\bar{w}^{-1}nx),\lambda+\rho_P)}e^{(H_{P'}(x),-w\lambda-\rho_{P'})}dn, \tag{10.6} \]

where the integral is over \( (N'(A_F) \cap \bar{w}N(A_F)\bar{w}^{-1}) \backslash N'(A_F) \). Here \( \bar{w} \in G(F) \) is any representative for \( w \). At the moment, this is just a formal definition. To make it rigorous, let \( K \leq G(A_F) \) be a maximal compact subgroup and assume that \( V \) is irreducible. Let

\[ \text{Ind}^{G}_{P}(V)^0 \leq \text{Ind}^{G}_{P}(V) \tag{10.7} \]

be the dense subspace consisting of \( \varphi \) that are smooth, \( K \)-finite, and have the property that

\[ m \mapsto \delta_P^{-1/2}(m)\varphi(mk) \]

is an element of \( \mathcal{A}(M) \) for all \( k \in K \) (see [MW95, §I.2.17]). When restricted to \( \text{Ind}^{G}_{P}(V)^0 \), the integral \( M(w, \lambda) \) converges for \( \lambda \) sufficiently large in a sense we now make precise. Let \( \Phi \) denote the set of roots of \( Z_{A_F}^0 \) in \( G \) and let \( \Delta \) be the base attached to the set of positive roots defined by \( P \) (see §1.9). Set

\[ (\mathfrak{a}_P)^+ := \{ \lambda \in \mathfrak{a}_M^* : \lambda(\alpha) > 0 \text{ for all } \alpha \in \Delta \}. \]

For the following proposition, see [MW95, §II.1.6].
Proposition 10.2.2 (Langlands) The intertwining operator $M(w, \lambda)$ converges absolutely for $\text{Re}(\lambda)$ in a suitable cone in $\mathfrak{a}_M^\ast$. If $V \leq L^2_{\text{cusp}}([M])$ then $M(w, \lambda)$ converges absolutely for $\text{Re}(\lambda) \in \rho_P + (\mathfrak{a}_P^\ast)^+$. \qed

It turns out that the operators $M(w \mapsto \lambda)$ admit meromorphic continuations \cite[§IV.1.8, §IV.1.10]{MW95}, \cite[Theorem 7.2]{Art05}:

Theorem 10.2.3 The operators $M(w \mapsto \lambda)$ admit meromorphic continuations to $\mathfrak{a}_M^\ast$ that are holomorphic and unitary on $ia_M$. If $P, P', P''$ are standard parabolic subgroups then the operators satisfy the functional equation

$$M(w_1 w_2, \lambda) = M(w_1, w_2 \lambda) \circ M(w_2, \lambda)$$

for $w_1 \in W(a_M, a_M')$ and $w_2 \in W(a_M, a_{M''})$. \qed

The subspace $ia_M \subseteq a_{P_C}$ plays a distinguished role, because the representations $I(\sigma, \lambda)$ are unitary for $\lambda \in ia_M$ (see Exercise 10.4).

10.3 Eisenstein series

Now $I(\sigma, \lambda)$ is a representation of $G(\mathbb{A}_F)$ on a space of functions on $G(\mathbb{A}_F)$, but the functions in $\text{Ind}_G^P(V)$ are not left invariant under $G(F)$. The functions in this space are left invariant under $P(F)$, however. We therefore do the most naïve thing possible to make them invariant under $G(F)$, namely average. Assume that

$$\varphi \in \text{Ind}_G^P(V)^0$$

where $V \leq L^2_{\text{disc}}([M])$ is an irreducible subrepresentation and $\text{Ind}_G^P(V)^0$ is defined as in (10.7). Let

$$E(x, \varphi, \lambda) := \sum_{\delta \in P(F) \setminus G(F)} \varphi(\delta x) e^{(H_P(\delta x), \lambda + \rho_P)}. \quad (10.8)$$

This is an Eisenstein series. It provides an intertwining map from $I(\sigma, \lambda)$ to the regular action of $G(\mathbb{A}_F)$ on functions on $G(\mathbb{A}_F)$:

$$E(xg, \varphi, \lambda) = E(x, I(\sigma, \lambda)(g) \varphi, \lambda) \quad (10.9)$$

(see Exercise 10.3) provided that the series $E(x, \varphi, \lambda)$ converges absolutely.

The problem is that $E(x, \varphi, \lambda)$ only converges for $\lambda$ sufficiently large \cite[§II.1.5]{MW95}:

Proposition 10.3.1 (Langlands) If $\text{Re}(\lambda)$ lies in a suitable cone in $\mathfrak{a}_M^\ast$ then $E(x, \varphi, \lambda)$ converges absolutely. If $\varphi$ is cuspidal $E(x, \varphi, \lambda)$ converges absolutely for $\text{Re}(\lambda) \in \rho_P + (\mathfrak{a}_P^\ast)^+$. \qed
On the other hand the representations $I(\sigma, \lambda)$ are unitary when $\lambda \in i a_M^*$, which is never in the plane of absolute convergence of the Eisenstein series. Instead, one has to analytically continue the Eisenstein series as a complex analytic function of $\lambda$ to the line $\lambda \in i a_M^*$ in order to construct pieces of $L^2([G])$. In the case $G = \text{GL}_2$, this was accomplished by Selberg [Sel56, Sel63]. Though important, this work offered only limited insight into the difficulties presented by the general case.

Langlands was the first to appreciate the difficulties presented by the general case, and was able to overcome them [Lan76]. It is hard to overestimate the impact of Langlands’ work on Eisenstein series to automorphic representation theory (and indeed much of mathematics, especially number theory). This work, for example, led to the discovery of Langlands reciprocity as enunciated in Langlands letter to Weil [Lan, Lan70], led to Langlands-Shahidi theory of automorphic $L$-functions [Lan71, Sha10], and also is required input into any general treatment of a trace formula, either in the Arthur-Selberg sense [Art78, Art80], or in Jacquet’s sense [Jac05a, JLR93].

If $P$ and $P'$ are associate standard parabolics, then the intertwining operators of the previous section give a relationship between representations induced from $P$ and representations induced from $P'$. This suggests a relationship between the associated Eisenstein series. The precise relationship is contained in the following theorem of Langlands (see [Art05, Theorem 7.2] and [MW95, §IV.1.10])

**Theorem 10.3.2 (Langlands)** The Eisenstein series $E(x, \varphi, \lambda)$ admits a meromorphic continuation to $a_{MC}^*$. It satisfies the functional equation

$$E(x, M(w, \lambda)\varphi, w\lambda) = E(x, \varphi, \lambda)$$

for $w \in W(a_M, a_{M'})$. □

Sometimes in applications one needs more precise information about the size of Eisenstein series and the intertwining operators. We refer to [GS01] for an important example where this is crucial. This paper refers to the equally important paper [M00], which proves that the Eisenstein series $E(x, \varphi, \lambda)$ is a quotient of functions of finite order under suitable assumptions on $\varphi$.

### 10.4 Decomposition of the spectrum

Assume for this section that $F$ is a number field. There is a decomposition

$$L^2([G]) = \bigoplus_P L^2_P([G])$$

(10.10)

where the (finite) direct sum is over all association classes $P$ of parabolic subgroups. In more detail, consider the Hilbert space of families $\mathcal{F}$ of measurable
functions
\[ F_P : i\mathfrak{a}_M^G \rightarrow \text{Ind}_P^G(L_{\text{disc}}^2([M])) \] (10.11)
indexed by \( P \in \mathcal{P} \) and satisfying the Weyl invariance condition
\[ F_P(w\lambda) = M(w,\lambda)F_P(\lambda) \]
for \( w \in W(\mathfrak{a}_M,\mathfrak{a}_{M'}) \).

The relevant inner product is
\[ (\mathcal{F}_1,\mathcal{F}_2) = \sum_{P \in \mathcal{P}} n_P^{-1} \int_{\mathfrak{a}_M^G} (F_{1P}(\lambda),F_{2P}(\lambda))d\lambda, \]
where \( F_{1P} \in \mathcal{F}_1 \) and \( F_{2P} \in \mathcal{F}_2 \) for \( P \in \mathcal{P} \). Here
\[ n_P := \sum_{\nu \in P} |W(\mathfrak{a}_M,\mathfrak{a}_{M'})|. \] (10.12)

Given such a family of functions, we can formally define
\[ \sum_{P \in \mathcal{P}} n_P^{-1} \int_{\mathfrak{a}_M^G} E(x,F_P(\lambda),\lambda)d\lambda \in L^2([G]). \] (10.13)

Assume that \( V \leq L_{\text{disc}}^2([M]) \) is an irreducible subrepresentation. Then the formal integral (10.13) is absolutely convergent when \( F_P \) is compactly supported as a function of \( \lambda \) and has image in \( \text{Ind}_P^G(V)^0 \), defined as in (10.7). The closure of the subspace of \( L^2([G]) \) spanned by the functions (10.13) under the assumption on \( F_P \) just mentioned is denoted
\[ L_P^2([G]). \]

We observe that when \( M = G \) one has that \( (\mathfrak{a}_M^G)^* = \{0\} \). Thus if \( \mathcal{G} \) is the association class of \( G \) (which just consists of \( G \) itself) we have
\[ L_\mathcal{G}^2([G]) = L_{\text{disc}}^2([G]). \]

**Theorem 10.4.1 (Langlands)** One has that
\[ L^2([G]) = \bigoplus_P L_P^2([G]) \]
where the sum is over association classes \( \mathcal{P} \) of parabolic subgroups of \( G \). \( \square \)

This is [Art05, Theorem 7.2], except that Arthur decomposes \( L^2(G(F)\backslash G(\mathbb{A}_F)) \) where as we have decomposed \( L^2([G]) \). Using Mellin inversion one can readily show the two statements are equivalent. The **continuous spectrum** is
Here we are using the fact that the association class of \( G \) is just \( G \). We then have

\[
L^2_{\text{disc}}([G]) \oplus L^2_{\text{cont}}([G]) = L^2([G]).
\] (10.15)

The residual spectrum is the orthogonal complement of \( L^2_{\text{cusp}}([G]) \) in \( L^2_{\text{disc}}([G]) \):

\[
L^2_{\text{res}}([G]) \oplus L^2_{\text{cusp}}([G]) := L^2_{\text{disc}}([G]).
\] (10.16)

We will see in §10.7 below that when \( G = \text{GL}_n \) the residual spectrum can be described in terms of residues of Eisenstein series attached to cusp forms. This is true in general [Lan76] but we will not discuss the general case further.

Let us end this section by describing the data appearing in Theorem 10.4.1 when \( G = \text{GL}_2 \). We take our minimal parabolic subgroup to be \( B_2 \), the Borel subgroup of upper triangular matrices, and fix the Levi subgroup \( M < B_2 \) of diagonal matrices. The discrete spectrum of \( L^2([M]) \) is the whole spectrum. In fact, by the Peter-Weyl theorem, Theorem 4.3.3, there is an isomorphism

\[
L^2([M]) \cong \bigoplus_{\chi \in \widehat{[M]}} C_{\chi}
\]

where the direct sum is over characters of \( [M] \) and \( C_{\chi} \) is a 1-dimensional space on which \( [M] \) acts through \( \chi \). There is only one association class of standard parabolic subgroups that is not \( \text{GL}_2 \) itself, and it consists of the single element \( B_2 \). The group \( W(\mathfrak{a}_M, \mathfrak{a}_M) \) is just \( \mathbb{Z}/2 \).

## 10.5 Local preparation for isobaric representations

In this section we collect some local preliminaries for our discussion of isobaric representations in §10.6. Thus for this section we assume that \( F \) is a local field.

We have already discussed in §4.9 and §8.4 how to classify admissible representations of \( G(F) \) for reductive \( F \)-groups \( G \) in terms of tempered representations. When \( G = \text{GL}_n \) and \( F \) is nonarchimedean, we have also discussed how to classify the tempered representations in terms of the square integrable representations. We now explain how these results can be combined to classify all admissible representations of \( \text{GL}_n(F) \) in terms of square integrable representations.

Assume \( G = \text{GL}_n \). Let \( n = \sum_{i=1}^k n_i \) and for each \( i \) let \( \pi_i \) be an irreducible essentially square integrable representation of \( \text{GL}_{n_i}(F) \). Let
\[
\text{GL}_{n_i}(F) := \ker(H_{\text{GL}_{n_i}} : \text{GL}_{n_i}(F) \to a_{\text{GL}_{n_i}}) \tag{10.17}
\]
where \(H_{\text{GL}_{n_i}}\) is defined as in (8.11). Thus
\[
\text{GL}_{n_i}(F) := \text{GL}_{n_i}(F)^1 \times A_{\text{GL}_{n_i}},
\]
where
\[
A_{\text{GL}_{n_i}} = \begin{cases} 
\mathbb{R}_{>0} I_{n_i} & \text{if } F \text{ is archimedean and} \\
\varpi I_{n_i} & \text{if } F \text{ is nonarchimedean.}
\end{cases}
\]
Here in the nonarchimedean case \(\varpi\) is a uniformizer for \(F\).

Let \(\pi_i\) be the unique irreducible representation of \(A_{\text{GL}_{n_i}} \setminus \text{GL}_{n_i}(F)\) such that
\[
\pi_i(g) = |\det g|^{\lambda_i} \pi_i^1(g) \tag{10.18}
\]
for some \(\lambda_i \in \mathbb{C}\). After changing the order of the partition, we can assume that
\[
\text{Re}(\lambda_1) \geq \cdots \geq \text{Re}(\lambda_k). \tag{10.19}
\]
Let \(P\) be the standard parabolic subgroup of type \((n_1, \ldots, n_k)\) with standard Levi subgroup \(M\). Let \(\lambda = (\lambda_1, \ldots, \lambda_k)\). Then we can form the induced representation
\[
I \left( \otimes_{i=1}^k \pi_i^1, \lambda \right).
\]

**Theorem 10.5.1** The induced representation \(I \left( \otimes_{i=1}^k \pi_i^1, \lambda \right)\) has a unique irreducible quotient \(J \left( \otimes_{i=1}^k \pi_i^1, \lambda \right)\). Assume that \(\{\pi_i'\}_{i=1}^{k'}\) is another collection of irreducible admissible essentially square integrable representations of \(\text{GL}_{n_i}(F)\) and \(\lambda_1', \ldots, \lambda_k' \in \mathbb{C}\) satisfy \(\text{Re}(\lambda_1') \geq \cdots \geq \text{Re}(\lambda_k')\). In the nonarchimedean (resp. archimedean) case there is an equivalence (resp. infinitesimal equivalence)
\[
J \left( \otimes_{i=1}^k \pi_i^1, \lambda \right) \cong J \left( \otimes_{i=1}^{k'} (\pi_i')^1, \lambda' \right) \tag{10.20}
\]
if and only if \(k = k'\) and there is a permutation \(\tau\) of \(\{1, \ldots, k\}\) such that \(\pi_i' \cong \pi_{\tau(i)}\) and \(\lambda_i' = \lambda_{\tau(i)}\).

**Proof.** This is consequence of Theorem 8.4.2 and the Bernstein-Zelevinsky classification discussed in §8.4 for the nonarchimedean case. A proof for the archimedean case follows from the discussion with references given after [Kn94, Theorem 1 and 4]. \(\square\)

Let \(\pi_1 \oplus \cdots \oplus \pi_k\) be the irreducible quotient of the theorem. In this notation we do not assume that the \(\pi_i\) are arranged so that \(\text{Re}(\lambda_1) \geq \cdots \geq \text{Re}(\lambda_k)\), so by definition
for any permutation $\tau$ of $\{1, \ldots, k\}$. Because of the theorem, this notation is unambiguous.

**Theorem 10.5.2** In the nonarchimedean (resp. archimedean) case any irreducible admissible representation of $\text{GL}_n(F)$ is equivalent (resp. infinitesimally equivalent) to $\boxtimes_{i=1}^{k} \pi_i$ for some set of essentially square integrable $\pi_i$.

**Proof.** Again, this is consequence of Theorem 8.4.2 and the Bernstein-Zelevinsky classification in §8.4 in the nonarchimedean case and the discussion following [Kna94, Theorem 1 and 4] in the archimedean case. $\square$

Given Theorem 10.5.1 and Theorem 10.5.2, we can define an irreducible admissible representation $\boxtimes_{i=1}^{k} \pi_i$ of $\text{GL}_n(F)$ for any tuple $\pi_1, \ldots, \pi_k$ of irreducible admissible representations of $\text{GL}_{n_i}(F)$ with $\sum_{i=1}^{k} n_i = n$. In particular, we can remove the assumption that the $\pi_i$ are essentially square integrable. For each $\pi_i$ we write $\pi_i = \boxtimes_{j=1}^{k_i} \pi_{ij}^{*}$ and then set

$$\boxtimes_{i=1}^{k} \pi_i = \boxtimes_{i=1}^{k} \left( \boxtimes_{j=1}^{k_i} \pi_{ij}^{*} \right).$$

The symbol $\boxtimes$ behaves something like a formal direct sum on the category of admissible representations, but we emphasize that $\boxtimes_{i=1}^{k} \pi_i$ is always irreducible.

It is useful to explicitly point out the relationship between the classification of unramified representations in terms of the Satake correspondence and the isobaric sum. For this let $T_n \leq \text{GL}_n$ be the maximal torus of diagonal matrices and let $B_n \leq \text{GL}_n$ be the Borel subgroup of upper triangular matrices.

**Theorem 10.5.3** Let $\chi_1, \ldots, \chi_n : F^\times \to \mathbb{C}^\times$ be $n$ unramified quasi-characters and let

$$\chi : T_n(F) \to \mathbb{C}^\times$$

$$(t_{ij}) \mapsto \prod_{i=1}^{n} \chi_i(t_{ii}).$$

The isobaric sum $\boxtimes_{i=1}^{n} \chi_i$ is then the unique unramified subquotient of the induced representation $\text{Ind}_{B_n}^{\text{GL}_n}(\chi)$.

**Proof.** We know that there is a unique unramified subquotient of $\text{Ind}_{B_n}^{\text{GL}_n}(\chi)$ by Lemma 7.6.5. On the other hand, the isobaric sum $\boxtimes_{i=1}^{n} \chi_i$ is unramified by [Mat13, Corollary 1.2]. $\square$

We warn the reader that when the $\pi_i$ are not generic and unitary the operation of taking the isobaric sum can have some unexpected properties. We give an example to illustrate.
Example 10.1. Assume $F$ is a nonarchimedean local field and let $B_2 \leq \text{GL}_2$ be the Borel subgroup of upper triangular matrices. Let 1 denote the trivial representation of $B_2(F)$. Consider

$$\pi_1 = I(1, \left(\frac{2}{3}, -\frac{1}{2}\right)) \quad \text{and} \quad \pi_2 = I(1, \left(\frac{1}{2}, -\frac{3}{2}\right)).$$

These representations of $\text{GL}_2(F)$ are irreducible by Theorem 8.4.4 and by the definition above

$$\pi_1 \boxtimes \pi_2 = |\cdot|^{3/2} \boxtimes |\cdot|^{1/2} \boxtimes |\cdot|^{-1/2} \boxtimes |\cdot|^{-3/2}$$

is the trivial representation of $\text{GL}_4(F)$ by [Zel80, Example 3.2]. But the trivial representation is not a quotient of $\text{Ind}_{\text{P}}^{\text{GL}_4} (\pi_1 \otimes \pi_2)$, where $P$ is the standard parabolic subgroup of $\text{GL}_4$ of type $(2, 2)$ (see Exercise 10.5).

Due to this subtlety, some care is required in applying Theorem 10.5.1 to the global setting. The issue is that it is not known that the local factors of a cuspidal automorphic representation of the general linear group are tempered, though this is conjectured to be true (see Conjecture 10.6.4 and the rest of §10.6 for more details). What is known is that these local components are unitary and generic in the sense of §11.3 (see Theorem 11.3.3). The reader can safely take the definition of generic as a “black box” to be opened in §11.3 for the discussion that follows.

Assume as above that we are given $n = \sum_{i=1}^{k} n_i$ and representations $\pi_i$ where the $\lambda_i$ defined as in (10.18). We assume that the $\lambda_i$ satisfy (10.19).

Theorem 10.5.4 For $1 \leq i \leq k$, suppose we are given essentially unitary generic representations $\pi_i$ of $\text{GL}_{n_i}(F)$. Then $\boxtimes_{i=1}^{k} \pi_i$ is a subquotient of

$$I(\otimes_{i=1}^{k} \pi_i^{-1}, \lambda).$$

Proof. We claim that each unitary generic representation $\pi$ of $\text{GL}_n(F)$ is of the form $I(\sigma, \lambda)$ where $\sigma$ is a square integrable representation of a Levi subgroup of $\text{GL}_n(F)$. Assuming the claim, the theorem follows from [MgW89, §1.2(3)]. If $F$ is nonarchimedean the claim is part of Theorem 8.4.4. If $F$ is archimedean, then the claim follows from Theorem 10.5.2 and [Jac09, Lemma 2.5].

10.6 Isobaric representations

Assume now that $F$ is a number field. So far we have explained how the entire spectrum of $L^2([G])$ can be described in terms of automorphic representations of Levi subgroups $M$ of parabolic subgroups $P$ of $G$. Let us state a less precise result that applies to all automorphic representations, not just automorphic representations in the $L^2$-sense. Given a cuspidal automorphic representation
10.6 Isobaric representations

\( \sigma \) of \( M(\mathbb{A}_F) \), we form the induced representation

\[
I(\sigma, \lambda)
\]

as in (10.1). For the proof of the following theorem, see the appendix to [BJ79] (see also [Lan79a]):

**Theorem 10.6.1 (Langlands)** Any irreducible subquotient of \( I(\sigma, \lambda) \) is an automorphic representation, and every automorphic representation is of this form.

This theorem applies to more general automorphic representations than those implicit in Theorem 10.4.1, but yields less information. We remark that in [BJ79, Appendix, Lemma 1] Langlands observes that the subquotients of \( I(\sigma, \lambda) \) can be described as follows. For all but finitely many \( v \) the representation \( \sigma_v \) is unramified and \( I(\sigma_v, \lambda) \) has a unique irreducible unramified subquotient \( \pi_v^0 \). The irreducible subquotients of \( I(\sigma, \lambda) \) are the representations of the form \( \otimes_v' \pi_v \) where \( \pi_v \) is an irreducible subquotient of \( I(\sigma_v, \lambda) \) for all \( v \) and \( \pi_v \cong \pi_v^0 \) for almost every \( v \).

Let us restrict \( G = \text{GL}_n \) and discuss refinements of this result. As a first step we discuss an operation on automorphic representations of \( \text{GL}_n \) that plays the role in automorphic representation theory of the direct sum in ordinary representation theory. In fact, under the conjectural Langlands correspondence (to be discussed in Chapter 12), this operation should be induced by taking direct sums of \( L \)-parameters.

Let \( n = \sum_{i=1}^{k} n_i \) and let \( \sigma_i \) is a cuspidal automorphic representation of \( A_{GL_{n_i}} \backslash GL_{n_i}(\mathbb{A}_F) \) for each \( i \). Let \( P \) be a standard parabolic subgroup of \( \text{GL}_n \) (i.e. a parabolic subgroup containing the Borel subgroup of upper triangular matrices) with Levi subgroup isomorphic to \( \prod_{i=1}^{k} \text{GL}_{n_i} \). Let \( \sigma = \otimes_{i=1}^{k} \sigma_i \).

Theorem 10.6.1 leaves open the question of whether the automorphic representation \( \pi \) of \( \text{GL}_n(\mathbb{A}_F) \) can arise as a subquotient of \( I(\sigma, \lambda) \) in two different manners. In fact, it can arise in essentially only one manner. More precisely, assume that

\[
\sum_{i=1}^{k} n_i = \sum_{j=1}^{k'} n_j' = n
\]

and \( \sigma_i \) (resp. \( \sigma_j' \)) are cuspidal automorphic representations of \( A_{GL_{n_i}} \backslash GL_{n_i}(\mathbb{A}_F) \) (resp. \( A_{GL_{n_j'}} \backslash GL_{n_j'}(\mathbb{A}_F) \)) for \( 1 \leq i \leq k \) and \( 1 \leq j \leq k' \). We let

\[
\sigma = \otimes_{i=1}^{k} \sigma_i \quad \text{and} \quad \sigma' = \otimes_{j=1}^{k'} \sigma_j'.
\]

We view these as representations of standard parabolics \( P \) and \( P' \) attached to

\[
\sum_{i=1}^{k} n_i \quad \text{and} \quad \sum_{j=1}^{k'} n_j',
\]
Theorem 10.6.2 (Jacquet and Shalika) Let $S$ be a finite set of places of $F$ including the infinite places such that $I(\sigma, \lambda)$ and $I(\sigma', \lambda')$ are unramified outside of $S$. If the irreducible spherical subquotients of $I(\sigma_v, \lambda)$ and $I(\sigma'_v, \lambda')$ are isomorphic for $v \not\in S$ then $k = k'$ and there is a permutation $\tau$ of the set of $k$ elements such that

$$\sigma_i \cong \sigma'_{\tau(i)} \text{ and } \lambda_i = \lambda'_{\tau(i)}.$$ 

This immediately implies the following corollary:

Corollary 10.6.3 If $\pi$ is an irreducible subquotient of $I(\sigma, \lambda)$ and $I(\sigma', \lambda')$ then the conclusion of Theorem 10.6.2 holds.

Given the Langlands classification of Theorem 4.9.2 and Theorem 8.4.2, it would be aesthetically pleasing if $\sigma_v$ were tempered for each $v$. Indeed, if this were the case, then upon rearranging the $\sigma_i$ we could assume that $\lambda$ is in the positive Weyl chamber. There would then be a canonical irreducible subquotient of $I(\sigma, \lambda)$, namely

$$\bigotimes_v J(\sigma_v, \lambda)$$

with the local factors $J(\sigma_v, \lambda)$ defined as in Theorem 4.9.1 and Theorem 8.4.1. An important open conjecture is that cuspidal representations always have tempered local factors:

Conjecture 10.6.4 (The Ramanujan conjecture) If $\pi$ is a cuspidal automorphic representation of $\mathbb{A}_{\text{GL}_n} \backslash \text{GL}_n(\mathbb{A}_F)$, then $\pi_v$ is tempered for all $v$.

A discussion of this conjecture and what is known towards it is contained in [Sar05].

Since the Ramanujan conjecture remains unproven to find a canonical irreducible subquotient of $I(\sigma, \lambda)$ we must proceed differently. It is convenient to adjust our notation slightly. Let $\pi_i$ be a cuspidal automorphic representation of $\text{GL}_{n_i}(\mathbb{A}_F)$ for $1 \leq i \leq k$ and let $n = \sum_{i=1}^k n_i$. Let $\pi^1_i$ be the unique cuspidal automorphic representation of $\mathbb{A}_{\text{GL}_{n_i}} \backslash \text{GL}_{n_i}(\mathbb{A}_F)$ such that

$$\pi_i(g) = |\det g|^{\lambda_i} \pi^1_i(g)$$

for some $\lambda_i \in \mathbb{C}$. After rearranging the $\pi_i$ we can and do assume (10.19) holds. Let $P$ be the standard parabolic subgroup of $\text{GL}_n$ of type $(n_1, \ldots, n_k)$ and let $\lambda = (\lambda_1, \ldots, \lambda_k)$. Then we can form the induced representation

$$I\left(\bigotimes_{i=1}^k \pi^1_i, \lambda\right).$$

By Theorem 10.6.1 any irreducible subquotient of this representation is automorphic. Recall the local isobaric sums
\[ \bigoplus_{i=1}^{k} \pi_i \]
defined in §10.5.

**Theorem 10.6.5** The representation

\[ \bigoplus_{i=1}^{k} \pi_i := \bigotimes_{\nu} \bigoplus_{i=1}^{k} \pi_{i\nu} \]  

(10.23)

is an irreducible subquotient of \( I \left( \bigotimes_{i=1}^{k} \pi_{i\nu} \right) \), hence automorphic.

**Proof.** The representations \( \pi_{i\nu} \) are generic in the sense of §11.3 by Theorem 11.3.3. Thus for all \( \nu \) the representation \( \bigoplus_{i=1}^{k} \pi_{i\nu} \) is an irreducible subquotient of \( I \left( \bigotimes_{i=1}^{k} \pi_{i\nu} \right) \) by Theorem 10.5.4, and it is unramified for all but finitely many \( \nu \) by Theorem 10.5.3. In view of the discussion of the irreducible subquotients of induced representations after Theorem 10.6.1, the proof follows. \( \square \)

**Definition 10.1.** An automorphic representation \( \pi \) of \( \text{GL}_n(\mathbb{A}_F) \) is isobaric if there are cuspidal automorphic representations \( \pi_i \) of \( \text{GL}_{n_i}(\mathbb{A}_F) \) with \( \sum_{i=1}^{k} n_i = n \) such that

\[ \pi \cong \bigoplus_{i=1}^{k} \pi_i. \]

Not all automorphic representations are isobaric. A concrete example is given in Exercise 10.9 below.

In view of Theorem 10.6.2, we can extend the definition of \( \bigoplus \) to arbitrary tuples of isobaric automorphic representations in a well-defined manner just as in (10.21). The operation \( \bigoplus \) on the set of isomorphism classes of isobaric representations behaves something like a direct product, but just as in the local case, isobaric sums are always irreducible representations. In fact, isobaric sums correspond to direct sums via the Langlands correspondence (see §13.2).

### 10.7 A theorem of Moeglin and Waldspurger

For this section, we assume that \( F \) is a number field. Let \( G \) be a reductive group over \( F \). The spectral decomposition of \( L^2(\Gamma) \) proved by Langlands leaves open the important question of how to describe the discrete spectrum of \( L^2([M]) \) for Levi subgroups \( M \) of \( G \). Some information on this question is obtained in the course of Langlands’ proof of Theorem 10.4.1, but nothing definitive.

In [Jac84] Jacquet gave a precise conjectural description of \( L^2_{\text{disc}}(\Gamma) \) which was later proven by Moeglin and Waldspurger in [MgW89]. We state this result in this section. Since the Levi subgroups of \( \text{GL}_n \) are products of general linear groups, by induction this gives a complete spectral decomposition of \( L^2(\Gamma) \).
Fix an integer \( n \). Recall that a parabolic subgroup \( P \leq \text{GL}_n \) standard if it contains the Borel subgroup of upper triangular matrices. For each factorization \( n = md \), there is a unique standard parabolic subgroup \( P \) with Levi subgroup \( M \) isomorphic to \( \text{GL}_m \cap \text{GL}_d(A_F) \). We can then form the tensor product representation \( \pi \otimes^m \) of \( A_M \setminus M(A_F) \). Let

\[
L^2_{\text{cusp}}(\pi \otimes^m) \leq L^2_{\text{cusp}}([M])
\]

be the \( \pi \otimes^m \)-isotypic subspace. We can then consider a form

\[
\varphi \in \text{Ind}_P^G(L^2_{\text{cusp}}(\pi \otimes^m))^0
\]

and the corresponding Eisenstein series

\[
E(x, \varphi, \lambda).
\]

Jacquet constructed [Jac84] an irreducible subrepresentation \( (\sigma, m) \) of \( L^2(\text{GL}_n) \) by taking an iterated residue of \( E(x, \varphi, \lambda) \). If we identify \( \mathfrak{a}_M^* = \mathbb{R}^m \) in such a way that

\[
\langle H_M \left( \begin{array}{c} x_1 \\ \vdots \\ x_m \end{array} \right), \lambda \rangle = \sum_{i=1}^m \lambda_i \log |\det x_i|
\]

then the iterated residue is taken at

\[
\left( \frac{m-1}{2}, \frac{m-3}{2}, \cdots, \frac{3-m}{2}, \frac{1-m}{2} \right).
\]

The representation \( (\sigma, m) \) is called a 

**Speh representation**, since Speh constructed the local analogues of these representations (see [Spe83]). In terms of isobaric sums, one has

\[
(\sigma, m) = \boxtimes_{i=1}^m \left( \sigma \otimes |.|^{(m+1)/2-i} \right)
\]

(see [MgW89, §I.11]).

To get a feel for these representations, it is useful to note that if \( v \) is a nonarchimedean place of \( F \) where \( \sigma \) is unramified then

\[
L(s, (\sigma, m)_v) = \prod_{i=1}^m L \left( s - i + \frac{m+1}{2}, \sigma_v \right).
\]

Here \( L(s, (\sigma, m)_v) \) is the \( L \)-function of \( (\sigma, m)_v \) (see §11.8). Jacquet then conjectured the following theorem, which was proven in [MgW89]:
10.7 A theorem of Moeglin and Waldspurger

**Theorem 10.7.1 (Moeglin and Waldspurger)** Any irreducible subrepresentation of $L^2([GL_n])$ is isomorphic to a Speh representation $(\sigma, m)$ for a unique factorization $md = n$ and a unique cuspidal automorphic representation $\sigma$ of $A_{GL_d} \backslash GL_d(\mathbb{A}_F)$.

To reduce possible confusion, we observe that a subrepresentation of $L^2([GL_n])$ is (by definition) a subspace of $L^2_{\text{disc}}([GL_n])$. The theorem gives a complete description of $L^2_{\text{disc}}([GL_n])$. More specifically, it implies that

$$L^2_{\text{disc}}([GL_n]) \cong \bigoplus_{d|n} \bigoplus_{\sigma} V(\sigma, n/d)$$

(10.24)

where the inner Hilbert space direct sum is over isomorphism classes of cuspidal automorphic representations of $A_{GL_d} \backslash GL_d(\mathbb{A}_F)$ and $V(\sigma, n/d)$ is the space of $(\sigma, m)$.

Theorem 10.7.1 forms the basis of much of what we know about the discrete spectrum for other groups. For example, it together with the Jacquet-Langlands correspondence is used to describe the discrete spectrum of inner forms of $GL_n$ in [Bad08]. As a consequence of the theory of twisted endoscopy, Arthur [Art13] and Mok [Mok15] have given a description of the discrete spectrum of quasi-split classical groups in terms of the parametrization of Moeglin and Waldspurger. This is explained in §13.8 below.

**Exercises**

10.1. Let $G$ be a reductive group. Give an example to show that the restriction map

$$X^*(G) \rightarrow X^*(Z_G^0)$$

need not be surjective. Show, however, that the restriction map

$$X^*(G) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow X^*(Z_G^0) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism.

10.2. Prove that the subspace $\text{Ind}^G_P(V)^0 \leq \text{Ind}^G_P(V)$ of (10.7) is dense.

10.3. Prove the intertwining relation (10.9).

10.4. Prove that $I(\sigma, \lambda)$ is unitary for $\lambda \in i\mathfrak{a}_M$.

10.5. With notation as in Example 10.1 prove that the trivial representation is not a quotient of $\text{Ind}^P_{\mathfrak{a}_P}^G(\pi_1 \otimes \pi_2)$.

10.6. Prove Theorem 10.5.1 and Theorem 10.5.2 using Theorem 8.4.2 and the results on the Bernstein-Zelevinsky classification stated in §8.4.
For the remaining problems, let $1$ denote the trivial representation of $GL_1(\mathbb{A}_F)$. Let

$$\pi := I(1, (\frac{1}{2}, -\frac{1}{2}));$$

it is a representation of $GL_2(\mathbb{A}_F)$.

10.7. Prove that every irreducible subquotient of $I(1, (\frac{1}{2}, -\frac{1}{2}))$ is automorphic and that the (unique) isobaric subquotient is the trivial representation of $GL_2(\mathbb{A}_F)$.

10.8. For all places $v$ of $F$, prove that $\pi_v$ has a composition series of length 2, the unique irreducible subrepresentation $St_v$ known as the **Steinberg representation**.

10.9. Prove that the irreducible subquotients of $\pi$ are in bijection with finite sets of places of $F$: the set $S$ of places corresponds to the representation $(\otimes_{v \in S} St_v) \otimes \pi^S$. 
Chapter 11
Rankin-Selberg $L$-functions

Artin and Hecke were together at Göttingen, but neither realized the intimate connection between the two different types of $L$-functions they were constructing. The moral of the story is to talk with your colleagues.

Abstract In this chapter we sketch the theory of generic representations and Rankin-Selberg $L$-functions.

11.1 Paths to the construction of automorphic $L$-functions

Let $F$ be a global field and let $\pi$ and $\pi'$ be a pair of cuspidal automorphic representations of $A_{\text{GL}_n} \backslash \text{GL}_n(\mathbb{A}_F)$ and $A_{\text{GL}_m} \backslash \text{GL}_m(\mathbb{A}_F)$ respectively. An important analytic invariant of this pair is the Rankin-Selberg $L$-function

$$L(s, \pi \times \pi').$$

This is a meromorphic function on the complex plane satisfying a functional equation whose poles can be explicitly described in terms of $\pi$ and $\pi'$. These $L$-functions play a crucial role in automorphic representation theory. In particular they are built into the statement of the local Langlands correspondence for $\text{GL}_n$. This will be discussed in §12.5. We refer to §13.6 for the interpretation of Rankin-Selberg $L$-functions in terms of Langlands $L$-functions (to be defined in §12.7).
In this chapter we describe one method by which Rankin-Selberg $L$-functions can be defined, namely the **Rankin-Selberg method**. A more detailed treatment is contained in [Cog07], which was our primary reference for this chapter. In preparation for the discussion, we define **generic representations** in §11.3 below. We have concentrated on the case of general linear groups in our treatment, but the Rankin-Selberg method can be applied in much greater generality. To learn about further developments, [Bum05] is a place to start.

There are other approaches to defining these $L$-functions. One is via the so-called **Langlands-Shahidi method**. This method also uses the notion of a generic representation. The idea is that Rankin-Selberg $L$-functions (and some more general $L$-functions) occur in the constant term of certain Eisenstein series. This approach was historically important because it suggested both the general definition of a Langlands $L$-function, discussed in §12.7, and the general formulation of Langlands functoriality. We do not discuss this construction. It is discussed at length in the book [Sha10].

In the special case where $\pi'$ is the trivial representation 1 of $GL_1$

$$L(s, \pi) := L(s, \pi \times 1)$$

is known as the **principal** or **standard** $L$-function of $\pi$. There is an alternate approach to defining these $L$-functions due to Godement and Jacquet [GJ72]. It is a direct generalization of Tate’s construction of the $L$-functions of Hecke characters in his famous thesis [Tat67], which in turn is an adelic formulation of the classical work of Hecke.

### 11.2 Generic characters

Let $G$ be a quasi-split reductive group over a global or local field $F$, let $B \leq G$ be a Borel subgroup with unipotent radical $N$, and let $T \leq B$ be a maximal split torus. Let $\Phi(G, T)$ be the set of roots of $T$ in $G$ and let $\Delta \subset \Phi(G, T)$ be the base associated to $B$ (see §1.9). For each $\alpha \in \Delta$, there is a corresponding root group $N_\alpha \leq N$ (see (1.31)).

**Definition 11.1.** If $F$ is a local field, a character $\psi : N(F) \to \mathbb{C}^\times$ is called **generic** if $\psi|_{N_\alpha(F)}$ is nontrivial for each simple root $\alpha \in \Delta$. If $F$ is global, a character $\psi : N(\mathbb{A}_F) \to \mathbb{C}^\times$ trivial on $N(F)$ is called **generic** if $\psi_v$ is generic for all places $v$.

**Example 11.1.** Take $G = GL_n$ and let $B_n \leq GL_n$ be the Borel subgroup of upper triangular matrices. Moreover let $N_n \leq B_n$ be the unipotent radical. If $F$ is local, any character of $N_n(F)$ is of the form
for some $m_1, \ldots, m_{n-1} \in F$, where $\psi : F \to \mathbb{C}^\times$ is a nontrivial character. Comparing this with the description of the root groups from (1.28), we see that the character is generic if and only if all of the $m_i$ are nonzero. If $F$ is global, any character of $N_n(A_F)$ trivial on $N_n(F)$ is of the same form, where the $m_i$ are in $F$ and $\psi : F \backslash A_F \to \mathbb{C}^\times$ is nontrivial. Again, the character is generic if and only if $\prod_{i=1}^{n-1} m_i \neq 0$. If all $m_i = 1$ we call this the standard character attached to $\psi$.

11.3 Generic representations

Let $G$ be a quasi-split reductive group over a local field $F$ and let $\psi : N(F) \to \mathbb{C}^\times$ be a character. Let $(\pi, V)$ be an admissible representation of $G(F)$.

**Definition 11.2.** Assume $F$ is nonarchimedean. A $\psi$-Whittaker functional on $V$ is a continuous linear functional $\lambda : V \to \mathbb{C}$ such that

$$
\lambda(\pi(n)\varphi) = \psi(n)\lambda(\varphi)
$$

for $\varphi \in V$ and $n \in N(F)$.

In this nonarchimedean setting, the term “continuous” means locally constant.

If $F$ is archimedean, we assume that $V$ is a Hilbert space and even that $\pi$ is unitary. For every $X \in U(g)$, the universal enveloping of the Lie algebra of $G$, we can define a seminorm $\| \cdot \|_X$ on $V_{sm}$ via

$$
\| \varphi \|_X := \| \pi(X)\varphi \|_2.
$$

Here $\| \cdot \|_2$ is the norm on the Hilbert space $V$. We can give $V_{sm}$ the structure of a Fréchet space via these seminorms, and it then makes sense to speak of continuous linear functionals. It is this notion of continuity we use in the following definition:

**Definition 11.3.** Assume $F$ is archimedean. A $\psi$-Whittaker functional on $V$ is a continuous linear functional $\lambda : V_{sm} \to \mathbb{C}$ such that

$$
\lambda(\pi(n)\varphi) = \psi(n)\lambda(\varphi)
$$
for \( \varphi \in V_{\text{sm}} \) and \( n \in N(F) \).

Assume, as above, that in the archimedean setting \((\pi, V)\) is unitary.

**Definition 11.4.** An irreducible admissible representation \((\pi, V)\) of \( G(F) \) is \( \psi\)-**generic** if it admits a nonzero \( \psi\)-Whittaker functional.

An irreducible admissible representation \((\pi, V)\) is simply said to be **generic** if there exists a Borel subgroup \( B \subseteq G \) with unipotent radical \( N \) and a generic character \( \psi : N(F) \to \mathbb{C}^\times \) such that \((\pi, V)\) is \( \psi\)-generic. We warn the reader that, in the older literature, generic representations were sometimes referred to as **nondegenerate representations**.

The following theorem [GK73, Sha74] has turned out to be extremely important:

**Theorem 11.3.1** Assume \( \psi \) is generic. The space of \( \psi\)-Whittaker functionals on an irreducible admissible representation of \( G(F) \) is at most 1-dimensional. \( \square \)

The assumption that \( \psi \) is generic cannot be removed (see Exercise 11.2).

We now define the related notion of a Whittaker function. The space of \( \psi\)-Whittaker functions on \( G(F) \) is defined as

\[
\mathcal{W}(\psi) := \{ W \in C^\infty(G(F)) : W(ng) = \psi(n)W(g) \}
\]

for all \((n, g) \in N(F) \times G(F)\). This space admits a natural action of \( G(F) \):

\[
G(F) \times \mathcal{W}(\psi) \longrightarrow \mathcal{W}(\psi)
\]

\[
(g, W) \longmapsto (x \longmapsto W(xg)).
\]

Let \((\pi, V)\) be an irreducible admissible representation of \( G(F) \). When \( F \) is archimedean, we assume \( \pi \) is unitary. A **\( \psi\)-Whittaker model** of \( \pi \) is the image of a nonzero \( G(F)\)-intertwining map

\[
A : V_{\text{sm}} \longrightarrow \mathcal{W}(\psi)
\]  

such that the functional

\[
V_{\text{sm}} \longrightarrow \mathbb{C}
\]

\[
\varphi \longmapsto A(\varphi)
\]

is continuous. Here we give \( V_{\text{sm}} \) the Fréchet topology described above when \( F \) is archimedean and when \( F \) is nonarchimedean, \( V = V_{\text{sm}} \) is given the discrete topology so the continuity condition is vacuous. We observe that (11.4) is a Whittaker functional. Conversely, a Whittaker functional \( \lambda \) gives rise to a Whittaker model via

\[
V_{\text{sm}} \longrightarrow \mathcal{W}(\psi)
\]
Thus a $\psi$-Whittaker model of $\pi$ exists if and only if $\pi$ is $\psi$-generic. Moreover we obtain the following corollary of Theorem 11.3.1:

**Corollary 11.3.2** If $\psi$ is generic and $\pi$ is $\psi$-generic, then it admits a unique $\psi$-Whittaker model of $\pi$.

Motivated by Corollary 11.3.2, if $\psi$ is generic and $(\pi, V)$ is $\psi$-generic, we let

$$ W(\pi, \psi) $$

be the image of any $A$ as in (11.3). We write the map explicitly as

$$ V_{\text{sm}} \rightarrow W(\pi, \psi) $$

$$ \varphi \mapsto W^\psi_{\psi}.$$

By Schur’s lemma, this map is well-defined up to multiplication by a nonzero complex number. Moreover it is an isomorphism. In the archimedean case, we are deliberately avoiding putting a topology on $W(\pi, \psi)$ other than the topology induced by $A$ via transport of structure. For one approach to introducing a more natural topology we refer to [Wal92, Chapter 15].

Now assume that $F$ is global, let $\pi$ be a cuspidal automorphic representation of $A_G \backslash G(\mathbb{A}_F)$ realized in a subspace $V \leq L^2_{\text{cusp}}([G])$, and let $\varphi \in V_{\text{sm}}$. Let $\psi : N(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$ be a character trivial on $N(F)$. We then define the **global $\psi$-Whittaker function**

$$ W^\psi_{\psi}(g) := \int_{[N]} \varphi(ng) \overline{\psi}(n) dn. $$

This definition depends on the choice of right invariant Radon measure $dn$ on $[N]$, but only up to scaling by a positive real number. We say that $(\pi, V)$ is **globally $\psi$-generic** if

$$ W^\psi_{\psi}(g) \neq 0 $$

for some $\varphi \in V$ and $g \in G(\mathbb{A}_F)$. We remark that a priori this notion depends on the realization of $\pi$ in $L^2_{\text{cusp}}([G])$ if it does not occur with multiplicity 1. It is not hard to check that if $\pi$ is globally $\psi$-generic and $\pi \cong \otimes'_v \pi_v$ then each $\pi_v$ is $\psi_v$-generic (see Exercise 11.3). As in the local setting, we simply say $(\pi, V)$ is **generic** if there is a Borel subgroup $B \leq G$ with unipotent radical $N$ and a generic character $\psi : N(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$ trivial on $N(F)$ such that $(\pi, V)$ is $\psi$-generic.

We can also define the notion of a global $\psi$-Whittaker model. Let

$$ W(\psi) := \{ W \in C^\infty(A_G \backslash G(\mathbb{A}_F)) : W(ng) = \psi(n)W(g) \}. $$
Here \((n, g) \in N(\mathbb{A}_F) \times A_G \backslash G(\mathbb{A}_F)\). Suppose that \((\pi, V)\) is globally \(\psi\)-generic. We have a \(G(\mathbb{A}_F)\)-intertwining map

\[
A : V_{sm} \rightarrow \mathcal{W}(\psi)
\]

\[
\varphi \mapsto W^\psi_\varphi.
\]

The \textbf{global} \(\psi\)-\textbf{Whittaker model} of \(\pi\) is the image \(\mathcal{W}(\pi, \psi)\) of this intertwining map.

The notion of a generic representation is only useful if we have an interesting supply of generic representations. Consider the case of \(\text{GL}_n\). Let \(\psi\) and \(\psi'\) be generic characters of \(N(\mathbb{A}_F)\) and \(N'(\mathbb{A}_F)\), respectively, where \(N\) and \(N'\) are the unipotent radicals of two Borel subgroups of \(\text{GL}_n\). By Exercise 11.1 a representation of \(\text{GL}_n\) is \(\psi\)-generic if and only if it is \(\psi'\)-generic. Therefore the following theorem implies that all cuspidal automorphic representations of \(A_{\text{GL}_n} \backslash \text{GL}_n(\mathbb{A}_F)\) are \(\psi\)-generic with respect to any generic character \(\psi\) of the unipotent radical of any Borel subgroup. Moreover they admit an expansion in terms of Whittaker functionals. Let \(N_n \leq \text{GL}_n\) be the unipotent subgroup of upper triangular matrices.

\textbf{Theorem 11.3.3} Let \(\varphi \in L^2_{\text{cusp}}([\text{GL}_n])\) be a smooth vector in the space of a cuspidal automorphic representation \(\pi\) of \(A_{\text{GL}_n} \backslash \text{GL}_n(\mathbb{A}_F)\). If \(\psi : N_n(\mathbb{A}_F) \rightarrow \mathbb{C}^\times\) is a generic character trivial on \(N_n(\mathbb{A}_F)\), then one has

\[
\varphi(g) = \sum_{\gamma \in N_{n-1}(F) \backslash \text{GL}_{n-1}(F)} W^\varphi_\psi ((\gamma_1) g) \quad (11.8)
\]

for a unique choice of left invariant measure on \([N_n]\).

The expression for \(\varphi\) in (11.8) is called its \textbf{Whittaker expansion}. It is a generalization (but not the only generalization) of the well-known Fourier expansion of a modular form.

We now prove this theorem in the special case \(n = 2\). Consider the function

\[
x \mapsto \varphi ((\begin{smallmatrix} 1 & \gamma \\ 0 & 1 \end{smallmatrix}) g).
\]

This is a continuous function on the compact abelian group \(F \backslash \mathbb{A}_F\), and hence admits a Fourier expansion. If we fix a nontrivial character \(\psi : F \backslash \mathbb{A}_F \rightarrow \mathbb{C}^\times\), then all other characters are of the form \(\psi_\alpha(x) := \psi(\alpha x)\) for \(\alpha \in F\) (see Lemma B.1.2). Thus the Fourier expansion of the function (11.9) is

\[
\sum_{\alpha \in F} \psi(\alpha x) \int_{F \backslash \mathbb{A}_F} \varphi ((\begin{smallmatrix} 1 & y_1 \\ 0 & 1 \end{smallmatrix}) g) \overline{\psi}(\alpha y) dy \quad (11.10)
\]

for a unique Haar measure \(dy\) on \(F \backslash \mathbb{A}_F\). Since \(\varphi\) is cuspidal, the \(\alpha = 0\) term vanishes identically. Taking a change of variables \(y \mapsto \alpha^{-1} y\) for each \(\alpha \in F^\times\) and using the left \(\text{GL}_2(F)\)-invariance of \(\varphi\), we see that (11.10) is equal to
\[ \sum_{\alpha \in F^*} \psi(ax) W_\psi^\varphi ((\begin{smallmatrix} \alpha & 1 \\ & 1 \end{smallmatrix}) g). \quad (11.11) \]

Setting \( x = 0 \) we obtain Theorem 11.3.3 when \( n = 2 \). We will not prove Theorem 11.3.3 in the higher rank case because we cannot improve on the exposition in [Cog07, §1.1]. The basic idea is to proceed as in the \( n = 2 \) case, using abelian Fourier analysis on certain abelian subgroup of \( N_n(\mathbb{A}_F) \). However, since \( N_n(\mathbb{A}_F) \) is no longer abelian for \( n > 3 \) one has to combine this with a clever inductive argument using the fact that \( \varphi \) is cuspidal.

Using Theorem 11.3.3, we obtain the following related theorem of Shalika [Sha74, Theorem 5.5]:

**Theorem 11.3.4 (Multiplicity one)** An irreducible admissible representation of \( \text{GL}_n(\mathbb{A}_F) \) occurs with multiplicity at most one in \( L^2_{\text{cusp}}([\text{GL}_n]) \).

We note that Theorem 11.3.4 is false for essentially every reductive group that is not a general linear group.

**Proof.** Let \((\pi_1, V_1)\) and \((\pi_2, V_2)\) with \( V_i \leq L^2_{\text{cusp}}([\text{GL}_n]) \) be two realizations of a given cuspidal automorphic representation \((\pi, V)\). Choose equivariant maps \( L_i : V \to V_i \). We then obtain Whittaker functionals

\[ \lambda_i : V_{\text{sm}} \to \mathbb{C}, \quad \varphi \mapsto W_{\psi_{\lambda_i}}^{L_i(\varphi)}(I_n), \]

where \( I_n \) is the identity element. They are nonzero by Theorem 11.3.3. By Theorem 11.3.1 one therefore has that \( \lambda_1 = c\lambda_2 \) for some \( c \in \mathbb{C}^\times \). Thus

\[ L_1(\varphi)(g) = \sum_{\gamma \in N_{n-1}(F) \backslash \text{GL}_{n-1}(F)} W_{\psi_{\lambda_1}}^{L_1(\varphi)}((\begin{smallmatrix} \gamma & 1 \\ & 1 \end{smallmatrix}) g) = c \sum_{\gamma \in N_{n-1}(F) \backslash \text{GL}_{n-1}(F)} W_{\psi_{\lambda_2}}^{L_2(\varphi)}((\begin{smallmatrix} \gamma & 1 \\ & 1 \end{smallmatrix}) g) = cL_2(\varphi)(g). \]

This implies that \( V_1 \) and \( V_2 \) have nonzero intersection and hence are equal. \( \square \)

### 11.4 Formulae for Whittaker functions

In this section we explain how to compute Whittaker functions, starting from the local unramified situation.

Until otherwise specified, let \( F \) be a nonarchimedean local field unramified over its prime field. Let \( \varpi \) be the uniformizer of \( F \) and let \( \psi : F \to \mathbb{C}^\times \) be a nontrivial unramified character. The results we will state in this section
are valid for any unramified reductive group over \( F \), but to simplify our discussion we will assume that \( G \) is a split reductive group. Let \( B \leq G \) be a Borel subgroup with split maximal torus \( T \leq B \). We let \( K \leq G(F) \) be a hyperspecial subgroup in good position with respect to \((B, T)\). We note that by the Iwasawa decomposition one has

\[
G(F) = \prod_{\mu \in X_*(T)} N(F) \mu(\varpi) K
\]

(11.12)

where \( N \) is the unipotent radical of \( B \).

Using the notation of \S 7.6, for \( \lambda \in \mathfrak{a}_C^* \) we can form the induced representation \( I(\lambda) \), where the induction is with respect to the Borel subgroup \( B \). Its unique unramified subquotient is denoted by \( J(\lambda) \). Any unramified representation is equivalent to such a \( J(\lambda) \). Assume that \( J(\lambda) \) is generic. Then the space

\[
W(J(\lambda), \psi)^K
\]

is 1-dimensional by Corollary 5.5.2. An element of the 1-dimensional space \( W(J(\lambda), \psi)^K \) is fixed by \( K \) on the right and transforms according to \( \psi \) under the action of \( N(F) \) on the left. Thus it is determined by its value on \( T(F) \).

The choice of \( T \) and \( B \), by duality, give rise to a maximal torus and Borel subgroup \( \tilde{T} \leq \tilde{B} \leq \tilde{G} \) as explained in \S 7.3. We have \( X^*(\tilde{T}) = X_*(T) \). For the purpose of Theorem 11.4.1 below, we say that \( \mu \in X_*(T) \) is \( \text{dominant} \) if the corresponding character in \( X^*(\tilde{T}) \) is a dominant weight with respect to \( \tilde{B} \). By Cartan-Weyl theory, if \( \mu \) is dominant there is a unique isomorphism class \( V(\mu) \) of irreducible representation of \( \tilde{G} \) with \( \mu \) as its highest weight. Let \( \chi_\mu \) be the character of \( V(\mu) \).

**Theorem 11.4.1** Any function \( W \) in the 1-dimensional space \( W(J(\lambda), \psi)^K \) satisfies \( W(I) \in \mathbb{C}^* \), where \( I \in G(F) \) is the identity element. If \( \mu \) is a dominant weight then

\[
\frac{W(\mu(\varpi))}{W(I)} = \delta_B^{1/2} \mu(\varpi) \chi_\mu(q^{-\lambda}).
\]

If \( \mu \) is not a dominant weight then \( W(\mu(\varpi)) = 0 \).

\[ \square \]

Here \( q^{-\lambda} \in \tilde{G}(\mathbb{C}) \) is the Langlands class of \( J(\lambda) \) as in Theorem 7.6.7.

Theorem 11.4.1 was originally proved in unpublished work of Kato after being conjectured by Langlands. A new proof together with a generalization to unramified groups occurs in [CS80]. The formula is commonly referred to as the **Casselman-Shalika formula**. In [FGKV98, Theorem 5.3] one can find the formula in the form stated in Theorem 11.4.1. To deduce the expression in Theorem 11.4.1 from the expression in [CS80] one uses the Weyl character formula.

It is instructive to write this formula down more explicitly when \( G = GL_n \). In this case it is due to Shintani [Shi76]. We assume that \( B = B_n \) is the Borel
subgroup of upper triangular matrices and $T = T_n \leq B_n$ is the maximal torus of diagonal matrices. In this case weights can be identified with tuples $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ as in Example 1.12. The cocharacter $\mu$ attached to such a tuple is recorded in Example 1.12. The dominant weights are those with

$$k_1 \geq \cdots \geq k_n.$$ 

The associated irreducible representation of $GL_n$ is $S_{k_1, \ldots, k_n}(V_{st})$, where $V_{st}$ is the standard representation and $S_{k_1, \ldots, k_n}$ is the Schur functor attached to the partition $k_1, \ldots, k_n$ of $\sum_{i=1}^{n} k_i$. Let $\chi_{k_1, \ldots, k_n}$ be its character.

**Corollary 11.4.2 (Shintani)** If $G = GL_n$ and $W \in \mathcal{W}(J(\lambda), \psi)^{GL_n(O_F)}$ is the unique vector satisfying $W(I_n) = 1$ then

$$W \begin{pmatrix} \pi^{k_1} \\ \vdots \\ \pi^{k_n} \end{pmatrix} = \begin{cases} \delta^{1/2}_{\mu_{\mathbb{C}}} (\mu(\varpi)) \chi_{k_1, \ldots, k_n}(q^{-\lambda}) & \text{if } k_1 \geq \cdots \geq k_n \\ 0 & \text{otherwise.} \end{cases}$$

We now return to the global setting. Let $(\pi, V)$ be a globally generic cuspidal representation of a quasi-split reductive group $G$, realized on a closed subspace $V \leq L^2([G])$. By Flath’s theorem (Theorem 5.7.1) and Theorem 6.6.3, we have a factorization

$$V_{sm} \cong V_{\infty} \otimes (\otimes_{v \mid \infty} V_v). \quad (11.13)$$

In the function field case, $V_{\infty} \cong \otimes_{v \mid \infty} V_v$, by Theorem 5.7.2. In the number field case, by [Fol95, Theorem 7.25] we have

$$V_{\infty} \cong \hat{\otimes}_{v \mid \infty} V_v$$

for some irreducible unitary representations $(\pi_v, V_v)$. Here the $V_v$ are Hilbert spaces and the tensor product is the tensor product of Hilbert spaces. It follows that

$$V_{sm} \cong \hat{\otimes}_{v \mid \infty} V_{v sm}, \quad (11.14)$$

where the completed tensor product is with respect to the Fréchet structure described after (11.2) above. Thus we have

$$\iota : V_{sm} \hat{\rightarrow} (\hat{\otimes}_{v \mid \infty} V_v) \otimes (\hat{\otimes}_{v \mid \infty} V_v), \quad (11.15)$$

where the hat should be regarded as merely an algebraic tensor product in the function field case.

Suppose that the spaces $V_v$ are subspaces of a space of functions

$$\{ \varphi_v : G(F_v) \rightarrow \mathbb{C} \}.$$
For example, in the unramified case they can all be realized as functions in a suitable principal series representation (see Theorem 7.6.7). We point out that the fact that the isomorphism (11.15) exists does not imply that we can write \( \varphi(g) = \prod_v \varphi_v(g_v) \) for some collection of \( \varphi_v \). However, remarkably, if we pass to the Whittaker model, an analogous statement turns out to be true as we now explain.

Let \( K = \prod_v K_v \leq G(\mathbb{A}_F) \) be a maximal compact subgroup. For almost all places, \( K_v \) is hyperspecial and \( V_v \) contains a unique line of \( K_v \)-fixed vectors. Hence the same is true of \( W(\pi_v, \psi_v) \) which is, after all, isomorphic to \( V_v \). There is then a unique \( W_0 \) such that \( W_0(I) = 1 \), where \( I \in G(F_v) \) is the identity. Here we are using the fact that any nonzero unramified vector \( \varphi_0 \in V_v^{K_v} \) for all but finitely many \( v \). By multiplying by a suitable elements of \( C \) we can and do assume that \( W_0 = W_{\psi_v} \) for all but finitely many \( v \). Here \( W_0 \) is defined as in (11.6).

Consider the restricted direct product
\[
\left( \otimes_v W(\pi_v, \psi_v) \right) \otimes \left( \otimes_v W(\pi_v, \psi_v) \right)
\]
with respect to the \( W_0 \). Here when \( F \) is a number field, the tensor product over the infinite places is completed with respect to the topologies on the \( W(\pi_v, \psi_v) \) obtained by transport of structure from \( V_v \). In the function field case, it should just be understood as an algebraic direct product. Consider the restricted direct product \( \otimes_v W(\pi_v, \psi_v) \) over all places, where we do not take the completed tensor product at the infinite places. The space \( \otimes_v W(\pi_v, \psi_v) \) is naturally a subspace of \( C^\infty(G(\mathbb{A}_F)) \) and one can check that the same is true of (11.16).

**Proposition 11.4.3** One has that
\[
W(\pi, \psi) = \left( \otimes_v W(\pi_v, \psi_v) \right) \otimes \left( \otimes_v W(\pi_v, \psi_v) \right).
\]
In particular, if \( \iota(\varphi) = \otimes_v \varphi_v \) then there is a \( c \in C^\times \) such that
\[
W_\psi(\varphi) = c \prod_v W_\psi(\varphi_v)
\]
for all \( g \in G(\mathbb{A}_F) \). Here \( \iota \) is defined as in (11.15).

Thus to compute the global Whittaker function \( W_\psi(\varphi) \), it suffices to compute the local Whittaker functions \( W_\psi(\varphi_v) \).

**Proof.** Given the isomorphism (11.15), to prove the first equality it suffices to prove the second. Thus assume \( \iota(\varphi) = \otimes_v \varphi_v \).
Upon replacing \( \varphi \) by \( R(g)\varphi \) for some \( g \in G(\mathbb{A}_F) \), we can and do assume that \( W_\psi^\varphi(I) \) and \( \prod_v W_\psi^\varphi_v(I) \) are nonzero. Let \( K \leq G(\mathbb{A}_F) \) be a maximal compact subgroup. Choose a finite set \( S \) of places of \( F \) large enough that \( K^S \) is hyperspecial and \( \varphi \) is fixed by \( K^S \). We claim that

\[
\frac{W_\psi^\varphi(g_{sI}^S)}{W_\psi^\varphi(I)} = \prod_{v \in S} \frac{W_\psi^\varphi_v(g_v)}{W_\psi^\varphi_v(I)}.
\]  

(11.17)

Since we are free to enlarge \( S \) and we have normalized \( \iota \) so that \( W_\psi^\varphi_v(I) = 1 \) for all but finitely many \( v \) the claim implies the proposition.

Choose \( v' \in S \). The linear functionals

\[ A_{1v'}, A_{2v'} : V_{v'} \rightarrow \mathbb{C} \]

given by

\[ A_{1v'}(\varphi') := W_{\psi_v}^{-1}(\varphi' \otimes (\otimes_{v \neq v'} \varphi_v))(I) \quad \text{and} \quad A_{2v'}(\varphi') := W_{\psi_v}^{\varphi_v}(I) \]

are both nonzero \( \psi_v \)-Whittaker functionals. Hence by Theorem 11.3.1

\[ A_{1v'} = c_{v'} A_{2v'} \]  

(11.18)

for some \( c_{v'} \in \mathbb{C}^\times \). Let

\[ A_j := \prod_{v \in S} A_{jv} : \otimes_{v \in S} V_v \rightarrow \mathbb{C}. \]

By (11.18), we have \( A_1 = c A_2 \) for some \( c \in \mathbb{C}^\times \) and (11.17) follows.  

11.5 Local Rankin-Selberg \( L \)-functions

Let \( F \) be a local field and let \( \psi : F \rightarrow \mathbb{C}^\times \) be a nontrivial character. Our goal in this section is to define local Rankin-Selberg \( L \)-functions for generic representations, at least when \( F \) is nonarchimedean. The fundamental idea here is that these \( L \)-functions are the smallest rational functions in \( q^{-s} \) that cancel the poles of a family of zeta functions. The definition of this family was originally given in [JPSS83] and the archimedean case was treated definitively in [Jac09]. We refer to these papers for proofs of the unproved statements we make below.

Let \( \pi \) and \( \pi' \) be irreducible admissible generic representations of \( GL_n(F) \) and \( GL_m(F) \), respectively. Let \( \psi : F \rightarrow \mathbb{C}^\times \) be an additive character. Use it to define a standard generic character of \( N_n(F) \) and \( N_m(F) \) as in Example 11.1. We denote these characters again by \( \psi \). Let
If $m < n$ let

$$
\Psi(s; W, W') := \int W \left( \begin{array}{c} h \\ I_{n-m} \end{array} \right) W'(h) |\det(h)|^{s-(n-m)/2} dh,
$$

$$
\tilde{\Psi}(s; W, W') := \int \int W \left( \begin{array}{c} h \\ I_{n-m} \end{array} \right) dxW'(h) |\det(h)|^{s-(n-m)/2} dh,
$$

(11.19)

where the top integral is over $N_m(F) \backslash GL_n(F)$, the outer integral on the bottom is over $N_m(F) \backslash GL_m(F)$, and the inner integral on the bottom is over $(n-m-1) \times m$ matrices with entries in $F$.

If $m = n$, then for each $\Phi \in S(F^n)$, the Schwartz space, let

$$
\Psi(s; W, W', \Phi) := \int_{N_n(F) \backslash GL_n(F)} W(g)W'(g)\Phi(\epsilon_n g)|\det g|^s dg.
$$

(11.20)

Here $\epsilon_n \in F^n$ is the elementary vector with 0's in the first $n-1$ entries and 1 in the last entry. The expressions in (11.19) and (11.20) are known as local Rankin-Selberg integrals.

**Proposition 11.5.1** Assume that $F$ is nonarchimedean.

(a) The integrals (11.19) and (11.20) converge for $\text{Re}(s)$ sufficiently large. For $\pi$ and $\pi'$ unitary, they converge absolutely for $\text{Re}(s) \geq 1$. For $\pi$ and $\pi'$ tempered, they converge absolutely for $\text{Re}(s) > 0$.

(b) Each integral is a rational function of $q^{-s}$.

(c) The $C$-linear span of the integrals in $C[q^s, q^{-s}]$ and the Schwartz function $\Phi$ when $m = n$ is a principal ideal $I(\pi, \pi')$ as $W$ and $W'$ vary. \hfill $\Box$

In this nonarchimedean case, one proves that there is a unique polynomial $P_{\pi, \pi'} \in C[x]$ satisfying $P_{\pi, \pi'}(0) = 1$ such that $P_{\pi, \pi'}(q^{-s})^{-1}$ is a generator of $I(\pi, \pi')$. One sets that

$$
L(s, \pi \times \pi') := P_{\pi, \pi'}(q^{-s})^{-1}.
$$

(11.21)

In the archimedean case, one actually defines $L(s, \pi \times \pi')$ using the local Langlands correspondence, which was established very early on in the theory by Langlands himself. We will explain this in more detail in §12.3 below. Right now we note the following important bound:

**Theorem 11.5.2** If $\pi$ and $\pi'$ are unitary, then $L(s, \pi \times \pi')$ is holomorphic and nonzero for $\text{Re}(s) \geq 1$. \hfill $\Box$

See [BR94a, §2] for a proof. Technically speaking they only state that the $L$-function is holomorphic for $\text{Re}(s) \geq 1$, but the argument actually proves that it is nonzero as well.

To discuss the archimedean case, we say that a holomorphic function $f(s)$ of $s \in C$ is **bounded in vertical strips** if it is bounded in
for all $\sigma_1 < \sigma_2 \in \mathbb{R}$. A meromorphic function $f(s)$ is bounded in vertical strips if for each $\sigma_1 < \sigma_2$ there is a polynomial $P_{\sigma_1,\sigma_2} \in \mathbb{C}[x]$ such that $P_{\sigma_1,\sigma_2}(s)f(s)$ is holomorphic and bounded in vertical strips in the original sense.

**Theorem 11.5.3** Assume $F$ is archimedean. The local Rankin-Selberg integrals (11.19) and (11.20) converge absolutely for $\Re(s)$ sufficiently large and admit meromorphic continuations to functions of $s$ that are bounded in vertical strips. \(\square\)

Let

$$w_n := \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \quad \text{and} \quad w_{m,n} := \begin{pmatrix} I_m \\ w_{n-m} \end{pmatrix}.$$ 

For functions $W$, $W'$ on $\text{GL}_n(F)$, $\text{GL}_m(F)$, respectively, let

$$\tilde{W}(g) := W(w_ng^{-t}),$$
$$\tilde{W}'(g) := W'(w_mg^{-t}),$$
$$\rho(w_{m,n})\tilde{W}(x) := \tilde{W}(xw_{m,n}).$$

**Theorem 11.5.4** There is a meromorphic function $\gamma(s, \pi \times \pi', \psi)$ such that if $m < n$ then

$$\Psi(1 - s; \rho(w_{m,n})\tilde{W}, \tilde{W}') = \omega'(-1)^{n-1}\gamma(s, \pi \times \pi', \psi)\Psi(s; W, W')$$

and if $m = n$ then

$$\Psi(1 - s; \tilde{W}, \tilde{W'}, \tilde{\Phi}) = \omega'(-1)^{n-1}\gamma(s, \pi \times \pi', \psi)\Psi(s; W, W', \Phi)$$

for all $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \overline{\psi})$. Here $\omega'$ is the central quasi-character of $\pi'$. If $F$ is nonarchimedean, then $\gamma(s, \pi \times \pi', \psi) \in \mathbb{C}(q^{-s})$. \(\square\)

Here

$$\tilde{\Phi}(x_1, \ldots, x_n) = \int_{F^n} \Phi(y_1, \ldots, y_n)\psi\left(\sum_{i=1}^n x_iy_i\right)dy_1 \ldots dy_n$$

is the Fourier transform of $\Phi$. Due to alternate notational conventions the functional equations in the archimedean case presented in [Jac09] are different.

We need one other factor, the local $\varepsilon$-factor:

$$\varepsilon(s, \pi \times \pi', \psi) := \frac{\gamma(s, \pi \times \pi', \psi)L(s, \pi \times \pi')}{L(1 - s, \pi' \times \pi''')}.$$  \hspace{1cm} (11.22)
Proposition 11.5.5 In the nonarchimedean case, \( \varepsilon(s, \pi \times \pi', \psi) \) is a function of the form \( cq^{-fs} \) for some real number \( f \) and \( c \in \mathbb{C}^\times \).

The \( \varepsilon \)-factor is a useful arithmetic invariant of the pair \( (\pi, \pi') \). Assume that \( \pi' \) is trivial. In this case \( f \in \mathbb{Z}_{\geq 0} \) and the ideal \( \mathfrak{o}^f \) is called the conductor of \( \pi \). One writes \( f := f(\pi) \), and calls it the exponent of the conductor, or sometimes just the conductor. Let

\[
K_1(\mathfrak{o}^{f(\pi)}) := \left\{ g \in \text{GL}_n(\mathcal{O}_F) : g \equiv \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & * \\ 0 & \cdots & 1 \end{pmatrix} \pmod{\mathfrak{o}^{f(\pi)}} \right\}.
\]

Then one has the following (see [Mat13]):

Theorem 11.5.6 The number \( f(\pi) \) is the smallest nonnegative integer such that the space \( V \) of \( \pi \) satisfies

\[
V^{K_1(\mathfrak{o}^{f(\pi)})} \neq 0.
\]

Moreover, \( \dim_{\mathbb{C}} V^{K_1(\mathfrak{o}^{f(\pi)})} = 1. \)

11.6 The unramified Rankin-Selberg \( L \)-functions

In this section we assume that \( F \) is nonarchimedean and unramified over its prime field, that \( \psi : F \to \mathbb{C}^\times \) is nontrivial and unramified, and that \( \pi \) and \( \pi' \) are irreducible unramified generic representations of \( \text{GL}_n(F) \) and \( \text{GL}_m(F) \), respectively. Let \( W \in \mathcal{W}(\pi, \psi) \) and \( W' \in \mathcal{W}(\pi', \overline{\psi}) \) be the unique spherical vectors satisfying \( W(I_n) = W'(I_m) = 1 \). We assume that \( \pi \cong J(\lambda) \) and \( \pi' \cong J(\lambda') \), where \( \lambda \in \mathfrak{a}_{I_n, \mathbb{C}}^* \) and \( \lambda' \in \mathfrak{a}_{I_m, \mathbb{C}}^* \).

Our aim in this section is to prove the following identity:

Theorem 11.6.1 For \( \text{Re}(s) \) sufficiently large, if \( n > m \) one has that

\[
\Psi(s; W, W') = \det(I_{mn} - q^{-s} q^{-\lambda} \otimes q^{-\lambda'})^{-1}.
\]

If \( m = n \) one has that

\[
\Psi(s; W, W', 1_{\mathcal{O}_p}) = \det(I_{n^2} - q^{-s} q^{-\lambda} \otimes q^{-\lambda'})^{-1}.
\]

Implicit in the definition of the local Rankin-Selberg integrals is a choice of measure on \( \text{GL}_m(F) \) and \( N_m(F) \) so that we can form a quotient measure on \( N_m(F) \backslash \text{GL}_m(F) \). In order for the equalities above to be valid, we must
choose the unique Haar measures that assign volume 1 to $GL_m(O_F)$ and $N_m(O_F)$, respectively.

**Proof.** For each $m \in \mathbb{Z}_{\geq 1}$ let $T^+(m)$ be the set of the tuples $\mu = (k_1, \ldots, k_m) \in \mathbb{Z}^m$ with $k_1 \geq \cdots \geq k_m \geq 0$. This can be identified with a subset of the dominant weights of $T_m$ (which in turn are cocharacters of $T_m$, see Example 11.12).

If $n > m$ let

$$T^+(m) \rightarrow T^+(n)$$

be the injection given by extending the tuple $k_1, \ldots, k_m$ by adding zeros.

Assume for the moment that $n > m$. Then by Corollary 11.4.2 we have

$$\Psi(s; W, W') = \sum_{\mu \in T^+(m)} W(\mu(\varpi)) I_{n-m} W'(\mu(\varpi)) | \det(\mu(\varpi))| s-(n-m)/2 \delta_{B_m}^{-1}(\mu(\varpi))$$

$$= \sum_{\mu \in T^+(m)} \chi_{k_1, \ldots, k_m} (q^{-\lambda^*}) \chi_{k_1, \ldots, k_m, 0, \ldots, 0} (q^{-\lambda}) q^{-|\mu|s}$$

$$= \det(I_{mn} - q^{-s} q^{-\lambda^*} \otimes q^{-\lambda})^{-1},$$

where we let $|\mu| = k_1 + \cdots + k_m$ and, as before, $\chi_{k_1, \ldots, k_m, 0, \ldots, 0}$ denotes the character associated to $S_{k_1, \ldots, k_m, 0, \ldots, 0}$. This last equality is known as the Cauchy identity [Bum13, Chapter 38]. Similarly, when $n = m$ we have

$$\Psi(s; W, W', 1_{O_F^2}) = \sum_{\mu \in T^+(n)} W(\mu(\varpi)) W'(\mu(\varpi)) 1_{O_F^2} (e_n \mu(\varpi)) | \det(\mu(\varpi))| s \delta_{B_m}^{-1} (\mu(\varpi))$$

$$= \sum_{\mu \in T^+(n)} \chi_{k_1, \ldots, k_n} (q^{-\lambda^*}) \chi_{k_1, \ldots, k_n} (q^{-\lambda^*}) q^{-|\mu|s}$$

$$= \det(I_{n^2} - q^{-s} q^{-\lambda} \otimes q^{-\lambda^*})^{-1}.$$  

Here, as before, $e_n \in F^n$ is the elementary vector with 0’s in the first $n - 1$ entries and 1 in the last entry. \qed

The calculation above implies that $\det(I_{mn} - q^{-s} q^{-\lambda} \otimes q^{-\lambda^*})$ divides $L(s, \pi \times \pi')^{-1}$ as a polynomial in $q^{-s}$. More is true (see [JPSS83]):

**Theorem 11.6.2** One has $L(s, \pi \times \pi') = \det(I_{mn} - q^{-s} q^{-\lambda} \otimes q^{-\lambda^*})^{-1}$. \qed

11.7 Global Rankin-Selberg $L$-functions

Let $\pi$ and $\pi'$ be cuspidal automorphic representations of $GL_n(\mathbb{A}_F)$ and $GL_m(\mathbb{A}_F)$, respectively.
We have previously defined the local Rankin-Selberg $L$-functions $L(s, \pi_v \times \pi'_v)$. The global Rankin-Selberg $L$-function is the product

$$L(s, \pi \times \pi') := \prod_v L(s, \pi_v \times \pi'_v). \quad (11.23)$$

If $\psi : F \backslash \mathbb{A}_F \to \mathbb{C}^\times$ is a nontrivial character we also define

$$\varepsilon(s, \pi \times \pi') := \prod_v \varepsilon(s, \pi_v \times \pi'_v, \psi_v). \quad (11.24)$$

The reason that $\psi$ is not encoded in to the left hand side of (11.24) is that the right hand side is in fact independent of the choice of $\psi$.

We observe that for $\text{Re}(s)$ sufficiently large

$$L(s, \pi \otimes | \det |^s_1 \times \pi' \otimes | \det |^s_2) = L(s + s_1 + s_2, \pi \times \pi').$$

Thus without loss of generality we can and do assume that $\pi$ (resp. $\pi'$) is trivial on $A_{\text{GL}_n}$ (resp. $A_{\text{GL}_m}$). The following theorem collects the basic facts about Rankin-Selberg $L$-functions:

**Theorem 11.7.1** The Rankin-Selberg $L$-function admits a meromorphic continuation to the plane, holomorphic except for possible simple poles at $s = 0, 1$. There are poles at $s = 0, 1$ if and only if $m = n$ and $\pi \cong \pi'^\vee$. One has a functional equation

$$L(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi')L(1 - s, \pi'^\vee \times \pi'^\vee). \quad (11.25)$$

One extremely important consequence of this theorem is the strong multiplicity one theorem:

**Theorem 11.7.2 (Strong multiplicity one)** Let $\pi$ and $\pi'$ be cuspidal automorphic representations of $\text{GL}_n(\mathbb{A}_F)$ and let $S$ be a finite set of places of $F$. If $\pi^S \cong \pi'^S$ then $\pi \cong \pi'$. □

We leave the proof as Exercise 11.7. It is a consequence of Proposition 11.5.1, Theorem 11.5.2 and Theorem 11.7.1. Despite its name, Theorem 11.7.2 does not imply the multiplicity one result of Theorem 11.3.4 above. More generally, one has the following (see Exercise 11.8).

**Corollary 11.7.3** Let $\pi$ and $\pi'$ be isobaric automorphic representations of $\text{GL}_n(\mathbb{A}_F)$ and let $S$ be a finite set of places of $F$. If $\pi^S \cong \pi'^S$ then $\pi \cong \pi'$. □

For a beautiful generalization of the strong multiplicity one theorem, we refer the reader to the work of Ramakrishnan on automorphic analogues of the Chebatarev density theorem [Ram15].

We outline the basic idea behind Theorem 11.7.1 in the case $m = n - 1$. We will not justify any of the convergence statements made in our outline.
Let $\varphi$ be in the space of $\pi$ and $\varphi'$ in the space in $\pi'$. We take

$$I(s; \varphi, \varphi') := \int_{\text{GL}_{n-1}(F) \setminus \text{GL}_{n-1}(A_F)} \varphi(h) \varphi'(h) |\det h|^{s-1/2} dh.$$  \hfill (11.26)

Let $\bar{\varphi}(g) := \varphi(g^{-t})$ and $\bar{\varphi}'(g) := \varphi'(g^{-t})$. Taking a change of variables $h \mapsto h^{-t}$ we see that

$$I(s; \varphi, \varphi') = I(1-s; \bar{\varphi}, \bar{\varphi'}).$$ \hfill (11.27)

This simple change of variables is what ultimately powers the proof of theorem 11.7.1.

Define

$$\Psi(s; W_\psi^\varphi, W_\psi^{\varphi'}) = \int_{N_{n-1}(A_F) \setminus \text{GL}_{n-1}(A_F)} W_\psi^\varphi(h) W_\psi^{\varphi'}(h) |\det h|^{s-1/2} dh.$$  

This is a global Rankin-Selberg integral.

**Theorem 11.7.4** For $\Re(s)$ sufficiently large, $I(s; \varphi, \varphi') = \Psi(s; W_\psi^\varphi, W_\psi^{\varphi'})$.

In particular, the functional equation (11.27) immediately implies an analogous one for the global Rankin-Selberg integral $\Psi(s; W_\psi^\varphi, W_\psi^{\varphi'})$.

**Proof.** Replacing $\varphi$ by its Whittaker expansion, we have that

$$I(s; \varphi, \varphi') = \int_{\text{GL}_{n-1}(F) \setminus \text{GL}_{n-1}(A_F)} \varphi(h) \varphi'(h) |\det h|^{s-1/2} dh$$

$$= \int_{\text{GL}_{n-1}(F) \setminus \text{GL}_{n-1}(A_F)} \sum_{\gamma} W_\psi^\varphi(\gamma_1) \varphi'(h) |\det h|^{s-1/2} dh,$$

where the inner sum is over $\gamma \in N_{n-1}(F) \setminus \text{GL}_{n-1}(F)$. Since $\varphi'(h)$ is left $\text{GL}_{n-1}(F)$-invariant and $|\det \gamma| = 1$ for $\gamma \in \text{GL}_{n-1}(F)$, we may interchange the order of summation and integration and obtain

$$I(s; \varphi, \varphi') = \int_{N_{n-1}(F) \setminus \text{GL}_{n-1}(A_F)} W_\psi^\varphi(h) \varphi'(h) |\det h|^{s-1/2} dh.$$  

This integral is absolutely convergent for $\Re(s) \gg 0$ which justifies the interchange.

Let us first integrate over $[N_{n-1}]$. View $N_{n-1} \hookrightarrow N_n$ as matrices of the form $(u \ 1)$. Then for $u \in N_{n-1}(A_F)$ one has $W_\psi^\varphi(ug) = \psi(u) W_\psi^\varphi(g)$. Thus

$$I(s; \varphi, \varphi')$$
\[
\begin{align*}
&\int_{N_{n-1}(\mathbb{A}_F)\setminus GL_{n-1}(\mathbb{A}_F)} W_{\psi}^{\pi}(u) \psi'(uh) du | \det h|^{s-1/2} dh \\
= &\int_{N_{n-1}(\mathbb{A}_F)\setminus GL_{n-1}(\mathbb{A}_F)} W_{\psi}^{\pi}(h) \psi(u) \phi'(uh) du | \det h|^{s-1/2} dh \\
= &\int_{N_{n-1}(\mathbb{A}_F)\setminus GL_{n-1}(\mathbb{A}_F)} W_{\psi}^{\pi}(h) W_{\psi'}^{\pi'}(h) | \det h|^{s-1/2} dh \\
= &\psi(s; W_{\psi}^{\pi}, W_{\psi'}^{\pi'}). \quad \Box
\end{align*}
\]

In the case \(m = n - 1\) above, Theorem 11.7.1 can be deduced from theorems 11.5.4 and 11.7.4; we leave this as Exercise 11.9.

### 11.8 The nongeneric case

In the theory above, crucial use was made of the fact that cuspidal automorphic representations are globally generic. Indeed, our definition of local \(L\)-functions was given in terms of the Whittaker model.

We now explain how to treat the general case. Assume first that \(F\) is a local field. From §10.6 we know that every pair of admissible irreducible representations \(\pi\) of \(GL_n(F)\) and \(\pi'\) of \(GL_m(F)\) can be written as an isobaric sum

\[
\pi \cong \bigoplus_{i=1}^k \pi_i \quad \text{and} \quad \pi' \cong \bigoplus_{j=1}^{k'} \pi'_j
\]

where the \(\pi_i\) and \(\pi'_j\) are essentially square integrable and hence generic by Theorem 8.4.4.

For nontrivial characters \(\psi : F\backslash \mathbb{A}_F \to \mathbb{C}^\times\) we then define

\[
L(s, \pi \times \pi') = \prod_{i=1}^k \prod_{j=1}^{k'} L(s, \pi_i \times \pi'_j),
\]

\[
\gamma(s, \pi \times \pi', \psi) = \prod_{i=1}^k \prod_{j=1}^{k'} \gamma(s, \pi_i \times \pi'_j, \psi),
\]

\[
\varepsilon(s, \pi \times \pi') = \prod_{i=1}^k \prod_{j=1}^{k'} \varepsilon(s, \pi_i \times \pi'_j).
\]

Of course, one has to check that this is consistent with our earlier definitions. In other words, we must know that if \(\pi\) and \(\pi'\) are generic, then this procedure yields the same result as if we used our original definition of the local factors. This can be done.
We adopt the analogous conventions in the global case and thereby obtain global Rankin-Selberg $L$-functions and $\varepsilon$-factors for isobaric automorphic representations. Their analytic properties can be deduced from the case where the isobaric representations are cuspidal.

Let $\pi_0, \pi_1, \ldots, \pi_k$ be a collection of cuspidal automorphic representations of $\text{GL}_n(\mathbb{A}_F)$. The multiplicity of $\pi_0$ in $\bigoplus_{i=1}^k \pi_k$ is

$$\# \{1 \leq i \leq k : \pi_i \cong \pi_0\}.$$ 

The reason we are being explicit about this definition is that $\bigoplus_{i=1}^k \pi_k$ is irreducible, although it is induced by the direct sum in a sense made precise in §13.2. Applying Theorem 11.7.1 we deduce the following analytic formula for the multiplicity:

**Lemma 11.8.1** Assume that $\pi_0, \pi_1, \ldots, \pi_k$ are a collection of cuspidal automorphic representations of $\text{GL}_n(\mathbb{A}_F)$. The order of the pole of

$$L(s, \pi_0 \times (\bigoplus_{i=1}^k \pi_k))$$

at $s = 1$ is the multiplicity of $\pi_0'$ in $\bigoplus_{i=1}^k \pi_k$. \qed

### 11.9 The converse theorem

Let $\pi$ be a cuspidal automorphic representation of $\text{GL}_n(\mathbb{A}_F)$. We have seen in Theorem 11.7.1 that for all cuspidal automorphic representations $\pi'$ of $\text{GL}_m(\mathbb{A}_F)$, the $L$-function $L(s, \pi \times \pi')$ is analytically well-behaved. The converse theorem asserts conversely that if an admissible representation $\pi$ of $\text{GL}_n(\mathbb{A}_F)$ has the property that $L(s, \pi \times \pi')$ is well-behaved for enough cuspidal automorphic representations $\pi'$ then $\pi$ is automorphic.

**Theorem 11.9.1 (The converse theorem)** Suppose that $\pi$ is an irreducible admissible representation of $\text{GL}_n(\mathbb{A}_F)$. Assume that, for all cuspidal automorphic representations $\pi'$ of $\text{GL}_m(\mathbb{A}_F)$ with $1 \leq m \leq \max(n - 2, 1)$, the $L$-functions $L(s, \pi \times \pi')$ and $L(s, \pi' \times \pi'')$ have analytic continuations to the plane that are holomorphic and bounded in vertical strips of finite width. Assume moreover that for all $\pi'$ as above

$$L(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi')L(1 - s, \pi'' \times \pi'').$$

Then $\pi$ is a cuspidal automorphic representation of $\text{GL}_n(\mathbb{A}_F)$. \qed

Here to define $\varepsilon(s, \pi \times \pi')$ we fix a nontrivial character $\psi : F \backslash \mathbb{A}_F \to \mathbb{C}^\times$ and formally define

$$\varepsilon(s, \pi \times \pi') := \prod_v \varepsilon(s, \pi_v \times \pi'_v, \psi_v).$$

(11.28)
We refer to [CPS99] for a proof of Theorem 11.9.1 and for variants.

The converse theorem has been used to establish cases of Langlands functoriality [CKPSS04]. In addition, it is philosophically important because it asserts, roughly, that an \( L \)-function that satisfies reasonable assumptions has to come from an automorphic representation. The reader is encouraged to revisit this comment after reading §12.8, which explains the conjectural relationship between Galois representations and automorphic representations.

We end this chapter with the following intriguing conjecture [CPS99, §8, Conjecture 2]:

**Conjecture 11.9.2 (Piatetski-Shapiro)** Let \( \pi \) be an irreducible admissible representation of \( \text{GL}_n(\mathbb{A}_F) \). Assume that for all characters

\[
\chi : \mathbb{A}_m^* F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^*
\]

the \( L \)-functions \( L(s, \pi \times \chi) \) and \( L(s, \pi^\vee \times \chi^{-1}) \) have analytic continuations to the plane that are holomorphic and bounded in vertical strips of finite width. Assume moreover that for all \( \chi \) as above

\[
L(s, \pi \times \chi) = \varepsilon(s, \pi \times \chi) L(1 - s, \pi^\vee \times \chi^{-1}).
\]

Then there is an automorphic representation \( \pi_0 \) of \( \text{GL}_n(\mathbb{A}_F) \) such that \( \pi_0^S \cong \pi^S \) for any finite set \( S \) of places of \( F \) including the infinite places and the places where \( \pi_0 \) and \( \pi \) are ramified. Moreover,

\[
L(s, \pi \times \chi) = L(s, \pi_0 \times \chi) \quad \text{and} \quad \varepsilon(s, \pi \times \chi) = \varepsilon(s, \pi_0 \times \chi).
\]

This conjecture is unknown even for \( n = 4 \).

**Exercises**

11.1. Let \( B_1 \) and \( B_2 \leq \text{GL}_n \) be Borel subgroups defined over a nonarchimedean local field \( F \). For \( 1 \leq i \leq 2 \), let \( N_i \) be the unipotent radical of \( B_i \) and let \( \psi_i \) be a generic character of \( N_i(F) \). Prove that an irreducible admissible representation \( \pi \) of \( \text{GL}_n(F) \) is \( \psi_1 \)-generic if and only if it is \( \psi_2 \)-generic.

11.2. Let \( T \leq \text{SL}_2 \) be the maximal torus of diagonal matrices, \( N_2 \leq \text{SL}_2 \) the unipotent radical of the Borel subgroup of upper triangular matrices, and let \( \lambda \in \mathfrak{a}_T^\vee \). Consider the unramified principal series representation \( I(\lambda) \) of \( \text{SL}_2 \) (see §7.6). Observe that \( \ell(\varphi) := \varphi(I_2) \) is an \( N_2(F) \)-invariant functional. Prove that for \( \text{Re}(\lambda) \) sufficiently large the functional

\[
\varphi \mapsto \int_{N_2(F)} \varphi \left( \left( \begin{smallmatrix} 1 & 1 \\ i & 1 \end{smallmatrix} \right) \left( \begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix} \right) \right) dt
\]
11.9 The converse theorem on the space (7.36) of $I(\lambda)$ is absolutely convergent, $N_1(F)$-invariant, and not a complex multiple of $\ell$. Conclude that the assumption that $\psi$ is generic in Theorem 11.3.1 cannot be removed.

11.3. Let $G$ be a split reductive group over a global field $F$ and assume that $\pi$ is a cuspidal globally $\psi$-generic representation of $G(\mathbb{A}_F)$. If $\pi \cong \otimes_v \pi_v$ prove that each $\pi_v$ is $\psi_v$-generic.

11.4. Let $F$ be a global field, $G$ a reductive group over $F$, and $v$ a place of $F$. Prove that a finite dimensional admissible representation of $G(F_v)$ is a quasi-character. Conclude that if $G$ is quasi-split and nonabelian and $(\pi, V)$ is a cuspidal globally $\psi$-generic representation of $G(\mathbb{A}_F)$ then $(\pi_v, V_v)$ is infinite dimensional for all $v$.

11.5. Prove Theorem 11.4.2 when $n = 2$.

11.6. Let $F$ be a nonarchimedean local field. Assume that $J(\lambda)$ is an irreducible unitary unramified generic representation of $GL_n(F)$ where $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Let $\psi : F \to \mathbb{C}^*$ be unramified, let $W \in W(\pi, \psi)$ and $W' \in W(\pi', \overline{\psi})$ be the unique spherical Whittaker functions satisfying $W(I_n) = W'(I_n) = 1$. Using the identity
\[
\det(I_n \otimes q^{-s} q^\lambda \otimes q^{-\lambda}) \Psi(s; W, W', 1_{\mathcal{O}_F}) = 1
\]
and Proposition 11.5.1, prove that every eigenvalue of $q^{-\lambda}$ lies in $(q^{-1/2}, q^{1/2})$.

11.7. If $\pi$ and $\pi'$ are cuspidal automorphic representations of $GL_n(\mathbb{A}_F)$ such that $\pi^S \cong \pi'^S$ for some finite set $S$ of places of $F$, prove that $\pi \cong \pi'$.

11.8. If $\pi$ and $\pi'$ are isobaric automorphic representations of $GL_n(\mathbb{A}_F)$ such that $\pi^S \cong \pi'^S$ for some finite set $S$ of places of $F$, prove that $\pi \cong \pi'$.

11.9. Prove Theorem 11.7.1 in the case $m = n - 1$ using Theorem 11.7.4 and the local functional equations of Theorem 11.5.4.
Chapter 12
Langlands Functoriality

The local Langlands conjecture is one of those hydra-like conjectures which seems to grow as it gets proved.

M. Harris and R. Taylor

Abstract Langlands functoriality has been a driving force in modern number theory, representation theory, and beyond since it was first posed in a famous letter of Langlands to Weil. We survey progress toward the local Langlands conjecture and state a rough form of the global Langlands functoriality conjecture in this chapter.

12.1 The Weil group

In this chapter we state the Langlands reciprocity and functoriality conjectures. Though the conjectures are mostly open, extremely important progress has been made.

We begin with reciprocity. Let $G$ be a reductive group which is unramified over a nonarchimedean local field $F$ (recall that by our conventions reductive groups are connected). By Corollary 7.5.2 one has a bijection

$$\{(\hat{G} \times \text{Fr}/\sim)(\mathbb{C})\} \leftrightarrow \left\{ \text{isomorphism classes of irreducible unramified representations of } G(F) \right\}.$$ 

This is the archetypal example of a Langlands reciprocity law or Langlands correspondence. Roughly speaking, Langlands reciprocity relates representations of generalized Galois groups with image in $L^cG$ and representations of the points of $G$ in locally compact rings. One refers informally to the Galois side of the correspondence and the automorphic side of the correspondence.
This terminology is standard, but a bit loose, because the Galois side really consists of representations of generalized Galois groups and the representations on the other side of the correspondence are only automorphic in the global setting. It is not difficult to see how the bijection above fits into this framework but we will postpone a discussion to §12.5.

We now describe more precisely the objects involved on the Galois side of the correspondence. We start by indicating why working with traditional Galois groups is not sufficient. Let $F$ be a global or local field, let $F_{\text{sep}}$ be a separable closure of $F$ and let

$$\text{Gal}_F := \text{Gal}(F_{\text{sep}}/F)$$

be the absolute Galois group of $F$. It is endowed with the profinite topology. For a prime $p$ of $O_F$ we denote by $F_{p} \in \text{Gal}_F$ a choice of geometric Frobenius element. It is well-defined up to conjugation and characterized by the fact that it induces the automorphism $x \mapsto x^{[O_F/p]-1}$ on the algebraic closure of $O_F/p$.

A continuous homomorphism $\text{Gal}_F \rightarrow \text{GL}_n(\mathbb{C})$ necessarily has finite image (see Exercise 12.1). On the other hand, there are many continuous homomorphisms $\text{Gal}_F \rightarrow \text{GL}_n(\mathbb{Q}_\ell)$ with infinite image.

**Example 12.1.** If $E$ is an elliptic curve over $\mathbb{Q}$ without CM, then the Tate module of $E$ gives a representation with image isomorphic to $\text{GL}_2(\mathbb{Z}_\ell)$ for almost all primes $\ell$ [Ser72].

**Example 12.2.** A more elementary example is given by the cyclotomic character, which is the character

$$\chi_\ell : \text{Gal}_\mathbb{Q} \rightarrow \mathbb{Z}^\times_\ell$$

defined as follows. If $\sigma \in \text{Gal}_\mathbb{Q}$ and $\zeta_n$ is a primitive $\ell^n$-th root of unity then $\sigma(\zeta_n) = \zeta_n^{a_{\sigma,n}}$ for some $a_{\sigma,n} \in (\mathbb{Z}/\ell^n)^\times$. We then define

$$\chi_\ell(\sigma) = \lim_{n \to \infty} a_{\sigma,n}.$$

One can check that if $p \neq \ell$ then $\chi_\ell(F_{p}) = p^{-1}$.

It would be nice to view these examples as complex valued representations of a single group. At least in the special case where we are only considering representations into $\text{GL}_1(\mathbb{C})$, Weil was able to accomplish this by constructing a modification of the Galois group. We discuss this modification following the exposition of [Tat79]. For finite extensions $E/F$ let

$$C_E := \begin{cases} E^\times & \text{if } F \text{ is local}, \\ E^\times \backslash A_E^\times & \text{if } F \text{ is global}. \end{cases}$$

By class field theory one has an **Artin reciprocity map**
Here and below \( ab \) denotes the maximal abelian Hausdorff quotient, that is, the quotient by the closure of the commutator subgroup. The Artin reciprocity map is not canonically defined. It depends on a choice of sign corresponding locally to whether uniformizers at a prime \( p \) are sent to \( \text{Fr}_p \) or its inverse. To differentiate between these two choices, the morphism \( \text{Fr}_p \) is referred to as the \textbf{geometric Frobenius} and the morphism \( \text{Fr}_p^{-1} \) is referred to as the \textbf{arithmetic Frobenius}. We normalize the Artin reciprocity map so that uniformizers at \( p \) are sent to \( \text{Fr}_p \), the geometric Frobenius. This agrees with the conventions of \([\text{Tat79}]\) and \([\text{HT01}]\).

**Definition 12.1.** Let \( F \) be a local or global field. Then a \textbf{Weil group} of \( F \) is a tuple \( (W_F, \phi, \{\text{Art}_E\}) \) where \( W_F \) is a topological group,

\[
\phi : W_F \rightarrow \text{Gal}_F
\]

is a continuous homomorphism with dense image, and for each finite extension \( E/F \)

\[
\text{Art}_E : C_E \rightarrow (\phi^{-1}(\text{Gal}_E))^{ab}
\]

(12.1)

is an isomorphism. Setting

\[
W_E := \phi^{-1}(\text{Gal}_E),
\]

these data are required to satisfy the following assumptions:

(a) For each finite extension \( E/F \), the composite

\[
C_E \xrightarrow{\text{Art}_E} W_E^{ab} \xrightarrow{\text{induced by } \phi} \text{Gal}_E^{ab}
\]

is the reciprocity map of class field theory.

(b) Let \( w \in W_F \) and \( \sigma = \phi(w) \in \text{Gal}_F \). For each \( E \), the following diagram

\[
\begin{array}{ccc}
C_E & \xrightarrow{\text{Art}_E} & W_E^{ab} \\
\downarrow{\sigma} & & \downarrow \\
C_{\sigma(E)} & \xrightarrow{\text{Art}_{\sigma(E)}} & W_{\sigma(E)}^{ab}
\end{array}
\]

commutes. Here the right arrow is induced by conjugation by \( w \).

(c) For \( E' \subseteq E \), the following diagram

\[
\begin{array}{ccc}
C_{E'} & \xrightarrow{\text{Art}_{E'}} & W_{E'}^{ab} \\
\downarrow{\text{inclusion}} & & \downarrow{\text{transfer}} \\
C_E & \xrightarrow{\text{Art}_E} & W_E^{ab}
\end{array}
\]
(d) The map
\[ W_F \longrightarrow \lim W_{E/F} \]
is an isomorphism, where \( W_{E/F} := W_F / W_E \), the bar denoting closure.

For the proof of the following theorem, we refer to [AT09]:

**Theorem 12.1.1** Let \( F \) be a local or global field. Then a Weil group of \( F \) exists and it is unique up to isomorphism. \( \square \)

In many cases the Weil group can be described explicitly:

**Example 12.5.** If \( F = \mathbb{C} \) then \( W_F \) is just \( \mathbb{C}^\times \), \( \phi \) is the trivial map and \( \text{Art}_F \) is the identity map.

**Example 12.6.** If \( F = \mathbb{R} \) then \( W_F \) is \( \mathbb{C}^\times \cup j\mathbb{C}^\times \) where \( j^2 = -1 \) and \( jcz^{-1} = \bar{c} \) for \( c \in \mathbb{C}^\times \). In this case \( \phi \) takes \( \mathbb{C}^\times \) to 1 and \( j\mathbb{C}^\times \) to the nontrivial element of \( \text{Gal}(\mathbb{C}/\mathbb{R}) \).

For number fields one does not have a nice intrinsic description like in the above examples. However, for all global fields, one has local-global compatibility analogous to local-global compatibility for Galois groups. In more detail, let \( v \) be a place of \( F \) and let \( \iota : F_{\text{sep}} \rightarrow F_v^\text{sep} \) be an \( F \)-homomorphism. Let \( E/F \) be a finite extension and let \( E_v := \iota(E)F_v \). Then there are commutative diagrams

\[
\begin{array}{ccc}
W_{F_v} & \xrightarrow{\phi_v} & \text{Gal}_{F_v} \\
\downarrow{\theta_v} & & \downarrow \\
W_F & \xrightarrow{\phi} & \text{Gal}_F
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
E_v^\times & \xrightarrow{\text{Art}_{E_v}} & W_{ab}^{E_v} \\
\downarrow & & \downarrow \\
E^{\times}\backslash \mathbb{A}_E^\times & \xrightarrow{\text{Art}_E} & W_E^{ab}
\end{array}
\tag{12.2}
\]

where the right horizontal arrows are both induced by \( \iota \) and the map \( E_v^\times \rightarrow E^{\times}\backslash \mathbb{A}_E^\times \) sends \( t \) to the class of the idele that is \( t \) in the \( v \)th place and 1 elsewhere. The morphism \( \theta_v \) is unique in the function field case and unique up to conjugation by an element of \( \ker(\phi : W_F \rightarrow \text{Gal}_F) \) in the number field case [Tat79, Proposition 1.6.1].

Almost by definition of the Weil group, one obtains the following correspondence:
Theorem 12.1.2 (The Langlands correspondence for $\text{GL}_1$) There is a bijection between isomorphism classes of automorphic representations of $\text{GL}_1(\mathbb{A}_F)$ and continuous representations $\chi : W_F \to \text{GL}_1(\mathbb{C})$.

Proof. An irreducible automorphic representation of $\text{GL}_1(\mathbb{A}_F)$ can be identified with a character of $F^\times \backslash \mathbb{A}_F^\times$, which is isomorphic to $W_F^{\text{ab}}$ by definition of the Weil group. \qed

12.2 The Weil-Deligne group and $L$-parameters

The Weil group has more complex representations than the Galois group. However, it still does not have enough complex representations to allow for a generalization of Theorem 12.1.2 (or even a local analogue) from $\text{GL}_1$ to general reductive groups $G$. The correct replacement of $W_F$ when $F$ is a number field should be the as yet hypothetical Langlands group $L_F$ (see §12.6 for a discussion of $L_F$). On the other hand, the correct replacement of $W_F$ when $F$ is a local field is known, so in this section we restrict to this case. We follow the treatment in [GR10].

The **Weil-Deligne group** of a local field $F$ is

$$W'_F := \begin{cases} W_F & \text{if } F \text{ is archimedean,} \\ W_F \times \text{SL}_2(\mathbb{C}) & \text{if } F \text{ is nonarchimedean.} \end{cases} \quad (12.3)$$

We have a homomorphism $\phi : W'_F \to \text{Gal}_F$ by definition of the Weil group. We again denote by $\phi$ the map

$$\phi : W'_F \to \text{Gal}_F \quad (12.4)$$

that is just $\phi$ in the archimedean case and it the composite of the natural quotient map $W'_F \to W_F$ with $\phi$ in the nonarchimedean case.

Let $G$ be a group scheme over $\mathbb{C}$ whose neutral component is a reductive group. If $F$ is archimedean, then a representation of $W'_F$ into $G(\mathbb{C})$ is simply a homomorphism $\rho : W'_F \to G(\mathbb{C})$.

Assume until stated otherwise that $F$ is nonarchimedean. We prepare to define a representation of $W'_F$. Let $k$ be the residue field of $F$, $\overline{k}$ a choice of algebraic closure, and $\text{Gal}_k := \text{Gal}(\overline{k}/k)$. The action of $\text{Gal}_F$ preserves the ring of integers and the (unique) prime ideal of the ring of integers of any finite extension field $E/F$ contained in $\overline{F}$. There is thus an exact sequence

$$1 \to I_F \to \text{Gal}_F \to \text{Gal}_k \to 1$$

where $I_F$ is the **inertia subgroup**, which can be defined as the kernel of the map $\text{Gal}_F \to \text{Gal}_k$.
The Weil group \( W_F \) is a subgroup of \( \text{Gal}_F \) containing \( I_F \). We thus have an exact sequence

\[ 1 \rightarrow I_F \rightarrow W_F \rightarrow \text{Gal}_k. \]

The last map is not surjective. Its image is isomorphic to \( \mathbb{Z} \), whereas \( \text{Gal}_k \) is isomorphic to \( \mathbb{Z} \), the profinite completion of \( \mathbb{Z} \). We deduce that

\[ W_F \cong I_F \rtimes (\text{Fr}) \]

where \( \text{Fr} \) is a geometric Frobenius element.

**Definition 12.2.** Let \( G \) be a group scheme over \( \mathbb{C} \) whose neutral component is a reductive group. If \( F \) is nonarchimedean, a representation of \( W_F' \) into \( G(\mathbb{C}) \) is a homomorphism

\[ \rho : W_F' \rightarrow G(\mathbb{C}) \]

such that \( \rho \) is trivial on an open subgroup of \( I_F \), \( \rho(\text{Fr}) \) is semisimple, and \( \rho|_{\text{SL}_2(\mathbb{C})} \) is induced by a morphism of group schemes \( \text{SL}_2 \rightarrow G \) over \( \mathbb{C} \).

By a representation of \( W_F' \) we mean a representation of \( W_F' \) into \( \text{GL}_n(\mathbb{C}) \) for some integer \( n \). There are various equivalent alternate definitions of a representation of the Weil-Deligne group in the literature (see [GR10] for more details). We also note that, in the nonarchimedean case, what we call a representation more properly ought to be called a Frobenius semisimple representation.

Recall that the \( L \)-group \( \tilde{L}G \) of a reductive group \( G \) over \( F \) admits a canonical map \( \tilde{L}G \rightarrow \text{Gal}_F \) given by taking the quotient by the neutral component. This map occurs in the following definition:

**Definition 12.3.** An \( L \)-parameter is a representation \( \rho \) of \( W_F' \) into \( \tilde{L}G \) such that the composite

\[ W_F' \xrightarrow{\rho} \tilde{L}G \rightarrow \text{Gal}_F \]

is the map \( \phi \) of (12.4). Two \( L \)-parameters are equivalent if they are conjugate by an element of \( \tilde{G}(\mathbb{C}) \).

Given an \( L \)-parameter \( \rho : W_F' \rightarrow \tilde{L}H \) and an \( L \)-map \( \tilde{L}H \rightarrow \tilde{L}G \), we note that the composite homomorphism

\[ W_F' \rightarrow \tilde{L}G \]

is an \( L \)-parameter.

When \( G \) is split, recall that the \( L \)-group of \( G \) is a direct product:

\[ \tilde{L}G := \tilde{G}(\mathbb{C}) \times \text{Gal}_F. \]

In particular in this case an \( L \)-parameter \( \rho : W_F' \rightarrow \tilde{L}G \) induces a representation into \( \tilde{G}(\mathbb{C}) \)
12.2 The Weil-Deligne group and L-parameters

\[ \rho : W_F' \longrightarrow L \longrightarrow \widehat{G}(\mathbb{C}) \]

and every representation of \( W_F' \) into \( \widehat{G}(\mathbb{C}) \) arises in this manner. Thus L-parameters may be identified with representations into \( \widehat{G}(\mathbb{C}) \) in this setting, and the notion of equivalence of L-parameters corresponds to \( \widehat{G}(\mathbb{C}) \)-conjugacy of the associated representations.

We now define L-functions, \( \varepsilon \)-factors, and \( \gamma \)-factors. We start by giving ourselves a representation \( \rho : W_F' \rightarrow \text{GL}(V) \). It is completely reducible because we have assumed that \( \rho(Fr) \) is semisimple. Recall that the isomorphism classes of irreducible representations of \( \text{SL}_2 \) are given by the symmetric power representations Sym\(^n\) of the standard representation. A (completely reducible) representation of the direct product \( W_F' = W_F \times \text{SL}_2(\mathbb{C}) \) decomposes as a direct sum of exterior tensor products of representations of \( W_F \) and \( \text{SL}_2(\mathbb{C}) \), so we have that

\[ V = \bigoplus_{n=0}^{\infty} V_n \otimes \text{Sym}^n. \]

Here \( V_n \) is a representation of \( W_F \) (that is zero for all but finitely many \( n \)). Thus \( V_n \otimes \text{Sym}^n \) is the Sym\(^n\)-isotypic component of \( V \), viewed as a representation of \( \text{SL}_2(\mathbb{C}) \) by restriction. We then define

\[ L(s, \rho) := \prod_{n=0}^{\infty} \det \left( 1 - \rho(Fr)q^{-(s+n/2)}|_{V_n} \right)^{-1}. \]  \hspace{1cm} (12.5)

Assume that \( E/F \) is a field extension of either archimedean or nonarchimedean fields. Let \( \rho : W_F' \rightarrow \text{GL}(V) \) be a representation. It is not hard to see that if we define the induction

\[ \text{Ind}_{E}^{F}(\rho) := \text{Ind}_{W_E}^{W_F}(\rho) : W_F' \longrightarrow \text{GL}(V^\oplus[W_F':W_E']) \]  \hspace{1cm} (12.6)

in the usual manner then it is a representation of \( W_F' \) into \( \text{GL}(V^\oplus[W_F':W_E']) \) in the sense of Definition 12.2. One checks that L-functions are invariant under induction in the following sense:

\[ L(s, \rho) = L(s, \text{Ind}_{E}^{F}(\rho)). \]  \hspace{1cm} (12.7)

Moreover, if \( \rho_i : W_F' \rightarrow \text{GL}(V_i), i \in \{1, 2\} \), are a pair of representations then the direct sum, defined in the usual manner, is a representation into \( \text{GL}(V_1 \oplus V_2) \) in the sense of Definition 12.2. One checks that L-functions are additive in the following sense:

\[ L(s, \rho_1 \oplus \rho_2) = L(s, \rho_1)L(s, \rho_2). \]  \hspace{1cm} (12.8)

We now assume until otherwise stated that \( F \) is archimedean. Part of the data of the Weil group is an isomorphism \( \text{Art}_F : F^\times \rightarrow W_F^{ab} \). Thus we can identify quasi-characters of \( W_F \) and quasi-characters of \( F^\times \). Define the
quasi-character

\[
\chi_{s,m} : F^\times \rightarrow \mathbb{C}^\times
\]
\[
z \mapsto |z|^s \frac{z^m}{(z\bar{z})^{m/2}}.
\] (12.9)

Here if \( F \) is complex, the bar denotes complex conjugation and \( m \) is an integer. If \( F \) is real, the bar is the identity map and we assume \( m \in \{0, 1\} \). In particular \( \chi_{0,1} \) is the sign character. We leave the proof of the following lemma as Exercise 12.8:

**Lemma 12.2.1** Every quasi-character of \( F^\times \) is of the form \( \chi_{s,m} \) for a unique \( s \in \mathbb{C} \) and \( m \in \mathbb{Z} \) where we assume \( m \notin \{0\} \). \( \square \)

Unless otherwise specified, when we discuss the character \( \chi_{s,m} \) we will always assume that \( s \in \mathbb{C} \), that \( m \in \mathbb{Z} \) and if \( F \) is real that \( m \in \{0, 1\} \).

Let

\[
\Gamma_F(s) := \begin{cases} 
\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) & \text{if } F = \mathbb{R}, \\
2(2\pi)^{-s} \Gamma(s) & \text{if } F = \mathbb{C}.
\end{cases}
\] (12.10)

Here \( \Gamma(s) \) is the usual \( \Gamma \)-function. We then define

\[
L(s, \chi_{s',m}) = \Gamma_F\left(s + s' + \frac{|m|}{|F:R|}\right).
\] (12.11)

Now \( W_C \) is just \( \mathbb{C}^\times \) as in Example 12.5 and \( W_R \) contains \( W_C \) as a normal subgroup of index 2. Using this fact the following proposition is not hard to prove (see Exercise 12.9):

**Proposition 12.2.2** When \( F \) is archimedean, representations of \( W_F \) are completely reducible. Irreducible representations of \( W_C \) are 1-dimensional and irreducible representations of \( W_R \) are 1- or 2-dimensional. The irreducible 2-dimensional representations of \( W_R \) are the induced representations

\[
\text{Ind}_C^R(\chi_{s,m})
\]
where \( m \neq 0 \). For \( m \neq 0 \) one has that

\[
\text{Ind}_C^R(\chi_{s,m}) \cong \text{Ind}_C^R(\chi_{s',m'})
\] (12.12)

if and only if \( s = s' \) and \( m = \pm m' \). \( \square \)

Motivated by (12.7) we define

\[
L(s, \text{Ind}_C^R(\chi_{s',m})) := L(s, \chi_{s',m}).
\]

By Proposition 12.2.2, a general representation \( \rho : W_F \to \text{GL}(V) \) is a direct sum of its irreducible subrepresentations. All these subrepresentations
12.2 The Weil-Deligne group and $L$-parameters

are either 1- or 2-dimensional and for such representations, we already have a definition of an $L$-function. Motivated by (12.8) we then define the $L$-function of such a representation to be the product of the $L$-functions of its irreducible subrepresentations. This completes the definition of $L(s, \rho)$ in the archimedean case.

Now we allow $F$ to be either archimedean or nonarchimedean and discuss $\varepsilon$-factors. Let $\psi : F \to \mathbb{C}^\times$ be a nontrivial additive character, $d^\times x$ be a Haar measure on $F^\times$, $dx$ be a Haar measure on $F$, and $\chi$ be a quasi-character, i.e., a homomorphism

$$\chi : W_F \to \text{GL}_1(\mathbb{C}).$$

We identify $\chi$ with a quasi-character of $F^\times$ using Theorem 12.1.2. Let

$$\varepsilon(s, \chi, \psi) := \varepsilon(s, \chi, \psi, dx)$$

to be the nonzero complex number such that

$$\int_{F^\times} \hat{f}(x) \chi(x)^{-1} |x|^{1-s} d^\times x = \varepsilon(s, \chi, \psi) \int_{F^\times} f(x) \chi(x) |x|^s d^\times x / L(s, \chi),$$

(12.13)

where $\hat{f}(y) := \int_F f(x) \psi(xy) dx$ is the Fourier transform of $f$. In certain cases $\varepsilon(s, \chi, \psi)$ can be explicitly computed. When $F$ is nonarchimedean and unramified over its prime field, $\chi$ and $\psi$ are unramified, and $dx$ is normalized so that $dx(O_F) = 1$, then $\varepsilon(s, \chi, \psi) = 1$.

Langlands and Deligne defined in general an $\varepsilon$-factor $\varepsilon(s, \rho, \psi)$ [Del73b, Del75]. The $\varepsilon$-factor encodes important arithmetic information about the representation $\rho$, including its Artin conductor [Tat79]. Finally we set

$$\gamma(s, \rho, \psi) := \frac{\varepsilon(s, \rho, \psi) L(1 - s, \rho^\vee)}{L(s, \rho)}.$$ (12.14)

Now given an $L$-parameter $\rho : W'_F \to {}^L G$ and a representation

$$r : {}^L G \to \text{GL}(V)$$

we obtain a representation $r \circ \rho : W'_F \to \text{GL}(V)$. We then define

$$L(s, \rho, r) := L(s, r \circ \rho),$$

$$\varepsilon(s, \rho, r, \psi) := \varepsilon(s, r \circ \rho, \psi),$$

$$\gamma(s, \rho, r, \psi) := \gamma(s, r \circ \rho, \psi).$$ (12.15)

In the case where $G = \text{GL}_n$ and the map $r : {}^L \text{GL}_n \to \text{GL}_n(\mathbb{C})$ is just the projection to the neutral component, $r$ is omitted from notation, i.e.

$$L(s, \rho) := L(s, \rho, r).$$
12.3 The archimedean Langlands correspondence

For this section, \( F \) is an archimedean local field. Over archimedean local fields the Langlands correspondence between \( L \)-packets of representations of \( G(F) \) and \( L \)-parameters into \( G \) was proven by Langlands [Lan89]. There is a substantial literature refining his result in various important ways, but we will not describe it.

Instead we will focus on the \( G = \text{GL}_n \) case and describe the Langlands correspondence explicitly. We recall from §10.5 that any irreducible admissible \( \pi \) is infinitesimally equivalent to an isobaric sum

\[
\pi \cong \oplus_{i=1}^k \pi_i
\]

where the \( \pi_i \) are irreducible and essentially square integrable. It turns out that the possibilities for essentially square integrable \( \pi \) are very limited. If \( G \) is a semisimple group over \( \mathbb{R} \) we say that \( G \) is equal rank if there is a maximal torus \( T \subseteq G \) that is anisotropic. In other words, \( T \leq G \) is maximal and \( T(\mathbb{R}) \) is compact.

Harish-Chandra’s fundamental theorem on the existence of discrete series representations follows:

**Theorem 12.3.1 (Harish-Chandra)** Let \( G \) be a reductive group over \( \mathbb{R} \). There are irreducible essentially square integrable representations of \( G(\mathbb{R}) \) if and only if \( G^{\text{der}} \) is equal rank.

We refer to [Wal88, §7.7.1] for the proof.

By the theorem, if \( F \) is complex then \( \text{GL}_n(F) \) admits irreducible essentially square integrable representations if and only if \( n = 1 \). In fact, 1-dimensional representations are always essentially square integrable. If \( F \) is real then \( \text{GL}_n(F) \) admits irreducible essentially square integrable representations if and only if \( 1 \leq n \leq 2 \) (see Exercise 12.12).

We now explicitly define the Langlands correspondence. The admissible representation corresponding to a 1-dimensional representation \( W_F \to \mathbb{C}^\times \) is the quasi-character defined by the composite

\[
\mathbb{C}^\times \overset{\text{Art}}{\longrightarrow} W_F^\text{ab} \longrightarrow \mathbb{C}^\times.
\]

If \( W_F \to \text{GL}_n(\mathbb{C}) \) is irreducible and \( n > 1 \) then \( n = 2 \) by Proposition 12.2.2. In this case the representation is of the form \( \text{Ind}_{W_F^\text{ab}}^{W_F}(\chi_{s,m}) \) by the same proposition. The associated irreducible admissible representation, by definition, is \( | \cdot |^s \pi_m \), where \( \pi_m \) is the discrete series representation (or limit of discrete series representation) of §4.7.

In general, if \( \rho : W_F \to \text{GL}_n(\mathbb{C}) \) is a representation, it is isomorphic to \( \oplus_{i=1}^k \rho_i \) where the \( \rho_i \) are all irreducible of dimension 1 or 2, and we let the corresponding irreducible admissible representation be \( \pi(\rho) := \oplus_{i=1}^k \pi(\rho_i) \), where \( \pi(\rho_i) \) is the irreducible admissible representation corresponding to \( \rho_i \).
Using Proposition 12.2.2 and the discussion of isobaric representations given in §10.5, we deduce the following special case of the archimedean local Langlands correspondence:

**Theorem 12.3.2** The map \( \rho \mapsto \pi(\rho) \) described above is a bijection between isomorphism classes of irreducible admissible representations of \( \text{GL}_n(F) \) and equivalence classes of semisimple representations \( W_F \to \text{GL}_n(\mathbb{C}) \).

Now suppose that \( m \hookrightarrow n \to \mathbb{Z} \) and we are given an irreducible admissible representation \( \pi \) of \( \text{GL}_n(F) \) and an irreducible admissible representation \( \pi' \) of \( \text{GL}_m(F) \) with associated representations \( \pi(\pi) : W_F \to \text{GL}_n(\mathbb{C}) \) and \( \pi(\pi') : W_F \to \text{GL}_m(\mathbb{C}) \). For any nontrivial character \( \psi : F \to \mathbb{C}^\times \) we let

\[
L(s, \pi \times \pi') = L(s, \pi(\pi) \otimes \pi(\pi')),
\]
\[
\varepsilon(s, \pi \times \pi', \psi) = \varepsilon(s, \pi(\pi) \otimes \pi(\pi'), \psi),
\]
\[
\gamma(s, \pi \times \pi', \psi) = \gamma(s, \pi(\pi) \otimes \pi(\pi'), \psi).
\]

This is the definition of the archimedean Rankin-Selberg \( L \)-factors. Given this indirect definition, it is remarkable that Jacquet, Piatetski-Shapiro and Shalika were able to prove that it satisfies the same analytic properties (e.g. functional equation) that the nonarchimedean local \( L \)-function does (see Theorem 11.5.4). The fact that they were able to prove this indicates that the archimedean Langlands correspondence given above has meaning. Without proving that some other property is preserved by the correspondence it is rather ad-hoc. Another indication of the utility of the correspondence is that it can be used to compute characters of representations. We refer to the foundational work of Shelstad and Langlands [LS87, She82] for more on this matter.

### 12.4 Local Langlands for \( \text{GL}_n \)

Let \( F \) be a local field. Let \( \Pi(\text{GL}_n) \) be the set of isomorphism classes of irreducible admissible representations of \( \text{GL}_n(F) \) and let \( \Phi(\text{GL}_n) \) be the set of \( \text{GL}_n(\mathbb{C}) \)-conjugacy classes of \( L \)-parameters \( \rho : W_F \to ^L\text{GL}_n \).

By a **local Langlands correspondence** for \( \text{GL}_n \) over \( F \) one means a collection of bijections

\[
\text{rec} := \text{rec}_F : \Pi(\text{GL}_n) \xrightarrow{\sim} \Phi(\text{GL}_n)
\]

such that

(a) If \( \pi \in \Pi(\text{GL}_1) \) then \( \text{rec}(\pi) = \pi \circ \text{Art}^{-1}_F \).

(b) If \( \pi_1 \in \Pi(\text{GL}_{n_1}) \) and \( \pi_2 \in \Pi(\text{GL}_{n_2}) \) then

\[
L(s, \pi_1 \times \pi_2) = L(s, \text{rec}(\pi_1) \otimes \text{rec}(\pi_2))
\]
and
\[ \varepsilon(s, \pi_1 \times \pi_2, \psi) = \varepsilon(s, \text{rec}(\pi_1) \otimes \text{rec}(\pi_2), \psi). \]

(c) If \( \pi \in \Pi(\text{GL}_n) \) and \( \chi \in \Pi(\text{GL}_1) \) then
\[ \text{rec}(\pi \otimes (\chi \circ \det)) = \text{rec}(\pi) \otimes \text{rec}(\chi). \]

(d) If \( \pi \in \Pi(\text{GL}_n) \) and \( \pi \) has central quasi-character \( \chi \) then
\[ \det(\text{rec}(\pi)) = \text{rec}(\chi). \]

(e) If \( \pi \in \Pi(\text{GL}_n) \) then \( \text{rec}(\pi^\vee) = \text{rec}(\pi)^\vee \) where \( \vee \) denotes the contragredient.

**Theorem 12.4.1 (Local Langlands correspondence)** There is a local Langlands correspondence for \( \text{GL}_n \) for every local field \( F \). \( \Box \)

When \( n = 1 \), this is essentially local class field theory and when \( F \) is archimedean the correspondence was constructed in \S 12.3. When \( F \) has positive characteristic (and hence is the completion of a global function field), the theorem is due to Laumon, Rapoport and Stuhler [LRS93]. In the mixed characteristic case, this was proved by Harris-Taylor first [HT01], then a simplified proof was found by Henniart [Hen00].

There are many additional desiderata that one might ask of the local Langlands correspondence. We mention a few. First, the correspondence sends isobaric sums to direct sums. In other words,
\[ \text{rec}(\pi_1 \boxplus \pi_2) = \text{rec}(\pi_1) \oplus \text{rec}(\pi_2). \]  

(12.17)

Second, supercuspidal representations of \( \text{GL}_n(F) \) correspond bijectively to irreducible representations of \( W_F^\prime \) that are trivial on the \( \text{SL}_2(\mathbb{C}) \)-factor. Finally, a representation \( \pi \) is essentially square integrable if and only if \( \text{rec}(\pi) \) is irreducible. Indeed, in the archimedean case, this is clear from our explicit description of the correspondence in \S 12.3. In the nonarchimedean case, the theorem of Bernstein (Theorem 8.4.3) asserts that a representation is essentially square integrable if and only if it is of the form \( Q(\sigma^n, \lambda_n) \) with notation as in Theorem 8.4.3 (here \( \sigma \) is supercuspidal). Such representations correspond to representations of \( W_F^\prime = W_F \times \text{SL}_2(\mathbb{C}) \) of the form \( \text{rec}(\sigma) \otimes \text{Sym}^n \).

All of these requirements are more or less built into the classification. In the nonarchimedean setting, the first paper to discuss how the local Langlands correspondence for \( \text{GL}_n \) could be reduced to the supercuspidal case was [Zel80].
12.5 The local Langlands conjecture

The (conjectural) Langlands correspondence for arbitrary reductive groups $G$ over local fields $F$ is far more complicated. The general shape involves a relationship between

\[ \Pi(G) := \left\{ \text{(infinitesimal) equivalence classes of irreducible admissible representations of } G(F) \right\} \]  \hspace{1cm} (12.18)

and

\[ \Phi(G) := \{ \hat{G}(\mathbb{C})\text{-conjugacy classes of } L\text{-parameters into } L^1 G \}. \]  \hspace{1cm} (12.19)

Here in (12.18) the adjective “infinitesimal” should be omitted unless $F$ is archimedean. We will state the conjecture for quasi-split reductive groups $G$ in this section. Our exposition follows that of [Kal16].

The conjectural correspondence is simplest to state in the tempered case and we will restrict to this setting. We have already defined the notion of a tempered representation in Definition 4.7. An $L$-parameter

\[ \rho : W'_k \longrightarrow L^1 G \]

is **tempered** if its image under the projection to $\hat{G}(\mathbb{C})$ is bounded, or equivalently has compact closure. Note that the projection $L^1 G \to \hat{G}(\mathbb{C})$ is just a set-theoretic projection; it is only a homomorphism if $G$ is split reductive. It is clear the notion of being tempered depends only on the equivalence class of the $L$-parameter. We denote by

\[ \Phi_t(G) \]  \hspace{1cm} (12.20)

the set of equivalence classes of tempered $L$-parameters. We denote by

\[ \Pi_t(G) \]

the set of isomorphism classes of tempered irreducible admissible representations of $G(F)$. The basic form of the local Langlands conjecture for quasi-split reductive groups is the following:

**Conjecture 12.5.1 (Local Langlands correspondence)** Assume that $G$ is quasi-split reductive.

(a) There is a surjective map

\[ \LL : \Pi_t(G) \longrightarrow \Phi_t(G) \]

with finite fibers \( \Pi(\rho) := \LL^{-1}(\rho) \).
(b) If one element of $\Pi(\rho)$ is essentially square integrable then they all are. An element of $\Pi(\rho)$ is essentially square integrable if and only if the image of $\rho$ is not contained in a proper parabolic subgroup of $^L G$.

c) If $\rho \in \Phi_\ell(G)$ is the image of $\rho_M \in \Phi_\ell(M)$ for a Levi subgroup $M \leq G$, then $\Pi(\rho)$ consists of the irreducible constituents of the representations that are parabolically induced from elements of $\Pi(\rho_M)$.

For the notion of a parabolic subgroup of $^L G$, we refer to §7.4. In the last assertion, we are implicitly using the fact that one can construct a morphism $^L M \to ^L G$ for every Levi subgroup $M \leq G$ as in §7.4. We will discuss the compatibility of the formulation above with the known case of $G = \mathrm{GL}_n$ at the end of this section. Of course, the conjecture as stated above is too imprecise to be useful. It is meant as a template into which one adds various desiderata. At bare minimum one should require that the correspondence satisfies certain character identities; one reference is [Kal16, §1.3-1.4].

For a general tempered $L$-parameter $\rho$, the isomorphism classes of representations in $\Pi(\rho)$ are by definition a \textbf{local $L$-packet}, or simply an $L$-packet. Let us describe the structure of an $L$-packet in an important special case.

**Definition 12.4.** An $L$-parameter is \textbf{unramified} if $F$ is nonarchimedean, $G$ is unramified and the parameter is trivial on $I_F \times \mathrm{SL}_2(\mathbb{C})$.

Let $\rho : W'_F \to ^L G$ be an unramified parameter. The class $\rho(\mathrm{Fr}) \in (\widehat{G} \rtimes \mathrm{Fr}/\sim)(\mathbb{C})$ and the choice of hyperspecial subgroup $K \leq G(F)$ give rise to a representation $J(\lambda, K)$ by (7.23), Lemma 7.5.6, and Theorem 7.6.7. In Theorem 7.6.7 we did not incorporate the $K$ into the notation for simplicity. We observe that if $K$ and $K'$ are $G(F)$-conjugate then $J(\lambda, K) \cong J(\lambda, K')$, but the converse is not in general true. We let $\Pi(\rho)$ be the set of equivalence classes of irreducible admissible representations of $G(F)$ isomorphic to some $J(\lambda, K)$ as $K$ varies over the hyperspecial subgroups of $G(F)$ (or equivalently the $G(F)$-conjugacy classes of hyperspecial subgroups of $G(F)$). The set $\Pi(\rho)$ is an \textbf{unramified $L$-packet}.

The conjecture, as stated, leaves open the question of the structure of the $L$-packet $\Pi(\rho)$. To describe a little of what is expected, let

$$S_\rho := C_{\widehat{G}(\mathbb{C})}(\rho(W'_F))$$

(12.21)

be the centralizer of $\rho(W'_F)$ in $\widehat{G}(\mathbb{C})$ and let

$$\mathfrak{S}_\rho := S_\rho / Z_{\widehat{G}(\mathbb{C})}^{\mathrm{Gal}_F}.$$ 

(12.22)

Here we are using the fact that $\widehat{G}$ comes equipped with an action of $\mathrm{Gal}_F$, the same action used to define $^L G$. 
**Lemma 12.5.2** For $G = \text{GL}_n$ the group $\mathcal{S}_\rho$ is connected.

**Proof.** It suffices to show that $S_\rho$ is connected. For irreducible $\rho$, Schur’s lemma implies that $S_\rho = \mathbb{G}_m I_n$ (the scalar matrices in $\text{GL}_n$). In general, the fact that $\rho$ is Frobenius semisimple implies that it is completely reducible (see Exercise 12.5) so $S_\rho$ is isomorphic to a product of $\mathbb{G}_m$’s, one for each irreducible subrepresentation of $\rho$. \[ \square \]

**Conjecture 12.5.3** For a tempered $L$-parameter $\rho$, there is an injection

$$\iota : \Pi(\rho) \rightarrow \text{Irr}(\pi_0(\mathcal{S}_\rho))$$

that is bijective if $F$ is nonarchimedean.

Here $\text{Irr}(\pi_0(\mathcal{S}_\rho))$ is the set of isomorphism classes of irreducible representations of the finite group $\pi_0(\mathcal{S}_\rho)$, the group of components of $\mathcal{S}_\rho$.

The theory of generic representations gives a means of isolating a basepoint in $\Pi(\rho)$. Consider the set of pairs $(B, \psi)$ where $B \leq G$ is a Borel subgroup and $\psi$ is a generic character of the $F$-points of its unipotent radical. There is a natural action of $G(F)$ on this set by conjugation and a Whittaker datum is a $G(F)$-orbit. Let $\mathfrak{w}$ be a Whittaker datum and say that an admissible irreducible representation $\pi$ is $\mathfrak{w}$-generic if it is $\psi$-generic for some (hence all) pairs $(B, \psi) \in \mathfrak{w}$.

The following is a version of a conjecture of Shahidi [Sha90, §9]:

**Conjecture 12.5.4 (Shahidi)** Every tempered $L$-packet $\Pi(\rho)$ contains a unique $\mathfrak{w}$-generic member.

Assume for this paragraph that conjectures 12.5.1, 12.5.3 and 12.5.4 are valid. Given a Whittaker datum $\mathfrak{w}$, we assume that the injection $\iota : \Pi(\rho) \rightarrow \text{Irr}(\pi_0(\mathcal{S}_\rho))$ can be normalized so that the generic element of the packet (which is unique by Conjecture 12.5.4) is sent to the trivial representation. We call this normalized injection $\iota_{\mathfrak{w}}$. Thus we have a pairing

$$\langle \, , \rangle : \Pi(\rho) \times \pi_0(\mathcal{S}_\rho) \rightarrow \mathbb{C}$$

$$(\pi, s) \mapsto \text{tr}(\iota_{\mathfrak{w}}(\pi)(s)).$$

(12.23)

It will play a crucial role in Conjecture 12.6.3 below.

Important progress towards conjectures 12.5.1, 12.5.3, and 12.5.4 has been made. We briefly indicate some results and refer to [Kal16] for a more detailed discussion. In the archimedean case, conjectures 12.5.1, 12.5.3, and 12.5.4 are completely known. They are due to Langlands and Shelstad. The reference [LS87] is a place to begin, though Shelstad has many other papers on the subject. We refer to [Kal16] for a list. Her work and her joint work with Langlands provided crucial insight motivating the formulation of the general conjectures.
There is also a great deal known for classical groups. The following is Arthur’s main theorem on the local Langlands correspondence [Art13, Theorem 1.5.1].

**Theorem 12.5.5** Assume the stabilization of the twisted trace formula and that $F$ is of characteristic zero. Parts (a) and (b) of Conjecture 12.5.1 and Conjecture 12.5.3 are valid for the split groups $\text{Sp}_{2n}$ and $\text{SO}_{2n+1}$. A slightly weaker version of these conjectures is valid for the quasi-split (or split) special orthogonal groups of even dimensional quadratic spaces. 

Let $G = \text{SO}_{2n}$, which we take to be either the split form of $\text{SO}_{2n}$ or a quasi-split inner form, and let $O_{2n}^\ast$ be the corresponding orthogonal group. In the theorem, the slightly weaker version of the correspondence is the correspondence obtained by replacing equivalence classes in $t(G)$ and $\hat{t}(G)$ by equivalence classes up to conjugation by $O_{2n}^\ast(C)$ and $O_{2n}^\ast(F)$, respectively.

We will not say precisely what is meant by the stabilization of the twisted trace formula. Moeglin and Waldspurger [MW16a, MW16b] have proven this stabilization under the assumption of the twisted weighted fundamental lemma. There does not seem to be any theoretical obstacle to obtaining this last result, but it has not been completed. Thus the work of Arthur is still conditional at this point.

There is an analogous statement for quasi-split unitary groups over local fields [Mok15, Theorem 2.5.1]:

**Theorem 12.5.6** Assume the stabilization of the twisted trace formula and that $F$ is of characteristic zero. Let $E/F$ be a quadratic extension and let $G$ be the corresponding quasi-split unitary group. Parts (a) and (b) of Conjecture 12.5.1 and Conjecture 12.5.3 are valid for $G$. 

Key identities that enter into the proof of Theorem 12.5.5 and Theorem 12.5.6 together with work of Konno [Kon02] imply cases of Conjecture 12.5.4:

**Theorem 12.5.7** Assume the stabilization of the twisted trace formula and that $F$ is of characteristic zero. If $G$ is the split group $\text{Sp}_{2n}$ or $\text{SO}_{2n+1}$, the quasi-split or split special orthogonal group on an even dimensional vector space, or is a quasi-split unitary group then Conjecture 12.5.4 is true. 

Finally we mention that Kaletha has constructed a set of supercuspidal representations from $L$-parameters trivial on $\text{SL}_2(\mathbb{C})$ under certain assumptions on the group $G$ [Kal19]. For example, it is sufficient that $G$ splits over a tamely ramified extension and the order of the residue field does not divide the order of the Weyl group of $G^\vee$. He then verifies that the construction satisfies many of the desiderata of the local Langlands correspondence. One
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expects that Kaletha’s work will lead to an explicit local Langlands correspondence provided that $G$ is, roughly, not too ramified. For any $G$, Fargues and Scholze have constructed a less explicit candidate for the restriction of LL to the set of supercuspidal representations [FS21]. They moreover prove many of its expected properties. These works indicate that the local Langlands conjecture will be proved in complete generality in the near future.

In this section we have limited our focus to the tempered case. In practice one needs to refine this in two qualitatively different manners. First, one often has to enlarge the set of equivalence classes of tempered $L$-parameters to a set of **almost tempered** $L$-parameters $\Phi_{at}(G)$ and define a set of **almost tempered** representations $\Pi_{at}(G)$ such that there is a local Langlands correspondence relating the two sets. This enlargement is necessary because we do not know that the local components of automorphic representations that are expected to be tempered are in fact tempered. This point will come up again in §12.6. For general representations, stating a version of the local Langlands correspondence requires the introduction of Arthur’s $A$-packets (see [Art89, Art90]).

We close this section by discussing the compatibility of conjectures 12.5.1, 12.5.3, and 12.5.4 and our discussion of the unramified correspondence with the local Langlands correspondence for $GL_n$ (Theorem 12.4.1). For $GL_n$ or $Res_{E/F}GL_n$ where $E/F$ is a field extension, take $\Phi(G)$ to be the set of all equivalence classes of $L$-parameters. Then for each $\rho$ we can let $\Pi(\rho)$ be the unique irreducible admissible representation of $G(F)$ corresponding to $\rho$ under the local Langlands correspondence. To check that, under the local Langlands correspondence, irreducible tempered representations are sent to tempered parameters is Exercise 12.10. The fact that the Langlands correspondence in this case is compatible with the unramified correspondence discussed above is a consequence of the fact that all hyperspecial subgroups of $GL_n(F)$ are $GL_n(F)$-conjugate to $GL_n(O_F)$ (see [Ser06, II.IV Appendix 1]). Condition (b) in Conjecture 12.5.1 is valid by the discussion at the end of §12.4. To check consistency with condition (c) of Conjecture 12.5.1, let $M$ be the Levi subgroup of $GL_n$, so $M \cong \prod_{i=1}^k GL_{n_i}$ is a product of general linear groups. We note that if

$$\rho_M \in \Phi_k(M) = \prod_{i=1}^k \Phi_i(GL_{n_i})$$

then the parabolic induction of the unique representation

$$\pi = \otimes_{i=1}^k \pi_{n_i} \in \Pi(\rho_M)$$

to $GL_n(F)$ is irreducible by Exercise 12.11 and equal to $\oplus_{i=1}^k \pi_{n_i}$. Given that the local Langlands correspondence sends isobaric sums to direct sums, we deduce compatibility of the local Langlands correspondence with requirement (c). To check compatibility with Conjecture 12.5.3, we use Lemma 12.5.2.
Finally, to check compatibility with Conjecture 12.5.4, we recall that every irreducible tempered representation of $GL_n(F)$ is generic by theorems 8.4.4 and 8.4.5.

### 12.6 Global Langlands functoriality

Let $F$ be a global field and let $G$ be a reductive group over $F$. As before, let $\text{Gal}_F := \text{Gal}(F^{\text{sep}}/F)$ and for all places $v$ of $F$ let

$$\text{Gal}_{F_v} := \text{Gal}(F_v^{\text{sep}}/F_v).$$

Then for each place $v$ of $F$ there is a $\text{Gal}_F$-conjugacy class of injective homomorphisms $\text{Gal}_{F_v} \rightarrow \text{Gal}_F$. Choosing such a homomorphism, one obtains an injection

$$L_{G_{F_v}} := \widehat{G}(\mathbb{C}) \rtimes \text{Gal}_{F_v} \longrightarrow \widehat{G}(\mathbb{C}) \rtimes \text{Gal}_F =: L_G.$$

Motivated by this, define

$$\Phi(G_{H_F}) := \left\{(\rho_v) \in \prod_v \Phi(G_{F_v}) : \rho_v \text{ is unramified for almost all } v \right\}. \quad (12.24)$$

Let $\Phi_t(G_{H_F}) \subset \Phi(G_{H_F})$ be the subset of $\rho = (\rho_v)$ such that $\rho_v \in \Phi_t(G_{F_v})$ for all $v$. We call $\Phi_t(G_{H_F})$ the set of adelic $L$-parameters into $L_G$ and call $\Phi_t(G_{H_F})$ the set of tempered adelic $L$-parameters into $L_G$. We will often drop the adjective “adelic.”

Let $H$ be another reductive group over $F$ and let

$$r : L_H \longrightarrow L_G$$

be an $L$-map. We then obtain a commutative diagram

$$
\begin{array}{ccc}
L_H & \longrightarrow & L_G \\
\downarrow & & \downarrow \\
L_{H_{F_v}} & \longrightarrow & L_{G_{F_v}}
\end{array}
$$

Given any $\rho \in \Phi(H_{H_F})$ we can define

$$r \circ \rho = (r \circ \rho_v) \in \Phi(G_{H_F}). \quad (12.26)$$

Thus given an $L$-map $L_H \rightarrow L_G$ we obtain a commutative diagram
Assume now that $G$ and $H$ are quasi-split and that we know the local Langlands conjecture for $G_{F_v}$ and $H_{F_v}$ for all places $v$. Thus, for example, we could take $G$ and $H$ to be general linear groups. Then given a tempered $L$-parameter $\rho \in \Phi_t(G_{A_F})$ we can form the global $L$-packet

$$\Pi(\rho) := \{ \pi_v = \otimes_v \pi_v : \pi_v \in \Pi(\rho_v) \text{ for all } v \}.$$  \hfill (12.27)

This is a set of irreducible admissible representations of $G(A_F)$. Two $L$-packets $\Pi(\rho)$ and $\Pi(\rho')$ are said to be equivalent if there is a bijection

$$j : \Pi(\rho) \to \Pi(\rho')$$

such that $j(\pi_v) \cong$ is isomorphic to $\pi_v^{\infty}$ and $j(\pi_v)$ is infinitesimally equivalent (resp. isomorphic) to $\pi_v$ if $F$ is a number field (resp. a function field). In general $L$-packets are infinite. The set of all $L$-packets of $G(A_F)$ up to equivalence is denoted $\Pi(G)$. We say a global $L$-packet is automorphic if one of its elements is an automorphic representation. The set of automorphic $L$-packets is denoted $\Pi_{\text{aut}}(G)$. There is a natural commutative diagram

$$\begin{array}{ccc}
\Pi_{\text{aut}}(G) & \longrightarrow & \Pi_{w,\text{aut}}(G) \\
\downarrow & & \downarrow \\
\Pi(G) & \longrightarrow & \Pi_w(G)
\end{array}$$

where the subscript $w$ denotes the weak $L$-packets of (7.41). The vertical arrows above are injective, but the horizontal arrows are not in general.

For the rest of this section, we assume that $G$ and $H$ are quasi-split and that conjectures 12.5.1, 12.5.3 and 12.5.4 are known for $G_{F_v}$ and $H_{F_v}$ for all places $v$. This is a reasonable working assumption, as the local Langlands correspondence is known in many cases as explained in §12.5. The remainder of this section is devoted to using the local Langlands correspondence to state the global Langlands functoriality conjecture. We begin with the simplest case. Assume that $G = \text{Res}_{E/F} \text{GL}_n$ for some field extension $E/F$ of finite degree. In this case local $L$-packets are singletons (see §12.5). Thus global $L$-packets are singletons in this setting and the following conjecture is natural:

**Conjecture 12.6.1** Suppose that $G = \text{Res}_{E/F} \text{GL}_n$. Let $\pi$ be an everywhere tempered cuspidal automorphic representation of $H(A_F)$ with $L$-parameter $\rho \in \Phi_t(H_{A_F})$. The unique admissible representation in $\Pi(r \circ \rho)$ is automorphic.
For general $G$, we could proceed by refining the weaker Conjecture 7.7.1 by replacing weak $L$-packets with adelic $L$-packets. Even if we did this, it is unclear how to isolate automorphic representations within an automorphic $L$-packet. To treat this problem seems to require the introduction of the conjectural Langlands group $\mathcal{L}_F$. This first appeared in [Lan79a] and is also discussed in [Art02], where a conjectural construction of the group is given. The theory of Tannakian categories also indicates that such a group ought to exist if we assume Langlands functoriality. For a discussion of this point of view and the obstacles one runs into if one pursues it, we refer to [Ram94].

The Langlands group $\mathcal{L}_F$ of a global field $F$ is supposed to be an extension of the Weil group $W_F$ by a locally compact group. For reductive $F$-groups $G$, one should be able to define global $L$-parameters $\mathcal{L}_F \to ^LG$ as in the local case; they ought to be continuous homomorphisms commuting with the projection to $\text{Gal}_F$ along the maps $\mathcal{L}_F \to W_F \to \text{Gal}_F$. Two global $L$-parameters are equivalent if they are $\hat{G}(\mathbb{C})$-conjugate. For every place $v$ one should have homomorphisms

$$W'_F \to \mathcal{L}_F,$$

unique up to conjugacy. We let

$$\Phi_t(G)$$

be the set of isomorphism classes of $L$-parameters $\mathcal{L}_F \to ^LG$ such that the image of the parameter under the set-theoretic projection map $^LG \to \hat{G}(\mathbb{C})$ is bounded. Similarly we let $\Pi_{t,\text{aut}}(G)$ be the set of isomorphism classes of automorphic representations of $G(\mathbb{A}_F)$ that are tempered at every place.

One is supposed to have a global analogue of the local parametrization of Conjecture 12.5.1:

**Conjecture 12.6.2 (Global Langlands reciprocity)** There is a surjective map

$$\Pi_{t,\text{aut}}(G) \to \Phi_t(G) \quad (12.28)$$

such that the diagram

$$\begin{array}{ccc}
\Pi_{t,\text{aut}}(G) & \longrightarrow & \Phi_t(G) \\
\downarrow & & \downarrow \\
\prod_v \Pi_t(G_{F_v}) & \longrightarrow & \Phi_t(G_{\mathbb{A}_F})
\end{array}$$

commutes.

Here the set $\prod_v \Pi_t(G_{F_v})$ is defined as follows: Fix a maximal compact subgroup $K = \prod_v K_v \leq G(\mathbb{A}_F)$. An element of $\prod_v \Pi_t(G_{F_v})$ is a formal product
\[ \prod_v \{ \pi_v \} \text{ where for } v \text{ nonarchimedean (resp. archimedean) } \{ \pi_v \} \text{ is the isomorphism class (resp. infinitesimal equivalence class) of the tempered admissible representation } \pi_v \text{ of } G(F_v). \]

The prime indicates that we require \( \pi_v \) to be \( K_v \)-unramified for all but finitely many \( v \).

In practice one hypothesizes the existence of \( L_F \), derives consequences from it, rephrases those consequences in terms that do not depend on the existence of \( L_F \), and then proves those consequences. For example, given an \( L \)-parameter \( \rho : L_F \to L^G \) one should obtain an automorphic \( L \)-packet \( \Pi(\rho) \).

As in the local setting, one defines

\[ S_\rho := C_{\widehat{G}(\mathbb{C})}(\rho(L_F)), \]

the centralizer of \( \rho(L_F) \) in \( \widehat{G}(\mathbb{C}) \), and sets

\[ \overline{S}_\rho := S_\rho/Z_{\widehat{G}(\mathbb{C})}^{G_{\text{al}}} \].

Then global \( L \)-packets ought to be controlled by the component group of \( \overline{S}_\rho \) as in the local setting. Remarkably, in [Art13] Arthur was able to use this yoga to give an unconditional definition of a group \( \pi_0(\overline{S}_\rho) \) which, if \( L_F \) exists, ought to be the component group of \( \overline{S}_\rho \). It then played a crucial role in the statement of his results.

Assume that \( \pi \) is a given tempered automorphic representation of \( H(\mathbb{A}_F) \) (which is to say that \( \pi_v \) is tempered for all \( v \)). Then there is a \( \rho \in \Phi_l(H_{\mathbb{A}_F}) \) such that \( \pi \in \Pi(\rho) \). It is natural to ask if there is an automorphic representation in \( \Pi(r \circ \rho) \), where \( r \circ \rho \) is defined as in (12.26). Langlands functoriality poses a conjectural answer to this question.

Let \( B \subseteq G \) be a Borel subgroup with unipotent radical \( N \) and let \( \psi : N(\mathbb{A}_F) \to \mathbb{C}^\times \) be a character trivial on \( N(F) \) such that \( \psi_v \) is generic for all \( v \). This provides us with a Whittaker datum \( w_v \) for all \( v \) and hence pairings

\[ (\cdot, \cdot)_v : \Pi(r \circ \rho_v) \times \pi_0(\overline{S}_{\rho_v}) \to \mathbb{C} \]

defined as in (12.23).

Let \( \pi' \) be the unique element of \( \Pi(r \circ \rho) \) such that \( \pi'_v \) is generic for all \( v \). Assume that \( \pi' \) occurs in \( L^2_{\text{disc}}([G]) \), the largest closed subspace on which the representation of \( G(\mathbb{A}_F) \) decomposes discretely. We assume moreover that for every \( v \) there is a homomorphism \( \pi_0(\overline{S}_{\rho_v}) \to \pi_0(\overline{S}_{\rho_{v'}}) \). A global \( L \)-parameter

\[ \rho : L_F \to L^G \]

is said to be \textbf{discrete generic} if it is semisimple, its projection to \( \widehat{G}(\mathbb{C}) \) is bounded and its image is not contained in a proper parabolic subgroup of \( L^G \).

For technical reasons, we assume that \( G \) satisfies the Hasse principal, which is to say that the map on Galois cohomology sets
\[ H^1(F, G) \to \prod_v H^1(F_v, G) \]
is injective. For more on Galois cohomology, we refer to §17.3.

**Conjecture 12.6.3** Under the assumptions above, the multiplicity of a \( \pi \in \Pi(r \circ \rho) \) in \( L^2_{\text{disc}}([G]) \) is

\[
\sum_\rho \frac{1}{|\pi_0(\mathfrak{F}_r \rho)|} \sum_{s \in \pi_0(\mathfrak{F}_r \rho)} \prod_v (\pi_v, s)_v,
\]

where \( \rho \) runs over the equivalence classes of discrete generic global parameters satisfying \( \pi_v \in \Pi(r \circ \rho_v) \) for all places \( v \) and \( \langle \cdot, \cdot \rangle_v \) is defined in (12.23).

This conjecture was first stated in [Kot84, §12], although we have followed the exposition of [Kal16].

Theoretically, one should be able to reduce whatever one wants to know about automorphic representations to the tempered case. The \( \mathcal{A} \)-packets of Arthur and his conjectures regarding them were introduced to make this precise [Art89, Art90] (see also [Clo07] and [Sha11] where the relationship between \( \mathcal{A} \)-packets and Conjecture 10.6.4, the Ramanujan Conjecture, is discussed). However, even when we expect representations to be tempered, in most cases we can only prove bounds that establish that they are close to being tempered. Moreover, the only means currently known to prove that they are indeed tempered seems to be using Langlands functoriality (compare [Lan70, Sar05]). To overcome this problem, as in the local case, instead of just dealing with tempered \( L \)-parameters and representations, one enlarges the set of parameters involved in the local Langlands correspondence to a set of almost tempered parameters and representations. This is supposed to be a set that is restrictive enough that the local Langlands correspondence is still correct, but general enough to include all representations that can occur as local components of the most tempered part of the discrete series of \( L^2([G]) \).

For examples of such an enlargement, we refer to [Art13].

### 12.7 Langlands \( L \)-functions

Assume that we are given a quasi-split reductive \( G \) over a local field \( F \) such that the local Langlands correspondence is known for \( G(F) \). Given a representation

\[ r : L^G \to \text{GL}(V), \]

an irreducible tempered representation \( \pi \) of \( G(F) \) and a nontrivial character \( \psi : F \to \mathbb{C}^\times \), one can then use (12.15) to define the local Langlands \( L \)-function, \( \varepsilon \)-factor, and \( \gamma \)-factor as follows:
By definition, admissible representations in the same $L$-packet necessarily have the same $L$-functions, $\varepsilon$-factors, and $\gamma$-factors. This is the reason for the terminology “$L$-packet.” Sometimes one says that elements in the same $L$-packet are $L$-indistinguishable. When $G = \text{GL}_n \times \text{GL}_m$ and

$$L^G \to \text{GL}_{nm}(\mathbb{C})$$

is the tensor product, Rankin-Selberg theory as discussed in Chapter 11 already furnishes us with a definition of these factors. Part of the content of the local Langlands correspondence for $\text{GL}_n$ (Theorem 12.4.1) is that the two definitions coincide.

Now assume that $F$ is a global field, $\psi : F\backslash \mathbb{A}_F \to \mathbb{C}^\times$ is a nontrivial character, and $G$ is a quasi-split reductive $F$-group such that the local Langlands correspondence for $G(F_v)$ is known for every place $v$. Assume moreover that we are given a representation

$$r : L^G \to \text{GL}(V).$$

The global Langlands $L$-function and global $\varepsilon$-factor are then

$$L(s, \pi, r) = \prod_v L(s, \pi_v, r),$$

$$\varepsilon(s, \pi, r) = \prod_v \varepsilon(s, \pi_v, r, \psi_v).$$

As the notation indicates, the global $\varepsilon$-factor is (conjecturally) independent of the choice of $\psi_v$. The basic expectations for these $L$-functions are recorded in the following conjecture:

**Conjecture 12.7.1** The $L$-function $L(s, \pi, r)$ is meromorphic as a function of $s$, is bounded in vertical strips, and satisfies the functional equation

$$L(s, \pi, r) = \varepsilon(s, \pi, r)L(1 - s, \pi^\vee, r).$$

This is in fact a consequence of Conjecture 12.6.1 and Theorem 11.7.1 together with some additional results and conjectures regarding the behavior of the Langlands correspondence under taking contragredients [AV16, Kal13]. Conversely, Conjecture 12.7.1 and the converse theorem, Theorem 11.9.1, can be combined to prove cases of Conjecture 12.6.1. Remarkably this strategy has been successfully executed in important examples. We refer to [CKPSS04, CFGK19] and Kim’s article in [CKM04].
In most cases Conjecture 12.7.1 is open. The most that is known in general is that $L(s, \pi, r)$ converges for Re$(s)$ sufficiently large (see [Lan71] and [Sha10, §2.5]).

12.8 Algebraic representations

Let $F$ be a number field and let $G$ be a reductive group over $F$. The hypothetical Langlands group $\mathcal{L}_F$ is expected to admit a morphism $\mathcal{L}_F \to \text{Gal}_F$.

In particular, suitable morphisms $\text{Gal}_F \to {}^LG$ commuting with the projection to $\text{Gal}_F$ should give rise to $L$-packets of automorphic representations of $G(F)$. It is natural to try and describe classes of $L$-packets that are of this form.

In [Clo90b], Clozel isolated a class of automorphic representations on $G = \text{GL}_n$ that he called “algebraic” and then conjectured that they are associated to particular types of Galois representations (not just representations of $L_F$). Buzzard and Gee [BG14] later extended Clozel’s conjectures to the case of arbitrary reductive $G$. We describe briefly Buzzard and Gee’s conjectures in this section, following the exposition in [BG14] closely.

Until otherwise specified, we work locally at an archimedean place $v$ of $F$ which we omit from notation, writing $F := F_v$. We identify $F = \mathbb{C}$. Let $B > T$ be a Borel subgroup and maximal torus of $G_\mathbb{C}$. The local Langlands correspondence is known in the archimedean case [Lan89]. It provides us with a set-theoretic map $\text{LL} : \Pi(G) \to \Phi(G)$

\[ \pi \mapsto \text{LL}(\pi) \]

with notation as in (12.18) and (12.19).

Fix a maximal torus $\hat{T}$ in $\hat{G}$ with an identification $X_*(\hat{T}) = X^*(T)$. We can and do assume that $\text{LL}(\pi)(\mathbb{C}^\times) \subseteq \hat{T}(\mathbb{C})$. Let

\[ \text{Hom}_\mathbb{R}(\mathbb{C}, \mathbb{C}) = \{\sigma, \tau\}. \]

Then one has that $\text{LL}(\pi)(z) = \sigma(z)^{\lambda_\sigma} \tau(z)^{\lambda_\tau}$ for $z \in \mathbb{C}^\times$ and for $\lambda_\sigma, \lambda_\tau \in X^*(T) \otimes \mathbb{C}$ such that $\lambda_\sigma - \lambda_\tau \in X^*(T)$. This follows from the fact that all quasi-characters of $\mathbb{C}^\times$ are of the form (12.9) (see Lemma 12.2.1).

The group
has a diagonal action of the Weyl group $W(G, T)(\mathbb{C})$. The action of $\text{Gal}_F$ on $X^*(T) \otimes \mathbb{C}$ given in the construction of the Langlands dual group in §7.3 is $W(G, T)(\mathbb{C})$-semilinear. In other words, for

$$\left((X^*(T) \otimes \mathbb{C})^{\text{Hom}(\mathbb{C}, \mathbb{C})}\right)^{\text{Gal}_F}$$

(12.30)

one has $\xi(wx) = \xi(w)\xi(x)$. If $F = \mathbb{R}$ and $\xi \in \text{Gal}_F$ is the nontrivial element then $\langle \xi(\lambda_r), \xi(\lambda_\sigma) \rangle$ is in the same $W(G, T)(\mathbb{C})$-orbit as $\langle \lambda_\sigma, \lambda_r \rangle$. The group $\text{Gal}_F$ also has an action of $\text{Gal}_F$ defined using both the action of $\text{Gal}_F$ on $X^*(T)$ and the action on $\text{Hom}_G(\mathbb{C}, \mathbb{C})$. Explicitly, if we think of an element of (12.30) as a morphism

$$\phi : \text{Hom}_G(\mathbb{C}, \mathbb{C}) \rightarrow X^*(T) \otimes \mathbb{C}$$

then $\xi \phi := (t \mapsto \xi \phi(t \circ \xi))$. These considerations imply that $(\lambda_\sigma, \lambda_r)$ gives us a well-defined element of

$$\left((X^*(T) \otimes \mathbb{C})^{\text{Hom}(\mathbb{C}, \mathbb{C})}/W(G, T)(\mathbb{C})\right)^{\text{Gal}_F}.$$

The $W(G, T)(\mathbb{C})$-orbit of $(\lambda_\sigma, \lambda_r)$ in $(X^*(T) \otimes \mathbb{C})^{\text{Hom}(\mathbb{C}, \mathbb{C})}$ is an invariant attached to $\rho_\sigma$.

**Definition 12.5.** An $L$-parameter $\rho : W_F \rightarrow L^G$ is $L$-algebraic if $\lambda_\sigma \in X^*(T)$. An irreducible admissible representation $\pi$ of $G(F)$ is $L$-algebraic if $\text{LL}(\pi)$ is $L$-algebraic.

Recall that we have fixed $T \leq B \leq G_{\mathbb{C}}$ and hence we have the notion of the positive roots. As usual, denote by $\rho_B$ the half-sum of positive roots.

**Definition 12.6.** An $L$-parameter $\rho : W_F \rightarrow L^G$ is $C$-algebraic if $\lambda_\sigma - \rho_B \in X^*(T)$. An irreducible admissible representation $\pi$ of $G(F)$ is $C$-algebraic if $\text{LL}(\pi)$ is $C$-algebraic.

The assertion $\lambda_\sigma \in X^*(T)$ is independent of the choice of $B$ and of the isomorphism $F \cong \mathbb{C}$. Likewise the assertion $\lambda_\sigma - \rho_B \in X^*(T)$ is also independent of such choices. If $\rho_B \in X^*(T)$, then two notions coincide.

The expectation (which will be made precise below) is that $L$-algebraic automorphic representations should correspond to representations of $\text{Gal}_F$ with image in $\hat{G}(\mathbb{Q}_p)$ (what we mean by this last symbol will be explained below). By Corollary 15.5.2 cohomological automorphic representations in the sense of Definition 15.6 are $C$-algebraic (hence the terminology $C$-algebraic). The known techniques for proving that $L$-algebraic representations correspond to representations of $\text{Gal}_F$ invariably involve cohomological representations, so $L$-algebraic and $C$-algebraic representations ought to be studied in tandem. The two definitions differ by the half-sum of positive roots. For $G = \text{GL}_n$,
the notion of $C$-algebraic coincides in the isobaric case with Clozel’s notation of algebraic used in [Clo90b, Clo16].

It is clear that the notion of $L$-algebraicity (resp. $C$-algebraicity) for an $L$-parameter $\rho$ depends only on the restriction of $\rho$ to $\mathbb{C}^\times$. Moreover the notion of $L$-algebraicity (resp. $C$-algebraicity) of an irreducible admissible representation $\pi$ of $G(F)$ depends only on the infinitesimal character of this representation. More precisely, letting

$$t^C := t \otimes_{\mathbb{R}} \mathbb{C}$$

we have the following:

**Lemma 12.8.1** An irreducible admissible representation $\pi$ of $G(\mathbb{R})$ is $L$-algebraic (resp. $C$-algebraic) if and only if the infinitesimal character $\chi_\lambda$ of $\pi$ satisfies that $\lambda \in X^*(T) \subset (t^C)^\vee$ (resp. $\lambda - \rho_B \in X^*(T)$).

**Proof.** In the notation of §4.6, the infinitesimal character of $\pi$ is $\chi_\lambda$ with $\lambda \in (t^C)^\vee$, unique defined up to the action of $W$. Now

$$(t^C)^\vee = X_*(T)_C = X^*(\widehat{T})_C$$

where the identifications are $W$-equivariant, so we may regard $\lambda$ as an element of $X^*(\widehat{T})$. The $W$-orbit of $\lambda$ contains $\lambda_\sigma$. For a sketch of the proof of this last assertion, see [Vog93, Proposition 7.4].

The restriction to $F = \mathbb{R}$ in the previous lemma is only for convenience. It is no loss of generality by an application of Weil restriction of scalars.

For the global definitions of algebraic representations, we let $G$ be a reductive group defined over a number field $F$. Fix an algebraic closure $\overline{F}$ of $F$ and form the $L$-group $L^G = \widehat{G} \rtimes \text{Gal}_F$. For each place of $v$ of $F$, fix an algebraic closure $\overline{F}_v$ of $F_v$ and an embedding $\overline{F} \hookrightarrow \overline{F}_v$. Let $\pi = \otimes_v \pi_v$ be an automorphic representations of $G(\mathbb{A}_F)$.

**Definition 12.7.** We say that $\pi$ is $L$-algebraic (resp. $C$-algebraic) if $\pi_v$ is $L$-algebraic (resp. $C$-algebraic) for all infinite places of $F$.

In the remainder of this section, we state an important conjecture on existence of Galois representations attached to $L$-algebraic automorphic representations and a related result (see Theorem 12.8.3 below).

To state the conjecture, we let $p$ be a fixed prime and fix an injection $\overline{\mathbb{Q}} \to \mathbb{C}$. We fix once and for all a choice of algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and isomorphism $\iota : \mathbb{C} \to \overline{\mathbb{Q}}_p$. Therefore the inclusion $\overline{\mathbb{Q}} \to \mathbb{C}$ together with $\iota$ induces an embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$.

We note that in the construction of $\widehat{G}$ and the action of $\text{Gal}_F$ on it, we could replace the base field $\mathbb{C}$ by any algebraically closed field. Thus we could consider $\widehat{G}$ as a group over $\overline{\mathbb{Q}} = \overline{\mathbb{Q}}_p$ for example and $\iota$ induces an isomorphism
12.8 Algebraic representations

\[ \tau : \widehat{G}(\mathbb{C}) \to \widehat{G}(\mathbb{Q}_p). \]

We can also form

\[ \widehat{G}(\mathbb{Q}_p) \times \text{Gal}_F \]  

and we have an isomorphism

\[ \tau : L^G \to \widehat{G}(\mathbb{Q}_p) \times \text{Gal}_F. \]

This construction is used in the following conjecture due to Buzzard and Gee. It is a generalization of a conjecture of Clozel in the case \( G = \text{GL}_n \).

**Conjecture 12.8.2** If \( \pi \) is \( L \)-algebraic, then there is a finite set \( S \) of places of \( F \) containing the infinite places, all places dividing \( p \), and all ramified places, and a continuous Galois representation

\[ \rho_\pi = \rho_{\pi,1} : \text{Gal}_F \to \widehat{G}(\mathbb{Q}_p) \times \text{Gal}_F \]

such that the composite of \( \rho_\pi \) and the projection to \( \text{Gal}_F \) is the identity and if \( v \notin S \) then the Frobenius semisimplification of \( \rho_\pi|_{W_F} \) is \( \widehat{G}(\mathbb{Q}_p) \)-conjugate to \( \tau(LL(\pi_v)) \).

There is additional important information about \( \pi \) at the infinite places and the places dividing \( p \) predicted in the full version of Conjecture 12.8.2 given in [BG14].

Though in general Conjecture 12.8.2 is open, there are important special cases that are known.

**Theorem 12.8.3** Suppose that \( F \) is CM or totally real, that \( G = \text{Res}_{F/\mathbb{Q}}\text{GL}_n \), and that \( \pi \) has the same infinitesimal character as an irreducible representation of \( \text{Res}_{F/\mathbb{Q}}\text{GL}_n \) and in particular is \( L \)-algebraic. Then Conjecture 12.8.2 is true. \( \square \)

There are in fact two different proofs of Theorem 12.8.3 available, one due to Harris, Lan, Taylor and Thorn [HLTT16] and another due to Scholze [Sch15].

**Exercises**

12.1. Prove that any continuous homomorphism from a profinite group to a connected Lie group has finite image.

12.2. With notation as in Example 12.2, prove that if \( p \neq \ell \) then \( \chi_\ell(\text{Fr}_p) = p^{-1} \).

12.3. Given a map \( G \to H \) of reductive groups with normal image, prove that there is an induced map \( \hat{H} \to \hat{G} \).
12.4. For reductive groups $H$ and $G$, give an example of a morphism $\hat{H} \to \hat{G}$ that is not induced by a map $H \to G$ as in Exercise 12.3.

12.5. Recall that we have assumed representations of $W_F'$ are Frobenius semisimple in the nonarchimedean case. With this in mind, prove that a representation $\rho : W_F' \to \text{GL}_n(\mathbb{C})$ is completely reducible.

12.6. Prove (12.7).

12.7. Prove (12.8).


12.9. Prove Proposition 12.2.2.

12.10. Prove that tempered representations are sent to tempered parameters under the local Langlands correspondence for $\text{GL}_n$.

12.11. For $1 \leq i \leq k$, let $\pi_i$ be an irreducible tempered representation of $\text{GL}_{n_i}(F)$. Let $\pi = \text{Ind}(\otimes_{i=1}^k \pi_i, 0)$. Prove that $\pi$ is irreducible.

12.12. Let $G$ be a semisimple group over $\mathbb{C}$. Show that $\text{Res}_{\mathbb{C}/\mathbb{R}} G$ is equal rank if and only if $G$ is trivial. Show that $\text{SL}_n$ (as a group over $\mathbb{R}$) is equal rank if and only if $n \leq 2$ (here we interpret $\text{SL}_1$ as the trivial group).
Chapter 13

Known Cases of Global Langlands Functoriality

...I had not recognized in 1966, when I discovered after many months of unsuccessful search a promising definition of automorphic $L$-function, what a fortunate, although, and this needs to be stressed, unforeseen by me, or for that matter anyone else, blessing it was that it lay in the theory of Eisenstein series.

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R. P. Langlands

Abstract In the number field case, the global Langlands functoriality conjecture is wide open. Despite this, several important cases have been established. We survey some of the hard earned progress in this chapter.

13.1 Introduction

What is now known as the Langlands functoriality conjecture was posed in a seventeen page handwritten letter [Lan] that Langlands wrote in 1967 to André Weil (see [Lan70]). In Chapter 12 we stated the conjecture with some degree of precision. There has been decisive progress on the conjecture in the local setting in both equal and mixed characteristics (see §12.4 and §12.5) and in the global setting when the base field $F$ is a function field (see §13.9). However for number fields the conjecture remains mostly open.

Despite this, there has been hard earned progress. We will survey some of this progress in the current chapter. Throughout $F$ is a global field with ring of adeles $\mathbb{A}_F$. The symbol $G$ will denote a reductive group over $F$. 

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13.2 Parabolic induction

Let $P$ be a parabolic subgroup of $G$. A choice of Levi subgroup $M \leq P$ yields a Levi decomposition $P = MN$ where $N$ is the unipotent radical of $P$. As in §7.4 we can realize $L^M$ as a subgroup of $L^P$ and thereby obtain an $L$-map

$$L^M \rightarrow L^G.$$  (13.1)

In the special case where $G = \text{GL}_n$, the conjugacy classes of Levi subgroups of parabolics are indexed by tuples $n_1, \ldots, n_d \in \mathbb{Z}_{>0}$ such that $\sum_{i=1}^d n_i = n$ as explained in (1.13). The map (13.1) is given by the identity on the Galois factor and the natural block diagonal embedding

$$\prod_{i=1}^d \text{GL}_{n_i}(\mathbb{C}) \rightarrow \text{GL}_n(\mathbb{C})$$

$$(x_1, \ldots, x_d) \mapsto \left( \begin{array}{c} x_1 \\ \vdots \\ x_d \end{array} \right)$$

on the neutral components. Thus given a collection of $L$-parameters into the $L^\text{GL}_n$ composition with (13.1) yields the direct sum of the parameters. The functorial transfer attached to the $L$-map (13.1) sends the automorphic representation $\pi_{x_1, \ldots, x_d}$ of $\text{GL}_n(\mathbb{A}_F)$ to the isobaric sum $\bigoplus_{i=1}^d \pi_i$.

For general $G$ and $M$, Langlands functoriality for the $L$-map (13.1) was essentially fully established by Langlands himself before Langlands even posed his functoriality conjecture. Making this precise is a matter of isolating the correct family of subquotients of the induced representations of Theorem 10.6.1.

13.3 $L$-maps into general linear groups

Let $G$ be a quasi-split reductive group over $F$ and let $E/F$ be a finite separable extension. Suppose we are given an $L$-map

$$r : L^G \rightarrow L^\text{Res}_{E/F} \text{GL}_n$$

for some $n$. Assume moreover that we know the local Langlands correspondence for $G$. Then for every cuspidal automorphic representation $\pi$ of $G(\mathbb{A}_F)$ and every place $v$ of $F$, we can associate an $L$-parameter

$$\rho(\pi_v) : W'_{F_v} \rightarrow L^G.$$
13.4 The strong Artin conjecture

We therefore obtain an $L$-parameter

$$r \circ \rho(\pi_v) : W_{F_v}^r \longrightarrow \text{Res}_{E/F} \text{GL}_n$$

for each $v$. By the local Langlands correspondence for $\text{Res}_{E/F} \text{GL}_n(F_v)$ (which is known, see §12.3 and §12.4) associated to $r \circ \rho(\pi_v)$, there is a unique irreducible admissible representation $r(\pi_v)$ with this $L$-parameter. We denote by

$$r(\pi) := \otimes_v^\prime r(\pi_v).$$

This is an irreducible admissible representation of $\text{GL}_n(\A_E)$. By the version of Langlands functoriality explicated in Conjecture 12.6.1, at least if $\pi$ is tempered, $r(\pi)$ ought to be automorphic. In practice one does not want to assume that $\pi$ is tempered, but merely that it is “almost tempered” in the sense that the matrix coefficients of $\pi_v$ are suitably close to being essentially square integrable. We refer to [Art13] for an example of how to make this precise.

The following definition is often useful:

**Definition 13.1.** An isobaric automorphic representation $\pi'$ of $\text{Res}_{E/F} \text{GL}_n(\A_F)$ is a **weak transfer** of $\pi$ with respect to $r$ if

$$r(\pi_v) \cong \pi'_v$$

for all $v$ not in some finite set $S$ of places of $F$.

Weak transfers are also often called **weak lifts**. If one wishes to be more specific, one can say that the functorial transfer of $\pi$ with respect to $r$ is compatible with the local Langlands correspondence outside of $S$. If a weak transfer exists then it is unique by strong multiplicity one for $\text{GL}_n$ (Theorem 11.7.3). In practice one constructs a weak transfer first and then checks compatibility of the transfer with the local Langlands correspondence at the places in $S$.

13.4 The strong Artin conjecture

Let $E/F$ be a (nontrivial) finite degree Galois extension of number fields. For any homomorphism

$$\rho : \text{Gal}(E/F) \longrightarrow \text{GL}_n(\C)$$

one can define the Artin $L$-function $L(s, \rho)$. Artin and Brauer proved that this $L$-function admits a meromorphic continuation to the plane, satisfies a functional equation, and has a finite number of poles [MM97]. Artin conjectured the following refinement of this statement:
Conjecture 13.4.1 (Artin) If \( \rho \) is nontrivial and irreducible then \( L(s, \rho) \) is holomorphic.

The only irreducible representation not covered by this conjecture is the trivial representation. If \( \rho \) is the trivial representation then \( L(s, \rho) = \Lambda_F(s) \), the completed Dedekind zeta function of \( F \). It is certainly not holomorphic; in fact it has simple poles at \( s \in \{0, 1\} \). Artin’s conjecture implies that this is the only irreducible Galois representation whose Artin \( L \)-function has a pole.

Let us reinterpret Artin \( L \)-functions from the point of view of Langlands functoriality. The representation \( \rho \) defines a representation

\[
r_{\rho} : L^1GL_1 \rightarrow GL_n(C)
\]

by tensoring with the trivial representation of \( L^1GL_1 = GL_1(C) \). Temporarily denote by 1 the trivial representation of \( GL_1(A_F) \), which is obviously automorphic. The Langlands functoriality conjecture predicts that the admissible representation \( r_{\rho}(1) \) is automorphic. If this is the case we say that \( \rho \) is automorphic. One has that

\[
L(s, r_{\rho}(1)) = L(s, 1, r_{\rho}) = L(s, \rho).
\]

Thus the Artin conjecture is implied by the Theorem 11.7.1 and the following conjecture, a refinement of the assertion that \( \rho \) is automorphic:

Conjecture 13.4.2 (Strong Artin) If \( \rho \) is irreducible then there exists a cuspidal representation \( \pi \) of \( GL_n(A_F) \) such that

\[
\pi_v = r_{\rho}(1)_v
\]

for all places \( v \).

This is in fact a much deeper conjecture than the Artin conjecture. One can make precise the difference between Conjecture 13.4.1 and Conjecture 13.4.2 using Theorem 11.9.1.

The strong Artin conjecture for nilpotent groups is known to be true [AC89]:

Theorem 13.4.3 (Arthur-Clozel) Assume that \( E/F \) is a Galois extension of number fields with nilpotent Galois group \( \text{Gal}(E/F) \). Then if \( \rho \) is any irreducible complex representation of \( \text{Gal}(E/F) \), the strong Artin conjecture is true for \( \rho \).

Even when \( \text{Gal}(E/F) \) is solvable the strong Artin conjecture is still not known in general.

In a different direction, one could ask for results towards the strong Artin conjecture for representations \( r : \text{Gal}(E/F) \rightarrow GL_n(C) \) for a fixed \( n \). The conjecture is of course true for \( n = 1 \) by class field theory. However even the case \( n = 2 \) is not known in general. To explain what is known it is convenient to recall the following theorem of Klein:
Theorem 13.4.4 (Klein) A finite subgroup of $\text{PGL}_2(\mathbb{C})$ is isomorphic to one of the following:

- a cyclic group,
- a dihedral group,
- the tetrahedral group $A_4$,
- the octahedral group $S_4$,
- the icosahedral group $A_5$.

A subgroup of $\text{GL}_2(\mathbb{C})$ mapping to a cyclic group in $\text{PGL}_2(\mathbb{C})$ acts reducibly on $\mathbb{C}^2$, so we can ignore these groups. We say a representation

$$\rho : \text{Gal}(E/F) \rightarrow \text{GL}_2(\mathbb{C})$$

is dihedral if the image of the composite map

$$\rho : \text{Gal}(E/F) \rightarrow \text{GL}_2(\mathbb{C}) \rightarrow \text{PGL}_2(\mathbb{C})$$

is a dihedral group. We use the analogous convention when dihedral is replaced by tetrahedral, octahedral, or icosahedral. In all but the icosahedral case, the strong Artin conjecture is known to be true:

Theorem 13.4.5 (Langlands-Tunnell) The strong Artin conjecture is true for dihedral, tetrahedral, or octahedral representations $\rho$. □

In [Lan80] Langlands derived the dihedral and tetrahedral cases from the theory of base change which will be reviewed in the following section. Tunnell completed the octahedral case in [Tun81]. Interestingly this theorem was used in Wiles’ proof of Fermat’s last theorem.

The key difference with the icosahedral case is that it is the only case where the $\text{Gal}(E/F)$ is nonsolvable. If one had the theory of base change for nonsolvable extensions, one could make decisive progress on this last case [Get12].

Though the theory of base change for nonsolvable extensions is unavailable at the time of this writing, one can still make progress towards the strong Artin conjecture under certain assumptions. Namely, assume that the base field $F$ is totally real. For every infinite place $v$ of $F$, complex conjugation defines a conjugacy class $c_v \subset \text{Gal}(\overline{F}/F)$. Any representation $\rho : \text{Gal}(E/F) \rightarrow \text{GL}_2(\mathbb{C})$ extends to $\text{Gal}(\overline{F}/F)$, and we say it is odd if $\det(\rho(c_v)) = -1$ for all infinite places $v$.

Theorem 13.4.6 If $F$ is a totally real field and $\rho : \text{Gal}(E/F) \rightarrow \text{GL}_2(\mathbb{C})$ is an odd irreducible representation then the strong Artin conjecture is true for $\rho$. □

This theorem was proven by Pilloni and Stroh [PS16b] after the case $F = \mathbb{Q}$ was settled by Khare and Wintenberger [KW09a, KW09b] as a consequence of...
their proof of Serre’s conjecture (Kisin made a decisive contribution [Kis09]).
There was a body of work prior to this spearheaded by Taylor; see [PS16b]
for references. The oddness assumption allows one to apply the powerful
machinery of Galois deformation theory that played so pivotal a role in Wiles’
proof of Fermat’s last theorem.

13.5 Base change

Let \( E/F \) be a finite degree Galois extension of number fields. Then

\[
\mathcal{L} \mathrm{Res}_{E/F} \mathcal{G} \mathcal{L}_n := \mathcal{G} \mathcal{L}_n(\mathbb{C})_{[E:F]} \rtimes \mathcal{G} \mathcal{L}_n
\]

where \( \mathcal{G} \mathcal{L}_n \) acts via its quotient \( \mathcal{G} \mathcal{L}(E/F) \), which in turn acts by permuting
the factors. There is an \( \mathcal{L} \)-map

\[
r_{E/F} : \mathcal{L} \mathcal{G} \mathcal{L}_n \longrightarrow \mathcal{L} \mathrm{Res}_{E/F} \mathcal{G} \mathcal{L}_n
\]

(13.4)
defined by stipulating that it is the diagonal embedding on the neutral com-
ponent and the identity on the Galois factor. Using the notation of (12.19),
suppose that \( \rho \in \Phi(\mathcal{G} \mathcal{L}_n F_v) \) for some place \( v \) of \( F \). Then

\[
\mathcal{L}(s, r_{E/F} \circ \rho) = \prod_{w | v} \mathcal{L}(s, \rho|_{\mathcal{W}_{E,w}^f})
\]

(13.5)

where the product is over all places \( w \) of \( E \) dividing \( v \). Here \( \mathcal{W}_{E,w}^f \) is the
Weil-Deligne group as (12.3). In other words, \( r_{E/F} \) corresponds to restric-
tion of Galois representations. The \( \mathcal{L} \)-map \( r_{E/F} \) and the functorial trans-
fers it induces (conjectural or not) are known as base change. It con-
jecturally allows us to relate automorphic representations of \( \mathcal{G} \mathcal{L}_n(\mathbb{A}_F) \) and
\( \mathrm{Res}_{E/F} \mathcal{G} \mathcal{L}_n(\mathbb{A}_F) := \mathcal{G} \mathcal{L}_n(\mathbb{A}_E) \). It is customary to write

\[
\pi_E := r_{E/F}(\pi).
\]

Consider the representation

\[
\rho_{\text{reg}} : \mathcal{G} \mathcal{L}_n \longrightarrow \mathcal{G} \mathcal{L}(E/F) \longrightarrow \mathcal{G} \mathcal{L}(\mathbb{C})_{[E:F]}
\]

where the first map is the quotient and the second is the regular representa-
tion. We define an \( \mathcal{L} \)-map

\[
\mathcal{A} l_{E/F} : \mathcal{L} \mathrm{Res}_{E/F} \mathcal{G} \mathcal{L}_n \longrightarrow \mathcal{L} \mathcal{G} \mathcal{L}_n(\mathbb{C})_{[E:F]}
\]

(13.6)

by stipulating that its composite with the quotient map \( \mathcal{L} \mathcal{G} \mathcal{L}_n(\mathbb{C}) \rightarrow
\mathcal{G} \mathcal{L}_n(\mathbb{C})_{[E:F]} \) is the tensor product of the standard representation of \( \mathcal{G} \mathcal{L}_n(\mathbb{C}) \)
with \( \rho_{\text{reg}} \). This \( \mathcal{L} \)-map and the functorial transfers it induces (conjectural or
not) are called **automorphic induction.** It allows us to relate automorphic representations of $\text{GL}_n(\mathbb{A}_E) = \text{Res}_{E/F} \text{GL}_n(\mathbb{A}_F)$ and $\text{GL}_n[E:F](\mathbb{A}_F)$. We note that

$$L(s, \pi, \text{AI}_{E/F}) = L(s, \pi).$$

(13.7)

Using the fact that the regular representation of $\text{Gal}(E\ll F)$ decomposes as

$$\bigoplus_{\rho} \rho^\otimes d(\rho)$$

where the sum is over the isomorphism classes of irreducible representations $\rho$ of $\text{Gal}(E/F)$ and $d(\rho)$ is the dimension of the space of $\rho$, one has that

$$L(s, \pi, r_{E/F}) = L(s, \pi, \text{AI}_{E/F} \circ r_{E/F}) = \prod_{\rho} L(s, \pi, r_{\text{st}} \otimes r_\rho)^{d(\rho)}.$$  

(13.8)

Here $r_{\text{st}}$ is the standard representation $r_{\text{st}} : L\text{GL}_n \to \text{GL}_n(\mathbb{C})$ and we associate to $\rho$ the representation $r_\rho : L\text{GL}_1 \to \text{GL}_{d(\rho)}(\mathbb{C})$ as in (13.3).

If $E/F$ is a prime degree cyclic extension, then the functorial transfers attached to $r_{E/F}$ and $\text{AI}_{E/F}$ are known to exist. The case $n = 2$ is due to Langlands [Lan80], who followed a method discovered by Saito [Sai75] that was rephrased as the representation theory by Shintani [Shi79]. Besides placing the theory in the correct level of generality, Langlands was able to use it and the converse theorem to establish the strong Artin conjecture for 2-dimensional Galois representations in many solvable cases (see Theorem 13.4.5).

For $\text{GL}_n$ the existence of the functorial transfers was established by Arthur and Clozel [AC89]. To be more precise, let $E/F$ be a prime degree Galois extension. Let $\theta$ be a generator of $\text{Gal}(E/F)$. It acts on the set of automorphic representations $\pi'$ of $\text{GL}_n(\mathbb{A}_E)$ via

$$\pi'^\theta(g) := \pi'(\theta(g)).$$

Let $N_{E/F} : \mathbb{A}_E^\times \to \mathbb{A}_F^\times$ be the norm map.

**Theorem 13.5.1 (Base change)** For every cuspidal automorphic representation $\pi$ of $\text{GL}_n(\mathbb{A}_F)$ the base change $\pi_E$ is an isobaric automorphic representation of $\text{GL}_n(\mathbb{A}_E)$. The base change $\pi_E$ is cuspidal if and only if $\pi \not\equiv \pi \otimes \eta$ for all characters $\eta : F^\times \backslash \mathbb{A}_E^\times / N_{E/F}(\mathbb{A}_E^\times) \to \mathbb{C}^\times$.

Conversely, if a cuspidal automorphic representation $\pi'$ of $\text{GL}_n(\mathbb{A}_E)$ satisfies $\pi' \cong \pi'^\theta$ then $\pi' = \pi_E$ for some cuspidal automorphic representation $\pi$ of $\text{GL}_n(\mathbb{A}_F)$. \(\square\)

**Theorem 13.5.2 (Automorphic induction)** If $\pi$ is a cuspidal automorphic representation of $\text{GL}_n(\mathbb{A}_F)$, then the automorphic induction $\text{AI}(\pi)$ is
an isobaric automorphic representation of $GL_{n|E:F}(\mathbb{A}_F)$. The representation $\text{AI}(\pi)$ is cuspidal if and only if $\pi \not\cong \pi^0$. A cuspidal automorphic representation of $GL_{n|E:F}(\mathbb{A}_F)$ is the automorphic induction of a cuspidal automorphic representation $\pi$ of $GL_n(\mathbb{A}_F)$ if and only if $\pi \cong \pi \otimes \eta$ for some character $\eta \in F^\times \backslash \mathbb{A}_F^\times / \mathbb{N}_{E/F}(\mathbb{A}_E^\times) \rightarrow \mathbb{C}^\times$.

Implicitly in this theorem we are using the local Langlands correspondence for the general linear group to define base change and automorphic induction. This was not available when Arthur and Clozel proved their result. Arthur and Clozel instead used the representation theoretic analogue defined by Shintani [AC89, Definition 6.1]. The compatibility of the two notions is checked in [HT01, Lemma VII.2.6].

By breaking an arbitrary solvable extension $E/F$ into a sequence of prime degree cyclic extensions, one deduces the existence of base changes of isobaric automorphic representations of $GL_n(\mathbb{A}_F)$ along $E/F$. Characterizing the image of the base change in this case is more subtle, however, see [LR98, Raj02].

The case of an arbitrary extension $E/F$ is open. The first author has described a possible approach to this case in [Get12, Get20] and explained how it implies the remaining open cases of the strong Artin conjecture for 2-dimensional complex representations of $\text{Gal}_F$, at least up to an abelian twist. The work in [Get12, Get20] can be seen as a refinement and explication of Langlands’ beyond endoscopy idea [Lan04] in a special case.

13.6 The Langlands-Shahidi method

As explained in §10.4, Langlands proved that $L^2([G])$ could be completely described in terms of cuspidal representations of Levi subgroups of $G$ by means of Eisenstein series. Some time after completing this, he realized that the constant terms of Eisenstein series (which naturally depend on certain complex parameters) admitted functional equations [Lan71]. This led him to the definition of the Langlands $L$-functions discussed in §12.7 and to posing his functoriality conjecture. Langlands only obtained coarse functional equations for these $L$-functions. His results were subsequently completed and refined by Shahidi [Sha10]. This method of proving the analytic continuation of $L$-functions is known as the Langlands-Shahidi method. Combining these results with the converse theorem (Theorem 11.9.1), one can obtain cases of Langlands functoriality that, at present, can not be obtained by any other method. The book [Sha10] and the surveys [CKM04] are useful references for this theory.

We record some important results obtained via the Langlands-Shahidi method. We assume throughout this section that $F$ is a number field. The first case concerns the symmetric powers of cuspidal automorphic representations of $GL_2$. More precisely, let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic
representation of $\text{GL}_2(\mathbb{A}_F)$ and let

$$\text{Sym}^k : {}^L\text{GL}_2 \longrightarrow {}^L\text{GL}_{k+1} \quad (13.9)$$

be the representation that is the identity on the Galois factors and the symmetric $k$th power representation on the neutral components. By the local Langlands correspondence explained in §12.3 and §12.4, $\text{Sym}^k(\pi_v)$ is a well-defined irreducible admissible representation of $\text{GL}_{k+1}(F_v)$ for all places $v$ of $F$. Hence

$$\text{Sym}^k(\pi) = \bigotimes_v \text{Sym}^k(\pi_v)$$

is an irreducible admissible representation of $\text{GL}_{k+1}(\mathbb{A}_F)$.

**Theorem 13.6.1** If $\pi$ is a cuspidal automorphic representation of $\text{GL}_2(\mathbb{A}_F)$ then the admissible representation $\text{Sym}^k(\pi)$ of $\text{GL}_{k+1}(\mathbb{A}_F)$ is automorphic for $k \leq 4$.

The proof of Theorem 13.6.1 was given by Gelbart-Jacquet [GJ78] for $k = 2$, by Kim-Shahidi [KS02] for $k = 3$, and by Kim [Kim03] for $k = 4$. As of this writing, the corresponding statement for $k \geq 5$ remains unproven. Gelbart and Jacquet’s work occurred before the local Langlands correspondence was proven. In [Kim03] Kim gives an alternate proof that includes compatibility with the local Langlands correspondence.

The second case we discuss is Langlands functoriality for Rankin-Selberg products of cuspidal automorphic representations of $\text{GL}_n$. For any $m, n \in \mathbb{Z}_{\geq 0}$, let

$$r_\otimes : {}^L(\text{GL}_m \times \text{GL}_n) \longrightarrow {}^L\text{GL}_{mn} \quad (13.10)$$

be the representation that is the tensor product on the neutral components and the identity map on the Galois factors. This family of $L$-maps is extremely important; taking tensor products is arguably the most basic procedure for constructing new representations out of old representations.

For every place $v$ of $F$ and every admissible representation $\pi_{1v} \otimes \pi_{2v}$ of $\text{GL}_m(F_v) \times \text{GL}_n(F_v)$, we have

$$L(s, \pi_{1v} \otimes \pi_{2v}, r_\otimes) = L(s, \pi_{1v} \times \pi_{2v})$$

where the $L$-function on the right is the Rankin-Selberg $L$-function appearing in §11.5. This identity is built into the local Langlands correspondence. It is customary to write

$$\pi_{1v} \boxtimes \pi_{2v} := r_\otimes(\pi_{1v} \otimes \pi_{2v}).$$

Thus for automorphic representations $\pi_1$ and $\pi_2$ of $\text{GL}_m(\mathbb{A}_F)$ and $\text{GL}_n(\mathbb{A}_F)$, respectively, we have an admissible representation

$$\pi_1 \boxtimes \pi_2 := \prod_v \pi_{1v} \boxtimes \pi_{2v}$$
of $\text{GL}_{mn}(\mathbb{A}_F)$. By Rankin-Selberg theory, specifically Theorem 11.7.1, the standard $L$-function of this representation behaves as predicted by Conjecture 12.7.1. In general this is not enough to deduce that $\pi_1 \boxtimes \pi_2$ is automorphic using the converse theorem (Theorem 11.9.1). However for small $m$ and $n$ a combination of the converse theorem and the Langlands-Shahidi method yields the following theorem:

**Theorem 13.6.2** For $1 \leq k \leq 3$, let $\pi_1$ and $\pi_2$ be cuspidal automorphic representations of $\text{GL}_2(\mathbb{A}_F)$ and $\text{GL}_k(\mathbb{A}_F)$, respectively. Then $\pi_1 \boxtimes \pi_2$ is automorphic. $\square$

For the $\text{GL}_2 \times \text{GL}_2$ case, this was proven in [Ram00a]. For the $\text{GL}_2 \times \text{GL}_3$ case, this is [KS02, Theorem 5.1]. The paper [Ram00a] was written before the proof of the local Langlands correspondence. Compatibility of the transfer with the local Langlands correspondence was checked in [Kim03].

### 13.7 Functoriality for the classical groups

We continue to assume that $F$ is a number field. Let

$$J_n := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \text{GL}_n(\mathbb{Z})$$

be the antidiagonal matrix and let

$$J'_{2n} := \begin{pmatrix} \bar{J}_n \\ J_n \end{pmatrix},$$

$$J'_{2n+1} := \begin{pmatrix} J_n \\ -\bar{J}_n \end{pmatrix}. \quad (13.11)$$

We take our “standard” orthogonal and symplectic groups to be the groups over $F$ whose points in an $F$-algebra $R$ are given by

$$\text{O}_n(R) := \{ g \in \text{GL}_n(R) : gJ_n g^t = J_n \},$$

$$\text{Sp}_{2n}(R) := \{ g \in \text{GL}_{2n}(R) : gJ'_{2n} g^t = J'_{2n} \}. \quad (13.12)$$

We also define some quasi-split groups as follows. Let $E/F$ be a quadratic extension. We then define

$$U_n(R) := \{ g \in \text{GL}_n(E \otimes_F R) : gJ'_n g^t = J'_n \},$$

where the bar denotes the action of the nontrivial element of $\text{Gal}(E/F)$. This is the quasi-split unitary group attached to the extension $E/F$. The extension
$E/F$ also defines a quasi-split orthogonal group as follows: View $E$ as a vector space over $F$ equipped with the quadratic form $N_{E/F}$. Choosing a basis, we obtain a symmetric matrix $\gamma_E \in \text{GL}_2(F)$ corresponding to the quadratic form in the usual manner. Then, for an $F$-algebra $R$, 

$$O_{2n}^*(R) := \left\{ g \in \text{GL}_n(R) : g \begin{pmatrix} J_{n-1} & \gamma_E \\ \gamma_E & J_{n-1} \end{pmatrix} g^t = \begin{pmatrix} J_{n-1} & \gamma_E \\ \gamma_E & J_{n-1} \end{pmatrix} \right\}.$$ 

We define $\text{SO}_n < O_n$ and $\text{SO}_n^* < O_n^*$ to be the subgroups of determinant 1 as usual. We refer to the groups above as the quasi-split classical groups of rank $n$ defined over $R$ and denote by $G_n$.

Following [CPSS11] we have the following table of groups and embeddings of $L$-groups:

<table>
<thead>
<tr>
<th>$G_n$</th>
<th>$r : L G_n \to L H_N$</th>
<th>$H_N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{SO}_{2n+1}$</td>
<td>$\text{Sp}_{2n}(\mathbb{C}) \times \text{Gal}<em>F \to \text{GL}</em>{2n}(\mathbb{C}) \times \text{Gal}_F$</td>
<td>$\text{GL}_{2n}$</td>
</tr>
<tr>
<td>$\text{SO}_{2n}$</td>
<td>$\text{SO}_{2n}(\mathbb{C}) \times \text{Gal}<em>F \to \text{GL}</em>{2n}(\mathbb{C}) \times \text{Gal}_F$</td>
<td>$\text{GL}_{2n}$</td>
</tr>
<tr>
<td>$\text{SO}_{2n}^*$</td>
<td>$\text{SO}_{2n}(\mathbb{C}) \times \text{Gal}<em>F \to \text{GL}</em>{2n}(\mathbb{C}) \times \text{Gal}_F$</td>
<td>$\text{GL}_{2n}$</td>
</tr>
<tr>
<td>$\text{Sp}_{2n}$</td>
<td>$\text{SO}_{2n+1}(\mathbb{C}) \times \text{Gal}<em>F \to \text{GL}</em>{2n+1}(\mathbb{C}) \times \text{Gal}_F$</td>
<td>$\text{GL}_{2n+1}$</td>
</tr>
<tr>
<td>$U_n$</td>
<td>$\text{GL}_n(\mathbb{C}) \times \text{Gal}_F \to \text{GL}_n(\mathbb{C})^2 \times \text{Gal}_F$</td>
<td>$\text{Res}_{E/F} \text{GL}_n$</td>
</tr>
</tbody>
</table>

We call the representation

$$r : L G_n \longrightarrow L H_N$$

listed above the standard representation of $L G_n$. The $L$-maps in the first, second, and fourth row are given by the natural inclusions. For the third, let

$$\tilde{w} := \begin{pmatrix} I_{n-1} & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & I_{n-1} \end{pmatrix}.$$ 

Then $\text{Gal}_F$ acts on $\text{SO}_{2n}(\mathbb{C})$ via its quotient $\text{Gal}(E/F)$, with the nontrivial element $\sigma$ of $\text{Gal}(E/F)$ acting by conjugation by $\tilde{w}$. The map

$$\text{SO}_{2n}(\mathbb{C}) \times \text{Gal}_F \to \text{GL}_{2n}(\mathbb{C}) \times \text{Gal}_F$$

is given by the natural embedding on the neutral component and sends $1 \times \sigma$ to $\tilde{w} \times \sigma$. Finally when $G_n = U_n$ and $H_N = \text{Res}_{E/F} \text{GL}_n$ the absolute
Galois group of $F$ again acts on $\text{GL}_n(\mathbb{C})$ via its quotient $\text{Gal}(E/F)$, with the nontrivial element $\sigma$ acting via

$$
LU_n^\infty = \text{GL}_n(\mathbb{C}) \rightarrow \text{GL}_n(\mathbb{C})
$$
$$
ge \mapsto J_n^{-1}g^{-t}J_n',
$$

and the embedding $r$ is given by

$$
r(g \times \sigma) = (g, J_n^{-1}g^{-t}J_n') \times \sigma. \quad (13.15)
$$

There is a great deal known about Langlands functoriality for this collection of $L$-maps. Definitive results were obtained for generic representations in [CKPSS04, CPSS11]. We focus on this case in the current section. Later Arthur [Art13] established the same result for arbitrary representations (in the orthogonal and symplectic cases) under the assumption that the twisted weighted trace formula can be stabilized (see §13.8 for details). The unitary case was completed by Mok [Mok15] under this same assumption.

The following theorem is [CPSS11, Theorem 1.1]. It uses the notation of the table above. It represents the culmination of an impressive body of work by Cogdell, Kim, Piatetski-Shapiro and Shahidi.

**Theorem 13.7.1** Let $\pi$ be an irreducible globally generic cuspidal automorphic representation of $G_n(\mathbb{A}_F)$. Then there exists a functorial transfer of $\pi$ to $H_N(\mathbb{A}_F)$ with respect to the $L$-map $r$.  

The main tools used in the proof of this theorem are the Langlands-Shahidi method and an analogue of the converse theorem stated in Theorem 11.9.1. Let $\pi$ be as in the statement of Theorem 13.7.1. For every integer $m \geq 1$ and every cuspidal automorphic representation $\tau$ of $H_m(\mathbb{A}_F)$, Langlands-Shahidi theory provides an $L$-function $L(s, \pi \times \tau)$ that admits a meromorphic continuation to the plane and a functional equation. One constructs an admissible representation $r(\pi)$ of $H_N(\mathbb{A}_F)$ such that

$$
L(s, \pi_v \times \tau_v) = L(s, r(\pi)_v \times \tau_v) \quad (13.16)
$$

for all archimedean places $v$ and nonarchimedean places $v$ where $F$ is unramified over its prime field and $\pi_v, \tau_v$ are unramified. It is harder to obtain information at the ramified places. For this one studies $\gamma$-factors $\gamma(s, \pi_v \times \tau_v, \psi_v)$ defined for the Langlands-Shahidi $L$-function $L(s, \pi_v \times \tau_v)$. They are the analogues for Langlands-Shahidi $L$-functions of the $\gamma$-factors defined in Theorem 11.5.4 in the context of Rankin-Selberg theory. When $m = 1$, so $\tau_v$ is a quasi-character, it can be shown that $\gamma(s, \pi_v \times \tau_v, \psi_v)$ depends only on the central character of $\pi_v$ if $\tau_v$ is sufficiently ramified. This result is known as stability of $\gamma$-factors. This allows one to prove (13.16) for all places $v$ and a range
of $m$ after replacing $\tau_v$ by a ramified twist. Applying a suitable converse theorem one deduces the automorphy of $r(\pi)$.

Assuming Theorem 13.7.1, Ginzburg, Rallis and Soudry [GRS01] had previously used their descent technique to characterize the image of the functorial transfer attached to $r$. In order to state their result, we digress to discuss a family of $L$-functions. As a first step we record a table of representations $r'_0$ and characters $\chi_G: [G_m]^\times \rightarrow \mathbb{C}^\times$. The representations $r'_0$ listed below are representations of $H_N$, where $H_N$ is the group attached to $G_n$ in the table above:

<table>
<thead>
<tr>
<th>$G_n$</th>
<th>$r'_0$</th>
<th>$\chi_G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SO_{2n+1}$</td>
<td>$\wedge^2$</td>
<td>1</td>
</tr>
<tr>
<td>$SO_{2n}$, $n \geq 2$</td>
<td>$\text{Sym}^2$</td>
<td>1</td>
</tr>
<tr>
<td>$SO_{2n}$</td>
<td>$\text{Sym}^2$</td>
<td>$\eta_{E/F}$</td>
</tr>
<tr>
<td>$Sp_{2n}$</td>
<td>$\text{Sym}^2$</td>
<td>1</td>
</tr>
<tr>
<td>$U_{2n}$</td>
<td>$As_{E/F} \otimes \eta_{E/F}$</td>
<td>1</td>
</tr>
<tr>
<td>$U_{2n+1}$</td>
<td>$As_{E/F}$</td>
<td>1</td>
</tr>
</tbody>
</table>

Here 1 denotes the trivial character, and $\eta_{E/F}$ is the character attached to $E/F$ by class field theory. The only representation that is not self-explanatory is the **Asai representation**

$$As_{E/F}: \text{Res}_{E/F}GL_n = GL_n^2(\mathbb{C}) \rtimes \text{Gal}_F \rightarrow GL_n(\mathbb{C})$$

that is defined by stipulating that, for $\xi \in \text{Hom}_F(E, F)$ and $\sigma \in \text{Gal}_F$,

$$As_{E/F}((g_\xi \times 1)(\otimes \xi g_\xi) = \otimes \xi g_\xi$$

and

$$As_{E/F}((1) \otimes (1) \otimes \sigma) = \otimes (1) \otimes (1) \otimes \sigma \otimes \xi$$

This plays an important role in §14.5.

Conjecture 12.7.1 on the analytic properties of Langlands $L$-functions is more or less known in the cases above. In more detail, let $\psi: F \backslash A_F \rightarrow \mathbb{C}^\times$ be a nontrivial character. Langlands-Shahidi theory produces $L$- and $\varepsilon$-factors

$$L_{LS}(s, \pi_v, r') \quad \text{and} \quad \varepsilon_{LS}(s, \pi_v, r', \psi_v)$$

for every place $v$ of $F$ that is equal to the $L$- and $\varepsilon$-factor

$$L(s, \pi_v, r') \quad \text{and} \quad \varepsilon(s, \pi_v, r', \psi_v)$$

when $v$ is archimedean or $v$ lies outside of some finite set $S$ of nonarchimedean places. Let
\[ L_{\text{LS}}(s, \pi, r) := \prod_v L_{\text{LS}}(s, \pi_v, r), \]
\[ \varepsilon_{\text{LS}}(s, \pi, r) := \prod_v \varepsilon_{\text{LS}}(s, \pi_v, r', \psi_v). \] (13.20)

The global \( \varepsilon \)-factor is again independent of \( \psi \). These factors are originally defined for \( \text{Re}(s) \) large. One has the following theorem, due to Shahidi following work of Langlands [Sha90, Theorem 7.7]:

**Theorem 13.7.2** The \( L \)-function \( L_{\text{LS}}(s, \pi, r) \) admits a meromorphic continuation to the plane to a function that is bounded in vertical strips. It satisfies the functional equation

\[ L_{\text{LS}}(s, \pi, r) = \varepsilon'_{\text{LS}}(s, \pi, r') L_{\text{LS}}(1 - s, \pi^\vee, r'). \]

The boundedness of the \( L \)-function in vertical strips is in fact a joint theorem of Gelbart and Shahidi [GS01].

Generalizations of the Rankin-Selberg theory discussed in §11.7 provide yet another definition of the \( L \)-functions and \( \varepsilon \)-factors and one can prove a theorem analogous to Theorem 13.7.2 using these \( L \)-functions in most cases (see [Bum05] for a survey). The definition of the \( L \)-functions in terms of Rankin-Selberg integrals is important because in practice it provides information about the poles of \( L \)-functions that is complementary to that obtained via the Langlands-Shahidi method.

Of course, one wants to know that all of these \( L \)- and \( \varepsilon \)-factors agree. This has been proven in important cases. For the agreement of Langlands-Shahidi and Rankin-Selberg \( L \)-factors in many cases, we refer to [Kap15]. We also have the following result [CST17]:

**Theorem 13.7.3** If \( r' \) is \( \wedge^2 \) or \( \text{Sym}^2 \), then one has that

\[ L_{\text{LS}}(s, \pi_v, r') = L(s, \pi_v, r') \quad \text{and} \quad \varepsilon_{\text{LS}}(s, \pi_v, r') = \varepsilon(s, \pi_v, r'). \]

After this digression on \( L \)-functions, we turn back to the characterization of the image of the functorial transfer. The group \( \text{Res}_{E/F} \text{GL}_n \) admits an outer automorphism \( \sigma \) induced by the action of the nontrivial element of \( \text{Gal}_{E/F} \). For irreducible admissible representations \( \pi' \) of \( \text{Res}_{E/F} \text{GL}_n(\mathbb{A}_F) \), let

\[ \pi'^\sigma(g) := \pi'(\sigma(g)) \]

and let

\[ \pi'^\varepsilon := (\pi'^\sigma)^\vee. \]

This is sometimes known as the conjugate dual representation. We say that \( \pi' \) is conjugate self-dual if \( \pi' \cong \pi'^\varepsilon \).
13.8 Endoscopic classification of representations

For the purposes of stating the following theorem, it is convenient to let
\( \pi^*: = \pi'^{\vee} \) when \( H_N = \mathrm{GL}_N \) and let \( \pi'^* \) be defined as above otherwise. Let \( \omega_{\pi'} \) be the central quasi-character of \( \pi' \). For the proof of the following theorem we refer to [GRS01, Sou05]:

**Theorem 13.7.4** Let \( \pi \) be a globally generic cuspidal automorphic representation of \( G_n(\mathbb{A}_F) \). The functorial lift \( r(\pi) \) satisfies \( \omega_{r(\pi)}|_{\mathbb{A}_F^\times} = \chi_{G_n} \) and is of the form

\[ \pi' = \text{Ind}(\pi'_1 \otimes \cdots \otimes \pi'_d) = \pi'_1 \oplus \cdots \oplus \pi'_d, \]

where each \( \pi'_i \) is a unitary cuspidal automorphic representation of \( H_{N_i}(\mathbb{A}_F) \) with \( \sum_i N_i = N \) satisfying

(a) \( \pi'^*_i \equiv \pi'_i \),

(b) \( \pi'^*_i \equiv \pi'^*_j \) if and only if \( i = j \),

(c) \( L^S(s, \pi'_i, r') \) has a pole at \( s = 1 \) for a sufficiently large finite set \( S \) of places of \( F \) including all infinite places.

Moreover, any \( \pi' \) as above satisfying \( \omega_{\pi'}|_{\mathbb{A}_F^\times} = \chi_{G_n} \) and (a)-(c) is of the form \( r(\pi) \) for some globally generic cuspidal automorphic representation \( \pi \) of \( G_n(\mathbb{A}_F) \).

Here \( \omega_{\pi'}|_{\mathbb{A}_F^\times} \) is just \( \omega_{\pi'} \) unless \( G_n = U_n \), in which case it is the restriction of \( \omega_{\pi'} \) to the proper subgroup

\[ \mathbb{A}_F^\times \hookrightarrow \mathbb{A}_E^\times \xrightarrow{\sim} Z_{\text{Res}_{E/F}}(\text{GL}_n(\mathbb{A}_F)). \]

Analogues of Theorem 13.7.1 and Theorem 13.7.4 are known for general spin groups by work of Asgari and Shahidi [AS06, AS14].

In this section we have concentrated on generic representations. This is because originally the approach explained above relied crucially on this assumption. Recently the genericity assumption has been removed in work of Cai, Friedberg, Ginzburg and Kaplan. We refer to [CFK18] (which is based on [CFGK19]) for a precise statement. This provides a new, unconditional, and independent proof of a portion of the endoscopic classification of representations on classical groups that we discuss in the next section.

### 13.8 Endoscopic classification of representations

In this section we continue to use the notational conventions of the previous section. In particular \( F \) is a number field. In [Art13], Arthur proves the existence of functorial transfer with respect to the \( L \)-maps \( r \) of the previous section (at least when \( G_n \neq U_n \)). In particular there is no genericity assumption in his work. Moreover, he gave a precise enough description of the fibers of the functorial transfer that he could classify the discrete spectrum of
$L^2([G_n])$ in terms of automorphic representations on $H_N$. We explain these results in the current section. The main tool used to establish Arthur’s result is the theory of twisted endoscopy (see §19.5), so one refers to Arthur’s work and subsequent refinements as the endoscopic classification of representations.

In his book [Art13] Arthur gives a careful account of how to replace objects attached to the conjectural global Langlands group by well-defined, unconjunctural objects. For the sake of clarity and brevity, we will not reproduce this discussion. Instead, we will try to state Arthur’s main results as directly as possible.

There are two points the reader should bear in mind. First, we have not stated all of Arthur’s results, and we have not included Mok’s results in the $G_n = U_n$ case since this would require more notation. We refer to [Mok15] instead. Second, all of the work in this section (and in Mok’s work) is conditional on the stabilization of the twisted weighted trace formula. Moeglin and Waldspurger have proven this under the assumption of the twisted weighted fundamental lemma [MW16a, MW16b]. The original fundamental lemma of Shelstad and Langlands was proved in the breakthrough work of Ngô in [Ngô10b] for which he received the Fields medal. Ngô’s work is based on the work of many over decades and his survey [Ngô10c] is a good place to obtain some historical perspective. Other versions of the fundamental lemma have been proven by Laumon and Chaudouard using nontrivial generalizations of Ngô’s techniques [CL10, CL12]. In principle there should be no new obstacle in proving the twisted weighted fundamental but as of this writing it is not complete.

For the remainder of this section, we assume $G_n \neq U_n$. The first main theorem is the following:

**Theorem 13.8.1** Every irreducible subrepresentation of $L^2([G_n])$ admits a functorial transfer to $H_N$ with respect to $r$. □

This result is implicit in the main theorems stated in [Art13]. Since irreducible subrepresentations of $L^2([G_n])$ need not be tempered, to make precise what one means by a functorial transfer, one needs more than the theory outlined in §12.5. We will not make this precise and instead refer the reader to [Art13].

There is a wrinkle which will continue to play a role below. When $G_n$ is $SO_{2n}$ or $SO^*_{2n}$, let $\theta : G_n \to G_n$ be an automorphism of order 2 preserving the “standard splitting” of $SO_{2n}$ [Art13, p. 41]. It is unique if $n > 3$. Then the functorial transfer given in Theorem 13.8.1 is insensitive to replacing a representation $\pi$ by $\pi^\theta$, where $\pi^\theta(g) := \pi(\theta(g))$. With this in mind let

$$\check{C}_c^\infty(G_n(\mathbb{A}_F))$$

be $C_c^\infty(G_n(\mathbb{A}_F))$ except in the special case where $G_n$ is $SO_{2n}$ or $SO^*_{2n}$, in which case it is the subalgebra of $C_c^\infty(G_n(\mathbb{A}_F))$ invariant under the outer automorphism $\theta$. 
Let
\[ L^2_{\text{disc}}([G_n]) < L^2([G_n]) \]
be the largest closed subspace that decomposes discretely under \( C^\infty_c(G_n(\mathbb{A}_F)) \).

In view of Theorem 13.8.1, it is natural to partition \( L^2_{\text{disc}}([G_n]) \) into packets consisting of fibers of the functorial transfer to \( H_N(\mathbb{A}_F) \) and then try to describe the fibers, preferably in terms of local data. This is precisely what Arthur accomplished.

We state the main classification result first. The remainder of the section is devoted to defining the notation that appears.

**Theorem 13.8.2 (Arthur)** There is an \( C^\infty_c(G_n(\mathbb{A}_F)) \)-module isomorphism
\[ L^2_{\text{disc}}([G_n]) \cong \bigoplus_{\psi \in \mathcal{P}_d(G_n)} \bigoplus_{\pi \in \hat{\Pi}_{\psi}(\tau_{\psi})} \pi^{\oplus m_{\psi}}, \]
where \( m_{\psi} = 1 \) or 2.

This is [Art13, Theorem 1.5.2]. For a cuspidal automorphic representation \( \tau \) of \( H_N(\mathbb{A}_F) \) and \( m \in \mathbb{Z}_{\geq 1} \), let \( (\tau, m) \) be the Spelh representation of \( \Sect 10.7 \).

One says that \( \tau \) is of **orthogonal type** if \( L^S(s, \tau, \Sym^2) \) has a pole at \( s = 1 \) and of **symplectic type** if \( L^S(s, \tau, \wedge^2) \) has a pole at \( s = 1 \). Here \( S \) is a finite set of places of \( F \) including the infinite places and all places where \( \tau \) is ramified. Since
\[ L^S(s, \tau, \Sym^2)L^S(s, \tau, \wedge^2) = L^S(s, \tau \times \tau) \]
it follows from Theorem 11.7.1 that \( \tau \) cannot be both orthogonal and symplectic, and it is orthogonal or symplectic if and only if \( \tau \cong \tau^\vee \). We define the type of the representation \( (\tau, m) \) as follows:

<table>
<thead>
<tr>
<th>Type</th>
<th>( m )</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>orthogonal</td>
<td>even</td>
<td>symplectic</td>
</tr>
<tr>
<td>symplectic</td>
<td>even</td>
<td>orthogonal</td>
</tr>
<tr>
<td>orthogonal</td>
<td>odd</td>
<td>orthogonal</td>
</tr>
<tr>
<td>symplectic</td>
<td>odd</td>
<td>symplectic</td>
</tr>
</tbody>
</table>

The set \( \mathcal{P}_d(G_n) \) is the set of automorphic representations of \( H_N(\mathbb{A}_F) \) of the form
\[ \boxplus_{i=1}^d (\tau_i, m_i) \]
where
(1) \( \tau_i \) is a cuspidal automorphic representation of \( H_{N_i}(\mathbb{A}_F) \),
(2) \( \sum_{i=1}^d N_i m_i = N \).
(3) $\tau_i' \cong \tau_i$ for all $i$,
(4) $\tau_i \cong \tau_j$ if and only if $i = j$.
(5) If $L G_n^\circ$ is orthogonal (resp. symplectic) then $(\tau_i, m_i)$ is orthogonal (resp. symplectic).

We note that condition (5) actually implies condition (3). The set $\tilde{\Psi}_2(G_n)$ is known as the set of discrete global $A$-packets of $G_n$ (see [Art13, page 33–34 in §1.4]). The $A$ in $A$-packets is in honor of Arthur, who first introduced these objects. The multiplicity $m_\psi$ in Theorem 13.8.2 is defined to be 1 unless $N$ is even, $L G_n^\circ = \text{SO}_N(\mathbb{C})$, and $N m_i$ is even for all $i$, in which case it is 2. The discrete global $A$-packet is said to be generic if $m_i = 1$ for all $i$. In this case we also refer to the packet as a discrete global generic $L$-packet.

For every $\psi \in \tilde{\Psi}_2(G_n)$, Arthur defines a finite 2-group $S_\psi$ and a character

$$\varepsilon_\psi : S_\psi \to \{\pm 1\}.$$ 

We omit the definition. For every place $v$ of $F$ and every $\psi \in \tilde{\Psi}_2(G_n)$, we define a representation

$$\psi_v : W_{F_v} \times \text{SL}_2(\mathbb{C}) \to L H_N$$

by

$$\psi_v := \oplus_{i=1}^d \text{rec}(\tau_{iv}) \otimes \text{Sym}^m$$

where rec is the local Langlands reciprocity map of (12.16) in §12.4. We note that $\psi_v$ is not an $L$-parameter, but one can obtain an $L$-parameter from it as follows:

$$\rho(\psi_v)(g) := \psi_v \left( g, \left( \frac{|g|}{|g|^{-1/2}} \text{sgn} g \right) \right).$$

The extra $\text{SL}_2$ factor occurring in $\psi_v$ is colloquially known as the Arthur-$\text{SL}_2$ and plays a role similar to the representation of $\text{SL}_2$ that appears in Hodge theory. One proves that $\psi_v$ factors through the $L$-map $r$ and hence defines a homomorphism

$$\psi_v : W_{F_v} \times \text{SL}_2(\mathbb{C}) \to L G_n.$$ 

These are examples of $A$-parameters, although we will not formally define this concept. One shows in addition the existence of maps

$$S_\psi \to \pi_0(\mathcal{S}_{\psi_v})$$

where

$$\mathcal{S}_{\psi_v} := C_{\hat{G}_n(\mathbb{C})}(\text{Im}(\psi_v)) / Z_{\hat{G}_n(\mathbb{C})}^{\text{Gal}_F}$$

as in (12.22). We warn the reader that the component group $\pi_0(\mathcal{S}_{\psi_v})$ is denoted by $S_{\psi_v}$ in [Art13].
Arthur then defines sets $\tilde{\Pi}(\psi_v)$ of irreducible admissible representations of $G_n(F_v)$ attached to each such parameter satisfying certain desiderata. This is already a substantial theorem as it amounts to a proof of the local Langlands classification for $G_n$. The sets $\tilde{\Pi}(\psi_v)$ are the local $A$-packets attached to $\psi_v$. The use of $\tilde{\Pi}(\psi_v)$ instead of the notation $\hat{\Pi}(\psi_v)$ of §12.5 is intentional. First, the tilde is a reminder that one is only classifying representations up to the action of the outer automorphism $\theta$ when $G_n$ is $\text{SO}_{2n}$ or $\text{SO}^\ast_{2n}$. Second, the packets $\tilde{\Pi}(\psi_v)$ are not $L$-packets. They are an enlargement of $L$-packets required due to the fact that the representations in question are often nontempered. In the special case where $\psi_v$ is trivial on the $\text{Arthur-SL}_2$, the packet $\tilde{\Pi}(\psi_v)$ is an (almost tempered) $L$-packet in the sense of §12.5. In particular $\psi_v$ is trivial on the Arthur-$\text{SL}_2$ for all $v$ for any discrete global generic $L$-packet $\tilde{\Pi}(\psi)$. As in §12.5, any $\pi_v \in \tilde{\Pi}(\psi_v)$ comes equipped with a character

$$\langle \cdot, \pi_v \rangle : \pi_0(\mathcal{S}_{\psi_v}) \to \mathbb{C}^\times$$

and thus we obtain a character

$$\langle \cdot, \pi \rangle := \prod_v \langle \cdot, \pi_v \rangle.$$ 

This allows us to form the global adelic $A$-packet

$$\tilde{\Pi}(\psi) := \left\{ \otimes_v \pi_v : \pi_v \in \tilde{\Pi}(\psi_v) \text{ and } \langle \cdot, \pi_v \rangle = 1 \text{ for almost all } v \right\}. \quad (13.22)$$

It consists of admissible representations of $G_n(\mathbb{A}_F)$. Not all of them are automorphic, let alone occur in $L^2_{\text{disc}}([G_n])$. The last piece of the classification theorem is a device for selecting which occur in $L^2([G_n])$. This is provided by $\varepsilon_\psi$. One defines

$$\tilde{\Pi}_\psi(\varepsilon_\psi) := \{ \pi \in \tilde{\Pi}(\psi) : \langle \cdot, \pi \rangle = \varepsilon_\psi \}.$$ 

This completes our discussion of the objects in Theorem 13.8.2.

### 13.9 The function field case

In the function field case much more is known. Let $F$ be a function field of characteristic $p$. For $\text{GL}_2$, the Langlands correspondence over $F$ was proved by Drinfeld [Dd80, Dd87b, Dd87a, Dd88]. The work earned Drinfeld a Fields’ medal. For $\text{GL}_r$ with $r$ arbitrary the Langlands correspondence over $F$ was proved by L. Lafforgue [La98]. The work earned him a Fields’ medal as well. Lafforgue’s argument follows Drinfeld’s, although there were substantial technical obstacles to overcome to adapt the techniques from the $\text{GL}_2$ case to higher rank.
L. Lafforgue constructed a bijection between the set of isomorphism classes of continuous irreducible representations \( \text{Gal}_F \to \text{GL}_r(\overline{\mathbb{Q}_p}) \) that are unramified almost everywhere and have finite order determinant and isomorphism classes of cuspidal automorphic representations of \( \text{GL}_r(\mathbb{A}_F) \) with finite order determinant. He moreover proved that the bijection is compatible with the local Langlands correspondence at all unramified places. This is enough to uniquely determine the correspondence by the Chebatarev density theorem on the Galois side and the strong multiplicity one assertion of Theorem 11.7.2 on the automorphic side. The proof involves comparing the Grothendieck-Lefschetz trace formula for the trace of Frobenius on a certain stack and the Arthur-Selberg trace formula. The Grothendieck-Lefschetz trace formula encodes Galois-theoretic information ultimately coming from the étale site and the Arthur-Selberg trace formula encodes automorphic information.

More recently, V. Lafforgue (L. Lafforgue’s brother) gave a decomposition of the space of cusp forms on an arbitrary reductive group over \( F \) in terms of \( L \)-parameters that is compatible with the local Langlands correspondence at almost all places [Laf18]. This can be viewed as the automorphic to Galois direction of the Langlands correspondence. Interestingly his proof does not involve the Arthur-Selberg trace formula.

All of the work above relies on objects known as shtukas (Russian for “thing”). The very definition of a shtuka involves the scheme

\[
\text{Spec}(\mathcal{O}_F \otimes_{\mathbb{F}_p} \mathcal{O}_F).
\] (13.24)

This sort of self product also comes up in Deligne’s proof of the Riemann hypothesis in the function field case. The analogue of this scheme does not exist in the number field case, since the various residue fields of the ring of integers of the number field all have different characteristics.

There has been a great deal of thought about what can be used to replace (13.24) in the number field case. The idea is that instead of \( \mathbb{F}_p \) one should use the field with one element. This remains a very interesting but mostly speculative prospect. The motivation comes not only from the link with Langlands functoriality but also from the link to the Riemann hypothesis.

**Exercises**

**13.1.** Suppose that \( G \) is a reductive group over a number field \( F \). Let \( r : L_1 G \to L_{\text{Res}_{E/F}} \text{GL}_n \) be an \( L \)-map and let \( \pi \) be a cuspidal automorphic representation of \( G(\mathbb{A}_F) \). Assume that a weak transfer \( \pi' \) of \( \pi \) to \( \text{Res}_{E/F} \text{GL}_n(\mathbb{A}_F) = \text{GL}_n(\mathbb{A}_E) \) with respect to \( r \) exists and \( \pi' \) is isobaric. Prove that \( \pi' \) is the unique isobaric automorphic representation of \( \text{GL}_n(\mathbb{A}_E) \) that is a weak transfer of \( \pi \) with respect to \( r \).

**13.2.** Prove (13.5).
13.3. Prove (13.7).

13.4. Prove that a continuous homomorphism

\[ \text{Gal}_F \rightarrow \text{GL}_n(\mathbb{C}) \]

has finite image.

13.5. Let \( \pi \) be a cuspidal automorphic representation of \( \text{A}_{\text{GL}_2} \backslash \text{GL}_2(\mathbb{A}_F) \). Let

\[ r_k := \text{Sym}^k \otimes (\text{Sym}^k) \]

be the representation given by \( \text{Sym}^k \otimes (\text{Sym}^k) \). Assume that \( L(s, \pi, r_k) \) converges absolutely for \( \text{Re}(s) > 1 \) for all \( k \geq 1 \). Prove the unramified Ramanujan conjecture for \( \pi \). In other words, for all places \( v \) of \( F \) such that \( \pi_v \) is unramified, say \( \pi_v \cong J(\lambda) \), the Langlands class (or Satake parameter) \( q^{\lambda} \in \text{GL}_2(\mathbb{C}) \) has eigenvalues of complex norm equal to 1.
Chapter 14
Distinction and Period Integrals

Abstract In this chapter we discuss the notion of a distinguished representation in local and global contexts. The global notion is defined in terms of period integrals and hence isolates a crucial link between automorphic representation theory and geometry of locally symmetric spaces.

14.1 Introduction

The notion of distinction of automorphic representations with respect to subgroups of the ambient group was introduced in a global setting by Harder, Langlands and Rapoport in their work on the Tate conjecture for Hilbert modular surfaces [HLR86]. Jacquet later developed a tool for studying distinction, namely the relative trace formula, which includes the usual trace formula as a special case. This will be treated in Chapter 18. In retrospect, the notion of distinction has been an integral part of representation theory for some time, as it often isolates models of representations of considerable interest. For example, the whole theory of generic representations can be thought of as a special case of the study of distinguished representations.

In this chapter we discuss the notion of distinction, starting from the local setting and moving to the global. The special case of spherical subgroups is discussed in §14.4 and important conjectures of Sakellaridis and Venkatesh are described in vague terms. We give a fairly thorough treatment of symmetric subgroups of the general linear group in §14.5 and then discuss the exciting prospect of generalizing this to classical groups in §14.6. A brief introduction to the Gan-Gross-Prasad conjecture follows in §14.7. This can be
viewed as a precursor to the Sakellaridis-Venkatesh conjectures. Finally we complete the chapter with some negative results in §14.8.

14.2 Distinction in the local setting

Let $G$ be a reductive group over a local field $F$ and let $H \leq G$ be a subgroup. Assume we are given a quasi-character $\chi : H(F) \to \mathbb{C}^\times$. Let $V_\chi$ be the space of $\chi$. In other words, it is the 1-dimensional complex vector space on which $H(F)$ acts via $\chi$.

**Definition 14.1.** An irreducible admissible representation $(\pi, V_\pi)$ of $G(F)$ is $(H, \chi)$-**distinguished** if

$$\text{Hom}_{H(F)}(V_\pi, V_\chi) \neq 0.$$ 

If $H$ and/or $\chi$ are understood then we often speak of $\chi$-distinguished, $H$-distinguished or simply distinguished representations. An element of

$$\text{Hom}_{H(F)}(V_\pi, V_\chi)$$

is a **relative character** or an $(H, \chi)$-**relative character** if we wish to be specific.

The definition above has the advantage that it is simple, but sometimes a slightly more sophisticated perspective is advantageous. Let

$$\mathcal{F}(H(F) \backslash G(F), \chi) := \{ f : G(F) \to \mathbb{C} : f(hg) = \chi(h)f(g) \text{ for } h \in H(F) \},$$

$$C^\infty(F(H(F) \backslash G(F), \chi) := \{ f \in C^\infty(G(F)) : f(hg) = \chi(h)f(g) \text{ for } h \in H(F) \}.$$ 

Thus the first space is just a space of functions, and the second is a space of smooth functions. The first space can be regarded as the space of sections on the line bundle over $H(F) \backslash G(F)$ defined by $\chi$ and the second can be regarded as the space of smooth sections.

**Lemma 14.2.1** There is a $\mathbb{C}$-linear isomorphism

$$\text{Hom}_{H(F)}(V_\pi, V_\chi) \cong \text{Hom}_{G(F)}(V_\pi, \mathcal{F}(H(F) \backslash G(F), \chi)). \quad (14.1)$$

**Proof.** To a relative character $\lambda$ and $\varphi \in V_\pi$, we associate the function

$$I_\lambda(\varphi)(g) := \lambda(\pi(g)\varphi).$$

To an intertwining map $I : V_\pi \to \mathcal{F}(H(F) \backslash G(F), \chi)$, we associate the relative character

$$\lambda_I(\varphi) := I(\varphi)(1)$$

where 1 is the identity of $G(F)$. \qed
Corollary 14.2.2 An irreducible admissible representation \((\pi, V_\pi)\) of \(G(F)\) is \((H(F), \chi)\)-distinguished if and only if there is a nonzero intertwining map
\[
V_\pi \rightarrow \mathcal{F}(H(F) \backslash G(F), \chi).
\]

\(\square\)

The lemma allows us to describe natural subspaces of the space of relative characters \(\text{Hom}_{H(F)}(V_\pi, V_\chi)\). For example, let
\[
\text{Hom}_{H(F), \text{sm}}(V_\pi, V_\chi)
\]
be the inverse image of \(C^\infty(H(F) \backslash G(F), \chi)\) under the bijection (14.1). We say that the elements of (14.2) are smooth relative characters. We say that \(\pi\) is smoothly \((H, \chi)\)-distinguished if (14.2) is nonzero. More sophisticated notions of smoothness must sometimes be employed in the archimedean case (see §11.3 for instance).

It is instructive to observe that one can view the theory of generic representations discussed in §11.3 as a special case of the more general concept of distinguished representations. Assume for simplicity that \(F\) is nonarchimedean. If \(G\) is quasi-split reductive over \(F\), \(N \leq G\) is the unipotent radical of a Borel subgroup, and \(\psi : N(F) \rightarrow \mathbb{C}^\times\) is a generic character then an irreducible admissible representation \(\pi\) of \(G(F)\) is \(\psi\)-generic if and only if it is smoothly \((N, \psi)\)-distinguished. The space \(\text{Hom}_{N(F), \text{sm}}(V_\pi, V_\psi)\) is the space of Whittaker functions, and \(\text{Hom}_{G(F)}(V_\pi, C^\infty(N(F) \backslash G(F), \psi))\) is the space of Whittaker models. Both are at most 1-dimensional. For many interesting cases, the phenomenon that (14.2) is at most 1-dimensional (or at least finite dimensional) persists. We discuss this in more detail in §14.4 below.

14.3 Global distinction and period integrals

The global version of distinction involves period integrals of cusp forms. Let \(G\) be a reductive group over a global field \(F\) and let \(H\) be a subgroup of \(G\). Let
\[
\chi : H(\mathbb{A}_F) \rightarrow \mathbb{C}^\times
\]
be a quasi-character trivial on \((A_G \cap H(\mathbb{A}_F))H(F)\). For the definition of the central subgroup \(A_G\), see (2.17). Thus \(\chi\) induces a function
\[
A_GH(F) \backslash A_GH(\mathbb{A}_F) = (A_G \cap H(\mathbb{A}_F))H(F) \backslash H(\mathbb{A}_F) \rightarrow \mathbb{C}^\times
\]
which we continue to denote by \(\chi\). If \(\varphi : [G] \rightarrow \mathbb{C}\) is a continuous function then we define the period integral

\[\int \cdots \, d\varphi.\]
\[ \mathcal{P}_\chi(\varphi) := \int_{AGH(F) \mathbin\setminus AGH(\mathbb{A}_F)} \varphi(h)\chi(h)dh \quad (14.3) \]

whenever this integral is absolutely convergent.

For a cuspidal automorphic representation \( \pi \) of \( AG \mathbin\setminus G(\mathbb{A}_F) \), let \( L^2_{\text{cusp}}(\pi) \) be the \( \pi \)-isotypic subspace of \( L^2_{\text{cusp}}([G]) \). We assume that \( \mathcal{P}_\chi(\varphi) \) is absolutely convergent for all smooth \( \varphi \in L^2_{\text{cusp}}([G]) \).

**Definition 14.2.** A cuspidal automorphic representation \( \pi \) of \( AG \mathbin\setminus G(\mathbb{A}_F) \) is \((H, \chi)\)-distinguished if \( \mathcal{P}_\chi(\varphi) \) is nonzero for some smooth \( \varphi \) in \( L^2_{\text{cusp}}(\pi) \).

If \( H \) and/or \( \chi \) are understood then we often speak of \( H \)-distinguished, \( \chi \)-distinguished or simply distinguished representations. If \( \chi \) is trivial, then we often write \( \mathcal{P} \) for \( \mathcal{P}_\chi \).

**Example 14.1.** If \( G \) is quasi-split, \( H = N \) is the unipotent radical of a Borel subgroup, and \( \psi : N(\mathbb{A}_F) \to \mathbb{C}^\times \) is a generic character, then a cuspidal representation is \( \psi \)-generic if and only if it is \((H, \psi)\)-distinguished.

The notion of distinction is also of interest for noncuspidal representations. However the definition is more complicated and usually involves truncation. See [JLR99, LR03, Off06a, Zyd19].

Before proceeding further let us treat a technical point:

**Lemma 14.3.1** Assume \( F \) is a number field. If \( H \subseteq G \) then

\[
AG \cap AH = AG \cap H(\mathbb{A}_F) = AZ_G \cap H.
\]

**Proof.** By applying Weil restriction of scalars we can and do assume that \( F = \mathbb{Q} \). Let \( T_G \) be the largest split subtorus of \( Z_G \) and \( T_H \) be the largest split subtorus in \( Z_H \). Then \( (T_G \cap H)^\circ \) is a subtorus of \( T_G \) [Mil17, §12.e] which is split by Exercise 14.2. Thus \( (T_G \cap H)^\circ \leq T_G \cap T_H \). On the other hand, \( T_G \cap H \geq T_G \cap T_H \), so we deduce

\[
(T_G \cap H)^\circ = (T_G \cap T_H)^\circ. \quad (14.4)
\]

As usual we say that subgroups \( G_1 \) and \( G_2 \) of \( G(\mathbb{R}) \) are commensurable and if \( G_1 \cap G_2 \mathbin\mid G_1 \) and \( G_1 \cap G_2 \mathbin\mid G_2 \) are finite. Write \( G_1 \sim G_2 \) if \( G_1 \) is commensurable to \( G_2 \); this is an equivalence relation. We have

\[
AG \cap H(\mathbb{A}_Q) = AG \cap H(\mathbb{R}) \sim T_G(\mathbb{R}) \cap H(\mathbb{R}) \sim (T_G \cap H)^\circ(\mathbb{R}).
\]

Thus, in view of (14.4),

\[
AG \cap H(\mathbb{A}_Q) \sim (T_G \cap T_H)^\circ(\mathbb{R}) \sim T_G(\mathbb{R}) \cap T_H(\mathbb{R}) \sim AG \cap AH. \quad (14.5)
\]

Thus \( AG \cap AH \mathbin\setminus AG \cap H(\mathbb{A}_Q) \) is finite.

Since the quotient \( AG \cap AH \mathbin\setminus AG \cap H(\mathbb{A}_Q) \) is finite, to show it is trivial it suffices to prove that \( AG \cap H(\mathbb{R}) \) is connected. The number of connected
components of $T_G(R) \cap H(R)$ in the real topology is finite by Theorem 4.4.1. Thus the same is true of $A_G \cap H(R)$. This implies that $A_G \cap H(R)$ is a real Lie subgroup of $A_G$ with only finitely many connected components. Now $A_G \cong \mathbb{R}^n_{>0}$ for some $n$ and the only closed real Lie subgroups of $\mathbb{R}^n_{>0}$ with finitely many components are connected by Exercise 14.1. Thus $A_G \cap H(R)$ is connected. This completes the proof of the first equality in the lemma.

We observe now that $(T_G \cap T_H)^\circ$ is the maximal $Q$-split torus in $Z_G \cap H$. Hence the neutral component of $(T_G \cap T_H)^\circ(R)$ as a real Lie group is $A_{Z_G \cap H}$. We have already shown in (14.5) that $A_G \cap A_H \simeq (T_G \cap T_H)^\circ(R)$ and hence $A_G \cap A_H \sim A_{Z_G \cap H}$. Since $A_G \cap A_H$ and $A_{Z_G \cap H}$ are both connected we deduce that $A_G \cap A_H = A_{Z_G \cap H}$. \hfill $\Box$

We now develop criteria for $P_\chi(\varphi)$ to be convergent. We begin with the following lemma:

**Lemma 14.3.2** If $H$ is reductive then the quotient $A_G H(F) \backslash A_G H(\mathbb{A}_F)$ is closed in $[G]$.

**Proof.** By definition of the quotient topology, we are to show that $A_G \backslash (G(F) \backslash A_G H(\mathbb{A}_F)) \subset A_G \backslash G(\mathbb{A}_F)$ is closed. For this it suffices to show that $A_G \backslash (G(F) \backslash A_G H(\mathbb{A}_F)/H(\mathbb{A}_F))$ is closed in $A_G \backslash G(\mathbb{A}_F)/H(\mathbb{A}_F)$. We will show that the image of $G(F)$ in the quotient $A_G \backslash G(\mathbb{A}_F)/H(\mathbb{A}_F)$ is discrete and closed in Lemma 17.6.4. \hfill $\Box$

Any smooth algebraic group $H \leq G$ has the property that $H^\circ$ is a semidirect product of a reductive and a unipotent group. We will assume this semidirect product is direct. In other words we assume that

$$H^\circ = H_r \times H_u \quad (14.6)$$

where $H_r$ is reductive and $H_u$ is unipotent.

A version of the following proposition is proven in [AGR93]:

**Proposition 14.3.3** Assuming (14.6), if $\varphi \in L^2_{\text{cusp}}([G])$ is smooth then

$$\int_{A_G H(F) \backslash A_G H(\mathbb{A}_F)} |\varphi(hg)\chi(h)|dh \ll_{\chi, \varphi} 1$$

for $g \in G(\mathbb{A}_F)$.

**Proof.** We first reduce the proposition to the case where $H$ is connected. Let $K^\infty \leq H(\mathbb{A}_F)$ be a compact open subgroup such that $\varphi$ and $\chi$ are left $K^\infty$-invariant. We recall that $H(\mathbb{A}_F)/H^\circ(\mathbb{A}_F)$ is compact by a theorem of Borel [Con12a, Proposition 3.2.1], and hence

$$H(F)H^\circ(\mathbb{A}_F) \backslash H(\mathbb{A}_F)/K^\infty$$
is finite. For appropriate choices of Haar measures $dh$ on $H(\mathbb{A}_F)$ and $dh^\circ$ on $H^\circ(\mathbb{A}_F)$, one has that

$$\int_{A_G H(F) \backslash A_G H(\mathbb{A}_F)} |\varphi(hg)\chi(h)| dh$$

$$= \int_{A_G H(F) \backslash A_G H(\mathbb{A}_F)/K_\infty} |\varphi(hg)\chi(h)| dh$$

$$= \sum_{h \in H(F)H^\circ(\mathbb{A}_F)/K_\infty} \int_{A_G H^\circ(F) \backslash A_G H^\circ(\mathbb{A}_F)} |\varphi(h_0 hg)\chi(h_0 h)| dh_0.$$

Thus we are reduced to the case where $H$ is connected.

We now assume $H$ is connected, so $H = H_F \times H_u$. We then have

$$\int_{A_G H(F) \backslash A_G H(\mathbb{A}_F)} |\varphi(hg)\chi(h)| dh$$

$$= \int_{A_G H_u(F) \backslash A_G H_u(\mathbb{A}_F)} \int_{\{H_u\}} |\varphi((h_r, h_u)g)\chi(h_r, h_u)| dh_r dh_u.$$

The set $[H_u]$ is compact by Theorem 2.6.3. Thus applying the Fubini-Tonelli theorem, we see that it suffices to establish the proposition in the special case where $H$ is reductive.

Assume first that we are in the function field case. In this setting, Theorem 9.5.1 asserts that any smooth function in $L^2_{\text{cusp}}([G])$ is compactly supported. Thus the proposition follows from Lemma 14.3.2.

Now assume that we are in the number field case. Let $T_{H0} \leq H^\text{der}$ and $T_0 \leq G^\text{der}$ be maximal $F$-split tori in $H^\text{der}$ and $G^\text{der}$, respectively. We can and do assume $T_0 \cap H^\text{der} = T_{H0}$. Let $P_H$ be a minimal parabolic subgroup of $H$ containing $T_{H0}$. It admits a Levi decomposition $M_H N_H = P_H$, where $M_H$ is the centralizer of $T_{H0}$ in $H$ (it is a Levi subgroup of $P_H$ [Bor91, Proposition 20.4] or [BT65, Theorem 4.15(b)]) and $N_H < P_H$ is the unipotent radical of $P_H$. By Theorem 2.7.2, there is an $0 < r < 1$ such that

$$H(\mathbb{A}_F) = A_H H(F) A_{T_{H0}(r)} \Omega K_H$$

where $\Omega$ is relatively compact subset of $N_H(\mathbb{A}_F) M_H(\mathbb{A}_F)^1$, $K_H \leq H(\mathbb{A}_F)$ is a maximal compact subgroup, and $A_{T_{H0}(r)}$ is defined with respect to the set of simple roots $\Delta_H$ of $T_{H0}$ in $H$ attached to the parabolic subgroup $P_H$. We refer to §2.7 for notation.

The Haar measure on $H(\mathbb{A}_F)$ decomposes as

$$d(\text{ann}mk) = d\phi d\psi dm dk$$

for $(a, n, m, k) \in A_H A_{T_{H0}(r)} \times N_H(\mathbb{A}_F) \times M_H(\mathbb{A}_F)^1 \times K_H$ by the Iwasawa decomposition of Theorem 2.7.1 and Proposition 3.2.1. Thus
14.3 Global distinction and period integrals

\[
\int_{A_G H(F) \setminus A_G H(A_F)} |\varphi(hg)\chi(h)| dh = \int_{(A_G \cap H) \setminus A_H A_{T_H(r)} \cap \Omega \times K_H} |\varphi(anmkg)\chi(anmk)| dadndmdk. \tag{14.7}
\]

Here we have used Lemma 14.3.1. Let \(\Delta\) be a set of simple roots for \(T_0\) in \(G\). We can then form the corresponding set

\[ A_{T_0}(r_0) \]

for \(r_0 \in \mathbb{R}_{>0}\). There is a Weyl chamber \(C \subset \text{Lie} A_{T_0}\) such that the closure of the image of \(C\) under the exponential map is \(A_{T_0}(1)\). Weyl chambers in \(\text{Lie} A_{T_0}\) are permuted simply transitively by \(W(G\hookrightarrow T)(F)\) \cite[Theorems 14.7, 21.2, and 21.6]{Bor91}. It follows that

\[ A_{T_0} = \bigcup_{w \in W(G,T)(F)} w A_{T_0}(1) w^{-1} \]

and the intersection of any two \(W(G,T)(F)\)-conjugates of \(A_{T_0}(1)\) is a set of measure zero with respect to any Haar measure on \(A_{T_0}\). Hence (14.7) is equal to

\[
\sum_{w \in W(G,T)(F)} \int |\varphi(anmkg)\chi(anmk)| dadndmdk,
\]

where the integral above is over

\[(A_G \cap A_H) \setminus A_H A_{T_H(r)} \cap w A_{T_0}(1) w^{-1} \times \Omega \times K_H.\]

Changing variables and using the left invariance of \(\varphi\) under \(w\), we see that this is

\[
\sum_{w \in W(G,T)(F)} \int |\varphi(aw^{-1}nmkg)\chi(aw^{-1}nk)| dadndmdk, \tag{14.8}
\]

where the integral is over

\[(A_G \cap w^{-1}A_H w) \setminus w^{-1} A_H A_{T_H(r)} w \cap A_{T_0}(1) \times \Omega \times K_H.\]

Using Theorem 4.2.6, write \(\varphi = \sum_{i=1}^n R(f_i)\varphi_i\) for some \(f_i \in C_c^\infty(A_G \setminus G(A_F))\). Then by Proposition 9.4.5 the integral (14.8) is bounded by a constant independent of \(g \in G(A_F)\). \(\square\)

The following is an immediate corollary of Proposition 14.3.3:

**Corollary 14.3.4** Assume (14.6). If \(\varphi \in L^2_{\text{cusp}}([G])\) is smooth then the integral defining \(P_\lambda(\varphi)\) is absolutely convergent and the functions
\[ g \mapsto \mathcal{P}_\chi (\pi(g) \varphi), \]
\[ g \mapsto \int_{A_G H(F) \backslash A_G H(\mathbb{A}_F)} |\varphi(hg)\chi(h)| dh \]

are continuous on \( G(\mathbb{A}_F). \) \( \Box \)

Assume \( F \) is a number field. Let \( K_\infty \leq G(F_\infty) \) be a maximal compact subgroup. It is often useful to work with \( K_\infty \)-finite functions in \( L^2_{\text{cusp}}(\pi) \) instead of merely smooth ones. For this purpose we prove the following lemma:

**Lemma 14.3.5** Assume \((14.6)\). Let \( \pi \) be a cuspidal automorphic representation of \( A_G \backslash G(\mathbb{A}_F) \). The representation \( \pi \) is \((H, \chi)\)-distinguished if and only if there is a \( K_\infty \)-finite smooth function \( \varphi \in L^2_{\text{cusp}}(\pi) \) such that

\[ \mathcal{P}_\chi (\pi(g) \varphi) \neq 0. \]

**Proof.** The “if” direction is obvious. We prove the “only if.” Let \( U \) be a neighborhood of 1 in \( A_G \backslash G(F_\infty) \) and let \( \varepsilon > 0 \). By the proof of Proposition 4.4.3, we can choose a nonnegative \( K_\infty \)-finite \( f_\infty \in C_\infty^c(A_G \backslash G(F_\infty)) \) with support in \( K_\infty U \) such that

\[ \int_{A_G \backslash G(F_\infty)} f(g) dg = 1 \quad \text{and} \quad \int_{A_G \backslash G(F_\infty) - U} f(g) dg < \varepsilon. \]

Choose a smooth function \( \varphi \) in \( L^2_{\text{cusp}}(\pi) \) such that \( \mathcal{P}_\chi (\varphi) \neq 0 \). It is fixed by some compact open subgroup \( K_\infty \leq G(\mathbb{A}_F) \). Let

\[ f := \text{meas}_{dg_\infty} (K_\infty)^{-1} f_\infty 1_{K_\infty}, \]

where \( 1_{K_\infty} \) is the characteristic function. Then \( \pi(f) \varphi \) is \( K_\infty \)-finite. We have

\[ |\mathcal{P}_\chi (\pi(f) \varphi) - \mathcal{P}_\chi (\varphi)| = \left| \int_{A_G \backslash G(\mathbb{A}_F)} f(g) (\mathcal{P}_\chi (\pi(g) \varphi) - \mathcal{P}_\chi (\varphi)) dg \right|. \]

Here we have used Corollary 14.3.4 to justify switching the order of integration. The above is bounded by

\[ \int_{A_G \backslash G(\mathbb{A}_F)} f(g) |\mathcal{P}_\chi (\pi(g) \varphi) - \mathcal{P}_\chi (\varphi)| dg. \quad (14.9) \]

Since \( \mathcal{P}_\chi (\pi(g) \varphi) \) is continuous as a function of \( g \in G(\mathbb{A}_F) \), \((14.9)\) can be made as small as we wish by adjusting \( U \) and \( \varepsilon \). \( \Box \)
14.4 Spherical varieties

Let $G$ be a reductive group over a field $F$ and let $H \leq G$ be a subgroup. We assume for this section that the characteristic of $F$ is zero because the majority of references restrict to this case.

For general $H$, one does not even have a conjectural answer to the question of which representations are $H$-distinguished. However, there is an important class of subgroups for which one can conjecturally characterize $H$-distinguished representations. Define the quotient scheme

$$X := G/H$$

as in §17.1. It may not be affine. Let $B \leq G_T$ be a Borel subgroup.

**Definition 14.3.** A subgroup $H \leq G$ is said to be spherical if $B$ has an open orbit on $G_T/H_T$.

More generally, let $X$ be a separated normal (not necessarily affine) scheme of finite type over $F$ equipped with an action of $G$. Again let $B \leq G_T$ be a Borel subgroup.

**Definition 14.4.** The scheme $X$ is a spherical variety if $B$ has an open orbit on $X_T$.

Thus if $H \leq G$ is a spherical subgroup then $G/H$ is a spherical variety. A spherical variety $X$ is of the form $G/H$ for some spherical subgroup $H$ if and only if it is homogeneous, that is, consists of a single $G$-orbit (see Proposition 17.1.2).

For any spherical variety $X$, Gaitsgory and Nadler [GN10] defined a dual group $\tilde{G}_X$ equipped with an embedding

$$\tilde{G}_X \times \text{SL}_2 \rightarrow \tilde{G}$$

of reductive groups over $\mathbb{C}$. When $X = G/H$ and $H$ is a reductive symmetric subgroup in the sense of §14.5 there are antecedents of the definition of $\tilde{G}_X$ in [Ric82, JLR93]. These antecedents do not include the $\text{SL}_2$-factor. This factor is crucial for understanding nontempered distinguished representations.

Assume that $X = G/H$ is a spherical variety where $G$ is split. Assume moreover that $F$ is a local field. Let $G_X$ be the split group over $F$ with $L G_X = \tilde{G}_X$. To avoid discussing $A$-parameters we assume that the $\text{SL}_2$-factor of (14.11) is trivial. Thus (14.11) yields an embedding

$$L G_X \rightarrow L G.$$  

(14.12)

Assume that the local Langlands correspondence is known for tempered representations of $G(F)$. In this case the basic local principle of Sakellaridis and Venkatesh is the following:
Principle 14.4.1 A tempered $L$-parameter $\rho : W'_F \to {}^L G$ factors through (14.12) if and only if some $\pi \in \Pi(\rho)$ is $H$-distinguished.

Due to work of Sakellaridis [Sak08, Sak13] we have a robust understanding of Principle 14.4.1 when the $L$-parameter is unramified. Generalizing Sakellaridis’ work to arbitrary unramified groups $G$ is an important (and probably accessible) task.

We have used “principle” instead of “conjecture” because it might be false as stated. One issue is that it might be necessary to enlarge the $L$-packet to a so-called Vogan $L$-packet that includes representations of pure inner forms of $G$. See §14.7 for the definition of pure inner forms. More serious issues can also occur. A case in point is when $H$ is an orthogonal group $O_n$ and $G$ is the general linear group $GL_n$ (see §14.5 below). We refer to [SV17, §16] for explicit conjectures.

Now assume that $F$ is a global field and $G$ is a reductive group over $F$. Let $H \leq G$ be a spherical subgroup. By Exercise 14.6, if $\pi = \bigotimes'_v \pi_v$ is a cuspidal automorphic representation of $G(\mathbb{A}_F)$ that is $H$-distinguished then $\pi_v$ is $H$-distinguished for all places $v$ of $F$. In general, the converse statement is false. There is usually an additional global constraint that can be phrased in terms of $L$-functions. We will illustrate this with examples in the next few sections.

Under suitable assumptions, in [SV17] Sakellaridis and Venkatesh gave a systematic conjectural description of this additional global constraint. Assume that $G$ is split, let $G_X$ be the split reductive group over $F$ with $^L G_X = \widetilde{G}_X$, and assume that the $SL_2$-factor in (14.11) is trivial. Thus we again have the embedding (14.12) in this global setting.

Principle 14.4.2 A tempered cuspidal automorphic representation $\pi$ of $G(\mathbb{A}_F)$ is $H$-distinguished if and only if

(a) it is a functorial transfer from $G_X(\mathbb{A}_F)$ with respect to (14.12) and
(b) a certain value or residue of a particular quotient of $L$-functions is nonzero.

Assuming Principle 14.4.1, condition (a) is essentially equivalent to $\pi_v$ being $H$-distinguished for all $v$. This is only a principle and not a conjecture for the same reasons mentioned in the local setting. We refer to [SV17, §17] for a precise conjecture.

There are antecedents of this principle in the special case of symmetric varieties in the work of Jacquet and his collaborators [JLR93, Jac05a]. These theorems together with the Gan-Gross-Prasad conjecture discussed in §14.7 were among the motivations for Principle 14.4.2.

In [SV17, §17] there is even a conjectural formula for the period integral $P(\varphi)$ for $\varphi$ in the space of $\pi$ in terms of $L$-functions. It is known as the Ichino-Ikeda conjecture. It was first posed in a special case in [II10]. The power and beauty of the conjectures of Sakellaridis and Venkatesh lies in the fact that they provide a uniform perspective on many proved and unproved
conjectures in local representation theory and period integrals. At the same time they suggest many new avenues of research. They are currently the most precise enunciation of what has become known as the \textbf{relative Langlands program}.

Before moving to examples of spherical subgroups in the next few sections, let us consider a general separated integral scheme $X$ of finite type over $F$ (not necessarily affine) equipped with an action of $G$. Knop and Schalke [KS17] have defined a dual group $\widehat{G}_X$ equipped with a morphism $\widehat{G}_X \to \widehat{G}$ in this generality. It would be interesting to investigate whether or not this generalized dual group can be used to generalize Principle 14.4.1.

14.5 Symmetric subgroups

Assume that $F$ is a characteristic zero field. Let $\sigma : G \to G$ be an automorphism over $F$ of order 2. For $F$-algebra $R$, let

$$G^\sigma(R) := \{ g \in G(R) : g^\sigma = g \}$$

be the subgroup fixed by $\sigma$ and denote by $H = (G^\sigma)^{\circ}$ its connected component.

In this case we refer to $X := G/H$ as a \textbf{symmetric variety} and $H$ as a \textbf{symmetric subgroup}. We have the following theorem of Vust [Vus74, §1.3]:

\textbf{Theorem 14.5.1} A symmetric subgroup is a spherical subgroup. \hfill $\Box$

To classify symmetric subgroups, it suffices to classify involutions of $G$. This question has been studied by several authors; we mention in particular the work of Helminck [Hel00]. The classification is somewhat complicated. To get a feel for what is going on, it is illuminating to consider the involutions of the classical groups over an algebraically closed field. When the field is $\mathbb{C}$, there is a nice discussion in [GW09, §11.3.4].

We now describe some results on distinction by symmetric subgroups of $\text{GL}_n$ or $\text{Res}_{E/F}\text{GL}_n$ when $E/F$ is a quadratic extension of number fields. There is a great deal known in this setting.

One of the motivations for Jacquet’s development of the relative trace formula was to prove the following theorem [Jac10]:

\textbf{Theorem 14.5.2} Let $E/F$ be a quadratic extension of number fields, let $G = \text{Res}_{E/F}\text{GL}_n$ and let $H$ be a quasi-split unitary group attached to the extension $E/F$. Then a cuspidal automorphic representation $\pi$ of $G(\mathbb{A}_F)$ is $H$-distinguished if and only if $\pi \cong \pi^\sigma$, where $\langle \sigma \rangle = \text{Gal}(E/F)$. \hfill $\Box$

See also [FLO12] for more refined versions of this theorem. This is sometimes known as the Jacquet-Ye case since its proof was more or less reduced to a local statement known as the Jacquet-Ye fundamental lemma in [JY96] and
later proved in two different manners by Ngô [Ngô99b, Ngô99a] and Jacquet [Jac04, Jac05b] (see §19.6 for more details). The condition that \( \pi \cong \pi^\sigma \) is equivalent to \( \pi \) being a base change lift from \( \text{GL}_n(\mathbb{A}_F) \), by work of Arthur and Clozel [AC89] (see Theorem 13.5.1). Thus this theorem is consonant with the principles laid out in §14.4, although the dual group of the relevant spherical variety is not defined here as \( G \) is not split.

Similarly, using a variant of the Rankin-Selberg theory we discussed in §11.7, Flicker and Zinoviev proved the following theorem [Fli88, FZ95]:

**Theorem 14.5.3** For \( G \) as in Theorem 14.5.2, a cuspidal automorphic representation \( \pi \) of \( \text{GL}_n(\mathbb{A}_F) \) is distinguished by \( \text{GL}_n \) if and only if the Asai \( L \)-function \( L(s, \pi, \text{As}_{E/F}) \) has a pole at \( s = 1 \).

Here the Asai \( L \)-function is the Langlands \( L \)-function attached to the Asai representation defined in (13.17). To briefly recall it, let \( E/F \) be an arbitrary (not just quadratic) extension with \( d := [E : F] \). Then the Asai representation

\[
\text{As}_{E/F} : \text{Res}_{E/F} \text{GL}_n = \text{GL}_n^d(\mathbb{C}) \times \text{Gal}_F \longrightarrow \text{GL}_n^d(\mathbb{C})
\]

is defined by stipulating that, for \( \tau \in \text{Hom}_F(E, \overline{F}) \),

\[
\text{As}_{E/F}((g_\tau \times 1))(\otimes \tau v_\tau) = \otimes \tau g_\tau v_\tau, \quad \text{As}_{E/F}((1) \times \sigma)(\otimes \tau v_\tau) = \otimes \tau v_{\sigma \tau}.
\]

A representation \( \rho : \text{Gal}_E \longrightarrow \text{GL}_n(\mathbb{C}) \) extends uniquely to a homomorphism

\[
\rho : \text{Gal}_F \longrightarrow \text{Res}_{E/F} \text{GL}_n
\]

commuting with the projections to \( \text{Gal}_F \) on the \( L \)-group side. Thus to each such \( \rho \) we can associate the representation

\[
\text{As}_{E/F}(\rho) := \text{As}_{E/F} \circ \rho : \text{Gal}_F \longrightarrow \text{GL}_n^d(\mathbb{C}).
\]

Note that for all field extensions \( L \geq E \) one has

\[
\text{As}_{E/F}(\rho)|_{\text{Gal}_L} \cong \otimes_{\tau \in \text{Hom}_F(E, F)} \rho^\tau|_{\text{Gal}_L}.
\]

Thus the Asai representation \( \text{As}_{E/F}(\rho) \) is a canonical extension of

\[
\otimes_{\tau \in \text{Hom}_F(E, F)} \rho^\tau
\]

to \( \text{Gal}_F \). Suppose now that \( E/F \) is quadratic and that \( \tau \) is the generator of \( \text{Gal}(E/F) \). Let \( \pi \) be a cuspidal automorphic representation of \( \text{Res}_{E/F} \text{GL}_n \backslash \text{Res}_{E/F} \text{GL}_n(\mathbb{A}_F) \). In this case the analytic properties of the Asai \( L \)-function are understood by Theorem 13.7.2. The discussion above (with
global Galois groups replaced by local Weil-Deligne groups) implies that

\[ L(s, \pi \times \pi^\tau) = L(s, \pi, \text{As}_{E/F}) L(s, \pi, \text{As}_{E/F} \otimes \eta_{E/F}) \]  

(14.14)

where \( \eta_{E/F} \) is the character attached to \( \text{Gal}(E/F) \) by class field theory. In particular, by Theorem 11.7.1 if \( L(s, \pi, \text{As}_{E/F} \otimes \eta_{E/F}^i) \) has a pole for some \( i \in \{0, 1\} \) then \( \pi \cong \pi^{\vee \tau} \). Moreover, if \( L(s, \pi, \text{As}_{E/F} \otimes \eta_{E/F}^i) \) has a pole at \( s = 1 \) then \( \pi \) is a functorial transfer from a quasi-split unitary group with respect to an \( L \)-map depending on the parity of \( n \) and \( i \) [Mok15, Theorem 2.5.4(a), Remark 2.5.5]. This is again consonant with the expectations in §14.4.

If \( n = 2, F = \mathbb{Q} \) and \( E \) is a real quadratic extension then Theorem 14.5.3 was proven in [HLR86]. It was then used to prove many cases of the Tate conjecture for Hilbert modular surfaces. As mentioned in the introduction, this paper is significant because it was where the notion of a distinguished representation was first defined. Extensions of Harder, Langlands and Rapoports’ results are contained in [Ram04, GH14].

We now move on to symmetric subgroups of \( \text{GL}_n \) (not \( \text{Res}_{E/F} \text{GL}_n \)). Let

\[ \sigma_{m,n} : \text{GL}_{m+n} \rightarrow \text{GL}_{m+n} \]

be conjugation by \( (I_m - I_n) \). Then let

\[ H := \text{GL}_{m+n}^\sigma = \text{GL}_m \times \text{GL}_n. \]

The following is [FJ93, Proposition 2.1]:

**Proposition 14.5.4** If \( m > n \) then no cuspidal automorphic representation of \( \text{GL}_{m+n}(\mathbb{A}_F) \) is distinguished by \( (\text{GL}_m, \chi \circ \det) \) for any quasi-character \( \chi : \mathbb{A}_F^{\times} \backslash \mathbb{A}_F^{\times} \rightarrow \mathbb{C}^{\times}. \)

Here we view \( \text{GL}_m \) as the subgroup \( \text{GL}_{m \times 1} \) of \( \text{GL}_m \times \text{GL}_n = H \). In particular, if \( m > n \) no cuspidal automorphic representation of \( \text{GL}_{m+n}(\mathbb{A}_F) \) is \( (H, \chi) \)-distinguished for any \( \chi \).

Thus from the point of view of cuspidal representations, the only interesting case is when \( m = n \). In this case, consider the quasi-character

\[ \mu_s : H(\mathbb{A}_F) \rightarrow \mathbb{C}^{\times} \]

\[ \left( \begin{array}{c} a \\ b \end{array} \right) \mapsto \frac{\det a}{\det b} \chi \left( \frac{\det a}{\det b} \right) \eta(\det b), \]

where \( \chi \) and \( \eta \) are characters of \( [\mathbb{G}_m] \). In particular they are trivial on \( \mathbb{A}_{\mathbb{C}_m} \). In this setting, Jacquet and Friedberg [FJ93, Theorem 4.1] prove the following:

**Theorem 14.5.5** Let \( \pi \) be a cuspidal automorphic representation of \( \text{GL}_{2n}(\mathbb{A}_F) \) with central character \( \omega_\pi \). Assume that \( \eta^n \omega_\pi = 1 \). The representation \( \pi \) is
(H, µn)-distinguished if and only if L(s, π, ∨ 2 ⊗ η) has a pole at s = 1 and L(s0, π ⊗ χ) ≠ 0.

In fact Friedberg and Jacquet prove that, for ϕ in the space of π, the period integral Pµs (ϕ) is a holomorphic multiple of L(s0, π ⊗ χ) (under the assumption that L(s, π, ∨ 2 ⊗ η) has a pole at s = 1). Ash and Ginzburg [AG94] use Theorem 14.5.5 to construct p-adic L-functions under a technical hypothesis, later proved in [Sun19].

If η = 1 and π has the trivial central character, the condition that L(s, π, ∨ 2) has a pole at s = 1 is equivalent to the statement that π is a lift of an automorphic representation of SO2n+1(AF) by Theorem 13.7.4. We note that SO2n+1 = Sp2n, and in this case the Lie algebra of G is sp2n (see [KS17, Table 3]). This again agrees with the expectations of §14.4.

If we take G = GLn and take σ(g) = J−1g−t J for a symmetric matrix J ∈ GLn (F) then Gσ is an orthogonal group O_n. In the case n = 2, one has a complete characterization of representations of GL2 distinguished by O2 via work of Waldspurger reinterpreted and reproved in [Jac86]. In general, the representations of GLn(AF) distinguished by On should be functorial lifts from the topological double cover of GLn(AF). This expectation is stated as a precise conjecture in [Mao98]. In the same paper, it is more or less reduced to a local conjecture known as the Jacquet-Mao fundamental lemma. The Jacquet-Mao fundamental lemma is now known [Do15, Do20] so it seems likely that the expected characterization of distinguished representations mentioned above will be proven in the near future. We point out that the topological double cover of GLn(AF) is not the adelic points of an algebraic group. Thus the rough version of the conjectures of Sakellaridis and Venkatesh mentioned in §14.4 must be modified to incorporate this setting.

If we take G = GL2n and take σ(g) := J−1g−t J for a skew-symmetric matrix J ∈ GL2n (F) then Gσ is a symplectic group Sp2n. It is not hard to check that all of these subgroups are conjugate under GL2n(F), so the choice of J is irrelevant (see Exercise 14.3). It is known that there are no cuspidal automorphic representations of GL2n(AF) that are distinguished by Sp2n by [JR92, Proposition 1]. However there are discrete automorphic representations of GL2n that are distinguished by Sp2n; these are classified in [Off06b].

We close by noting that, in the discussion above, we have treated every symmetric subgroup H of GLn up to conjugation by GLn(AF) [GW09, §11.3.4]. Of course, this is not the same as conjugation up to GLn(F), but we omit a discussion for brevity.
14.6 Relationship with the endoscopic classification

We continue to let $F$ be a number field. When the locally symmetric spaces attached to a reductive $F$-group $G$ are hermitian, one can exploit the link between algebraic and arithmetic geometry and distinction explained in §15.6 to produce many interesting results. As recalled in the introduction to this chapter, this was the original motivation for the introduction of the notion of distinction. From this point of view, the results in §14.5 on distinction of cuspidal automorphic representations of $GL_n(\mathbb{A}_F)$ and $Res_{E/F}GL_n(\mathbb{A}_F)$ with respect to symmetric subgroups suffer a serious drawback: the locally symmetric spaces attached to $GL_n$ are never hermitian for $n > 2$.

In order to obtain results on distinction relevant to algebraic and arithmetic geometry, the theory of Shimura varieties exposed in §15.8 suggests investigating automorphic representations of classical groups. Of course this is also of interest independently of applications to algebraic or arithmetic geometry.

The first author and Wambach [GW14] provide one means of relating distinction of automorphic representations of $GL_n$ to distinction of automorphic representations on unitary groups (including those defining hermitian locally symmetric spaces). It is suggestive of broader phenomena that merit study.

To state the result, let $E/F$ be a quadratic extension of totally real number fields, let $\langle \sigma \rangle = \text{Gal}(E/F)$, and let $M/F$ be a CM extension. Let $H$ be a quasi-split unitary group in $n$ variables attached to the extension $M/F$ and let $G := Res_{E/F}H$. It is the restriction of scalars down to $F$ of a quasi-split unitary group attached to the extension $ME/E$. The following is the main theorem of [GW14] under some simplifying assumptions:

**Theorem 14.6.1** Let $\pi$ be a cuspidal automorphic representation of $G(\mathbb{A}_F)$. Suppose that $\pi$ satisfies the following assumptions:

(a) There is a finite dimensional representation $V$ of $G_{F,\infty}$ such that $\pi$ is $V$-cohomological.
(b) There are finite places $v_1 \neq v_2$ of $F$ totally split in $ME/F$ such that $\pi_{v_1}$ and $\pi_{v_2}$ are supercuspidal.
(c) For all places $v$ of $F$ such that $ME/F$ is ramified, one of the extensions $M/F$ or $E/F$ is split.

Then the representation $\pi$ admits a base change $\pi'$ to $Res_{M/E}G(\mathbb{A}_E) := GL_n(\mathbb{A}_{ME})$ with respect to the $L$-map

$$r : L^G \longrightarrow L^{Res_{M/E}GL_n}.$$

If the partial Asai $L$-function $L^S(s, \pi', As_{ME/F})$ has a pole at $s = 1$ then some cuspidal automorphic representation $\pi''$ of $G(\mathbb{A}_F)$ in the $L$-packet of $\pi'$ is $H$-distinguished. Moreover, we can take $\pi''$ to be $V$-cohomological. \[\square\]
Here $S$ is a sufficiently large finite set of places of $M$ including the infinite places. The notion of $V$-cohomological is explained in §15.5 below.

The heart of the proof is based on a particular case of a general principle which we now explain. Let $G$ be a reductive group, let $\langle \tau \rangle = \text{Gal}(M/F)$, and let

$$\tilde{G} := \text{Res}_{M/F}G.$$ 

Let $\sigma$ be an automorphism of $G$ of order 2 and let $H := G^\sigma \leq G$ and $\tilde{G}^\sigma \leq \tilde{G}$ be the subgroups fixed by $\sigma$. Let $\theta = \sigma \circ \tau$, viewed as an automorphism of $\tilde{G}$, and let $\tilde{G}^\theta \leq \tilde{G}$ be the subgroup fixed by $\theta$. Let $\pi$ be a cuspidal automorphic representation of $G(\mathbb{A}_F)$. Assume that $\pi$ admits a weak transfer $\pi'$ to $\tilde{G}(\mathbb{A}_F)$.

In favorable circumstances, the machinery developed in [GW14] together with suitable fundamental lemmas should imply the general principle:

**Principle 14.6.2** The representation $\pi'$ is both $\tilde{G}^\sigma$-distinguished and $\tilde{G}^\theta$-distinguished if and only if a cuspidal automorphic representation in the $L$-packet of $\pi$ is $H$-distinguished.

We have not asserted this principle as a conjecture because it may be false as stated. In particular one might need to work with, for instance, $(H, \chi)$-distinguished representations for characters $\chi$ or incorporate pure inner forms of the groups involved. Moreover there might be an additional constraint involving $L$-functions. In Theorem 14.6.1, the group $G$ is a unitary group and $\tilde{G}$ is the restriction of scalars of a general linear group. Even restricting $G$ to this setting, there are more interesting cases to consider. For example, one apply Principle 14.6.2 to study the unitary analogue of Theorem 14.5.5. This case was originally considered in the PhD thesis of J. Polák [Pol16, Pol15]. Part of the proof of Principle 14.6.2 in this case was announced by W. Zhang, and another part has been completed by Leslie [Les19, Les20].

It is reasonable to expect a generalization of Principle 14.6.2 to the setting where $\sigma$ and $\tau$ are arbitrary commuting involutions of an arbitrary reductive group $\tilde{G}$ and $G = (\tilde{G}^\tau)^\circ$ is the neutral component of the fixed points under $\tau$. Since classical groups are, roughly, the neutral components of the fixed points of involutions on $\text{GL}_n$, this would allow one to classify the distinguished representations of classical groups in terms of their transfer to general linear groups. Thus one would have an analogue of the twisted endoscopic classification of the automorphic discrete spectrum of classical groups in terms of the general linear group (see §13.8 and [Art13, Mok15]). We remark that in many cases one direction of the principle can be obtained as a consequence of known results on Langlands functoriality and the so-called residue method. For a family of examples, we refer to [PWZ21].
14.7 Gan-Gross-Prasad type period integrals

The Gan-Gross-Prasad conjecture characterizes representations of a pair of groups distinguished by a subgroup embedded diagonally \([GGP12a, GGP12b]\). It can be viewed as a special case of the Sakellaridis-Venkatesh conjectures mentioned in §14.4, though the original version of the conjecture, due to Gross and Prasad, is older \([GP92]\).

For simplicity, we will only state the conjectures of Gan, Gross and Prasad in a restricted setting, referring to \([GGP12a, GGP12b]\) for the more general setup. Let \(F\) be a local or global field and let \(E\) be an étale \(F\)-algebra of degree 1 or 2. In other words \(E = F, E = F \oplus F, \) or \(E/F\) is a quadratic field extension. We let 
\[
\sigma : E \to E
\]
be the generator of \(\text{Gal}(E/F)\). Let \(V\) be a finite rank free \(E\)-module. We let \(V_0\) be the fixed points of \(\sigma\) acting on \(V\); it is an \(F\)-vector space. An \(E\)-Hermitian form \((\cdot, \cdot)\) on \(V\) is a pairing 
\[
(\cdot, \cdot) : V \times V \to E
\]
that is \(E\)-linear in the first variable and satisfies 
\[
(\cdot, \cdot) = \sigma((\cdot, \cdot)). \tag{14.15}
\]
We assume that \((\cdot, \cdot)\) is nondegenerate. We let \(\tilde{G}_V\) be the group whose points in an \(F\)-algebra \(R\) are given by
\[
\tilde{G}_V(R) := \{g \in \text{GL}_V(E \otimes_F R) : (gv, gw) = (v, w) \text{ for all } v, w \in V \otimes_F R\}.
\]
Let \(G_V\) be the neutral component of \(\tilde{G}_V\). Then \(G_V\) is a classical group. We list the possibilities in the following table:

<table>
<thead>
<tr>
<th>(E)</th>
<th>(G_V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E = F)</td>
<td>(SO_V)</td>
</tr>
<tr>
<td>(E = F \oplus F)</td>
<td>(\text{GL}_{V_0})</td>
</tr>
<tr>
<td>(E/F) quadratic</td>
<td>(U_V)</td>
</tr>
</tbody>
</table>

Here \(SO_V\) and \(U_V\) are the special orthogonal and unitary group of \(V\), respectively.

In order to proceed, it is useful to use the definition of a pure inner form of an algebraic group \(G\). We will give two equivalent definitions, one using Galois cohomology, and an ad-hoc definition using the specific features of our
Two algebraic groups $G$ and $G'$ over $F$ are said to be forms of each other if $G_{F_{\text{sep}}} \cong G'_{F_{\text{sep}}}$. The forms $G$ and $G'$ are said to be equivalent if $G \cong G'$ (as algebraic groups over $F$). For an algebraic group $H$ over a field $k$, we let

$$H^1(k, H) := H^1(\text{Gal}_k, H(k_{\text{sep}}))$$

(14.16)

denote the usual Galois cohomology set. The set of equivalence classes of forms of $G$ is in bijection with $H^1(F, \text{Aut}(G))$. The so-called inner forms are those forms corresponding to the subgroup of inner automorphisms. This means that the corresponding cocycles are in the image of the map

$$H^1(F, G/Z_G) \rightarrow H^1(F, \text{Aut}(G)),$$

where $Z_G$ is the center of $G$. The pure inner forms are those corresponding to cocycles in the image of the map

$$H^1(F, G) \rightarrow H^1(F, G/Z_G) \rightarrow H^1(F, \text{Aut}(G)).$$

Two pure inner forms are said to be equivalent if the corresponding cocycles in $H^1(F, G)$ are equivalent. It turns out that the pure inner forms of $G_V$ are all of the form $G'_V$ for free $E$-modules $V'$ with $E$-rank equal to $V$. More specifically, if $E$ and $V$ are fixed, the side conditions on $V'$ that are equivalent to $G'_V$ being a pure inner form of $G_V$ are given in the following table (see [GGP12b, §2]):

<table>
<thead>
<tr>
<th>$E$</th>
<th>conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E = F$</td>
<td>$\text{rank}_E V' = \text{rank}_E V$ and $\text{disc}(V) = \text{disc}(V')$</td>
</tr>
<tr>
<td>$E = F \oplus F$</td>
<td>$\text{rank}_E V' = \text{rank}_E V$</td>
</tr>
<tr>
<td>$E/F$ quadratic</td>
<td>$V' \cong V$ as hermitian spaces</td>
</tr>
</tbody>
</table>

We state the following:

**Proposition 14.7.1** If $G$ and $G'$ are reductive groups over a global field $F$ and are inner forms of each other, then there exists a finite set $S$ of places of $F$ such that for $v \notin S$ one has an isomorphism $G_{F_v} \cong G'_{F_v}$.

**Proof.** There is a finite set $S$ of places of $F$ such that $G_{F_v}$ and $G'_{F_v}$ are quasi-split reductive for $v \notin S$ [Spr79, Lemma 4.9]. On the other hand, two inner forms of the same quasi-split reductive group are necessarily isomorphic [Mil17, Corollary 23.53].

Thus for $S$ as in the proposition, we can unambiguously identify isomorphism classes of representations of $G(\mathbb{A}_F)$ and $G'(\mathbb{A}_F)$. We say that automorphic
representations $\pi$ of $G(\mathbb{A}_F)$ and $\pi'$ of $G'(\mathbb{A}_F)$ are nearly equivalent if $\pi^S \cong \pi'^S$ for any finite set $S$ of places as above. By design, the notion of near equivalence is unchanged if we enlarge $S$.

Let $W < V$ be a subspace nondegenerate with respect to $\langle \cdot, \cdot \rangle$ such that $\text{rank}_E V/W = 1$. Then the restriction of $\langle \cdot, \cdot \rangle$ to $W$ is naturally an $E$-Hermitian form and we can form the group $G_W$ as above. Moreover one has a natural embedding $i : G_W \hookrightarrow G_V$ and hence an embedding of $G_W$ into

$$G := G_V \times G_W$$

given on points in an $F$-algebra $R$ by

$$G_W(R) \rightarrow G_V(R) \times G_W(R)$$
$$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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if and only if there is a pure inner form \((G_{W'}, G_{V'})\) of \((G_W, G_V)\) and a cuspidal automorphic representation \(\pi'\) of \(G'(\mathbb{A}_F)\) nearly equivalent to \(\pi\) that is \(H'\)-distinguished.

There is a local version of the conjecture which we omit. The import of the local version for the global Conjecture 14.7.2 is that it allows one to detect fairly explicitly which \(\pi'\) in the near equivalence class of \(\pi\) is distinguished. Stating this in a reasonable form seems to require one to collect representations on the adelic points of all the pure inner forms of \(G\) into a so-called Vogan \(L\)-packet. We prefer not to make this precise and instead refer the reader to the lucid explanation in [GGP12b]. Assuming certain expected properties of Vogan \(L\)-packets, the local Gan-Gross-Prasad conjecture has been proven for generic \(L\)-packets of representations on orthogonal groups over nonarchimedean local fields of characteristic zero by Mœglin and Waldspurger [MW12]. For tempered \(L\)-packets of representations of unitary groups over characteristic zero local fields (either archimedean or nonarchimedean), it has been proven by Beuzart-Plessis [BP16b].

We state now what is known about the global conjecture. When \(E = F \oplus F\), the groups \(G_V\) and \(G_W\) are general linear groups. In this case, the conjecture is true (see Exercise 14.9). In fact the general linear group admits only one isomorphism class of pure inner form. Thus in the statement of the conjecture we may take \(G = G'\) and \(H = H'\) in this case.

Under suitable assumptions on \(G\) (in particular that it is orthogonal), Ginzburg, Jiang and Rallis proved one implication, namely if some representation in the near equivalence class of \(\pi\) is distinguished, then the central \(L\)-value is nonzero [GJR05, Theorem A] (see also [GJR04]). The proof is elegant in its simplicity. It is a clever application of the theory of automorphic descent which explicitly constructs the inverse of the functorial lift from classical groups to the general linear group. It is this theory that underlies one proof of Theorem 13.7.4.

Currently the most definitive result on the global Gan-Gross-Prasad conjecture is in the unitary case:

**Theorem 14.7.3** Assume that \(E/F\) is a quadratic extension of number fields so that \(G\) is a product of unitary groups. Then Conjecture 14.7.2 is true.

The method of proof is quite beautiful and was suggested by Jacquet and Rallis [JR11]. One reduces the unitary case of the Gan-Gross-Prasad conjecture to the general linear case, which, as mentioned above, is known, using a comparison of relative trace formulas. The comparison relies on a local conjecture called the Jacquet-Rallis fundamental lemma that was proven by Z. Yun [Yun11]. The approach is an antecedent of Principle 14.6.2 enunciated in §14.6. It is interesting to note that, unlike in the settings considered in §14.5 and in Theorem 14.6.1, allowing pure inner forms is necessary for Theorem 14.7.3 to be true. It is expected that including pure inner forms will be necessary in proving cases of Principle 14.6.2.
14.8 Necessary conditions for distinction

Theorem 14.7.3 is the culmination of the work of quite a few mathematicians. After Z. Yun proved the Jacquet-Rallis fundamental lemma, W. Zhang established another important local result (the smooth transfer) in the nonarchimedean case, and then proved a slightly weaker version of Theorem 14.7.3 [Zha14]. Smooth transfer in the archimedean case was proved by H. Xue [Xue17]. Other necessary local results were completed by Bleuzart-Plessis [BP16a]. Finally, as with most trace formula comparisons, to obtain a complete result requires delicate arguments to deal with the fact that the geometric and spectral sides of trace formulae are rarely convergent. In the current setting, this work was undertaken by Chaudouard and Zydor. The theorem in the form stated above is [BPCZ20, Theorem 1.1.5.1] (see also [BPLZZ19]). Since the proof of Theorem 14.7.3 does not proceed via the twisted trace formula, it is not contingent on the stabilization of the twisted trace formula (see §13.8 for a discussion).

The work above does not generalize in an obvious way to the orthogonal case of Conjecture 14.7.2. However, an approach to this case has recently been proposed [Kri19].

We close this section by mentioning an exciting analogue of Conjecture 14.7.2 that (in certain cases) gives a formula for $L'(\frac{1}{2}, \pi, r_V \otimes r_W)$ when $L(\frac{1}{2}, \pi, r_V \otimes r_W) = 0$. The derivative can in some cases be related to heights of algebraic cycles, and the relationship represents an important link between representation theory and arithmetic geometry. A formula of this type was conjectured in [GGP12b] and is known as the arithmetic Gan-Gross-Prasad conjecture. It generalizes the famous Gross-Zagier formula [GZ86], which is a key input into the best known results on the Birch and Swinnerton-Dyer conjecture in the number field case. In the unitary setting, the arithmetic Gan-Gross-Prasad conjecture was essentially reduced to a local conjecture known as the arithmetic fundamental lemma by W. Zhang [Zha12b]. Z. Yun and W. Zhang together used related ideas to prove the best known results towards the Birch and Swinnerton-Dyer conjecture in the function field case [YZ17, YZ19].

14.8 Necessary conditions for distinction

Assume for the moment that $F$ is a global field. One should be aware that for general reductive groups $G$ and connected subgroups $H \subseteq G$, there do not exist cuspidal automorphic representations of $G(\mathbb{A}_F)$ that are $H$-distinguished. A somewhat trivial example is given by taking $H$ to be the unipotent radical of a parabolic subgroup. In this case there are no cuspidal automorphic representations of $G(\mathbb{A}_F)$ that are $H$-distinguished by the very definition of a cuspidal representation.

A more substantial result was obtained by Ash, Ginzburg, and Rallis in [AGR93]:

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Theorem 14.8.1 If \((G, H)\) is one of the following pairs of reductive groups over a number field \(F\), then there are no cuspidal automorphic representations of \(G(F)\) that are \(H\)-distinguished:

- \((\text{GL}_{n+k}, \text{SL}_n \times \text{SL}_k)\) for \(n \neq k\),
- \((\text{GL}_{2n}, \text{Sp}_{2n})\),
- \((\text{GL}_{2n+1}, \text{Sp}_{2n})\),
- \((\text{SO}(n, n), \text{SL}_n)\) for \(n\) odd,
- \((\text{Sp}_{2(n+k)}, \text{Sp}_{2n} \times \text{Sp}_{2k})\),
- \((\text{Sp}_{2n}, \text{Sp}_{2(n-\ell)})\) with \(4\ell < n\).

There is an additional case in [AGR93] which we omit since it involves cusp forms on (disconnected) orthogonal groups and we have not defined cusp forms on such groups. We also point out that the theorem is stated in the case \(F = \mathbb{Q}\) in [AGR93] but this is not used in the proofs. We refer to [AGR93] for the definition of the embeddings of \(H\) into \(G\).

Let \(F\) be a nonarchimedean local field of characteristic not equal to 2 and let \(G\) be a quasi-split reductive group over \(F\) equipped with an automorphism

\[ \sigma : G \rightarrow G \]

of order 2. Let

\[ H := \langle (G^\sigma)^{\text{der}} \rangle. \]

In this setting, Prasad [Pra19, Theorem 1] gave a beautifully simple criterion for there to exist generic representations of \(G(F)\) distinguished by \(H(F)\):

Theorem 14.8.2 If there is an irreducible admissible generic representation of \(G(F)\) distinguished by \(H(F)\), then there is a Borel subgroup \(B < G^\sigma\) such that \(B \cap \sigma(B)\) is a maximal torus of \(G^\sigma\).

One also has the following partial converse [Pra19, Proposition 11]:

Proposition 14.8.3 If there is a Borel subgroup \(B < G\) such that \(B \cap \sigma(B)\) is a maximal torus of \(G\), then there exists an irreducible admissible generic representation of \(G(F)\) distinguished by \(G^\sigma(F)\).

Exercises

14.1. Let \(H \leq \mathbb{R}^n\) be a closed real Lie subgroup of \(\mathbb{R}^n\) with finitely many connected components. Prove that \(H \cong \mathbb{R}^m\) for some \(m \leq n\).

14.2. Let \(T\) be a split torus over a characteristic zero field \(k\). Prove that any connected subgroup of \(T\) is a split torus.
14.3. Let $G$ be a reductive group over a local or global field $F$ and let $H, H' \leq G$ be subgroups that are $G(F)$-conjugate. If $F$ is local, prove that an irreducible admissible representation of $G(F)$ is $H$-distinguished if and only if it is $H'$-distinguished. If $F$ is global and $H$ is the direct product of a reductive and a unipotent group, prove that a cuspidal automorphic representation $\pi$ of $G(\mathbb{A}_F)$ is $H$-distinguished if and only if it is $H'$-distinguished.

14.4. Let $H$ be a reductive group over the nonarchimedean local field $F$, viewed as a subgroup of $G := H \times H$ via the diagonal embedding. Show that an irreducible admissible representation $\pi_1 \times \pi_2$ of $G(F)$ is $H$-distinguished if and only if $\pi_1 \cong \pi_2'$.

14.5. Let $H$ be a reductive group over the global field $F$ viewed as a subgroup of $G := H \times H$ via the diagonal embedding. Let $\pi_1$ and $\pi_2$ be cuspidal automorphic representations of $A_H \backslash H(\mathbb{A}_F)$ and let $\pi = \pi_1 \otimes \pi_2$ be their exterior tensor product, viewed as a cuspidal automorphic representation of $A_G \backslash G(\mathbb{A}_F)$. Show that $\pi_1 \otimes \pi_2$ is $H$-distinguished if and only if $\pi_1 \cong \pi_2'$.

14.6. Let $G$ be a reductive group over a global field $F$, let $H \leq G$ be a subgroup, and let $\chi : H(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$ be a quasi-character trivial on $H(F)$. Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $G(\mathbb{A}_F)$ that is $(H, \chi)$-distinguished. Prove that for all nonarchimedean places $v$, the representation $\pi_v$ is smoothly $(H, \chi_v)$-distinguished.

14.7. Give an example of a subgroup $H \leq \text{GL}_2 \times \text{GL}_2$ over a global field $F$, a quasi-character $\chi : H(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$, and a smooth function $\varphi$ in a cuspidal automorphic representation of $(A_{\text{GL}_2} \backslash \text{GL}_2(\mathbb{A}_F))^2$ such that the integral

$$\int_{A_G \backslash H(F) \backslash A_G H(\mathbb{A}_F)} |\varphi(h)\chi(h)| dh$$

diverges.


14.9. Prove the Gan-Gross-Prasad conjecture when $E = F \oplus F$ (and hence $G_V$ and $G_W$ are general linear groups) using Rankin-Selberg theory.
Chapter 15
The Cohomology of Locally Symmetric Spaces

We are in a forest whose trees will not fall with a few timid hatchet blows. We have to take up the double-bitted axe and the cross-cut saw, and hope that our muscles are equal to them.

R. P. Langlands

Abstract We discuss how automorphic representation theory controls the cohomology of arithmetic locally symmetric spaces.

15.1 Introduction

A fruitful application of the theory of automorphic representations is to give the decomposition of the cohomology of locally symmetric spaces as a module under certain algebras of correspondences. We survey this application in this chapter. The primary technical tool one uses to make the connection is $(\mathfrak{g}, K_\infty)$-cohomology. The definition is given in §15.4.

One case that is particularly important is when the locally symmetric space is the $\mathbb{C}$-points of a quasi-projective scheme over a number field. Shimura was the first to study this case systematically, and in his honor, such varieties are called Shimura varieties. We state Deligne’s reformulation of the notion of a Shimura variety at the end of this chapter. Part of the reason that this case is so important is that it is one of the few cases where there is a direct relationship between automorphic representations and Galois representations. We point the reader to some of the vast body of work that has been done elucidating this relationship in §15.8.

Our real motivation for this chapter, however, is to explicitly discuss the relationship between distinguished representations and cycles on locally sym-
metric spaces. This was in fact the original motivation for Harder, Langlands, and Rapoport’s introduction of the notion of a distinguished automorphic representation \[HLR86\]. In this paper, they related period integrals of automorphic representations to algebraic cycles on Hilbert modular varieties. Using this relation, they proved cases of the Tate conjecture for these varieties. There has been work generalizing their results, but even very basic questions remain. We point out one such question explicitly in §15.6. We hope that in the near future, the knowledge one now has about the étale cohomology of Shimura varieties can be combined with the relative trace formula to systematically study the Tate and Beilinson-Bloch conjectures for these varieties. It is beyond the scope of this book to make this completely precise, but we mention some promising results that have been obtained in §15.8.

Understanding the material in this chapter requires some knowledge of sheaves and cohomology. In §15.8 we even briefly discuss étale cohomology of Shimura varieties. The results will not be used in the remainder of the book, so the reader should feel free to skip this chapter on a first reading. However, we encourage the reader to at some point investigate the fascinating interplay between arithmetic geometry and automorphic forms that is the subject of Langlands’ quote in the epigraph.

15.2 Locally symmetric spaces

Unless otherwise specified, throughout this chapter \(G\) is a reductive group over \(\mathbb{Q}\). We let \(K_\infty \leq G(\mathbb{R})\) be a compact subgroup containing the maximal connected compact subgroup (in the real topology) and, as before, let

\[A_G \leq G(\mathbb{R})\]

be the identity component in the real topology of the maximal \(\mathbb{Q}\)-split torus in the center of \(G\). To ease notation, we write \(K\) for \(K_\infty \leq G(\mathbb{A}_\mathbb{Q}^\infty)\), a compact open subgroup. Finally we set

\[X := X_G := A_G \backslash G(\mathbb{R})/K_\infty.\]

Any connected component of \(X\) is a symmetric space. One can essentially take this to be the definition of a symmetric space, although it is not the most natural definition. We let

\[\text{Sh}^K := \text{Sh}(G,X)^K := G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_\mathbb{Q}^\infty))/K\]

(15.1)

and refer to it as a Shimura manifold. As in explained (2.20), this is a finite union of locally symmetric spaces. Indeed, if we take a set of representatives \((g_l)_{l \in I}\) for the finite set \(G(\mathbb{Q}) \backslash G(\mathbb{A}_\mathbb{Q}^\infty)/K\), then
\[
\prod_{i \in I} \Gamma_i \backslash X \rightarrow \text{Sh}^K
\]
(15.2)

where \( \Gamma_i = g_i K g_i^{-1} \cap G(\mathbb{Q}) \).

To relate this setting to the traditional framework of locally symmetric spaces, it is useful to have in mind the notion of arithmetic and congruence subgroups. These were already mentioned in Definition 2.10 and in §2.6, but we recall the definition for the convenience of the reader:

**Definition 15.1.** A subgroup \( \Gamma \leq G(\mathbb{Q}) \) is **arithmetic** if there is some faithful representation \( \rho : G \rightarrow \text{GL}_n \) such that \( \rho(\Gamma) \) is commensurable with \( \rho(G(\mathbb{Q})) \cap \text{GL}_n(\mathbb{Z}) \). It is **congruence** if it is of the form \( G(\mathbb{Q}) \cap K \) for some compact open subgroup \( K \leq G(\mathbb{A}_\mathbb{Q}) \).

Congruence subgroups are arithmetic (see Exercise 15.3). The famous congruence subgroup problem asks whether every arithmetic subgroup is congruence. The answer is in general no, but the failure of arithmetic subgroups to be congruence can be controlled via techniques that originated in the influential paper \[BMS67\] (see also \[Men65\]). In particular, every arithmetic subgroup of \( \text{SL}_n\mathbb{Q} \) for \( n \geq 3 \) and \( \text{Sp}_{2n}\mathbb{Q} \) for \( n \geq 2 \) is congruence (see loc. cit.). The survey by Prasad and Rapinchuk in \[Mil10\] is a source for more up to date information.

A useful assumption on a subgroup \( \Gamma \leq G(\mathbb{Q}) \) is that it is torsion free. This implies, for example, that

\[
\Gamma \backslash X
\]
(15.3)
is a manifold, as opposed to an orbifold (see Exercise 15.2). Unfortunately, the property of being torsion free is not preserved under certain constructions, such as intersecting with a parabolic subgroup and mapping to the Levi quotient. A more robust condition is that of being neat:

**Definition 15.2.** An element \( g \in \text{GL}_n(\mathbb{Q}) \) is **neat** if the subgroup of \( \mathbb{Q}^\times \) generated by its eigenvalues is torsion free. An arithmetic subgroup \( \Gamma \leq G(\mathbb{Q}) \) is **neat** if given any (equivalently, one faithful) representation \( \rho : G \rightarrow \text{GL}_n \), \( \rho(g) \) is neat for all \( g \in \Gamma \).

Clearly if \( \Gamma \) is neat then \( \Gamma \) is torsion free. Moreover, if \( \Gamma \) is neat, then all its subgroups and homomorphic images are neat.

**Lemma 15.2.1** Any congruence subgroup \( \Gamma \leq G(\mathbb{Q}) \) contains a congruence subgroup of finite index that is neat.

**Proof.** When \( G = \text{GL}_n \), we can take the neat subgroup to be of the form

\[
\{ g \in \text{GL}_n(\mathbb{Z}) : g \equiv I \pmod{N} \}
\]
for a sufficiently divisible integer \( N \). The general case follows from this. \( \Box \)
**Definition 15.3.** A compact open subgroup $K \leq G(\mathbb{A}_Q^\infty)$ is **neat** if $G(\mathbb{Q}) \cap K$ is neat.

An elaboration of the proof of Lemma 15.2.1 implies the following (see Exercise 15.4):

**Lemma 15.2.2** A compact open subgroup $K < G(\mathbb{A}_Q^\infty)$ contains a neat subgroup of finite index. If $K$ is neat then so is $gKg^{-1}$ for all $g \in G(\mathbb{A}_Q^\infty)$. □

Our motivation for introducing this notion is the following:

**Lemma 15.2.3** If $K$ is neat then $\text{Sh}^K$ is a smooth manifold.

**Proof.** Since $K$ is neat, $gKg^{-1} \cap G(\mathbb{Q})$ is torsion free for all $g \in G(\mathbb{A}_Q^\infty)$ by Lemma 15.2.2. It follows that the $\Gamma_i$ occurring in the decomposition (15.2) are all torsion free. As mentioned above, this implies that the $\Gamma_i \backslash X$ are all manifolds. □

In §5.2 we discussed the Hecke algebra $C_c^\infty(G(\mathbb{A}_Q^\infty) \backslash K)$. We recall that this is the space of $\mathbb{C}$-linear combinations of characteristic functions of double cosets $KgK$ for $g \in G(\mathbb{A}_Q^\infty)$. We now explain how these operators can be realized geometrically as correspondences. Let $K$ and $K'$ be compact open subgroups of $G(\mathbb{A}_Q^\infty)$, and $g \in G(\mathbb{A}_Q^\infty)$ be such that $K' < gKg^{-1}$. Then we have a map

$$T_g : \text{Sh}^{K'} \longrightarrow \text{Sh}^{K}$$

$$G(\mathbb{Q})(x, hK') \longrightarrow G(\mathbb{Q})(x, hgK).$$

(15.4)

These are finite étale maps if $K$ and $K'$ are neat.

The geometric realization of the characteristic function of $KgK$ is the correspondence

$$\xymatrix{ \text{Sh}^{K \cap gKg^{-1}} & \text{Sh}^{K} \ar[l]_{T_I} \ar[r]^{T_g} & \text{Sh}^{K}. }$$

(15.5)

It acts on functions via pullback $T_{I^*}^*$ along $T_I$ followed by pushforward $T_{g^*}$ along $T_g$. In other words, if we set $K' := K \cap gKg^{-1}$ then for functions $\varphi$ on $\text{Sh}^{K}$ and $\varphi'$ on $\text{Sh}^{K'}$ we have that

$$T_I^*(\varphi)(x, hK') : = \varphi(x, hK),$$

$$T_{g^*}(\varphi')(x, hK) : = \sum_{k' \in K/K'} \varphi'(x, hk'gK').$$

Set
Then for any function \( \varphi : \text{Sh}^K \to \mathbb{C} \),

\[
T(g)\varphi = R(1_{KgK})\varphi,
\]

where

\[
R(1_{KgK})\varphi(x) := \int_{G(\mathbb{A}^\infty)} 1_{KgK}(g)\varphi(xg)dg
\]

with the Haar measure \( dg \) normalized so that \( dg(K) = 1 \).

### 15.3 Local systems

A well-known classical fact is that if \( \Gamma \) is a congruence subgroup of \( \text{GL}_2\mathbb{Q} \) contained in the subgroup of matrices with positive determinant and \( \mathfrak{H} \) is the upper half-plane, then the cohomology group

\[
H^1(\Gamma \backslash \mathfrak{H}, \mathbb{C})
\]

can be decomposed as a direct sum of three summands, one isomorphic to the space of weight 2 cusp forms on \( \Gamma \) namely \( S_2(\Gamma) \) (see §6.7 for the definition), one isomorphic to the space

\[
\{ z \mapsto f(\bar{z}) : f \in S_2(\Gamma) \}
\]

of antiholomorphic forms, and one isomorphic to a certain space of Eisenstein series. In order to give a geometric interpretation of modular forms of weight bigger than 2, it is necessary to introduce certain local systems. We now explain their construction. Our treatment is largely modeled on that of the thesis of S. Morel [Mor08].

Until further notice, \( \Gamma \leq G(\mathbb{Q}) \) is a neat arithmetic subgroup. Let \( V_0 \) be a representation of the algebraic group \( G \), let \( k \) be a characteristic zero field and let

\[
V := V_0(k).
\]

It is a representation of \( G(k) \) and hence \( G(\mathbb{Q}) \) by restriction. We equip it with the discrete topology and form the quotient

\[
V^\Gamma := \Gamma \backslash (V \times X)
\]

by the diagonal action of \( \Gamma \) on the product. We say that the diagram given by the natural projection
is a local system. This is the total space of a locally constant sheaf of vector spaces. For each open set \( U \subseteq \Gamma \backslash X \), we let

\[
\mathcal{V}^f|_U := \{ \text{locally constant sections } s : U \to \mathcal{V}^f \}
\]  

be the abelian group of sections of the map (15.10). Then the functor

\[
\{ \text{open } U \subseteq \Gamma \backslash X \} \to \mathsf{Ab}
\]

\[
U \mapsto \mathcal{V}^f|_U
\]

from the category of open sets of \( \Gamma \backslash X \) with morphisms given by inclusions to the category of abelian groups \( \mathsf{Ab} \) is a sheaf. It is called the sheaf of sections of \( \mathcal{V}^f \). Its sheaf cohomology is denoted

\[
H^\bullet(\Gamma \backslash X, \mathcal{V}^f).
\]  

We now turn to the adelic setting. Let \( \mathcal{V} \) be as above. For \( K \subseteq G(\mathbb{A}_\mathbb{Q}^\infty) \) a compact open neat subgroup, define

\[
\mathcal{V}^K := G(\mathbb{Q}) \backslash (V \times X \times G(\mathbb{A}_\mathbb{Q}^\infty)/K),
\]

where for \( \gamma \in G(\mathbb{Q}) \),

\[
\gamma(v, x, gK) = (\gamma.v, \gamma.x, \gamma gK).
\]  

We thus have a natural map \( \mathcal{V}^K \to \mathsf{Sh}^K \). This is again a local system and we define the associated sheaf of sections as above.

The relationship between the two constructions given above can be described as follows. Fix \( g \in G(\mathbb{A}_\mathbb{Q}^\infty) \) and let \( \Gamma = gKg^{-1} \cap G(\mathbb{Q}) \). We then have an embedding

\[
\iota : \Gamma \backslash X \to \mathsf{Sh}^K
\]

\[
\Gamma x \mapsto G(\mathbb{Q})(x,g)K.
\]

One has that \( \iota^{-1}\mathcal{V}^K \cong \mathcal{V}^\Gamma \) where \( \iota^{-1}\mathcal{V}^K \) is the inverse image sheaf.

We now describe how to associate to each \( g \in G(\mathbb{A}_\mathbb{Q}^\infty) \) a homomorphism

\[
T(g) : H^\bullet(\mathsf{Sh}^K, \mathcal{V}^K) \to H^\bullet(\mathsf{Sh}^K, \mathcal{V}^K).
\]  

We first note that for any neat compact open subgroups \( K, K' \subseteq G(\mathbb{A}_\mathbb{Q}^\infty) \) with \( K' \leq gKg^{-1} \), one has an isomorphism of sheaves

\[
\theta : \mathcal{V}^{K'} \xrightarrow{\cong} T^{-1}_g \mathcal{V}^K = \mathcal{V}^K \times_{\mathsf{Sh}^K} \mathsf{Sh}^{K'}
\]

\[
(v, (x, hK')) \mapsto ((v, (x, hgK)), (x, hK')).
\]  

(15.17)
Here $T^{-1}V^K$ is the inverse image sheaf and the object on the right is the fiber product over the canonical projection $V^K \to \text{Sh}^K$ and the morphism $T_g : \text{Sh}^{K'} \to \text{Sh}^K$. The isomorphism $\theta$ is called a **lift of the correspondence**. This isomorphism canonically induces the morphism \((15.16)\). Concretely the procedure is to take a cocycle on $\text{Sh}^K$ with values in $V^K$, pull it back to $\text{Sh}^{K'}$ and then sum over the fibers of $T^{-1}V^K \to V^K$ to obtain a cocycle on $\text{Sh}^K$.

Alternately, one can work at the level of sheaves as follows. By generalities on sheaf cohomology [Har11, (4.33)], one has a homomorphism

$$H^\bullet(\text{Sh}^K, V^K) \to H^\bullet(\text{Sh}^{K'}, T^{-1}V^K) = H^\bullet(\text{Sh}^{K'}, V^{K'}). \quad (15.18)$$

Here $I$ is the identity in $G(\mathbb{A}_Q^\infty)$ and the equality is a consequence of the identification $T^{-1}V^K = V^{K'}$. The isomorphism $\theta$ induces an isomorphism

$$\theta : H^\bullet(\text{Sh}^{K'}, V^{K'}) \to H^\bullet(\text{Sh}^{K'}, T^{-1}V^K). \quad (15.19)$$

To finish, we construct a “trace” homomorphism

$$H^\bullet(\text{Sh}^{K'}, T^{-1}V^K) \to H^\bullet(\text{Sh}^K, V^K). \quad (15.20)$$

Composing \((15.18)\), \((15.19)\), and \((15.20)\) then gives

$$T(g) : H^\bullet(\text{Sh}^K, V^K) \to H^\bullet(\text{Sh}^K, V^K). \quad (15.21)$$

To construct \((15.20)\), we follow [Ive86, §VII.4], to which we refer for more details. We note that the assertions below all heavily depend on the fact that $T_g$ is a covering map. For any sheaf $F$ of abelian groups on $\text{Sh}^K$ and any $y \in \text{Sh}^K$, one has

$$(T_gT^{-1}F)_y = \bigoplus_{x \in T_g^{-1}(y)} F_y.$$ 

One can check that there is a unique morphism of sheaves

$$\text{tr} : T_gT^{-1}F \to F$$

given on the fiber over $y$ by taking the sum. Now let

$$V^K \to I^\bullet$$

be an injective resolution of $V^K$. We then obtain a morphism

$$\text{tr} : T_gT^{-1}I^\bullet \to I^\bullet. \quad (15.22)$$

We apply the “cohomology of global sections” functor to obtain a map

$$H^\bullet(I^\bullet(\text{Sh}^K, T_gT^{-1}I^\bullet)) \to H^\bullet(\text{Sh}^K, V^K).$$
But $T_g^{-1}$ is exact and transforms injectives into injectives. Hence

$$H^\bullet \Gamma(\text{Sh}^K, T_g T_g^{-1} I^*) = H^\bullet (\text{Sh}^{K'}, T_g^{-1} V^K).$$

### 15.4 $(\mathfrak{g}, K_\infty)$-cohomology

A fundamental tool for describing $H^\bullet (\text{Sh}^K, V^K)$ as a Hecke module is $(\mathfrak{g}, K_\infty)$-cohomology. We give its definition in this section. The basic reference is [BW00].

Let $\mathfrak{g} \geq \mathfrak{k}$ be the Lie algebras of $G_\mathbb{R}$ and $K_\infty$ respectively. Let $\mathcal{A}$ be a $\mathfrak{g}$-module (not necessarily of finite dimension). For $q \in \mathbb{Z}_{\geq 0}$, let

$$C^q(\mathfrak{g}; \mathfrak{k}; \mathcal{A}) = \text{Hom}_k(\wedge^q (\mathfrak{g} \triangleleft \mathfrak{k}), \mathcal{A}).$$

Here the action of $\mathfrak{k}$ on $\wedge^q (\mathfrak{g} \triangleleft \mathfrak{k})$ is the adjoint action. More precisely, $\text{Hom}_k(\wedge^q (\mathfrak{g} \triangleleft \mathfrak{k}), \mathcal{A})$ is the subspace of $\text{Hom}_k(\wedge^q (\mathfrak{g} \triangleleft \mathfrak{k}), \mathcal{A})$ consisting of functions $f$ satisfying

$$\sum_{i=1}^q f(x_1, \ldots, [x, x_i], \ldots, x_q) = x.f(x_1, \ldots, x_q)$$

for $x \in \mathfrak{k}$, where $[\ , \ ]$ is the Lie bracket of $\mathfrak{g}$. We define a differential $d : C^q(\mathfrak{g}; \mathfrak{k}; \mathcal{A}) \longrightarrow C^{q+1}(\mathfrak{g}; \mathfrak{k}; \mathcal{A})$ on the complex by

$$(df)(x_0, \ldots, x_q) = \sum_{i=1}^q (-1)^i x_i \cdot f(x_0, \ldots, \hat{x}_i, \ldots, x_q)$$

$$+ \sum_{1 \leq i < j \leq q} (-1)^{i+j} f([x_i, x_j], x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_q).$$

Here a hat denotes an omitted argument. One checks that this is well-defined (i.e. independent of the choice of representative of the cosets $x_i + \mathfrak{k}$) and that $d \circ d = 0$. We then let $H^\bullet(\mathfrak{g}; \mathfrak{k}; \mathcal{A})$ denote the cohomology of this complex:

$$H^q(\mathfrak{g}; \mathfrak{k}; \mathcal{A}) := H^q(C^\bullet(\mathfrak{g}; \mathfrak{k}; \mathcal{A}), d) := \frac{\ker d : C^q(\mathfrak{g}; \mathfrak{k}; \mathcal{A}) \longrightarrow C^{q+1}(\mathfrak{g}; \mathfrak{k}; \mathcal{A})}{\text{im} d : C^{q-1}(\mathfrak{g}; \mathfrak{k}; \mathcal{A}) \longrightarrow C^q(\mathfrak{g}; \mathfrak{k}; \mathcal{A})}.$$
15.5 The cohomology of Shimura manifolds

For this action, define
\[ C^\bullet(g, K_\infty; A) = C^\bullet(g, \mathfrak{k}; A)^{K_\infty}. \]
Denote by \( H^\bullet(g, K_\infty; A) \) the cohomology of this complex.

**Definition 15.4.** The \((g, K_\infty)\)-cohomology of the \((g, K_\infty)\)-module \( A \) is \( H^\bullet(g, K_\infty; A) \).

**15.5 The cohomology of Shimura manifolds**

Until otherwise stated we assume that \( V \) is a complex vector space (that is, \( k = \mathbb{C} \) in (15.8)). We now relate \((g, K_\infty)\)-cohomology and the cohomology of \( \text{Sh}^K \) with coefficients in \( \mathcal{V}^K \). We refer to [BW00, §VII.2] for proofs.

Let \( m := \dim_{\mathbb{R}} X \) for \( X = A_G \backslash G(\mathbb{R})/K_\infty \). Fix a basis \( \omega^i, 1 \leq i \leq m \), of left \( G(\mathbb{R}) \)-invariant 1-forms on \( X \). For \( I = \{i_1, \ldots, i_q\} \subseteq \{1, \ldots, m\} \) with \( i_j < i_n \) for \( j < n \), set
\[ \omega^I := \omega^{i_1} \wedge \cdots \wedge \omega^{i_q}. \]
Let
\[ A^K := C^\infty(G(\mathbb{Q}) \backslash G(\mathbb{A}_G)/K). \]
Denote by
\[ A^q(\text{Sh}^K, \mathcal{V}^K) \]
the space of differential \( q \)-forms on \( \text{Sh}^K \) with coefficients in \( \mathcal{V}^K \). Any element of \( A^q(\text{Sh}^K, \mathcal{V}^K) \) can be written as
\[ \eta = \sum_I f_I \omega^I \]
with \( f_I \in (A \otimes V)^{A_G} \). Let
\[ a_G := \text{Lie } A_G. \quad (15.24) \]

The discussion above yields an identification
\[ A^\bullet(\text{Sh}^K, \mathcal{V}^K) = C^\bullet(a_G \backslash g, K_\infty; (A^K \otimes V)^{A_G}) \quad (15.25) \]
commuting with the differentials, which in turn yields an isomorphism
\[ H^\bullet(\text{Sh}^K, \mathcal{V}^K) \cong H^\bullet(a_G \backslash g, K_\infty; (A^K \otimes V)^{A_G}). \quad (15.26) \]
This gives an explicit link between cohomology and automorphic representations. This isomorphism is Hecke equivariant in the sense that for \( g \in G(\mathcal{A}_\mathbb{Q}) \), the action of the correspondence \( T(g) \) is intertwined with the action of \( \mathbb{1}_{K\mathcal{O}_K} \).

We leave the proof of this Hecke equivariance assertion to the reader as Exercise 15.7.

This whole construction motivates the following definition:

**Definition 15.5.** A vector \( \varphi \in \mathcal{A}_K \) is **cohomological** if there is a \( v \in V \) and \( \omega^I \) on \( X \) such that

\[
\varphi \omega^I \otimes v \in C^*(a_G \backslash K; (\mathcal{A}_K \otimes V)^{A_G})
\]
defines a nonzero class in \( H^*(a_G \backslash g, K^\infty; (\mathcal{A}_K \otimes V)^{A_G}) \).

If we wish to specify the representation \( V \) we say that \( \varphi \) is \( V \)-**cohomological**.

Now \( \mathcal{A} \) is naturally a \( (\mathcal{A}_K \otimes V)^{A_G} \) module. Let us decompose it under the action of \( G(\mathcal{A}_\mathbb{Q}) \).

There is a quasi-character \( \xi : A_G \rightarrow \mathbb{R}_{>0} \) such that the restriction of \( V \) to \( A_G \) is \( \xi^{-1} \). Let

\[
\mathcal{A}(\xi)^K := \{ \varphi \in \mathcal{A}_K : \varphi(ag) = \xi(a)\varphi(g) \text{ for } (a, g) \in A_G \times G(\mathcal{A}_\mathbb{Q}) \}.
\]

Then

\[
(\mathcal{A}_K \otimes V)^{A_G} = \mathcal{A}(\xi)^K \otimes V.
\]

As usual \([G] := A_G G(\mathbb{Q}) \backslash G(\mathcal{A}_\mathbb{Q}) \). There is a \( G(\mathcal{A}_\mathbb{Q})^1 \)-intertwining map

\[
\mathcal{A}(\xi)^K \rightarrow C^\infty([G])^K
\]

\[
\varphi \mapsto (g \mapsto \xi'(g)\varphi(g))
\]

(15.27)

for an appropriate quasi-character \( \xi' : G(\mathcal{A}_\mathbb{Q}) \rightarrow \mathbb{R}_{>0} \). We let

\[
\mathcal{A}(\xi)_{\text{cusp}}^K
\]

be the inverse image of \( L^2_{\text{cusp}}([G])^K \cap C^\infty([G])^K \) under the map (15.27).

By Corollary 9.1.2,

\[
\mathcal{A}(\xi)_{\text{cusp}}^K = \bigoplus_{\pi} (\pi^K)^{m(\pi)}
\]

where the sum is over isomorphism classes of cuspidal automorphic representations \( \pi \) of \( G(\mathcal{A}_\mathbb{Q}) \) and \( m(\pi) \) is the multiplicity of \( \pi \) in \( \mathcal{A}(\xi)_{\text{cusp}}^K \). Hence

\[
H^*(\text{Sh}^K, \mathcal{V}^K) > H^*_{\text{cusp}}(\text{Sh}^K, \mathcal{V}^K) := H^*(a_G \backslash g, K^\infty; \mathcal{A}(\xi)_{\text{cusp}}^K \otimes V) \quad (15.28)
\]

\[
= \bigoplus_{\pi} H^* (a_G \backslash g, K^\infty; \pi^K \otimes V)^{m(\pi)}.
\]
The group $H^\bullet_{cusp}(\text{Sh}^K, \mathcal{V}^K)$ is known as the \textbf{cuspidal cohomology}. Its complement is described in terms of so called \textbf{Eisenstein cohomology}; for some information about the decomposition of the whole of the cohomology, see [BLS96]. It follows that, as a module under $C_c^\infty(G(\mathbb{A}_Q) \backslash K)$, one has

$$H^\bullet_{cusp}(\text{Sh}^K, \mathcal{V}^K) = \bigoplus_{\pi} \left( H^\bullet(\mathfrak{a}_G \backslash \mathfrak{g}, K_\infty; (\pi_\infty \otimes V) \otimes (\pi_\infty^\infty)^K)^{m(\pi)} \right). \quad (15.29)$$

This decomposition is sometimes called \textbf{Matsushima’s formula}. For a cuspidal automorphic representation $\pi$ of $G(\mathbb{A}_Q)$ we refer to

$$H^\bullet_{cusp}(\text{Sh}^K, \mathcal{V}^K)(\pi_\infty) := \bigoplus_{\pi': \pi_\infty \cong \pi_\infty^\infty} \left( H^\bullet(\mathfrak{a}_G \backslash \mathfrak{g}, K_{\infty}; (\pi_\infty' \otimes V) \otimes (\pi_\infty^\infty)^K)^{m(\pi')} \right)$$

as the $\pi_\infty$-isotypic component of the cohomology.

\textbf{Definition 15.6.} A cuspidal automorphic representation $\pi$ of $G(\mathbb{A}_Q)$ is \textbf{cohomological} if there is a representation $V$ of as above such that

$$H^\bullet(\mathfrak{a}_G \backslash \mathfrak{g}, K_{\infty}; \pi_\infty \otimes V) \neq 0.$$

A vector $\varphi$ in the space of $\pi$ is \textbf{cohomological} if there exists an embedding from the space of $\pi$ into $L^2_{cusp}([G])$ such that the image of $\varphi$ is cohomological.

Thus if there is a vector in the space of $\pi$ that is cohomological, then $\pi$ itself is cohomological. If we wish to be more specific, we could speak of $V$-cohomological representations or $V$-cohomological vectors.

The condition that $\pi$ is cohomological is very restrictive. An easy way to quantify this is given by the following theorem [BW00, Corollary I.4.2]:

\textbf{Theorem 15.5.1} \textit{If $H^\bullet(\mathfrak{a}_G \backslash \mathfrak{g}, K_{\infty}; \pi_\infty \otimes V) \neq 0$ then the infinitesimal character of $\pi_\infty^\vee$ is equal to the infinitesimal character of $V$.} \hfill $\square$

A more or less equivalent way of phrasing this assertion is the following corollary:

\textbf{Corollary 15.5.2} \textit{A cohomological cuspidal representation $\pi$ of $G(\mathbb{A}_Q)$ is $C$-algebraic in the sense of Definition 12.7.}

\textit{Proof.} This follows from Theorem 15.5.1 and Lemma 12.8.1. \hfill $\square$

The link between geometry and representation theory given by Matsushima’s formula is key to a great deal of beautiful mathematics. It is of interest to see if this link can be generalized. There is an important generalization involving cohomology with coefficients in coherent sheaves in the special case where the symmetric space $X$ is hermitian [Har90, BHR94]. For applications to Galois representations, see [PS16a, GK19], and for vanishing results see [Lan16]. In this theory the space under consideration does not change. It is always the Shimura manifold $\text{Sh}^K$. 
We would like to point out two possible generalizations where the space
does change. First, one could consider period domains. From an adelic
perspective, these are spaces of the form

\[ G(\mathbb{Q}) \backslash (A_G \backslash G(\mathbb{R})) / K_{\infty} \times G(\mathbb{A}_Q^\infty) / K \]  

(15.30)

where \( G \) is a reductive group over \( \mathbb{Q} \) and \( K_{\infty} < G(\mathbb{R}) \) is a compact subgroup
satisfying certain assumptions. In particular \( K_{\infty} \) is not necessarily a maxi-
mal compact subgroup. A possible reference for period domains is [CMS17].
Carayol has obtained some suggestive results in this setting [Car05, Car15].
Second, one could consider the pseudo-Riemannian locally symmetric spaces
investigated by Kassel and Kobayashi [KK20]. They can also be brought into
the form (15.30) for suitable compact (but not necessarily maximal compact)
\( K_{\infty} \) [KK20, (5.1)]. In either case it would be interesting to see if some ana-
logue of Matsushima’s formula holds, possibly employing a generalization of
\( \langle g, K_{\infty} \rangle \)-cohomology.

15.6 The relation to distinction

Usually in the literature, one finds references to cohomological representations
but no references to cohomological vectors. Despite this, the notion of a
cohomological vector is of great importance.

Suppose that \( H \leq G \) are reductive \( \mathbb{Q} \)-groups. We assume for simplicity
that the corresponding symmetric spaces can be chosen so that the inclusion
of \( H(\mathbb{R}) \) into \( G(\mathbb{R}) \) induces an injection

\[ X_H \hookrightarrow X := X_G. \]

Then there is an embedding

\[ \text{Sh}(H, X_H)^{K \cap H(\mathbb{A}_Q^\infty)} \hookrightarrow \text{Sh}(G, X)^K =: \text{Sh}^K. \]

For the remainder of this section, we assume for simplicity that \( \text{Sh}^K \) is com-
 pact, which is to say that \( G^{\text{der}} \) has no nontrivial \( \mathbb{Q} \)-split subtorus by Theorem
2.6.3. This implies that every automorphic representation of \( G(\mathbb{A}_Q) \) is cuspi-
dal, and hence

\[ H^\bullet_{\text{cusp}}(\text{Sh}^K, \mathcal{Y}^K) = H^\bullet(\text{Sh}^K, \mathcal{Y}^K). \]

We also assume \( \text{Sh}(H, X_H)^{K \cap H(\mathbb{A}_Q^\infty)} \) and \( \text{Sh}^K \) are orientable.

Let \( V \) be a representation as in (15.8) and let \( V^\vee \) be the dual representa-
tion. Let

\[ (V^\vee)^K := G(\mathbb{Q}) \backslash (V^\vee \times X \times G(\mathbb{A}_Q^\infty) / K) \]  

(15.31)
be the corresponding local system. We denote by $V_H$ the local system on $\text{Sh}(H, X_H)^{K \cap H(\mathbb{A}^\infty)}$ attached to $V|_{H(\mathbb{Q})}$. Let $n$ be the real dimension of $\text{Sh}^K$ and let $q$ be the real codimension of $\text{Sh}(H, X_H)^{K \cap H(\mathbb{A}^\infty)}$ in $\text{Sh}^K$. There is a cycle class map

$$H^0(\text{Sh}(H, X_H)^{K \cap H(\mathbb{A}^\infty)}, V_H) \to H^q(\text{Sh}^K, V^K) \quad (15.32)$$

defined as follows. For a class $[Z] \in H^0(\text{Sh}(H, X_H)^{K \cap H(\mathbb{A}^\infty)}, V_H)$, we have a linear functional

$$H^{n-q}(\text{Sh}^K, (V^K)_K) \to \mathbb{C}$$

$$\eta \mapsto \int_{[Z]} \eta.$$  \hspace{1cm} (15.33)

Poincaré duality [Har11, Theorem 4.8.9] furnishes a perfect pairing

$$H^{n-q}(\text{Sh}^K, (V^K)_K) \times H^q(\text{Sh}^K, V^K) \to \mathbb{C} \quad (15.34)$$

so $[Z]$ defines an element of $H^q(\text{Sh}^K, V^K)$, yielding the cycle class map (15.32).

Describing the $C_c^\infty(G(\mathbb{A}_\mathbb{Q}^\infty) \parallel K)$-span of the image of (15.32) is an important (and mostly open) problem. This is especially true in the situation where $\text{Sh}^K$ and $\text{Sh}(H, X_H)^{K \cap H(\mathbb{A}^\infty)}$ are Shimura varieties, not just manifolds, due to the connection with the Tate conjecture (see §15.8).

We now explain the precise connection between the image of (15.32) and distinguished representations and point out an important technical problem that deserves more study. By Matsushima’s formula (15.29), to describe the $C_c^\infty(G(\mathbb{A}_\mathbb{Q}^\infty) \parallel K)$-span of the image of (15.32), it suffices to describe its projection to

$$H^*(\text{Sh}^K, V^K)(\pi^\infty)$$

for all automorphic representations $\pi'$ of $G(\mathbb{A}_\mathbb{Q})$.

Unwinding the considerations of §15.5 and the definition of the cycle class map, we see that the projection of the image of (15.32) to $H^*(\text{Sh}^K, V^K)(\pi^\infty)$ is nonzero if and only if there is

- a (cuspidal) automorphic representation $\pi$ of $G(\mathbb{A}_\mathbb{Q})$ with $\pi^\infty \cong \pi^\infty K$,
- $(v^\vee, v) \in V^\vee \times H^0(\text{Sh}(H, X_H)^{K \cap H(\mathbb{A}^\infty)}, V_H)$,
- a $V^\vee$-cohomological vector $\varphi$ in the space of $\pi$ and an $\omega'$ on $X$ such that

$$\int_{[\text{Sh}(H, X_H)^{K \cap H(\mathbb{A}^\infty)}]} \varphi \omega' v^\vee (v) \neq 0.$$  \hspace{1cm} (15.35)

This motivates the following question:

**Question 15.6.1** If an automorphic representation $\pi$ of $G(\mathbb{A}_\mathbb{Q})$ is $H$-distinguished and $V^\vee$-cohomological, then is there
(v', v) \in V^\vee \times H^0(\text{Sh}(H, X_H)^{K\cap H(\Lambda_H^\infty)}, V_H),

an \omega^f on X, and a \nu^\vee\text{-cohomological vector } \varphi in the space of \pi such that (15.35) holds?

Recall that \pi is \ H\text{-distinguished if and only if }

\int_{A_G H(\mathbb{Q}) \backslash A_G H(\mathbb{A})} \varphi(h) dh \neq 0

for some smooth \varphi \in L^2(\pi), the \pi\text{-isotypic subspace of } L^2([G]). A priori, it could happen that no such \varphi satisfies the additional stipulations of the question. For example, it could be that no such \varphi is \nu^\vee\text{-cohomological.}

We do not know, even conjecturally, the answer to Question 15.6.1 in any degree of generality. In important special cases, the question above can be answered in the affirmative (see [KS15, Sun17, Sun19] for example).

In this section we have assumed that \text{Sh}^K is compact. The considerations above are of course still of interest in the noncompact case, but even defining the cycle class map (15.32) in any level of generality is complicated. Extending the cycle class map to the noncompact case is another open problem that has not received the scrutiny it deserves. We refer to [GG12] for an example where the difficulties caused by noncompactness of \text{Sh}^K are overcome in a very special case.

15.7 More on \((g, K_\infty)\)-cohomology

We have seen that the question of whether or not a given automorphic representation contributes to the cohomology of a Shimura manifold with coefficients in a local system is completely determined by the \((g, K_\infty)\)-cohomology of its factor at infinity. Fortunately \((g, K_\infty)\)-cohomology is a very pleasant object with which to work. In this section, we list some properties of these groups; the canonical reference is [BW00].

For \(i = 1, 2\), let \(G_i\) be a reductive group over \(\mathbb{R}\) and let \(K_i \leq G_i(\mathbb{R})\) be a maximal compact subgroup. Moreover let \(A_i\) be an admissible \((g_i, K_i)\)-module. Then one has a Küneth formula [BW00, §I.1.3]:

**Theorem 15.7.1 (Küneth formula)** For any \(n \in \mathbb{Z}_{\geq 0}\) one has natural isomorphisms

\[ H^n(g_1 \oplus g_2, K_1 \times K_2; A_1 \otimes A_2) = \bigoplus_{p+q=n} H^p(g_1, K_1; A_1) \otimes H^q(g_2, K_2; A_2). \]

For the statement of the next theorem, let \(G\) be a reductive group over \(\mathbb{Q}\) (we only use the \(\mathbb{Q}\)-structure to define \(A_G\)). Let \(g\) be the Lie algebra of \(G_\mathbb{R}\).
and let $K_\infty \leq G(\mathbb{R})$ be a maximal compact subgroup. Let $\mathcal{A}$ be an admissible $(a_G, K_\infty)$-module. The following is [BW00, Proposition 1.7.6]:

**Theorem 15.7.2 (Poincare duality)**

One has that

$$H^q(a_G \setminus g, K_\infty; \mathcal{A}) \cong H^{(\dim \mathbb{R}) - q}(a_G \setminus g, K_\infty; \mathcal{A}^\vee).$$

Combining theorems 15.7.1 and 15.7.2 we see that to compute the $(g, K_\infty)$-cohomology of a $(g, K_\infty)$-module, it suffices to understand the case where $g$ is simple over $\mathbb{R}$ and when $q \leq \dim \mathbb{R} X/2$.

We now consider the cohomology of admissible representations of the form

$$\pi_\infty \otimes V$$

where $\pi_\infty$ is an irreducible unitary $(g, K_\infty)$-module and $V$ is irreducible. Here when we say $\pi_\infty$ is unitary we mean that $\pi_\infty$ is the $(g, K_\infty)$-module underlying a unitary representation of $G(\mathbb{R})$. Such a unitary representation is uniquely determined up to isomorphism by Theorem 4.4.6. Let

$$\theta : g \rightarrow g$$

be the Cartan involution. Its fixed point subalgebra is $\mathfrak{k}$, the Lie algebra of $K_\infty$. Let $\mathfrak{p}$ be its $-1$ eigenspace.

When $\pi_\infty$ is trivial, we have the following proposition [BW00, Corollary II.3.2]:

**Proposition 15.7.3** Assume as above that $V$ is irreducible. One has that

$$H^\bullet(g, K_\infty; V) = \begin{cases} 
0 & \text{if } V \text{ is nontrivial}, \\
(\wedge \mathfrak{p})_{K_\infty} & \text{if } V \text{ is trivial}.
\end{cases}$$

We can interpret this proposition as saying that much of the cohomology vanishes when $V$ is nondegenerate. This phenomenon persists for other $\pi_\infty$. We now state a vanishing theorem for cohomology in this context. Combined with Poincaré duality, it implies that the cohomology of $\text{Sh}^K$ is concentrated around the degree equal to $\frac{1}{2} \dim \text{Sh}^K$.

For each $\theta$-stable parabolic subalgebra $q \leq g \otimes \mathbb{C}$, let $\mathfrak{n}(q)$ be its nilradical. Let

$$\mathcal{P}(V) := \{\text{$\theta$-stable parabolic subalgebras } q : \dim V^\mathfrak{n}(q) = 1\}$$

and let

$$c(V) := \min\{\dim(\mathfrak{n}(q) \cap \mathfrak{p}) : q \in \mathcal{P}(V)\}.$$}

The following is [BW00, Theorem II.1.10]:
Theorem 15.7.4 Assume as above that $\pi_{\infty}$ is an irreducible unitary $(\mathfrak{g}, K_{\infty})$-module and that $V$ is irreducible. If $\mathfrak{g}$ is semisimple and the kernel of $\pi_{\infty}$ is contained in $\mathfrak{k}$ then

$$H^i(\mathfrak{g}, K_{\infty}; \pi_{\infty} \otimes V)$$

vanishes for $i < c(V^\vee)$.

We close this section by pointing out that Vogan and Zuckerman have provided a classification of all $(\mathfrak{g}, K_{\infty})$-modules with nonzero cohomology and computed the cohomology of these modules (see [VZ84]).

15.8 Shimura varieties

In this section we define the notion of a Shimura variety. Although much of the theory of Shimura varieties was established by Shimura, a simpler formulation was introduced by Deligne in [Del71]. Deligne’s formulation has been almost universally adopted in the literature and we will use it here. An introduction to [Del71] is contained in [Mil05]. Proofs for the results we state below can be found in these references.

For a reductive group $G$, let $G^{\text{ad}} := G/Z_G$ be the adjoint group. The Deligne torus $S := \text{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$.

Note that $S(\mathbb{R}) = \mathbb{C}^\times$. The following definition is due to Deligne:

Definition 15.7. Let $G$ be a reductive group over $\mathbb{Q}$ and let $X$ be a $G(\mathbb{R})$-conjugacy class of homomorphisms $h : S \to G_{\mathbb{R}}$. The pair $(G, X)$ is a Shimura datum if

(a) For $h \in X$, only the characters $z/\mathbb{Z}$, $1$, and $\mathbb{Z}/z$ occur in the representation of $S$ on $(\text{Lie} G^{\text{ad}})_{\mathbb{C}}$ defined by $h$,
(b) $\theta = \text{ad}(h(\sqrt{-1}))$ is a Cartan involution of $G^{\text{ad}}$,
(c) $G^{\text{ad}}$ has no $\mathbb{Q}$-factors on which the projection of $h$ is trivial.

The condition (b) is equivalent to the statement that

$$\{g \in G^{\text{ad}}(\mathbb{C}) : g = \theta(\overline{g})\}$$

is compact. Conditions (a) and (b) together imply that $X$ is a Hermitian symmetric space for $G$.

Since $\text{Sh}^K$ is a complex manifold in this case, a natural question is whether $\text{Sh}^K$ can be realized as the complex points of some variety. This is indeed the case by a basic theorem of Baily and Borel. It states that if $K$ is neat then $\text{Sh}^K$ can be given the structure of the complex points of smooth quasi-projective scheme over $\mathbb{C}$ in a canonical manner. One can say even more.
For each $x \in X$, we have a cocharacter

$$u_x(z) := h_{x_C}(z, 1),$$

where $x_C$ denotes the base change of $x$ to $\mathbb{C}$. Its image in

$$(G \backslash X_\ast(G))(\mathbb{C})$$

is independent of the choice of $x$. Here $G$ acts on $X_\ast(G)$ via conjugation and $G \backslash X_\ast(G)$ is the quotient in the sense of geometric invariant theory as in §17.1.

Let $E := E(G, X) \subset \mathbb{C}$ be the field of definition of $u_x$. It is a number field, independent of the choice of $x \in X$. It is called the reflex field of $(G \hookrightarrow X)$.

**Theorem 15.8.1** For each neat $K \leq G(\mathbb{A}_\mathbb{Q}^\infty)$, there exists a smooth quasi-projective variety $M(G, X)^K$ defined over the reflex field $E$ of $(G, X)$ such that

$$\text{Sh}^K = M(G, X)^K(\mathbb{C})$$

and all the correspondence $T(g)$ are defined over $E$. Furthermore, there is a canonical such model characterized by the Galois action on certain special points. $\square$

**Definition 15.8.** The Shimura variety attached to $(G, X)$ is the projective limit

$$M := M(G, X) := \biglim_K M(G, X)^K$$

of the canonical models of $\text{Sh}^K$.

**Definition 15.9.** A morphism of Shimura data $(G, X) \to (G', X')$ is a morphism of algebraic groups $G \to G'$ such that the induced map $G(\mathbb{R}) \to G'(\mathbb{R})$ sends $X$ to $X'$.

**Definition 15.10.** A morphism of Shimura varieties $M(G, X) \to M(G', X')$ is an inverse system of regular maps compatible with the action of Hecke correspondences.

**Theorem 15.8.2** A morphism of Shimura data induces a morphism of Shimura varieties over the compositum $E(G, X)E(G', X')$. Moreover, it is a closed immersion if $G \to G'$ is injective. $\square$

**Example 15.1.** Let $G = \text{GL}_2$ and set $h$ to be $h(a + b\sqrt{-1}) = \left( \begin{array}{cc} a & b \\ -b & a \end{array} \right)$. In this case, each $\text{Sh}(G, X)^K$ is a finite union of modular curves (see the end of §2.6).

**Example 15.2.** Let $G = \text{Res}_{F/\mathbb{Q}}\text{GL}_2$, where $F/\mathbb{Q}$ is totally real. Take

$$h(a + b\sqrt{-1}) = \prod_{\sigma:F \to \mathbb{R}} \left( \begin{array}{cc} \sigma(a) & \sigma(b) \\ -\sigma(b) & \sigma(a) \end{array} \right).$$
Then the associated Shimura varieties are known as **Hilbert modular varieties**.

*Example 15.3.* Let \( G = \text{GSp}_{2n} \). For a \( \mathbb{Z} \)-algebra \( R \hookrightarrow \) we have that

\[
\text{GSp}_{2n}(R) = \{ g \in \text{GL}_{2n}(R) : g^t J g = c(g) J \text{ for some } c(g) \in R^\times \}
\]

for \( J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \). Take \( h \) to be \( h(a + b\sqrt{-1}) = \begin{pmatrix} aI & bJ \\ -bJ & aI \end{pmatrix} \). The associated Shimura varieties are known as **Siegel modular varieties**. This can be generalized to totally real number fields as in the previous example, yielding Hilbert-Siegel modular varieties.

Let \( D \) be an associative algebra with unit over a field \( F \). We do not require \( D \) to be commutative. An involution of \( D \) is an \( F \)-linear bijection \( \ast : D \to D \) such that \( (yx)^\ast = x^\ast y^\ast \).

*Example 15.4.* Let \( F \) be a totally real number field and \( k \subset F \) a totally imaginary quadratic extension. Let \( D \) be a simple algebra with center \( k \) and assume \( D \) admits an involution \( \ast \) that induces the nontrivial automorphism of \( k \subset F \).

For \( \mathbb{Q} \)-algebras \( R \hookrightarrow \) set

\[
G(R) := \{ x \in D \otimes_{\mathbb{Q}} R : xx^\ast \in R^\times \}.
\]

Assume that we are given a homomorphism

\[
h_0 : \mathbb{C} \to D \otimes_{\mathbb{Q}} \mathbb{R}
\]

such that \( h_0(z)^\ast = h_0(\overline{z}) \). The map

\[
i : D \to D
\]

\[
x \mapsto h_0(\sqrt{-1})^{-1} x^\ast h_0(\sqrt{-1})
\]

is then an involution of \( D \). We assume it is positive in the sense that \( \text{tr}_{D/\mathbb{Q}}(\ast x(x)) > 0 \) for \( x \in D - \{0\} \). The restriction of \( h_0 \) to \( \mathbb{C}^\times \) defines

\[
h : S \to G_{\mathbb{R}}.
\]

Let \( X \) be the \( G(\mathbb{R}) \)-conjugacy class of \( h \). Then \( (G, X) \) is a Shimura datum. The associated Shimura varieties are sometimes called **Kottwitz varieties** since Kottwitz employed them to fantastic effect in the construction of Galois representations attached to automorphic representations [Kot92]. There are two key properties that make these varieties pleasant objects to consider. First, if \( D \) is chosen appropriately, then \( \text{Sh}^K \) is compact. Second, the group \( G \) is a form of a unitary similitude group, and thus for essentially half the primes \( p \), the group \( G_{\mathbb{Q}_p} \) is a product of general linear groups. This fact was exploited in the original proofs of the local Langlands correspondence for \( \text{GL}_n \) [HT01, Hen00].
15.8 Shimura varieties

Suppose that \((G, X)\) is a Shimura datum with the reflex field \(E\). Let \(K \leq G(\mathbb{A}^\infty_{\mathbb{Q}})\) be a neat compact open subgroup. For simplicity, assume that \(\text{Sh}^K\) is compact. To ease notation, let

\[
M^K := M(G, X)^K,
\]

where \(M(G, X)^K\) is as in Theorem 15.8.1. We can then consider the étale cohomology groups

\[
H^\bullet_{\text{ét}}(M^K, \overline{\mathbb{Q}}_\ell) := H^\bullet_{\text{ét}}(M^K \times_E \overline{E}, \overline{\mathbb{Q}}_\ell). \tag{15.38}
\]

We can also consider versions of these groups with coefficients in a local system, but let us take trivial coefficients for simplicity. We refer the reader to [Mil80] for proofs of the properties of étale cohomology we use below.

These groups are modules under the Hecke algebra \(C^1_c(G(\mathbb{A}^\infty_{\mathbb{Q}})/\mathcal{K})\) and \(\text{Gal}_E\). Upon choosing an isomorphism \(\overline{\mathbb{Q}}_\ell \cong \mathbb{C}\) and an embedding \(E \hookrightarrow \mathbb{C}\), one obtains a comparison isomorphism

\[
H^\bullet_{\text{ét}}(M^K, \overline{\mathbb{Q}}_\ell) \cong H^\bullet(\text{Sh}^K, \mathbb{C}) \tag{15.39}
\]

that is \(C^\infty_c(G(\mathbb{A}^\infty_{\mathbb{Q}})/K)\)-equivariant.

Combining this with Matsushima’s formula (15.29), we obtain a decomposition

\[
H^i_{\text{ét}}(M^K, \overline{\mathbb{Q}}_\ell) = \bigoplus_{\pi^\infty} H^i_{\text{ét}}(\pi^\infty) \otimes \pi^\infty^K
\]

as \(\text{Gal}_E \times C^\infty_c(G(\mathbb{A}^\infty_{\mathbb{Q}})/K)\)-modules. Here \(H^i_{\text{ét}}(\pi^\infty)\) is a \(\text{Gal}_E\)-representation on which \(C^\infty_c(G(\mathbb{A}^\infty_{\mathbb{Q}})/K)\) acts trivially and \(\text{Gal}_E\) acts trivially on \(\pi^\infty^K\). The sum is over all irreducible admissible \(G(\mathbb{A}^\infty_{\mathbb{Q}})\)-representations \(\pi^\infty\) such that there is an irreducible admissible representation \(\pi^\infty\) of \(A_G \backslash G(\mathbb{R})\) such that \(\pi^\infty \otimes \pi^\infty\) occurs in \(L^2([G])\). The \(\text{Gal}_E\)-representation \(H^i_{\text{ét}}(\pi^\infty)\) is of dimension

\[
\sum_{\pi^\infty} m(\pi^\infty \otimes \pi^\infty) \dim H^i(\mathfrak{a}_G \backslash \mathfrak{g}, K_\infty; \pi^\infty)
\]

where the sum is over irreducible admissible representations of \(A_G \backslash G(\mathbb{R})\) and \(m(\pi^\infty \otimes \pi^\infty)\) is the multiplicity of \(\pi^\infty \otimes \pi^\infty\) in \(L^2([G])\).

One has a conjectural description of the \(\text{Gal}_E \times C^\infty_c(G(\mathbb{A}^\infty_{\mathbb{Q}})/K)\)-modules \(H^i_{\text{ét}}(\pi^\infty) \otimes \pi^\infty^K\) in terms of the \(L\)-parameter of \(\pi\). A nice survey is given in [BR94b]. In many cases, these conjectures have now been proven (see [Shi11, CHLN11] for instance). This whole picture has an analogue when \(M^K\) is not assumed to be compact. However the technical complications in the noncompact setting are enormous and have only been overcome in key cases [Mor08, Mor10].

The emphasis so far has been on using these results to relate Galois representations to automorphic representations. As we mentioned in §12.6, one
expects packets of tempered automorphic representations of $G(\mathbb{A}_Q)$ to be parameterized by $L$-parameters $\mathcal{L}_F \rightarrow L^G$. The work on Shimura varieties described above can sometimes allow one to establish this correspondence when the $L$-parameter is $C$-algebraic in the sense of §12.8.

Lest the reader be misled, the description of the étale cohomology group $H^\bullet_{\text{ét}}(M_{\mathcal{L} E} \hookrightarrow \mathbb{Q}_\ell)$ in terms of automorphic representations is useful for more than proving cases of Langlands functorality. In fact, this description is only the beginning of the story. These étale cohomology groups are intimately tied to two of the deepest conjectures in algebraic and arithmetic geometry, namely the Tate [Tat94] and Beilinson-Bloch conjectures [Sch88, Nek94].

Let $\text{CH}^p(X)_{\mathbb{Q}_\ell}$ denote the Chow group of codimension $p$ cycles on $X$ (with $\mathbb{Q}_\ell$ coefficients). There is a cycle class map

$$\text{cl}: \text{CH}^p(X)_{\mathbb{Q}_\ell} \rightarrow H^{2p}_{\text{ét}}(X_{\mathbb{Q}_\ell}(p))^\text{Gal}_E.$$ (15.40)

Here $(p)$ is the Tate twist.

**Conjecture 15.8.3 (Tate)** The map (15.40) is surjective.

The Beilinson-Bloch conjecture gives a description of the kernel of $\text{cl}$. We will not make the conjecture precise. To indicate the depth of the conjecture, we point out that it contains the Birch and Swinnerton-Dyer conjecture as a special case.

Now for general $X$, it is difficult to compute $H^{2p}_{\text{ét}}(X_{\mathbb{Q}_\ell}(p))^\text{Gal}_E$, and even more difficult to exhibit any elements in $\text{CH}^p(X)_{\mathbb{Q}_\ell}$. Shimura varieties provide an example where one can compute the Galois representation $H^{2p}_{\text{ét}}(X_{\mathbb{Q}_\ell}(p))$ and produce at least some elements of $\text{CH}^p(X)_{\mathbb{Q}_\ell}$ as we now explain.

Assume that $(H, X_H)$ is a Shimura datum with reflex field $E_H := E(H, X_H)$ that is equipped with a morphism

$$(H, X_H) \rightarrow (G, X).$$

For simplicity, assume that $H$ is a subgroup of $G$ and the map $X_H \rightarrow X$ is injective and induced by the inclusion $H(\mathbb{R}) \rightarrow G(\mathbb{R})$. Write $n$ for the real codimension of $\text{Sh}(H, X_H)^K_H$ in $\text{Sh}^K$ for $K < G(\mathbb{A}_Q^\infty)$ and for $K_H := K \cap H(\mathbb{A}_Q^\infty)$. Note that $n$ is independent of the choice of $K$. In the étale setting, we again have a cycle class map

$$H^0_{\text{ét}}(M(H, X_H)^{K_H}_{E_H}, \mathbb{Q}_\ell) \rightarrow H^n_{\text{ét}}(M^K_{E}, \mathbb{Q}_\ell(n/2))^\text{Gal}_E.$$

(15.41)

The image of this map is contained in $\text{cl}(\text{CH}^n/2(X_{E_H})_{\mathbb{Q}_\ell})$. The map is compatible with the cycle class map of (15.32) under the relevant comparison isomorphisms [DMOS82, §I]. Motivated by the Tate conjecture, one is lead to the following question:

**Question 15.8.4** What part of the cohomology of
15.8 Shimura varieties

\[ H^n_{\text{ét}}(M^K, \mathcal{T}_n(n/2))^\text{Gal}_{E/H} \]

is in the \( C^\infty_c(G(\mathbb{A}_{\mathbb{F}}) \backslash K) \)-span of the image of (15.41)?

As explained in §15.6, this question can be rephrased in terms of the existence of distinguished representations satisfying certain local assumptions. This was, in fact, the original motivation for the definition of a distinguished representation as mentioned in the beginning of this chapter, and the answer is still not well-understood. However, many interesting cases have been studied [GH14, GW14, Kud97, MR87, Ram04, Yun11, Zha14].

There is also a connection between distinguished representations and the Beilinson-Bloch conjecture, though it is not as direct. For many years, S. Kudla and his collaborators have investigated the conjecture for certain Shimura varieties using a version of the Weil representation [Kud04, KRY06]. Though the theory of distinguished representations is not emphasized in their work, it is lurking in the background. In a different setting related to the Gan-Gross-Prasad conjecture W. Zhang conjectured a more direct connection between distinguished representations and the Beilinson-Bloch conjecture using a relative trace formula [Zha12a, Zha12b]. W. Zhang has since made decisive progress towards establishing his conjecture [Zha21]. In the function field setting, analogues of the relative trace formula can even be used to give the strongest results known towards the Birch and Swinnerton-Dyer conjecture [YZ17, YZ19].

Exercises

15.1. Let \( Y \) be a Hausdorff topological space and let \( \Gamma \) be a discrete group acting freely and properly discontinuously on it. Prove that there is a unique topology on the set theoretic quotient \( \Gamma \backslash Y \) such that the quotient map \( Y \to \Gamma \backslash Y \) is a local homeomorphism. In other words, for every \( y \in Y \), there is an open neighborhood of \( y \) mapping homeomorphically onto an open neighborhood of \( \Gamma y \).

15.2. Let \( \Gamma \) and \( X \) be as in (15.3). Deduce that there is a unique manifold structure on \( \Gamma \backslash X \) such that the quotient map \( X \to \Gamma \backslash X \) is a local homeomorphism (see Exercise 15.1 above for the definition of the local homeomorphism).

15.3. Prove that a congruence subgroup is arithmetic.

15.4. Prove Lemma 15.2.2.

15.5. Compute the \( (\mathfrak{gl}_2, O_2(\mathbb{R})) \)-cohomology of all irreducible admissible \( (\mathfrak{gl}_2, O_2(\mathbb{R})) \)-modules.

15.6. Prove the formula (15.7).
15.7. Prove that the isomorphism (15.26) is Hecke equivariant. 

15.8. Prove that the pairs \((G, X)\) in examples 15.1, 15.2, 15.3, and 15.4 are Shimura data.
Chapter 16
Spectral Sides of Trace Formulae

A felicitous but unproved conjecture may be of much more consequence for mathematics than the proof of many a respectable theorem.

A. Selberg

Abstract An important tool for the study of automorphic forms is the Arthur-Selberg trace formula, or more generally, the relative trace formula. We investigate the spectral side of these identities in this chapter.

16.1 The automorphic kernel function

The remaining chapters of this book are devoted to stating and proving simple versions of trace formulae and sketching some applications. These formulae all have a geometric side involving integrals of a test function along certain orbits and a spectral side involving period integrals of automorphic forms.

The first step is to employ a fundamental idea applied to the study of automorphic forms by Selberg [Sel56] which we now describe. Let $G$ be a reductive group over a global field $F$ and as in §2.6 let

$$[G] := A_G G(F) \backslash G(\mathbb{A}_F).$$

(16.1)

For $x \in A_G \backslash G(\mathbb{A}_F)$, one has the regular representation

$$R(x) : L^2([G]) \rightarrow L^2([G])$$

$$\varphi \mapsto (g \mapsto \varphi(gx)).$$

(16.2)

For
Consider the smooth version of the regular representation:

\[ R(f) : L^2([G]) \rightarrow L^2([G]) \]

\[ \varphi \mapsto \left( g \mapsto \int_{A_G \backslash G \backslash \mathbb{A}_F} f(x) \varphi(gx) dx \right). \]

Selberg observed that there is a simple formula for the kernel function of this operator. Before we give the formula, let us make precise what we mean by a kernel function. Let \( V \subseteq L^2([G]) \) be a closed subspace and let

\[ A : V \rightarrow V \]

be a \( \mathbb{C} \)-linear operator. We say a function \( K \in C^\infty([G] \times [G]) \) is a **kernel function** for \( A \) if \( y \mapsto K(x, y) \) is in \( L^2([G]) \) for all \( x \in [G] \) and

\[ \int_{[G]} K(x, y) \varphi(y) dy = A \varphi(x) \]

for all \( \varphi \in V \) and almost every \( x \in [G] \).

**Lemma 16.1.1** If a \( \mathbb{C} \)-linear operator \( A : V \rightarrow V \) admits a kernel function then the kernel function is unique.

**Proof.** Suppose that \( K_1, K_2 \in C^\infty([G] \times [G]) \) are kernel functions for \( A \). Let \( y_0 \in [G] \) and let \( \varphi_n \in C^\infty_c([G]) \) converge weakly to the Dirac distribution supported at \( y_0 \). We assume that the support of \( \varphi_n \) is contained in a compact subset of \( [G] \) independent of \( n \). Then

\[ K_1(x, y_0) = \lim_{n \to \infty} \int_{[G]} K_1(x, y) \varphi_n(y) dy \]

\[ = \lim_{n \to \infty} \int_{[G]} K_2(x, y) \varphi_n(y) dy \]

\[ = K_2(x, y_0) \]

for all \( x \in [G] \). \( \Box \)

In view of the lemma, it makes sense to speak of “the” kernel function of an operator. Selberg observed the following:

**Lemma 16.1.2** The function

\[ K_f(x, y) := \sum_{\gamma \in G(F)} f(x^{-1} \gamma y). \]

(16.5)

is the kernel function for \( R(f) \).
Proof. One has that
\[ R(f)\varphi(x) = \int_{A_G \backslash G(\mathbb{A}_F)} f(y)R(y)\varphi(x)dy \]
\[ = \int_{A_G \backslash G(\mathbb{A}_F)} f(y)\varphi(xy)dy \]
\[ = \int_{A_G \backslash G(\mathbb{A}_F)} f(x^{-1}y)\varphi(y)dy \]
\[ = \int_{[G]} \sum_{\gamma \in G(F)} f(x^{-1}\gamma y)\varphi(y)dy. \]

Here to justify the last step, one uses the unfolding Lemma 9.2.4. To check that \( K_f(x, y) \) is smooth as a function of \( x \) and \( y \), we show that for \( (x, y) \) in a fixed compact set, the sum is finite. Let
\[ \Omega_1 \times \Omega_2 \subset A_G \backslash G(\mathbb{A}_F) \times A_G \backslash G(\mathbb{A}_F) \]
be compact subsets and let
\[ (x, y^{-1}) \in \Omega_1 \times \Omega_2. \]
Then the only nonzero summands in \( K_f(x, y) \) correspond to \( \gamma \) satisfying \( \gamma \in \Omega_1 \text{Supp}(f)\Omega_2. \)

The function \( K_f(x, y) \) is the automorphic kernel function attached to \( f \).

The kernel \( K_f(x, y) \) can also be expanded spectrally. It is easiest to describe the contribution of the cuspidal spectrum as we now explain. We will show in Theorem 16.2.3 below that the restriction \( R_{\text{cusp}}(f) \) of \( R(f) \) to \( L^2\text{cusp}([G]) \) admits a kernel function
\[ K_{f, \text{cusp}}(x, y) = \sum_{\pi} K_{\pi(f)}(x, y), \quad (16.6) \]
where the sum is over isomorphism classes of cuspidal automorphic representations \( \pi \) of \( A_G \backslash G(\mathbb{A}_F) \) and \( K_{\pi(f)}(x, y) \) is the kernel function of the restriction \( \pi(f) \) of \( R(f) \) to \( L^2\text{cusp}(\pi) \), the \( \pi \)-isotypical subspace of \( L^2\text{cusp}([G]) \). Moreover, the same theorem states that \( K_{\pi(f)}(x, y) \in L^2([G] \times [G]) \), the sum in (16.6) converges in \( L^2([G] \times [G]) \) and the \( L^2 \)-expansion of \( K_{\pi(f)}(x, y) \) is given by
\[ K_{\pi(f)}(x, y) = \sum_{\varphi \in \mathcal{B}_\pi} \pi(f)\varphi(x)\bar{\varphi}(y) \quad (16.7) \]
for \( \mathcal{B}_\pi \) an orthonormal basis of the space of \( \pi \). We note that \( \pi(f) \) need not be of finite rank in general, but it is if \( f \) is \( K_\infty \)-finite by admissibility of \( \pi \) (see Exercise 16.1).

We thus arrive at the identity that underlies all trace formulae:
where the sum on the right is over isomorphism classes of cuspidal automorphic representations \( \pi \) of \( A_G \backslash G(\mathbb{A}_F) \) and the \(*\) denotes the contribution of the orthogonal complement of \( L^2_{\text{cusp}}([G]) \) in \( L^2([G]) \). This contribution will be made precise in §16.3. The left hand side is the geometric expansion of the kernel. The right hand side is the spectral expansion of the kernel.

The key point here is that the right hand side manifestly contains all automorphic representations of \( A_G \backslash G(\mathbb{A}_F) \) whereas the left hand side, at least a priori, does not involve automorphic representations at all. It packages automorphic information in an entirely different manner. The goal is to play these two different manners of encoding automorphic representations off of each other.

One method of extracting information from (16.8) is taking traces as we now explain. Theorem 16.2.3 implies that \( K_{\pi(f)}(x, y) \) is smooth in \((x, y)\). Moreover it is the kernel function of a trace class operator by Theorem 9.1.1. Thus by [Bri91, Corollary 3.2] one has

\[
\sum_{\pi} \int_{[G]} K_{\pi(f)}(x, x) dx = \sum_{\pi} m(\pi) \text{tr} \pi(f),
\]

where \( m(\pi) \) is the multiplicity of \( \pi \) in \( L^2_{\text{cusp}}([G]) \).

Thus (16.9), in principle, is equal to the integral of the left hand side of (16.8) over the diagonal (minus the contribution of \(*\)). We say in principle because the integral of the left hand side over the diagonal is rarely convergent. In any case, at the expense of a massive amount of work, one can make sense of the integral over the diagonal of the left hand side of (16.8) and one obtains the Arthur-Selberg trace formula.

For the moment we leave the geometric expansion of the kernel and focus on the spectral side. We return to the geometric side in §18.2 when we discuss trace formulae in simple settings.

### 16.2 Relative traces

Let \( \pi \) be a cuspidal automorphic representation of \( A_G \backslash G(\mathbb{A}_F) \). As just mentioned, the integral of \( K_{\pi(f)}(x, y) \) over the diagonal copy of \([G]\) gives the trace of the operator \( m(\pi) \pi(f) \), where \( m(\pi) \) is the multiplicity of \( \pi \) in \( L^2_{\text{cusp}}([G]) \).

There is no need to just integrate over the diagonal copy of \([G]\) however. Jacquet was the first to systematically study the integrals of \( K_{\pi(f)}(x, y) \) over subgroups other than the diagonal copy of \([G]\) (apart from twisted versions of the diagonal embedding that appear in the twisted trace formula) [Jac97,
These new integrals are called relative traces. They are introduced in this section.

Let $\mathcal{HS}$ denote the $C$-vector space of Hilbert-Schmidt operators on $L^2([G])$. It becomes a Hilbert space with respect to the pairing

$$\mathcal{HS} \times \mathcal{HS} \to \mathbb{C}$$

$$(A, B) \mapsto \text{tr}(A \circ B^*) .$$

Here $B^*$ is the adjoint of $B$ and the pairing is well-defined because the convolution of two Hilbert-Schmidt operators is trace class by Exercise 9.2.

Lemma 16.2.1

The map

$$\mathcal{HS} \to L^2([G] \times [G])$$

$$A \mapsto \sum_{\varphi} A\varphi(x)\overline{\varphi}(y)$$

is an isometric isomorphism, where the sum is over an orthonormal basis of $L^2([G])$. The map is independent of the choice of basis.

Proof. If $A$ is of finite rank, then the sum $\sum_{\varphi} A\varphi(x)\overline{\varphi}(y)$ has only finitely many nonzero terms. With this in mind, it is easy to check that the map is an isometry when restricted to the subspace of finite rank operators. On the other hand the space of finite rank operators is dense in $\mathcal{HS}$, so the map extends by continuity to an isometry. This implies in particular that the sum over $\varphi$ converges in $L^2([G] \times [G])$. The inverse of the map is given by sending $K \in L^2([G] \times [G])$ to the operator

$$\varphi \mapsto \left( x \mapsto \int_{[G]} K(x, y)\varphi(y)dy \right) ;$$

this operator is Hilbert-Schmidt by Exercise 9.4. The map in the lemma is independent of the choice of orthonormal basis because its inverse is independent of the choice of orthonormal basis.

For $f \in C_\infty^\infty(A_G \setminus G(A_F))$, let

$$f^\vee(g) := f(g^{-1}) \quad \text{and} \quad f^*(g) := \overline{f(g^{-1})} .$$

(16.10)

Lemma 16.2.2

Let $V \leq L^2([G])$ be a closed $A_G \setminus G(A_F)$-invariant subspace such that the restriction $R(f)|_V$ to $V$ is Hilbert-Schmidt. Then

$$(R(f)|_V)^* = R(f^*)|_V .$$

Proof. Let $B_V$ be an orthonormal basis of $V$. If $\varphi_1, \varphi_2 \in B_V$ then

$$\int_{[G] \times [G]} \sum_{\varphi \in B_V} R(f)\varphi(x)\overline{\varphi}(y)\overline{\varphi}_2(x)\varphi_1(y)dxdy$$
\[ \int_{[G]} R(f)\varphi_1(x)\overline{\varphi_2(x)}\,dx = \int_{A_G \setminus G(A_F)} \int_{[G]} f(g)\varphi_1(xg)\overline{\varphi_2(x)}\,dx\,dg \]

\[ = \int_{A_G \setminus G(A_F)} \int_{[G]} \varphi_1(x)f(g)\overline{\varphi_2(xg^{-1})}\,dx\,dg \]

\[ = \int_{[G]} \varphi_1(x)R(f')\overline{\varphi_2(x)}\,dx \]

\[ = \int_{[G] \times [G]} \sum_{\varphi \in B_V} \varphi(x)\overline{\varphi(y)}R(f')\overline{\varphi_2(x)}\varphi_1(y)\,dxdy. \]

Here the manipulation of integrals is justified by the fact that the sums converge in \( L^2([G] \times [G]) \) by Lemma 16.2.1. We deduce that

\[ \sum_{\varphi \in B_V} R(f)\varphi(x)\overline{\varphi(y)} = \sum_{\varphi \in B_V} \varphi(x)R(f')\overline{\varphi(y)} \quad (16.11) \]

almost everywhere on \([G] \times [G]\). Therefore the lemma follows from Lemma 16.2.1.

We now prove a theorem already employed in §16.1. Recall that \( R_{\text{cusp}}(f) \) is of trace class by Theorem 9.1.1, and in fact the restriction of \( R(f) \) to the whole discrete spectrum of \( L^2([G]) \) is of trace class by Theorem 9.1.3. Moreover, recall from Exercise 9.3 that trace class operators are Hilbert-Schmidt.

**Theorem 16.2.3** Let \( V \leq L^2([G]) \) be a closed \( A_G \setminus G(A_F) \)-invariant subspace such that the restriction \( R(f)|_V \) to \( V \) is Hilbert-Schmidt and let \( B_V \) be an orthonormal basis of this subspace. For \( f \in C_\infty^\infty(A_G \setminus G(A_F)) \) the restriction \( R(f)|_V \) admits a kernel function

\[ K_{f,V}(x,y) \in L^2([G] \times [G]) \cap C_\infty([G] \times [G]) \]

with \( L^2 \)-expansion

\[ K_{f,V}(x,y) = \sum_{\varphi \in B_V} R(f)\varphi(x)\overline{\varphi(y)}. \quad (16.12) \]

Here when we say that (16.12) is an \( L^2 \)-expansion we mean that the sum on the right converges in \( L^2([G] \times [G]) \) to \( K_{f,V}(x,y) \).

**Proof.** Since \( V \) is closed in \( L^2([G]) \) we can extend the operator by 0 to obtain a Hilbert-Schmidt operator on \( L^2([G]) \). Thus we can apply Lemma 16.2.1. It follows that the equality (16.12) is valid if we understand the left hand side as an \( L^2 \)-function and we understand the right-hand side as converging in the \( L^2 \)-sense. With this in mind it is easy to check that \( K_{f,V}(x,y) \) is a kernel function for \( R(f)|_V \) provided it is smooth.
16.2 Relative traces

By the Dixmier-Malliavin lemma (Theorem 4.2.7) and its obvious analogue in the nonarchimedean case, we can write $f$ as a finite sum of functions of the form

$$f_1 * f_2 * f_3$$

for $f_1, f_2, f_3 \in C^\infty(G(\mathfrak{A}_F))$. It clearly suffices to prove the lemma for $f$ of this special form, so we assume that $f = f_1 * f_2 * f_3$. Thus using (16.11) one has that

$$K_{f,V}(x, y) = \sum_{\varphi \in \mathcal{B}_V} R(f_2 * f_3) \varphi(x) \overline{\varphi(y)} = \sum_{\varphi \in \mathcal{B}_V} R(f_2 * f_3) \varphi(x) R(f_3^\vee) \overline{\varphi(y)} = (R(f_2) \times R(f_3^\vee)) \sum_{\varphi \in \mathcal{B}_V} R(f_3) \varphi(x) \overline{\varphi(y)}.$$

The function in the last equality of (16.13) is smooth as a function of $(x, y)$ by Theorem 4.2.2, Proposition 4.2.5, and their obvious analogues in the nonarchimedean case. Hence $K_{f,V}(x, y)$ is smooth. \qed

We now prepare to define relative traces. Let

$$H \leq G \times G$$

be a subgroup. We assume that the neutral component of $H$ is the direct product (not semidirect product) of a unipotent and a reductive group. We set

$$A_{G,H} := H(\mathfrak{A}_F) \cap (A_G \times A_G).$$

(16.14)

Then the natural map

$$A_{G,H} \backslash H(\mathfrak{A}_F) \longrightarrow (A_G \times A_G) \backslash (A_G \times A_G) H(\mathfrak{A}_F)$$

is an isomorphism. Implicit in the definition of $R(f)$ is a choice of measure on $A_G \backslash G(\mathfrak{A}_F)$. To define relative traces, we additionally choose a Haar measure $d(h_L, h_R)$ on $A_{G,H} \backslash H(\mathfrak{A}_F)$. It induces a right $H(\mathfrak{A}_F)$-invariant Radon measure on $A_{G,H} H(F) \backslash H(\mathfrak{A}_F)$.

Let

$$\chi : H(\mathfrak{A}_F) \longrightarrow \mathbb{C}^\times$$

be a quasi-character trivial on $A_{G,H} H(F)$. We define the relative trace of $\pi(f)$ with respect to $H$ and $\chi$ to be

$$rtr_{H,\chi} \pi(f) = \int_{A_{G,H} H(F) \backslash H(\mathfrak{A}_F)} K_{\pi(f)}(h_L, h_R) \chi(h_L, h_R) d(h_L, h_R).$$

(16.15)

This is well-defined by the following lemma:
Lemma 16.2.4 For any closed subspace \( V \leq L^2_{\text{cusp}}([G]) \), one has that
\[
\int_{A^GH(F)\backslash H(A_F)} |K_{\pi,V}(h_\ell,h_r)x(h_\ell,h_r)|d(h_\ell,h_r) < \infty.
\]

Proof. This is an immediate consequence of Theorem 16.2.3 and Corollary 14.3.4.

The key property of relative traces is contained in the following proposition:

Proposition 16.2.5 Suppose that \( \pi \) is a cuspidal automorphic representation of \( A_G \backslash G(A_F) \). The representation \( \pi \otimes \pi' \) of \( (A_G \backslash G(A_F))^2 \) is \((H,\chi)\)-distinguished if and only if \( \text{rtr}_{H,\chi}(f) \neq 0 \) for some \( f \in C_\infty^c(A_G \backslash G(A_F)) \).

Proof. Recall the definition of the period integral \( P_\chi \) from (14.3). One has that
\[
P_\chi(K_{\pi(f)}) = \text{rtr}_{H,\chi}(f),
\]
where the convergence is justified by Lemma 16.2.4. Certainly \( K_{\pi(f)} \) lies in the \( \pi \otimes \pi'\)-isotypic subspace of \( L^2_{\text{cusp}}([G] \times [G]) \) so we deduce the “if” direction.

Now assume that \( \pi \otimes \pi' \) is \((H,\chi)\)-distinguished. By Lemma 16.3.5 and Exercise 16.4, there are \( K_\infty \)-finite smooth functions \( \varphi, \varphi' \in L^2_{\text{cusp}}(\pi) \) such that \( P_\chi(\varphi \otimes \overline{\varphi}) \neq 0 \). Using Proposition 4.5.5 and Exercise 5.7, choose \( f \in C_\infty^c(A_G \backslash G(A_F)) \) such that \( \pi(f) \) maps \( \varphi \) to \( \varphi' \) and sends any vector in \( L^2_{\text{cusp}}(\pi) \) orthogonal to \( \varphi \) to zero. Then
\[
\text{rtr}_{H,\chi}(f) = P_\chi(\varphi' \otimes \overline{\varphi}).
\]

Thus investigating which representations of \( (A_G \backslash G(A_F))^2 \) are \((H,\chi)\)-distinguished is equivalent to studying the linear functionals
\[
C_\infty^c(A_G \backslash G(A_F)) \to \mathbb{C}
\]
\[f \mapsto \text{rtr}_{H,\chi}(f).
\]

The main result of this chapter relates the kernel function \( K_{f,\text{cusp}}(x,y) \) to these linear functionals. It amounts to the computation of the cuspidal contribution to the spectral side of the relative trace formula:

Theorem 16.2.6 Let \( f \in C_\infty^c(A_G \backslash G(A_F)) \). One has that
\[
\int_{A^G H(F)\backslash H(A_F)} K_{f,\text{cusp}}(h_\ell,h_r)x(h_\ell,h_r)d(h_\ell,h_r) = \sum_{\pi} \text{rtr}_{H,\chi}(\pi(f)),
\]
where the sum on the right is over isomorphism classes of cuspidal automorphic representations \( \pi \) of \( A_G \backslash G(A_F) \). Moreover, the integral on the left and
the sum on the right are absolutely convergent. In particular, if \( R(f) \) has cuspidal image then

\[
\int_{A_{G,F} \backslash H(F) / H(F)} K_f(h, h) \chi(h, h) h(h, h) = \sum_\pi \text{tr} \pi(f).
\]

After the proof of Theorem 16.2.6, we describe the general spectral expansion of \( K_f(x, y) \) and then, in \( \S \) 16.4, describe one method for constructing functions \( f \) such that \( R(f) \) has cuspidal image.

**Proof.** We assume that \( F \) is a number field. The proof in the function field case is easier and is left as Exercise 16.2.

By Theorem 9.1.1, for any \( f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F)) \) the operator \( R_{\text{cusp}}(f) \) is of trace class, hence Hilbert-Schmidt, and the same is true of \( \pi(f) \) for any cuspidal automorphic representation \( \pi \) of \( A_G \backslash G(\mathbb{A}_F) \). These facts will be used several times below.

By the Dixmier-Malliavin lemma (Theorem 4.2.7) we can and do assume \( f = f_1 * f_2 * f_3 \). Using (16.11) we have

\[
\sum_\pi |K_\pi(f)(x, y)| = \sum_\pi |\pi(f_2) \times \pi(f_3) K_\pi(f_1)(x, y)| \\
\leq \sum_\pi |\pi(f_2) \times \pi(f_3) K_\pi(f_1)(x, y)|,
\]

(16.16)

where the sum is over isomorphism classes of cuspidal automorphic representations of \( A_G \backslash G(\mathbb{A}_F) \). Let \( \mathcal{S}(t) \) be a Siegel domain in \( G(\mathbb{A}_F) \) such that \( G(F)A_G \mathcal{S}(t) = G(\mathbb{A}_F) \). By Proposition 9.4.5 and with the notation of that proposition, for \( (x, y) \in \mathcal{S}(t) \times \mathcal{S}(t) \) the sum (16.16) is bounded by a constant depending on \( \mathcal{S}(t) \), \( B_1, B_2 \in \mathbb{R}_{> 0} \) and \( f_1, f_2 \) times

\[
\sum_\pi \max_{a_1, a_2 \in \Delta} a_1(s_x)^{-B_1} a_2(s_y)^{-B_2} ||K_\pi(f_1)(x, y)||_2 \\
= \max_{a_1, a_2 \in \Delta} a_1(s_x)^{-B_1} a_2(s_y)^{-B_2} \sum_\pi \text{tr} \pi(f_3 \ast f_3).
\]

(16.17)

Here the identity follows from lemmas 16.2.1 and 16.2.2. Since the operator \( R_{\text{cusp}}(f_3 \ast f_3) \) is of trace class, this sum converges absolutely. Thus the sum

\[
\sum_\pi K_\pi(f)(x, y)
\]

converges absolutely and uniformly in \( (x, y) \in [G] \times [G] \) and hence is a continuous function on \( [G] \times [G] \). On the other hand, \( K_{f, \text{cusp}}(x, y) \) is smooth and equal to (16.18) almost everywhere by Theorem 16.2.3. We conclude that \( K_{f, \text{cusp}}(x, y) \) is equal to (16.18) pointwise.

In the argument above we proved the estimate
\[
\sum_{\pi} |K_{\pi(f)}(x, y)| \ll \max_{\alpha_1, \alpha_2 \in \Delta} \alpha_1(s_x)^{-B_1} \alpha_2(s_y)^{-B_2} \sum_{\pi} \text{tr} \pi(f_3^* \ast f_3)
\]

where the implied constant depends on \(\mathcal{S}(t), f_1, f_2, B_1, B_2\). Therefore by using this estimate and the argument proving Proposition 14.3.3, one deduces the result. \(\Box\)

16.3 The full expansion of the automorphic kernel

Using Langlands’ decomposition of the entire spectrum of \(L^2([G])\), we now give the full spectral expansion of \(K_f(x, y)\), not just its restriction to the cuspidal subspace. Assume for this section that \(F\) is a number field.

We will use the notation and terminology developed in Chapter 10. In particular we fix a minimal parabolic subgroup \(P_0\) of \(G\) with Levi decomposition \(P_0 = M_0N_0\) and assume that the Levi decomposition \(P = MN\) of a standard parabolic subgroup \(P\) is chosen so that \(M \geq M_0\).

Let \(f \in C_c^\infty(A_G \backslash G(A_F))\). The automorphic kernel function \(K_f(x, y)\) has an expansion

\[
K_f(x, y) = \sum_P n_P^{-1} \sum_{\sigma} \int_{I(a_P)} \sum_{\varphi \in B(\sigma)} E(x, I(\sigma, \lambda)(f) \varphi, \lambda) \overline{E(y, \varphi, \lambda)} d\lambda \tag{16.19}
\]

where the sum on \(P\) is over all standard parabolic subgroups of \(G\), the sum on \(\sigma\) is over all isomorphism classes of irreducible \(A_M \backslash M(A_F)\)-subrepresentations of \(L^2_{\text{disc}}([M])\), and \(B(\sigma)\) is an orthonormal basis of \(\text{Ind}_P^G(L^2_{\text{disc}}([M])(\sigma))\) consisting of vectors in \(\text{Ind}_P^G(L^2_{\text{disc}}([M])(\sigma))\) which is defined as (10.7). A priori, this expression only converges in the sense that for each \(x\) it is an \(L^2\)-expansion of \(K_f(x, y)\) as a function of \(y\). However, one can make sense of it pointwise by an analogue of the argument in the proof of Theorem 16.2.6. See [Art78, §4] or below [Art05, Theorem 7.2].

16.4 Functions with cuspidal image

The general spectral expansion (16.19) of \(K_f(x, y)\) is formidable, and it becomes more serious when one tries to integrate the kernel. In particular, it is not integrable over the diagonal copy of \([G]\). To remedy this, Arthur introduced truncated versions of the kernel that are integrable over the diagonal and can be given spectral interpretations [Art05]. This work was used in the construction of the functorial transfers discussed in §13.5 and 13.8.

As we have seen, if we assume that \(R(f)\) has cuspidal image then the spectral expansion of \(K_f(x, y)\) is much simpler. Happily, there are now several
fairly flexible methods available to construct $R(f)$ with cuspidal image [LV07, BPLZZ19]. We will content ourselves with explaining an older but simpler method. Although it has limitations that will be explained below, it is still quite useful in practice.

**Definition 16.1.** Let $v$ be a nonarchimedean place of a global field $F$. A function $f_v \in C_c^\infty(G(F_v))$ is said to be **supercuspidal** if
\[
\int_{N(F_v)} f_v(gnh)dn = 0
\]
for all proper parabolic subgroups $P < G_{F_v}$ (defined over $F_v$) with unipotent radical $N$ and all $g, h \in G(F_v)$.

**Lemma 16.4.1** Let $v$ be a finite place of $F$, let $f^v \in C_c^\infty(A_G \backslash G(A_{F_v}))$, and let $f_v \in C_c^\infty(G(F_v))$ be supercuspidal. Let $f(g) = f_v(g_v)f^v(g^v)$, so that $f \in C_c^\infty(A_G \backslash G(A_{F_v}))$. Then $R(f)$ has cuspidal image.

**Proof.** Let $P < G$ be a proper parabolic subgroup with unipotent radical $N$. As usual, let $[N] := N(F) \backslash N(\mathbb{A}_F)$. For $\varphi \in L^2([G])$, $R(f)\varphi$ is smooth and hence can be integrated over any compact subset. For all $x \in A_G \backslash G(\mathbb{A}_F)$, unfolding as in Lemma 9.2.4 we have
\[
\int_{[N]} R(f)\varphi(nx)dn = \int_{[N]} \int_{A_G \backslash G(\mathbb{A}_F)} f(g)\varphi(nxg)dgdn
\]
\[
= \int_{[N]} \int_{A_G \backslash G(\mathbb{A}_F)} f(x^{-1}n^{-1}g)\varphi(g)dgdn
\]
\[
= \int_{[N]} \int_{A_G(N(F)) \backslash G(\mathbb{A}_F)} \sum_{\delta \in N(F)} f(x^{-1}n^{-1}\delta)\varphi(g)dgdn
\]
\[
= \int_{A_G(N(F)) \backslash G(\mathbb{A}_F)} \int_{[N]} \sum_{\delta \in N(F)} f(x^{-1}n^{-1}\delta^{-1}g)\varphi(g)dndg
\]
\[
= \int_{A_G(N(F)) \backslash G(\mathbb{A}_F)} f(x^{-1}n^{-1}g)\varphi(g)dndg
\]
\[
= 0,
\]
where the last equality follows from the fact that the inner integral
\[
\int_{N(\mathbb{A}_F)} f(x^{-1}n^{-1}g)\varphi(g)dn
\]
vanishes since $f_v$ is supercuspidal. 

Essentially all examples of supercuspidal functions are obtained using the following lemma (see Exercise 16.5):
Lemma 16.4.2 Assume that $Z_G(F_v)$ is compact for some $v$ and that $(\pi_v, V)$ is an irreducible supercuspidal representation of $G(F_v)$. If $f_v$ is a matrix coefficient of $\pi_v$ then $f_v$ is supercuspidal.

The assumption that $Z_G(F_v)$ is compact is not essential; see the discussion of truncated matrix coefficients in [HL04]. This lemma indicates the intrinsic limitation of supercuspidal functions: supercuspidal functions can only be used to study representations that are supercuspidal at some place. For a precise statement, see exercises 16.5 and 16.6.

Proof. Let $P$ be a proper parabolic subgroup of $G_{F_v}$ with unipotent radical $N$. Let $V_N$ and $V'_N$ be the Jacquet modules defined as in (8.13) in §8.3. Assume that

$$\int_{N(F_v)} f_v(gnh)dn \neq 0$$

for some $g, h \in G(F_v)$. Realize $V \otimes V'$ as a subspace of $C_c^\infty(G(F_v))$ as in the proof of Proposition 8.5.3. We then obtain a nonzero map

$$V \otimes V' \rightarrow V_N \otimes V'_N$$

$$f_v \mapsto (g, h) \mapsto \int_{N(F_v)} f_v(g^{-1}nh)dn$$

contradicting the supercuspidality of $\pi_v$. \qed

Exercises

16.1. Let $K_\infty$ be a maximal compact subgroup of $G(F_\infty)$. If $f$ is $K_\infty$-finite, then prove that $\pi(f)$ has finite image.

16.2. Prove Theorem 16.2.6 when $F$ is a function field.

16.3. Let $v$ be a nonarchimedean place of a global field $F$. A function $f_v \in C_c^\infty(G(F_v))$ is said to be $F$-supercuspidal if

$$\int_{N(F_v)} f_v(gnh)dn = 0$$

for all proper parabolic subgroups $P < G$ (defined over $F$) with unipotent radical $N$ and all $g, h \in G(F_v)$. Prove that the conclusion of Lemma 16.4.1 remains valid if we assume $f_v$ is $F$-supercuspidal.

16.4. For $1 \leq i \leq 2$, let $G_i$ be a reductive group over a number field $F$, let $K_{i, \infty} \leq A_{G_i} \backslash G_i(F_\infty)$ be maximal compact subgroup, and let $\pi_i$ be a cuspidal automorphic representation of $A_{G_i} \backslash G_i(A_F)$. Then a smooth function in
16.4 Functions with cuspidal image

\[ L^2_{\text{cusp}}([G_1] \times [G_2])(\pi_1 \otimes \pi_2) \]  

(16.20)

need not be in the algebraic tensor product

\[ L^2_{\text{cusp}}([G_1])(\pi_1) \otimes L^2_{\text{cusp}}([G_2])(\pi_2). \]

Despite this, show that a \( K_{1\infty} \times K_{2\infty}\)-finite smooth function in (16.20) is a finite sum of functions of the form \( \varphi_1 \otimes \varphi_2 \) where \( \varphi_i \) is a \( K_{i\infty}\)-finite smooth function in \( L^2_{\text{cusp}}([G_i])(\pi_i) \) for \( 1 \leq i \leq 2 \).

16.5. Let \( v \) be a finite place of the number field \( F \) and let \( G \) be a reductive group over \( F \). Assume that \( Z_G(F_v) \) is compact. Prove that any supercuspidal function is a finite sum of matrix coefficients of supercuspidal representations of \( G(F_v) \).

16.6. Under the assumptions of Exercise 16.5, let \( f = f_v f^v \in C^\infty_c(G(\mathbb{A}_F)) \) where \( f_v \) is a matrix coefficient of an irreducible supercuspidal representation \( \pi'_v \). Prove that if \( \pi \) is a cuspidal automorphic representation of \( G(\mathbb{A}_F) \) such that \( K_{\pi(f)}(x, y) \neq 0 \), then \( \pi_v \cong \pi'_v \).

16.7. Let \( K_\infty \) be a maximal compact subgroup of \( G(F_\infty) \). If \( f \) is \( K_\infty\)-finite, prove that

\[ \text{rtr}_{H,\chi} \pi(f) = \sum_{\varphi \in B_\pi} P_\chi(\pi(f) \varphi \times \varphi), \]

where we take the basis \( B_\pi \) to consist of \( K_\infty\)-finite vectors (which are smooth by Proposition 4.4.3).
Chapter 17
Orbital Integrals

When I suggested the term endoscopy to Shelstad I didn’t know it had a medical meaning.

A. Ash

Abstract We define and study orbital integrals. In the next chapter, we explain how the geometric sides of trace formulae are written in terms of these relative orbital integrals.

17.1 Group actions, orbits, and stabilizers

This chapter requires more serious algebraic geometry and Galois cohomology than other chapters in this book. Moving forward, the main concepts that we will require are various constructions and definitions related to relative classes contained in the current section and the definition of relative orbital integrals, given in §17.4 in the local setting and §17.7 in the adelic setting. If desired, the other sections of this chapter can be omitted, although they will be used in the proofs of results in §17.4 and §17.7.

We briefly describe the notion of an algebraic group action and some constructions involving it. These show up constantly in the study of trace formulae, implicitly or explicitly, so we have summarized some useful results. Our basic references for the geometric constructions in this chapter are [MFK94, Mil17, Poo17].

Let $k$ be a Noetherian ring, let $H$ be a smooth (affine) group scheme over $k$, and let $X$ be an affine scheme of finite type over $k$. A morphism

$$a : X \times H \rightarrow X$$

(17.1)

is a (right) action of $H$ on $X$ if the following diagram
\[
\begin{array}{c}
X \times H \times H \xrightarrow{a \times 1_H} X \times H \\
\downarrow{a \times m} \quad \ \downarrow{a} \\
X \times H \quad \quad \rightarrow \quad X
\end{array}
\]

commutes and the composite

\[
X \xrightarrow{1_X \times e} X \times H \xrightarrow{a} X
\]

is the identity. Here \( m \) is the multiplication map and \( e : \text{Spec}(k) \rightarrow H \) is the identity section. These assumptions imply, in particular, that for every \( k \)-algebra \( R \),

\[
a : X(R) \times H(R) \rightarrow X(R)
\]

is a right action. One can formulate the notion of a left action in the analogous manner. Let \( a : X \times H \rightarrow X \) and \( a' : X' \times H \rightarrow X' \) be right actions of \( H \) on affine schemes \( X \) and \( X' \) of finite type over \( k \), respectively. An \( H \)-equivariant morphism is a morphism of schemes \( b : X \rightarrow X' \) such that

\[
\begin{array}{c}
X \times H \xrightarrow{a} X \\
\downarrow{b \times 1_H} \quad \downarrow{b} \\
X' \times H \xrightarrow{a'} X'
\end{array}
\]

commutes. It is an isomorphism if \( b \) is an isomorphism.

For \( \gamma \in X(k) \), we have a morphism

\[
a(\gamma, \cdot) : H \rightarrow X.
\]

We denote by \( H_\gamma \) the stabilizer of \( \gamma \). Thus \( H_\gamma \) is the subgroup scheme of \( H \) given by the fiber product

\[
H_\gamma = \text{Spec}(k) \times_X H
\]

where the map \( H \rightarrow X \) is \( a(\gamma, \cdot) \) and the map from \( \text{Spec}(k) \) to \( X \) is given by \( \gamma \). In terms of points, for \( k \)-algebras \( R \), one has that

\[
H_\gamma(R) = \{ h \in H(R) : a(\gamma, h) = \gamma \}. \quad (17.2)
\]

When \( k \) is a field, the map \( \text{Spec}(k) \rightarrow X \) defined by \( \gamma \) is a closed immersion, and hence \( H_\gamma \) is a closed subgroup scheme of \( H \).

Assume for the remainder of this section that \( k \) is a field. If \( I \leq H \) is a subgroup, then one can always define a quotient scheme \( I \backslash H \). It is a separated scheme of finite type over \( k \) [Mil17, Theorem 7.18]. In fact it is quasi-projective in the sense that it admits an immersion into projective space by the proof of [Mil17, Theorem 7.18]. It admits a basepoint \( b \in (I \backslash H)(k) \) and is equipped with a morphism given on points in a \( k \)-algebra \( R \) by
The nonempty fibers of this morphism are the right cosets of $I(R)$ in $H(R)$ [Mil17, §5(c)]. It is important to point out that the map (17.3) need not be surjective for a given $R$. For more information, we refer to Proposition 17.1.8. Of course we can also form a right quotient $H / I$ in the analogous manner.

**Example 17.1.** If $I$ is a parabolic subgroup of a reductive group $H$ then $I / H$ is projective. When $H = \text{GL}_2$ and $I = \mathcal{B}$ is a Borel subgroup

$$B \backslash \text{GL}_2 \cong \mathbb{P}^1.$$  

The following is [Mil17, Proposition 7.17]:

**Proposition 17.1.1** The natural map

$$H \backslash H \longrightarrow X$$

is an immersion. \hfill \Box

For affine schemes $Y = \text{Spec}(A)$ for a ring $A$, denote by $|Y|$ the set of prime ideals of $A$ equipped with the Zariski topology. In other words $|Y|$ is the underlying topological space of $Y$. The morphism $a(\gamma, \cdot)$ induces a continuous map of topological spaces $|H| \rightarrow |X|$. The image

$$a(\gamma, |H|) \subseteq |X|$$

is open in its closure [Mil17, Proof of Proposition 1.65]. We define

$$O(\gamma) \subset X$$

(17.4)

to be the subscheme with this image as its underlying topological space, given the reduced induced scheme structure. The subscheme $O(\gamma) \subset X$ is the $H$-orbit of $\gamma$. Let $\overline{k}$ be an algebraic closure of $k$.

**Definition 17.1.** A subscheme $O \subseteq X$ with the property that $O_{\overline{k}} = O(\gamma)$ for some $\gamma \in X(\overline{k})$ is an $H$-orbit.

It is important to note that though $O(\overline{k})$ is nonempty for any $H$-orbit $O$, it may very well happen that $O(k)$ is empty.

Recall that we have assumed $H$ is smooth.

**Proposition 17.1.2** The map $a(\gamma, \cdot)$ induces an isomorphism

$$H \gamma \backslash H \longrightarrow O(\gamma).$$

The scheme $O(\gamma)$ is smooth. Moreover, the map

$$H \longrightarrow O(\gamma)$$
is smooth if $H^\gamma$ is smooth.

Proof. See [Mil17, Corollary 7.13 and Proposition 7.17] for the first assertion and see [Mil17, Proposition 7.4 and Proposition 7.15] for the last two assertions. \hfill \Box

For technical reasons later, we will require the following complementary results:

**Lemma 17.1.3** The quotient $H^\circ \backslash H$ is finite étale.

Proof. See [Mil17, Proposition 5.58]. Milne takes étale to mean finite étale [Mil17, §A(i)]. \hfill \Box

**Lemma 17.1.4** If $I \leq H$ is a subgroup then the morphism

$$I^\circ \backslash H \to I \backslash H$$

is finite and flat and in particular proper.

Proof. This follows from Lemma 17.1.3 and [Mil17, Proposition 7.15]. \hfill \Box

**Definition 17.2.** If $O(\gamma) \subset X$ is closed, we say that $\gamma$ is **relatively semisimple**. We say that $\gamma$ is **relatively regular** if the dimension of $O(\gamma)$ is maximal among all $\gamma \in X(k)$.

Here the adjective relatively is short for relative to the action of $H$.

Relative semisimplicity has important consequences for $H_\gamma$:

**Theorem 17.1.5** Assume that $H^\circ$ is reductive. The orbit $O(\gamma)$ is affine if and only if $(H_\gamma)^\circ$ is reductive. In particular, if $O(\gamma)$ is closed, then $(H_\gamma)^\circ$ is reductive. \hfill \Box

This was proven by Matsushima in characteristic zero [Mat60] and Haboush [Hab78] and Richardson [Ric77] in arbitrary characteristic. We point out that, in general, if the neutral component of $H_\gamma$ is reductive, then it is not the case that $O(\gamma)$ is closed (see Exercise 17.2).

**Example 17.2.** Let $X = H$ and let $H$ act on $X$ via conjugation. Thus for $k$-algebras $R$, the action is given by

$$X(R) \times H(R) \to X(R)$$

$$(\gamma, h) \mapsto h^{-1} \gamma h.$$

We refer to this as the **group case**. In this setting, $O(\gamma)$ is the conjugacy class of $\gamma$ and $H_\gamma$ is the centralizer of $\gamma$. Assume that $k$ is perfect. Then an element $\gamma \in X(k)$ is relatively semisimple if and only if it is semisimple in the usual sense of Definition 1.13 (see [Ste65, Corollary 6.13]). Moreover, $\gamma$ is relatively regular if and only if $\gamma$ is regular in the sense of [Ste65]. This is the reason for the terminology relatively semisimple and relatively regular.
Example 17.3. Let \( X = \mathfrak{h} \), where \( \mathfrak{h} := \text{Lie} \, H \), and let \( H \) act on \( X \) via the adjoint action. Even if one is primarily interested in the group case, it has proven crucial to use this infinitesimal model of conjugation. For example, the proof of the fundamental lemma was reduced to a Lie algebra version by Waldspurger [Wal97, Wal06, Wal08, Wal09a, Wal09b]. It was this version that was proved by Ngô [Ngô10b].

Example 17.4. Let \( G \) be a reductive group, let \( X = G \), and let \( H \leq G \times G \) be a subgroup. Then we have a natural action

\[
G(R) \times H(R) \longrightarrow G(R)
\]

\[(g, (h_\ell, h_r)) \longmapsto h_\ell^{-1}gh_r.\]

(17.5)

We can recover (17.2) by taking \( H \) to be \( G \) embedded diagonally into \( G \times G \).

Definition 17.3. Let \( R \) be a \( k \)-algebra. A \textbf{class} is an element of

\[
\Gamma(R) := X(R)/H(R).
\]

A \textbf{relatively semisimple class} is an element of

\[
\Gamma_{ss}(k) := \{\gamma \in X(k) : \gamma \text{ is relatively semisimple}\}/H(k).
\]

(17.6)

We use the notation \([\gamma]\) for the class of \( \gamma \).

From the point of view of algebraic geometry, the notion of a class is too refined. In fact it turns out to be too refined even for some purposes in automorphic representation theory. One needs to consider the coarser notion of a geometric class. This entails some understanding of quotients of affine schemes by reductive group actions, so we digress to discuss this topic.

The action of \( H \) on \( X \) corresponds to a morphism of \( k \)-algebras

\[
\bar{\alpha} : \mathcal{O}(X) \longrightarrow \mathcal{O}(X) \otimes_k \mathcal{O}(H),
\]

(17.7)

where \( \mathcal{O}(Y) \) is the coordinate ring of the affine scheme \( Y \) (see §1.2). In other words, \( \alpha \) defined in (17.1) is just the map of affine schemes induced by the map \( \bar{\alpha} \) of \( k \)-algebras. Call an element \( r \in \mathcal{O}(X) \) \textbf{invariant} if \( \bar{\alpha}(r) = r \otimes 1 \). The set of invariant elements is a \( k \)-subalgebra, and by abuse of notation, we let

\[
\mathcal{O}(X)^H \leq \mathcal{O}(X)
\]

(17.8)

be this subalgebra.

Assume for the moment that \( k \) is algebraically closed and that \( X \) is integral. We can then identify \( X(k) \) with the closed points of the underlying topological space of \( X \) [GW10, Proposition 3.33]. Moreover we can identify \( \mathcal{O}(X) \) with the \( k \)-algebra \( \mathcal{O}(X(k)) \) of regular functions on the affine algebraic
set $X(k)$ [GW10, §3.13]. The group $H(k)$ acts on $X(k)$ and this induces an action on $\mathcal{O}(X(k))$. In this case we have

$$\mathcal{O}(X(k))^{H(k)} = \mathcal{O}(X)^H.$$  \hfill (17.9)

If $H^\circ$ is reductive, we set

$$X/H := \text{Spec}(\mathcal{O}(X)^H).$$  \hfill (17.10)

It is an affine scheme of finite type over $k$ [MFK94, Theorem 1.1]. This is the Geometric Invariant Theory quotient of $X$ by $H$, or GIT quotient for short. The inclusion (17.8) induces a morphism

$$p : X \rightarrow X/H.$$ Often in the literature, the GIT quotient is denoted by $X \sslash H$ and the stack theoretic quotient is denoted by $X/H$ or $[X/H]$. Since we only use GIT quotients in this work and the doubleslash could be confused with notation we use for double quotients (see §5.5 for example), we adopt the notation above.

Let us explain why $X/H$ deserves to be regarded as a quotient. Let $p_1 : X \times H \rightarrow X$ denote the projection to the first factor. A categorical quotient of $X$ by $H$ is a $k$-scheme $Y$ and a morphism $p : X \rightarrow Y$ such that

$$\begin{array}{cccc}
X \times H & \xrightarrow{\alpha} & X \\
p_1 \downarrow & & \downarrow p \\
X & \xrightarrow{p} & Y
\end{array}$$  \hfill (17.11)

commutes and for any morphism $q : X \rightarrow Z$ from $X$ to another $k$-scheme $Z$ such that

$$\begin{array}{cccc}
X \times H & \xrightarrow{\alpha} & X \\
p_1 \downarrow & & \downarrow q \\
X & \xrightarrow{q} & Z
\end{array}$$  \hfill (17.12)

commutes, there is a unique morphism $\chi : Y \rightarrow Z$ such that $q = \chi \circ p$. Concretely, the assertion that (17.12) commutes is a way to make precise the assertion that $q$ is constant on $H$-orbits. The universal property states that any morphism constant on $H$-orbits factors through the categorical quotient. As usual with universal properties, the definition immediately implies that the categorical quotient is unique up to isomorphism if it exists.

As the reader probably has guessed, the GIT quotient is a categorical quotient [MFK94, Theorem 1.1]. Thus it is reasonable to call it a quotient. We remark that the quotients $I \backslash H$ are also categorical quotients [Mil17, Definition B.6, Theorem B.37]. The formation of these quotient commutes with base change in the following sense. If $k'/k$ is a field extension then there
are canonical identifications
\[(X/H)_{k'} = X_{k'}/H_{k'} \quad \text{and} \quad (I\setminus H)_{k'} = I_{k'}/H_{k'}. \quad (17.13)\]

The first identification is part of [MFK94, Theorem 1.1]. The second identification follows from the construction of the quotient \(I\setminus H\) in terms of sheaves [Mil17, Theorem B.37].

For our purposes, we need to understand the points of \(X\setminus H\). This is somewhat complicated. Let \(k \leq k^{\text{sep}} \leq \kbar\) be a separable closure and an algebraic closure of \(k\), respectively. We recall that the closed points of the topological space underlying a scheme \(Y\) of finite type over \(k\) can be identified with \(\text{Aut}_k(k)-\text{orbits of elements of } Y(k)\) [GW10, Proposition 5.4]. Using this fact, we can give a fairly concrete description of the points of \(X\setminus H\) over \(k\):

**Proposition 17.1.6** Assume that \(H^\circ\) is reductive. There is a bijection
\[(X/H)(k) \xrightarrow{\sim} \left\{ \gamma \in X(k^{\text{sep}}) \text{ is relatively semisimple and } O(\gamma)(k^{\text{sep}}) \text{ is } \text{Gal}_k\text{-invariant} \right\} \]
given by sending \(x \in (X/H)(k)\) to the closed \(H\)-orbit \(O\) in \(X\) such that \(O(k^{\text{sep}})\) is the unique closed \(H_{k^{\text{sep}}}\)-orbit in the fiber of \(p : X_{k^{\text{sep}}} \to (X/H)_{k^{\text{sep}}}\) over \(x\).

Under this identification, the map \(X(k) \to (X/H)(k)\) sends an \(H(k)\)-orbit \(\gamma H(k)\) to the unique closed orbit in the Zariski closure of \(O(\gamma)\) in \(X\).

Here we are using the notation \(O\) for an orbit in \(X\). Thus \(O = O(\gamma)\) for some \(\gamma \in X(k)\). Part of the content of the proposition is that we can always take \(\gamma\) to be an element of \(X(k^{\text{sep}})\).

The following fact will be used in the proof of Proposition 17.1.6 and repeatedly below [Sta16, Lemma 32.25.6]:

**Lemma 17.1.7** If \(Y\) is a smooth scheme over a field \(k\), then \(Y(k^{\text{sep}})\) is dense in the underlying topological space of \(Y\). \(\square\)

**Proof of Proposition 17.1.6:** Assume that \(k = k^{\text{sep}}\). The morphism \(p : X \to X/H\) is surjective (even universally submersive) [MFK94, Chapter 1.2, Theorem 1.1]. We claim that the induced map
\[p : X(k^{\text{sep}}) \to (X/H)(k^{\text{sep}}) \quad (17.14)\]
is also surjective. An element of \((X/H)(k^{\text{sep}})\) defines a closed point \(x\) of \((X/H)_{k^{\text{sep}}}\). Consider the fiber \(F\) of the morphism \(p : X_{k^{\text{sep}}} \to (X/H)_{k^{\text{sep}}}\) over \(x\). Since \(F(\kbar)\) is nonempty there is an \(H\)-orbit \(O \subseteq F\); in particular \(O(\kbar)\) is nonempty. Thus \(O(\kbar)\) is smooth by Proposition 17.1.2, which implies \(O\) is smooth. Hence \(O(k^{\text{sep}}) \subseteq F(k^{\text{sep}})\) is nonempty by Lemma 17.1.7. We conclude that (17.14) is surjective.

Now let \(x \in (X/H)(k)\) and let \(F\) denote the fiber of \(p\) over \(x\) as above. It is a closed subscheme of \(X\) and is a union of \(H\)-orbits. Let \(O\) be the orbit of
smallest dimension. We claim that it is closed. Indeed, let \( \overline{O} \) be the closure of \( O \). Then \( \overline{O} - O \), if nonempty, is a union of \( H \)-orbits of dimension strictly smaller than the dimension of \( O \). Thus \( O \) is closed. Any two closed \( H \)-orbits in \( X \) can be separated by an element of \( \mathcal{O}(X)^H \) (see Chapter 1.2, Corollary 1.2 and Appendix 1.C, Corollary A.1.3 of [MFK94]). Thus \( O \) is the unique closed \( H \)-orbit in \( \mathcal{F} \). This yields a bijection

\[
(X/H)(k) \to \{ \text{closed } H \text{-orbits } O \subseteq X \} \quad (17.15)
\]
given by sending \( x \in (X/H)(k) \) to the unique closed \( H \)-orbit in the fiber of \( p : X \to X/H \) over \( x \). By Lemma 17.1.7 and Proposition 17.1.2 each orbit on the right of (17.15) is of the form \( O(\gamma) \) for some \( \gamma \in X(k) \). This implies the first assertion of the proposition when \( k = k_{\text{sep}} \). The second assertion follows from the proof of the first.

Using the action of \( \text{Gal}_k \) (see [GW10, §5.2] for example), it is easy to deduce the proposition in general from the case where \( k = k_{\text{sep}} \). \( \square \)

We now drop our running assumption that \( H \) has reductive neutral component. The following is a useful analogue of Proposition 17.1.6:

**Proposition 17.1.8** For any smooth affine algebraic subgroup \( I \leq H \),

\[
(I \backslash H)(k) = \{ I(k_{\text{sep}})h \in I(k_{\text{sep}}) \triangleleft H(k_{\text{sep}}) : h \xi(h^{-1}) \in I(k_{\text{sep}}) \text{ for all } \xi \in \text{Gal}_k \}.
\]

**Proof.** Using the action of \( \text{Gal}_k \) (see [GW10, §5.2]), one easily reduces the proposition to the case where \( k = k_{\text{sep}} \).

We henceforth assume that \( k = k_{\text{sep}} \). The morphism \( H \to I \backslash H \) is faithfully flat and hence surjective [Mil17, Proposition 7.4(b)]. Arguing as in the proof of Proposition 17.1.6, we deduce that

\[
p : H(k_{\text{sep}}) \to (I \backslash H)(k_{\text{sep}}) \quad (17.16)
\]
is also surjective. Since the fibers are the cosets of \( I(k_{\text{sep}}) \) in \( H(k_{\text{sep}}) \) (see the discussion around (17.3)), we deduce the proposition. \( \square \)

**Definition 17.4.** A geometric class is a nonempty set of the form \( O(k) \) where \( O \subseteq X \) is an \( H \)-orbit.

We observe that \( \gamma_1, \gamma_2 \in X(k) \) are in the same geometric class if and only if \( O(\gamma_1) = O(\gamma_2) \). The condition that \( \gamma \) is relatively semisimple depends only on the geometric class of \( \gamma \) by Exercise 17.17. It therefore makes sense to speak of the set

\[
\Gamma_{\text{geo,ss}}(k) \quad (17.17)
\]
of relatively semisimple geometric classes.

By Proposition 17.1.6, when \( H \) has reductive neutral component, one has a bijection
\[ \Gamma_{\text{geo,ss}}(k) \longrightarrow \text{Im}(X(k) \longrightarrow (X/H)(k)). \quad (17.18) \]

Still assuming \( H \) has reductive neutral component, one has a natural sequence of maps

\[ \Gamma_{\text{ss}}(k) \longrightarrow X(k)/H(k) \longrightarrow \Gamma_{\text{geo,ss}}(k). \quad (17.19) \]

If \( k = k^{\text{sep}} \), then the composite is an isomorphism by Proposition 17.1.6. However, in general the first map is not surjective and the second map is not injective (even if \( k = k^{\text{sep}} \)) (see Exercise 17.4). If \( k \neq k^{\text{sep}} \) then the second map is not in general surjective.

For use in §17.7, we record two lemmas.

**Lemma 17.1.9** If \( \gamma \in X(k) \) is relatively semisimple with respect to \( H^\circ \) then it is relatively semisimple with respect to \( H \).

*Proof.* We can and do assume \( k = \overline{k} \). Let \( O(\gamma) \) be the \( H \)-orbit of \( \gamma \) in \( X \) and let \( O(\gamma)' \) be the \( H^\circ \)-orbit of \( \gamma \) in \( X \). Then using \( |\cdot| \) to denote underlying topological spaces,

\[ |O(\gamma)| = \bigcup_{h \in H^\circ(k) \setminus H(k)} |O(\gamma)' h|. \]

Therefore \( |O(\gamma)| \) is a finite union of closed subspaces of \( |X| \) and hence is closed. \( \square \)

Assume that

\[ H^\circ := H_r \times H_u \quad (17.20) \]

with \( H_r \) reductive and \( H_u \) unipotent.

**Lemma 17.1.10** An element \( \gamma \in X(k) \) is relatively semisimple with respect to the action of \( H \) if it is relatively semisimple with respect to the action of \( H_r \).

*Proof.* We can and do assume \( k = \overline{k} \). By Lemma 17.1.9, it suffices to treat the case \( H = H^\circ \). Throughout this proof, \( \gamma \) is relatively semisimple. We first deal with the case where \( \gamma \) is relatively regular with respect to the action of \( H_r \).

The subset \( Z \subseteq X(k) \) of points that are relatively regular with respect to the action of \( H_r \) can be identified with the set of closed points in an open subset of the underlying topological space of \( X \). This is a consequence of the upper semicontinuity of the dimension of stabilizers (see the discussion above [MFK94, Definition 0.9]). Let

\[ X^{\text{reg}} \subset X \]
be the open subscheme of $X$ with underlying topological space $Z$ given the reduced induced subscheme structure. Thus $X^{\text{reg}}(k)$ is the set of closed points of $Z$. It is easy to see that $X^{\text{reg}}$ is an $H$-invariant subscheme.

Assume for the moment that $\gamma \in X^{\text{reg}}(k)$. Now consider the GIT quotient

$$p : X \longrightarrow X/H_r.$$ 

Using the universal property of categorical quotients and the fact that $H_u$ and $H_r$ commute, we see that the action of $H_u$ on $X$ induces an action of $H_u$ on $X/H_r$.

Let $O(p(\gamma))$ be the $H_u$-orbit of $p(\gamma)$. Since $\gamma$ is relatively regular and relatively semisimple, the same is true of all elements of $O(p(\gamma))$. Hence for every $x \in O(p(\gamma))(k)$, Proposition 17.1.6 implies that $p^{-1}(x)(k) = xH_r(k)$. It follows that $p^{-1}(O(p(\gamma)))(k) = \gamma H(k)$ and thus $p^{-1}(O(p(\gamma)))$ is the orbit of $\gamma$ under $H$. On the other hand, by [Mil17, Theorem 17.64], the orbit $O(p(\gamma))$ is closed. Hence $p^{-1}(O(p(\gamma)))$ is closed.

Thus we have proven the proposition for $\gamma \in X^{\text{reg}}(k)$, in other words, when $\gamma$ is relatively regular with respect to $H_r$. We now reduce the general case to this one. Let $X^0 = X$, $X^1 = X - X^{\text{reg}}$, and for $i > 1$, let $X^i = X^{i-1} - (X^{i-1})^{\text{reg}}$. These subschemes are all preserved by $H$. Thus we have a sequence of closed subschemes

$$X = X^0 \supseteq X^1 \supseteq \cdots.$$ 

Since $X$ is Noetherian, this sequence stabilizes. It follows that there exists an $n$ such that $(X^n)^{\text{reg}} = X^n$. Thus our sequence is

$$X = X^0 \supseteq X^1 \supseteq \cdots \supseteq X^n \supseteq \emptyset$$

where for each $i$, one has $X^{i-1} - X^i = (X^{i-1})^{\text{reg}}$. Now let $\gamma \in X(k)$ be relatively semisimple with respect to $H_r$. Let $i$ be the largest index such that $\gamma \in X^i(k)$. Then $\gamma$ is relatively regular in $X^i(k)$ with respect to the action of $H_r$. Let $O(\gamma)$ be the $H_r$-orbit of $\gamma$ in $X$. Its intersection with $X^i$ is $O(\gamma)$ itself, and it is closed (being the intersection of two closed subschemes). Moreover the $H$-orbit of $\gamma$ in $X$ is contained in $X^i$. Thus we have reduced to the case where $\gamma$ is relatively regular with respect to $H_r$, as desired. \qed

### 17.2 Luna’s slice theorem and relative conjugacy classes

Assume for this section that $H$ is a smooth algebraic group over a field $k$ with reductive neutral component that acts (on the right) on an affine $k$-scheme $X$. The structure of $X/H$ around sufficiently nice points $\gamma \in X(k)$ has a simple structure. Luna’s slice theorem, Theorem 17.2.1 below, makes this precise. After stating this theorem we discuss the special case of quotients...
associated to conjugacy classes and relative conjugacy classes. In these cases a key corollary, Theorem 17.2.2, of Luna’s slice theorem can be refined to show that the relevant GIT quotients are isomorphic to quotients of tori by Weyl groups. This fact plays a crucial role in comparisons of trace formulae, to be discussed in §19.3.

If $H$ acts on an affine scheme $Y$ of finite type over $k$, we let

$$X \wedge^H Y := (X \times Y)/H. \quad (17.21)$$

Here the action of $H$ on $X \times Y$ is the diagonal action. The affine scheme $X \wedge^H Y$ is known as the **contracted product** of $X$ and $Y$. This construction will appear again in Definition 17.12 below in a slightly different setting. Other common notations for the contracted product are $X \times^H Y$ or $X \times_H Y$, but we have opted for the notation in (17.21) to avoid confusion with fiber products.

Let $\gamma \in X(k)$ be relatively semisimple and let $S \subset X$ be an $H_{\gamma}$-stable locally closed integral affine subscheme of finite type over $k$ such that $\gamma \in S(k)$. We then have a right action

$$S \times H \times H_{\gamma} \times H \to S \times H \quad (17.22)$$

given on points by $(s, h) \cdot (h_0, h') = (sh_0, h_0^{-1}hh')$. By Theorem 17.1.5, $H_{\gamma}$ is reductive. Assume additionally that $H_{\gamma}$ is smooth; this is automatic in characteristic zero. Then we can form the categorical quotient $S \wedge^{H_{\gamma}} H$. The natural action map $S \times H \to X$ is $H \times H$-equivariant and constant on $H$-orbits and hence induces a map

$$\varphi : S \wedge^{H_{\gamma}} H \to X.$$

Moreover, the map

$$S \times H \to S$$

given by projection to the first factor is $H_{\gamma}$-equivariant. It is even $H$-equivariant if we let $H$ act trivially on the codomain $S$. By the universal property of categorical quotients we deduce a morphism

$$(S \wedge^{H_{\gamma}} H)/H \to S/H_{\gamma} \quad (17.23)$$

which is in fact an isomorphism by Exercise 17.10. We therefore have a diagram

$$\begin{array}{ccc}
S \wedge^{H_{\gamma}} H & \xrightarrow{\varphi} & X \\
\downarrow & & \downarrow \\
(S \wedge^{H_{\gamma}} H)/H \cong S/H_{\gamma} & \xrightarrow{\varphi/H} & X/H.
\end{array}$$

Here $\varphi/H$ is just the morphism induced by $\varphi$. A morphism $X \to Y$ of affine $k$-schemes of finite type is **étale** if and only if it is smooth of relative dimension
0 (which is to say that the dimension of \( X \) and \( Y \) are equal). One should think of an étale map as an analogue in algebraic geometry of a covering map.

**Definition 17.5.** We say that \( S \) as above is an **étale slice** of the action of \( H \) on \( X \) at \( \gamma \) if

(a) The morphism \( \varphi \) is étale with open image \( U \).

(b) The morphism \( \varphi/H \) is étale and the induced morphism

\[
S \times^H H \rightarrow S/H \times_U U/H
\]

is an isomorphism.

Luna’s slice theorem asserts the existence of étale slices under mild hypotheses [MFK94, Appendix 1.D]:

**Theorem 17.2.1 (Luna’s slice theorem)** Assume that \( k \) is a characteristic zero field. Étale slices \( S \) exist for any \( \gamma \in X(k) \) with closed orbit along which \( X \) is normal. Moreover if \( X \) is smooth at \( \gamma \) then \( S \) can be taken to be smooth.

One can use this theorem to give a nice description of the quotient \( X/H \) itself. In more detail, for \( k \)-algebras \( R \) and subgroups \( H' \leq H \), let

\[
X^{H'}(R) := \{ x \in X(R) : xh = x \text{ for } h \in H'(R') \text{ and } R\text{-algebras } R' \}.
\]

This is a closed subscheme of \( X \) [Mil17, Theorem 7.1].

We have the following theorem, which is an amalgam of results of Luna and Richardson [MFK94, Appendix 1.D]:

**Theorem 17.2.2** Assume that \( X \) is normal and that \( k \) has characteristic zero. Let \( H' \leq H \) be a subgroup with reductive neutral component. Then for \( \gamma \in X^{H'}(k) \), the orbit of \( \gamma \) under \( H \) is closed if and only if the orbit of \( \gamma \) under the normalizer \( N_H(H') \) is closed. The map

\[
X^{H'}/N_H(H') \rightarrow X/H
\]

is finite. If \( \gamma \) is a representative for a generic closed orbit in \( X \), \( H' = H_\gamma \), and \( X^{H'} \) is irreducible, then the map (17.24) an isomorphism.

There is a slightly more precise description of what is meant by a generic closed orbit in [LR79], but yet more precision is needed in practice. This can be achieved in a nice family of examples. We elaborate in the remainder of the section.

When we consider the action of \( H \) on itself (\( X = H \)) via conjugation, the **Chevalley restriction theorem** provides a definitive description of the quotient \( X/H \) [Ste65, §6]:
Theorem 17.2.3 Assume $H$ is reductive and let $T \leq H$ be a maximal torus. Then the inclusion $T \rightarrow H$ induces an isomorphism

$$T/W(H,T) \rightarrow H/H,$$

where the action of $H$ on $H$ is via conjugation. \qed

In this case, the stabilizer $H_\gamma$ of $\gamma \in H(k)$ is called the centralizer of $\gamma$ and is denoted $C_{\gamma,H}$. We refer to §1.7 for the definition of the Weyl group $W(H,T)$ of $T$ in $H$.

The following is [Ste65, 2.11]:

Theorem 17.2.4 Assume $H$ is reductive and that $\gamma \in H(k)$ is semisimple. Then $\gamma$ is regular if and only if $C_{\gamma,H}$ is a maximal torus in $H$. \qed

In practice it is useful to know that one can place assumptions on $H$ so that $C_{\gamma,H}$ itself is connected. To make this precise, we recall some foundational work of Steinberg. Let

$$\sigma : H \rightarrow H$$

be an automorphism.

Definition 17.6. The automorphism $\sigma$ is semisimple if there is a smooth algebraic group $G$ containing $H_\mathfrak{g}$ and a semisimple $\gamma \in N_G(H)(\mathbb{K})$ such that $\sigma$ is equal to the restriction of conjugation by $\gamma$ to $H_\mathfrak{g}$.

This is an intuitive definition, but the following more technical definition is also quite useful:

Definition 17.7. The automorphism $\sigma$ is quasi-semisimple or quass if $\sigma$ fixes a Borel subgroup $B$ of $H_\mathfrak{g}$ and a maximal torus of $H_\mathfrak{g}$ contained in $B$.

In both of these definitions, the conditions on $\sigma$ are phrased in terms of the automorphism it induces on $H_\mathfrak{g}$. A semisimple automorphism is quass by [Ste68, Theorem 7.5]. Consider

$$H^\sigma(R) := \{ h \in H(R) : \sigma(h) = h \}. \quad (17.25)$$

This is an algebraic subgroup of $H$. We have the following basic result of Steinberg [Ste68, Theorem 9.1]:

Theorem 17.2.5 If $H$ is a semisimple simply connected group and $\sigma$ is a quass automorphism of $H$ then $H^\sigma$ is reductive (so it is connected). \qed

Corollary 17.2.6 If $H$ is a reductive group with simply connected derived group and $\gamma \in H(k)$ then $C_{\gamma,H}$ is connected.

The Jordan decomposition is badly behaved over imperfect fields. Therefore we will make the following convention for the remainder of this section: we say that an element $\gamma \in H(k)$ is semisimple if its image under the canonical map $H(k) \rightarrow H(\mathbb{k})$ is semisimple.
Proof. This is clear from Theorem 17.2.5 if $H$ is semisimple. In the general case, let $H^\text{der} \leq H$ be the derived group of $H$, let $Z_H \leq H$ be the center of $H$, and let $Z_{H,t}$ be the maximal torus contained in $Z_H$. We then have $Z_{H,t}H^\text{der} = H$ [Mil17, Proof of Proposition 21.60(c)]. Assume that $\gamma \in H^\text{der}(k)$. Then one checks that

$$C_{\gamma,H} = Z_{H,t}C_{\gamma,H^\text{der}}$$

and the corollary follows in this case. In general, we have $\gamma = z\gamma'$ where $(z,\gamma') \in Z_{H,t}(\overline{k}) \times H^\text{der}(\overline{k})$ and one checks that

$$(C_{\gamma,H})_{\pi} = C_{\gamma',H^\text{der}}$$

so we deduce the corollary in this case as well. □

Without any additional assumptions on $\sigma$, we can at least deduce the following result:

**Theorem 17.2.7** If $\sigma$ is semisimple, then $(H^\sigma)^\circ$ is reductive.

**Proof.** We may assume that $k = \overline{k}$. Choose $G$ and $\gamma$ as in the definition of a semisimple automorphism in Definition 17.6. The orbit of $\gamma$ under conjugation by $H$ is closed in $G$ by [BT65, Théorème 10.2(i)]. The stabilizer of $\gamma$ is $H^\gamma$, so we conclude by Theorem 17.1.5. □

**Corollary 17.2.8** Assume that $\sigma$ has finite order and that either the characteristic of $k$ is zero or the order of $\sigma$ is coprime to the characteristic of $k$. Then $(H^\sigma)^\circ$ is reductive.

**Proof.** The automorphism $\sigma$ is semisimple (see Exercise 17.6). □

Assume for the remainder of this section that the characteristic of $k$ is not 2. Let $G$ be a reductive group over $k$ and let $\sigma : G \rightarrow G$

be an automorphism of order 2. There is an action of $H := G^\sigma \times G^\sigma$ on $X = G$ given on points in a $k$-algebra $R$ by

$$X(R) \times H(R) \rightarrow X(R)$$

$$\quad (g,(g_t,g_r)) \mapsto g^{-1}_t g g_r.$$ (17.26)

In this case, the set $\Gamma(R)$ is often referred to as the set of **relative conjugacy classes**. This is a generalization of the notion of conjugacy classes in a strict sense (see Exercise 17.9).

There is a great deal of helpful geometry available in this special case that aids in the study of the set of classes $\Gamma(R)$ of Definition 17.3. It is useful to introduce the map

$$B_\sigma : G \rightarrow G$$
given on points by $g \mapsto g^{-\sigma} g$. We denote by $Q$ the scheme theoretic image of $B_\sigma$. Then $Q$ is a closed affine $k$-subscheme of $G$.

We have an isomorphism [Ric82, Lemma 2.4] of $k$-schemes

$$B_\sigma : G^\sigma \backslash G \longrightarrow Q \tag{17.27}$$

which intertwines the right action of $G$ with the following action of $G$ on $Q$:

$$Q(R) \times G(R) \longrightarrow Q(R)$$

$$(g, q) \mapsto g^{-\sigma} q g. \tag{17.28}$$

In particular, one has an isomorphism of affine $k$-schemes

$$X/H = G/H = G^\sigma \backslash G/G^\sigma \cong Q/G^\sigma,$$

where the action on the right is given by (17.28). In particular, when restricted to $G^\sigma$, the action (17.28) is just conjugation, explaining why in this setting $\Gamma(R)$ is referred to as the set of relative conjugacy classes.

For $\gamma \in G(R)$ let

$$C_{\gamma, G^\sigma} := C_{\gamma, G} \cap G^\sigma.$$

We leave the following lemma as an exercise (see Exercise 17.11).

**Lemma 17.2.9** For a $k$-algebra $R$, there is an isomorphism

$$H_\gamma(R) \longrightarrow C_{B_\sigma(\gamma), G^\sigma}(R)$$

$$(g_1, g_2) \mapsto g_2.$$

We then have the following proposition [Ric82, §7]:

**Proposition 17.2.10** An element $\gamma \in X(k) = G(k)$ is relatively semisimple with respect to the action of $H = G^\sigma \times G^\sigma$ as in (17.26) if and only if $B_\sigma(\gamma)$ is semisimple in the usual sense. □

We have seen that the notion of a maximal torus in a reductive group is absolutely crucial. In the case of symmetric subgroups, the following definition is a substitute:

**Definition 17.8.** A torus $T \subseteq G$ is said to be $\sigma$-split if for all $k$-algebras $R$ and all $t \in T(R)$, one has $t^{-1} = t^\sigma$.

Beware that in [Ric82] a $\sigma$-split torus is called a “$\sigma$-anisotropic torus.”

It is not hard to see that if $T$ is any $\sigma$-split torus then $T \leq Q$. Indeed, every element of $T(\ol{k})$ is in the image of the isogeny $t \mapsto t^2 = t^{-\sigma} t$. Moreover, $\sigma$-split tori exist (see [Ric82, (2.5)] and [Vus74, §1]).

Let $T_\sigma \leq Q$ be a maximal $\sigma$-split torus. We have a corresponding Weyl group
\[ W(G, T_\sigma) = N_G(T_\sigma)/Z_G(T_\sigma), \]  

(17.29)

sometimes called the **little Weyl group**. Here \( N_G(T_\sigma) \) (resp. \( Z_G(T_\sigma) \)) is the normalizer (resp. centralizer) of \( T_\sigma \) in \( G \). As recalled in §1.7 this is a finite étale group scheme over \( k \). We have the following generalization of the Chevalley restriction theorem [Ric82, Corollary 11.5]:

**Theorem 17.2.11 (Richardson)** Let \( T_\sigma \subseteq G \) be a maximal \( \sigma \)-split torus. The inclusion \( T_\sigma \rightarrowtail Q \) induces an isomorphism

\[ T_\sigma / W(G, T_\sigma) \rightarrowtail Q / G^\sigma. \]

\[ \square \]

### 17.3 Local geometric classes

Let \( k \) be a field. Suppose that \( \gamma \in X(k) \) has a closed orbit under \( H \) and that \( H_\gamma \) is smooth (we do not assume that \( H \) has reductive neutral component). We then have a sequence

\[ 1 \rightarrow H_\gamma(k) \rightarrow H(k) \rightarrow (H_\gamma \backslash H)(k) \]

and we can identify the last object with \( O(\gamma)(k) \) by means of Proposition 17.1.2. We now briefly explain how to compute the image using a little Galois cohomology. The reader should feel free to omit this discussion and refer back to it later if necessary. Our primary reference is [Ser02, §I.5].

For a smooth affine algebraic group \( H \) over \( k \), one defines \( Z^1(k, H) \), the set of 1-cocycles of \( \text{Gal}_k \) in \( H(k^{\text{sep}}) \), to be the set of maps

\[ c : \text{Gal}_k \rightarrow H(k^{\text{sep}}) \]

\[ \sigma \mapsto c(\sigma) \]

that satisfy the **cocycle condition**

\[ c(\sigma_1 \sigma_2) = c(\sigma_1) c(\sigma_2) \]  

(17.30)

for all \( \sigma_1, \sigma_2 \in \text{Gal}_k \). Two 1-cocycles \( c \) and \( c' \) are called **cohomologous** if there is a \( h \in H(k^{\text{sep}}) \) such that \( c'(\sigma) = h^{-1} c(\sigma) h \) for all \( \sigma \in \text{Gal}_k \). This is an equivalence relation and the set of equivalence classes is denoted by

\[ H^1(k, H). \]

(17.31)

It is the **first Galois cohomology set** of \( \text{Gal}_k \) in \( H \) (or more precisely in \( H(k^{\text{sep}}) \)). This is not a group unless \( H \) is commutative, but it does come with a distinguished element, called the **neutral element**. It is the class of
the 1-cocycle sending every element of Gal$_k$ to 1. The cohomology sets are functorial in $H$ in the obvious sense. We also mention, though this is not at all obvious from the definition, that the sets $H^1(k,H)$ can be computed in many interesting cases. We refer to [Ser02] for more details.

When $H$ is commutative, the set $H^1(k,H)$ has a group structure, and using this fact, Borovoi was able to give a group structure on $H^1(k,H)$ for all reductive $H$ (at least if $k$ is of characteristic zero) [Bor98b]. Though we will not use Borovoi’s work, we point out that it can be used to give an elegant treatment of the stabilization of the trace formula [Lab99]. In fact, Borovoi’s constructions borrow heavily from Kottwitz’s work on the stabilization of the trace formula [Kot84, Kot86b].

If $I \to H$ is a morphism of smooth group schemes over $k\hookrightarrow$, let
\[ D(k\hookrightarrow I\hookrightarrow H) := \ker(H^1(k,I) \to H^1(k,H)). \] (17.32)
Here the kernel simply means all classes that are sent to the neutral element of $H^1(k,H)$.

Now suppose that $I \subseteq H$ is a smooth subgroup scheme. We leave the following lemma as Exercise 17.12:

**Lemma 17.3.1** The map
\[ \text{cl} : (I\setminus H)(k) \to D(k,I,H) \]
\[ I(k^{\text{sep}})h \mapsto (\sigma \mapsto h\sigma(h^{-1})) \]
defines a bijection between the $H(k)$-orbits in $(I\setminus H)(k)$ and $D(k,I,H)$. \qed

We refer to the map cl in the lemma as the **class map**. The lemma implies that there is an exact sequence of pointed sets
\[ 1 \to I(k\setminus H)(k) \to (I\setminus H)(k) \to D(k,I,H) \to 1. \] (17.33)

In other words, the set of classes in the geometric class of $\gamma$ is in bijection with $D(k,I,H)$. This observation is the beginning of the theory of endoscopy, first introduced by Langlands in [Lan83]. The groundwork of the theory was laid by Kottwitz in [Kot84, Kot86b] and more recent useful references include [KS99, Lab99].

To proceed further, we require a description of the other $H(k)$-orbits in $(I\setminus H)(k)$. By Proposition 17.1.8, any element of $(I\setminus H)(k)$ is a coset $I(k^{\text{sep}})h$ for some $h \in H(k^{\text{sep}})$ satisfying $h\sigma(h^{-1}) \in I(k^{\text{sep}})$ for all $\sigma$. Let $I_h$ be the stabilizer of $I(k^{\text{sep}})h$ under the action of $H$. Then tautologically we have
\[ (I\setminus H)(k) = \bigsqcup I(h) \setminus H(k) \to (I\setminus H)(k), \] (17.34)
where the disjoint union is over the set of $H(k)$-orbits in $(I\setminus H)(k)$ and the implicit maps are given by
\[ I_h(k) \setminus H(k) \to (I \setminus H)(k) \]
\[ I_h(k)x \to I(k^{\text{sep}})hx. \]

We note that
\[
\text{cl}(\text{Im}(I_h(k) \setminus H(k)) \to (I \setminus H)(k)))
\]
(17.35)
is simply the class of the 1-cocycle \( h \sigma(h^{-1}) \).

The groups \( I_h \) are not isomorphic as \( h \) varies, but they are related in an important manner. Two smooth algebraic groups \( Q \) and \( \tilde{Q} \) are said to be \textbf{forms} of each other if there is an isomorphism
\[
\varphi : Q_{k^{\text{sep}}} \to \tilde{Q}_{k^{\text{sep}}}.
\]
(17.36)
The concept of a form arises naturally in the classification of algebraic groups; one can proceed in many circumstances by classifying groups of a certain type over \( k^{\text{sep}} \) and then classifying all of the groups over \( k \) that base change to a given group over \( k^{\text{sep}} \). A pair \((\tilde{Q}, \varphi)\) consisting of an algebraic group \( \tilde{Q} \) over \( k \) and an isomorphism \( \varphi \) as in (17.36) is called an \textbf{inner form} of \( Q \) if for all \( \sigma \in \text{Gal}_k \), the automorphism \( \varphi^{-1} \circ \sigma \circ \varphi \circ \sigma^{-1} \) of \( Q_{k^{\text{sep}}} \) is an inner automorphism. Usually the isomorphism \( \varphi \) is omitted from notation. It is easy to check that the \( I_h \) are all inner forms of \( I \) (see Exercise 17.13).

Now suppose that \( k \) is a local field. In keeping with our usual notation for local and global fields, we let \( k = F \). We record the following useful result (see [Poo17, Proposition 3.5.73]):

\textbf{Proposition 17.3.2} If \( X \to Y \) is a smooth map of separated schemes of finite type over \( F \), then \( X(F) \to Y(F) \) is open.

Using this proposition, we prove the following lemma:

\textbf{Lemma 17.3.3} Let \( I \leq H \) be a smooth algebraic subgroup. The map
\[
\text{cl} : (I \setminus H)(F) \to D(F, I, H)
\]
is continuous if we give \( D(F, I, H) \) the discrete topology. In particular, the image of \( I(F) \setminus H(F) \) in \( (I \setminus H)(F) \) is closed.

\textit{Proof.} The last assertion follows from the first and Lemma 17.3.1. To prove the first assertion, it suffices to show that the preimage under \( \text{cl} \) of any point in \( D(F, I, H) \) is open. By Lemma 17.3.1 and (17.34), this is equivalent to the assertion that the maps
\[
I_h(F) \setminus H(F) \to (I \setminus H)(F)
\]
of (17.35) have open image. To check that these maps have open image, it suffices to prove that the map \( H(F) \to (I \setminus H)(F) \) given by the action of \( H(F) \) on the orbit of \( I(k^{\text{sep}})h \) is open. This map is induced by a morphism
of schemes $\phi : H \to I \backslash H$, so by Proposition 17.3.2 it suffices to check that $\phi$ is smooth. Since $I_\gamma$ is an inner form of $I$, it is smooth, so $\phi$ is smooth by Proposition 17.1.2.

The following theorem gives conditions under which $D(F, I, H)$ is finite:

**Theorem 17.3.4** Assume $I$ is a smooth algebraic group over the local field $F$. If $I$ is reductive or $F$ has characteristic zero, then $H^1(F, I)$ is finite.

*Proof.* If $F$ has characteristic zero, this is contained in [Ser02, §III.4.2 and 4.3]. In the reductive case, see [Ser02, §III.4.3, Remarks].

### 17.4 Local relative orbital integrals

Let $F$ be a local field and let $H$ be a smooth algebraic group defined over $F$ acting on a smooth affine scheme $X$. We define local relative orbital integrals in this section.

Assume we are given a quasi-character $\chi : H(F) \to \mathbb{C}^\times$.

**Definition 17.9.** A relatively semisimple element $\gamma \in X(F)$ is $\chi$-relevant if $\chi$ is trivial on $H_\gamma(F)$.

If $\chi$ is understood, we often omit it from notation and speak simply of relevant elements. We require the notion of relevance to define local orbital integrals, and only relevant $\gamma$ will contribute to the global trace formula (see Theorem 17.7.4 below). If $\chi$ is trivial then all relatively semisimple elements are relevant.

Since we have assumed $X$ is affine and smooth over $F$, if $F$ is archimedean then $X(F)$ is a smooth manifold, so $C_c^\infty(X(F))$ is defined. If $F$ is nonarchimedean, we let $C_c^\infty(X(F))$ be the space of compactly supported locally constant functions on $X(F)$.

We require one more assumption on $\gamma \in X(F)$ before defining local relative orbital integrals.

**Definition 17.10.** We say $\gamma \in X(F)$ is relatively unimodular if $\delta_H|_{H_\gamma} = \delta_{H_\gamma}$.

See §3.2 for definition of the modular character $\delta_H$. In particular, if $H^\circ$ is reductive then relatively semisimple $\gamma$ are unimodular by Theorem 17.1.5.

Now suppose we are given $f \in C_c^\infty(X(F))$ and a relevant relative semisimple and relatively unimodular $\gamma \in X(F)$. We fix right Haar measures $d_rh$ on $H(F)$ and $d_rh_\gamma$ on $H_\gamma(F)$. Because $\gamma$ is relatively unimodular, one can then form the quotient measure $\frac{d_rh}{d_rh_\gamma}$ (see Theorem 3.2.2). We can then form the local relative orbital integral

$$\text{RO}_\gamma(f) := \text{RO}_\gamma(f) = \int_{H_\gamma(F) \backslash H(F)} f(\gamma h)\chi(h) \frac{d_rh}{d_rh_\gamma}.$$  \hspace{1cm} (17.37)
Note that we have used the fact that $\gamma$ is $\chi$-relevant to define this integral. This integral depends on the choice of measures, but traditionally they are not encoded in the notation. In some settings it may be necessary to replace $H_\gamma$ by some subgroup of $H_\gamma$ containing $(H_\gamma)^\circ$.

**Theorem 17.4.1** If $\gamma \in X(F)$ is relevant, relatively unimodular and relatively semisimple and $H_\gamma$ is smooth then the integral defining $\text{RO}_\gamma(f)$ is absolutely convergent.

**Proof.** Since the measure $\frac{dh}{d_x h_\gamma}$ is a Radon measure on $H_\gamma(F)\backslash H(F)$, to show the integral is well-defined and absolutely convergent, it is enough to construct a pull-back map

$$C_c(X(F)) \longrightarrow C_c(H_\gamma(F)\backslash H(F))$$

attached to the map

$$H_\gamma(F)\backslash H(F) \longrightarrow X(F)$$

$$H_\gamma(F)h \longmapsto \gamma h. \quad (17.38)$$

Thus it suffices to show that this map is proper. The orbit $O(\gamma)$ of $\gamma$ is closed in $X$. Hence $O(\gamma)(F)$ is closed in $X(F)$ by Theorem 2.2.1.

It therefore suffices to treat the case where $X = O(\gamma) = H_\gamma \backslash H$. In this case, we invoke Lemma 17.3.3 to see that the image of $H_\gamma(F)\backslash H(F)$ in $(H_\gamma \backslash H)(F)$ is closed. \hfill $\square$

### 17.5 Torsors

In this section, we discuss torsors, which play a key role in §17.6. It is convenient to return to the more general setting of §17.1. Thus $H$ is a smooth group scheme over a Noetherian ring $k$ acting (on the right) on an affine scheme $X$ of finite type over $k$.

For example, one could take $X = H$ with the natural right $H$-action. This is known as the **trivial $H$-torsor**. This example can be profitably generalized:

**Definition 17.11.** The scheme $X$ is an $H$-torsor if it is faithfully flat over $k$ and the map

$$1_X \times a : X \times H \longrightarrow X \times X$$

is an isomorphism.

An equivalent (and more intuitive) way to define an $H$-torsor is to say that it is a scheme faithfully flat and of finite type over $k$ equipped with an action of $H$ that becomes isomorphic to the trivial $H$-torsor after a faithfully flat finite type base extension. For more details, we refer the reader to [Poo17, §6.5.1].
Our definition is the specialization of the definition in [Poo17, §6.5.1] to the case of affine Noetherian base schemes. The set of all isomorphism classes of $H$-torsors is denoted by

$$\tilde{H}^1_{\text{fppf}}(k, H) = \tilde{H}^1_{\text{fppf}}(\text{Spec}(k), H). \quad (17.39)$$

It is a pointed set with the class of the trivial torsor as neutral element. The \textit{fppf} stands for \textit{fidèlement plat de présentation finie}, since torsors are required to be faithfully flat and locally of finite presentation over $k$. Since we have assumed $H$ is smooth, $H$-torsors will in fact be smooth over Spec$(k)$ [Poo17, Remark 6.5.2]. Assume that $k'$ is a noetherian $k$-algebra. Then one has a natural map

$$\tilde{H}^1_{\text{fppf}}(k, H) \to \tilde{H}^1_{\text{fppf}}(k', H) \quad (17.40)$$

given by sending an $H$-torsor $Y$ to its base extension $Y_{k'}$.

Suppose that $I \leq H$ is a smooth closed subgroup scheme. Let $Y$ be an $I$-torsor. One has an action of $I \times H$ on $Y \times H$ given on points in a $k$-algebra $R$ by

$$Y(R) \times H(R) \times I(R) \times H(R) \to Y(R) \times H(R)$$

$$(y, h, i, h') \mapsto (yi, i^{-1}hh'). \quad (17.41)$$

The actions of $I$ and $H$ commute.

**Definition 17.12.** The \textbf{contracted product of $Y$ and $H$}

$$Y \wedge^I H := (Y \times H)/I$$

is the quotient of $Y \times H$ by the action of $I$.

The contracted product $Y \wedge^I H$ is an $H$-torsor, where the action is inherited from the action of $H$ in (17.41). We will not enter into a detailed discussion of what is meant by a quotient in this context. Briefly, the action of $I$ on $Y \times H$ defines a quotient which is a presheaf of sets for the fppf site on Spec$(k)$. A useful reference for this concept is [Poo17, Chapter 6]. The corresponding sheaf is representable [Poo17, Theorem 6.5.10]. It follows from the construction of the quotient via fppf sheaves that $Y \wedge^I H$ is a categorical quotient of $Y \times H$ by $I$. Moreover, for any Noetherian $k$-algebra $k'$ there is a canonical identification

$$(Y \wedge^I H)_{k'} = Y_{k'} \wedge^{I_{k'}} H_{k'}. \quad (17.42)$$

We point out the notion of a contracted product makes sense in greater generality as long as one knows how to interpret the quotient. See (17.21) for the case where $k$ is a field and $H^0$ is reductive. The contracted product provides us with a morphism of pointed sets
\[
\tilde{H}_{\text{fppf}}^1(k, I) \rightarrow \tilde{H}_{\text{fppf}}^1(k, H) \tag{17.43}
\]

sending the isomorphism class of an \(I\)-torsor \(Y\) to the isomorphism class of \(Y \wedge^I H\).

Assume momentarily that \(k\) is a field. Then there is a canonical bijection
\[
\tilde{H}_{\text{fppf}}^1(k, H) \rightarrow H^1(k, H). \tag{17.44}
\]

It is constructed as follows. Suppose we are given an \(H\)-torsor \(X\). Then, since \(X\) is smooth, we can choose \(x \in X(k_{\text{sep}})\) by Lemma 17.1.7. Since the map \(1_X \times a : X \times H \rightarrow X \times X\) is an isomorphism, we see that the map \(a(x, \cdot) : H(k_{\text{sep}}) \rightarrow X(k_{\text{sep}})\) is a bijection. In particular, for each \(\sigma \in \text{Gal} k\), there is a \(c(\sigma) \in H(k_{\text{sep}})\) such that \(\sigma(x) = x.c(\sigma)\). One checks that \(c(\sigma)\) is a 1-cocycle and that replacing \(X\) by an equivalent \(H\)-torsor \(X'\) and \(x\) by \(x' \in X'(k_{\text{sep}})\) yields a cohomologous 1-cocycle. The map (17.44) sends the equivalence class of \(X\) to the class of \(c\). We omit the construction of the inverse map (see [Poo17, §5.12.4]). If \(c\) is a 1-cocycle in \(H(k_{\text{sep}})\) constructed from \(X\) and a choice of \(x \in X(k_{\text{sep}})\) as above we say that \(c\) is a 1-cocycle in \(H(k_{\text{sep}})\) attached to \(X\). Using (17.44), we occasionally allow ourselves to identify \(\tilde{H}_{\text{fppf}}^1(k, H)\) and \(H^1(k, H)\) when \(k\) is a field.

**Lemma 17.5.1** Assume \(k\) is a field and that \(I\) is a smooth closed subgroup of \(H\). One has a commutative diagram
\[
\begin{array}{ccc}
\tilde{H}_{\text{fppf}}^1(k, I) & \rightarrow & \tilde{H}_{\text{fppf}}^1(k, H) \\
\sim & & \sim \\
H^1(k, I) & \rightarrow & H^1(k, H),
\end{array}
\]

where the vertical arrows are given by the bijection (17.44), the top arrow is given by (17.43) and the bottom arrow is given by observing that a 1-cocycle with coefficients in \(I(k_{\text{sep}})\) has coefficients in \(H(k_{\text{sep}})\).

**Proof.** The universal property of categorical quotients gives a \(\text{Gal} k\)-equivariant map
\[
(Y(k_{\text{sep}}) \times H(k_{\text{sep}}))/I(k_{\text{sep}}) \rightarrow (Y \wedge^I H)(k_{\text{sep}}). \tag{17.46}
\]

We claim that this map is bijective.

To see this we employ the useful technique of reduction to the trivial torsor. Choose \(y \in Y(k')\) for some finite separable extension \(k'/k\) using Lemma 17.1.7. There is then an isomorphism \((I \times H)_{k'} \rightarrow (Y \times H)_{k'}\) given on points in a \(k'\)-algebra \(R\) by
17.6 Adelic classes

Let $F$ be a global field and let $H$ be a smooth algebraic group over $F$ acting on an affine scheme $X$ of finite type over $F$ on the right. Thus we are in the setting of §17.1 in the special case where $F = k$. We now develop topological results on $H(A_F)$-classes in $X(A_F)$ that are necessary to study global relative orbital integrals (see Proposition 17.7.1). Our exposition is based on the treatment in [GH15] and generalizes some of the results of loc. cit.

**Theorem 17.6.1** Suppose that $\gamma$ is relatively semisimple and $H_\gamma$ is smooth. Then the map

$$H_\gamma(A_F) \backslash H(A_F) \longrightarrow X(A_F)$$

is proper.

For $\gamma$ as in the statement of the theorem, one has that $O(\gamma)(A_F)$ is closed in $X(A_F)$ by Theorem 2.2.1. It is therefore no loss of generality to assume that $X = O(\gamma)$, which is to say that there is a smooth subgroup $I$ of $H$ such that $X = I \backslash H$. Recall that for any finite set $S$ of places of $F$ including the infinite places, we set

$$I(R) \times H(R) \overset{\sim}{\rightarrow} Y(R) \times H(R)$$

$$(i, h) \mapsto (yi, i^{-1}h).$$

This intertwines the right action of $I_{k'}$ on $(I \times H)_{k'}$ induced by its right action on $I_{k'}$ with the right action of $I_{k'}$ on $(Y \times H)_{k'}$. Using the universal property of categorical quotients we deduce that there is a commutative diagram

$$\begin{array}{ccc}
I_k \times H_k & \overset{\sim}{\longrightarrow} & Y_k \times H_k \\
\downarrow & & \downarrow \\
I_{k'} \times H_{k'}/I_{k'} & \overset{\sim}{\longrightarrow} & Y_{k'} \wedge I_{k'}/H_{k'}
\end{array}$$

where the horizontal arrows are the quotient maps. The fact that (17.46) is bijective now follows from (17.42) and Proposition 17.1.8 applied to $I_{k'} \times H_{k'}/I_{k'}$.

Let $y \in Y(k_{\text{sep}})$ and for each $\sigma \in \text{Gal}_k$, let $c(\sigma) \in I(k_{\text{sep}})$ be the element such that $\sigma(y) = yc(\sigma)$. Thus $c$ is a 1-cocycle in $I(k_{\text{sep}})$ attached to $Y$. Now $(y \times 1)I(k_{\text{sep}}) \subseteq (Y \wedge I)(k_{\text{sep}})$ and

$$\sigma(y \times 1)I(k_{\text{sep}}) = (yc(\sigma) \times 1)I(k_{\text{sep}}) = (y \times c(\sigma))I(k_{\text{sep}}).$$

Thus $c$ is also a 1-cocycle in $H(k_{\text{sep}})$ attached to $Y \wedge I$. \qed
\[ \hat{O}_F^S = \prod_{v \in S} \mathcal{O}_{F_v} \quad \text{and} \quad \mathcal{O}_F^S := F \cap \hat{O}_F^S. \]

Thus \( \mathcal{O}_F^S \subset F \) is the set of elements of \( F \) that are integral outside of the finite places in \( S \). We also recall the notion of a model of an affine scheme reviewed in §2.4. For a sufficiently large set \( S \) of places of \( F \), we can choose models \( \mathcal{H} \) and \( \mathcal{X} \) of \( H \) and \( X = I \backslash H \) over \( \mathcal{O}_F^S \), respectively, such that there is a map

\[ \mathcal{H} \to \mathcal{X} \]

whose generic fiber is the canonical map \( H \to X \) [Poo17, Theorem 3.2.1]. We let \( \mathcal{I} \) be the schematic closure of \( I \) in \( \mathcal{H} \).

The map \( \hat{O}_F^S \to \mathbb{A}_F^S \) is an open topological embedding so

\[ \mathcal{X}(\hat{O}_F^S) \to \mathcal{X}(\mathbb{A}_F^S) = X(\mathbb{A}_F^S) \quad (17.47) \]

is an open topological embedding by Exercise 2.2. Similarly we obtain, for every finite place \( v \), open topological embeddings

\[ \mathcal{X}(\mathcal{O}_{F_v}) \to \mathcal{X}(F_v) = X(F_v). \quad (17.48) \]

By Proposition 2.4.7 we have a homeomorphism of topological spaces

\[ X(\mathbb{A}_F) \cong \prod_v X(F_v), \quad (17.49) \]

where the restricted direct product is taken with respect to the open sets \( \mathcal{X}(\mathcal{O}_{F_v}) \). This discussion is also valid with \( X \) and \( \mathcal{X} \) replaced by \( H \) (resp. \( I \)) and \( \mathcal{H} \) (resp. \( \mathcal{I} \)). In fact we discussed the algebraic group case in §2.3. Moreover, there are again isomorphisms of topological groups

\[ H(\mathbb{A}_F) \cong \prod_v H(F_v) \quad \text{and} \quad I(\mathbb{A}_F) \cong \prod_v I(F_v), \quad (17.50) \]

where the restricted direct product is taken with respect to the subgroups \( H(\mathcal{O}_{F_v}) \) (resp. \( \mathcal{I}(\mathcal{O}_{F_v}) \)) for all finite places \( v \). These isomorphisms are compatible with the group actions \( X(F_v) \times H(F_v) \to X(F_v) \) and \( X(\mathbb{A}_F) \times H(\mathbb{A}_F) \to X(\mathbb{A}_F) \) because the generic fiber of \( H \to \mathcal{X} \) is \( H \to X \). It is tempting to try to define the quotient \( \mathcal{I} \backslash \mathcal{H} \) and relate it to \( \mathcal{X} \), but we do not wish to discuss representability of quotients over Dedekind rings. Fortunately, coming to grips with this issue is unnecessary for our purposes.

**Lemma 17.6.2** Assume that \( I \) is connected. For a large enough finite set \( S \) of places of \( F \) including the infinite places, the inverse image of \( \mathcal{X}(\hat{O}_F^S) \) in \( H(\mathbb{A}_F^S) \) is equal to \( I(\mathbb{A}_F^S) \mathcal{H}(\hat{O}_F^S) \). The morphism \( \mathcal{H} \to \mathcal{X} \) induces a bijection

\[ \mathcal{I}(\hat{O}_F^S) \backslash \mathcal{H}(\hat{O}_F^S) \to \mathcal{X}(\hat{O}_F^S). \]
Proof. Since $H$ is connected and the map $H \to X$ is smooth by Proposition 17.1.2, the map

$$H(\mathbb{A}_F) \to X(\mathbb{A}_F)$$

is open by [Con12b, Theorem 4.5]. For any finite set $S$ of places of $F$ containing the infinite places, the subset $\mathcal{H}(\mathcal{O}_F^S) \subset H(\mathbb{A}_F^S)$ is open and so its image is open in $X(\mathbb{A}_F^S)$. Upon enlarging $S$, we see that this image must be $X(\mathbb{A}_F^S)$ by definition of the restricted direct topology on $X(\mathbb{A}_F)$.

Let $v \notin S$ be a place of $F$. If $h \in H(F_v)$ is in the inverse image of $X(O_{F_v})$ then there is some $h' \in \mathcal{H}(O_{F_v})$ and an $x \in I(F_v)$ such that $xh = h'$ by Proposition 17.1.8 which implies that $x = hh'^{-1} \in I(F_v)$. Thus we deduce the first assertion of the lemma.

Again applying Proposition 17.1.8, if $h, h' \in \mathcal{H}(O_{F_v})$ are mapped to the same point of $X(O_{F_v})$ then there is some $x \in I(F_v)$ such that $x = h = h'$. It follows that $x = h'^{-1} \in I(F_v)$, and we deduce the second assertion of the lemma.

Proof of Theorem 17.6.1: As mentioned below the statement of Theorem 17.6.1, we can and do assume that $X = O(\gamma) = H \setminus H$. Since the map

$$H^c \setminus H \to H \setminus H$$

is proper by Lemma 17.1.4, the map

$$(H^c \setminus H)(\mathbb{A}_F) \to (H \setminus H)(\mathbb{A}_F)$$

is proper by [Con12b, Proposition 4.4]. Here we have also used the fact, mentioned above, that for any smooth algebraic subgroup $I \leq H$, the scheme $I \setminus H$ is separated and of finite type over $F$ [Mil17, Theorem 7.18].

Thus we are reduced to showing that for any connected smooth subgroup $I \leq H$, the map

$$I(\mathbb{A}_F) \setminus H(\mathbb{A}_F) \to (I \setminus H)(\mathbb{A}_F)$$

is proper. To keep notation consistent with Lemma 17.6.2, we write $X = I \setminus H$ and let $\mathcal{X}$ and $\mathcal{H}$ be defined as above that lemma.

Let $\Omega \subset X(\mathbb{A}_F)$ be a compact set. We must show its inverse image in $I(\mathbb{A}_F) \setminus H(\mathbb{A}_F)$ is compact. Let $S$ be a finite set of places of $F$ including the infinite places. It suffices to treat the case $\Omega = \Omega_S \times \Omega^S$ where $\Omega_S \subset X(F_S)$ is compact and $\Omega^S = \mathcal{X}(\mathcal{O}_F^S)$ since any compact subset admits a finite cover by open sets whose closure is of this form.

Taking $S$ sufficiently large and invoking Lemma 17.6.2, we see that the inverse image of $\Omega^S$ in $I(\mathbb{A}_F^S) \setminus H(\mathbb{A}_F^S)$ is $I(\mathbb{A}_F^S) \mathcal{H}(\mathcal{O}_F^S)$, hence compact. On the other hand, the inverse image of $\Omega_S$ in $I(F_S) \setminus H(F_S)$ is compact by Lemma 17.3.3.

We observe the following:
Lemma 17.6.3 If $Y$ is an affine scheme of finite type over $F$ then $Y(F)$ is discrete and closed in $Y(\mathcal{A}_F)$.

Proof. The set $F < \mathcal{A}_F$ is discrete and closed by Lemma 2.1.3 and hence we conclude by Exercise 2.2. □

We pause to prove a result used earlier in Lemma 14.3.2:

Lemma 17.6.4 Let $G$ be a reductive group over $F$ and let $H \leq G$ be a reductive closed subgroup. The image of $G(F)$ in $A_G \backslash G(\mathcal{A}_F)/H(\mathcal{A}_F)$ is discrete and closed.

Here, of course, we have given $A_G \backslash G(\mathcal{A}_F)/H(\mathcal{A}_F)$ the quotient topology.

Proof. We begin by showing that the image of $G(F)$ in $G(\mathcal{A}_F)/H(\mathcal{A}_F)$ is discrete and closed. The set $(G/H)(F)$ is discrete and closed in $(G/H)(\mathcal{A}_F)$ by Lemma 17.6.3. The natural map

$$G(\mathcal{A}_F)/H(\mathcal{A}_F) \longrightarrow (G/H)(\mathcal{A}_F)$$

is a continuous injection, so the inverse image of $(G/H)(F)$ in $G(\mathcal{A}_F)/H(\mathcal{A}_F)$ is again discrete and closed. But this inverse image contains the image of $G(F)$ in $G(\mathcal{A}_F)/H(\mathcal{A}_F)$, hence $G(F)$ is discrete and closed in $G(\mathcal{A}_F)/H(\mathcal{A}_F)$.

Define $G(\mathcal{A}_F)^1$ as in (2.16); thus $A_G G(\mathcal{A}_F)^1 \leq G(\mathcal{A}_F)$ is a closed subgroup of finite index and is all of $G(\mathcal{A}_F)$ if $F$ is a number field by Lemma 2.6.2. Let

$$A' := A_G \cap H(\mathcal{A}_F).$$

The subset

$$A' G(\mathcal{A}_F)^1/H(\mathcal{A}_F) \subseteq G(\mathcal{A}_F)/H(\mathcal{A}_F)$$

is closed and contains the image of $G(F)$ in $G(\mathcal{A}_F)/H(\mathcal{A}_F)$. Hence the image $G(F)H(\mathcal{A}_F)$ of $G(F)$ in $A' G(\mathcal{A}_F)^1/H(\mathcal{A}_F)$ is discrete and closed. We have natural maps

$$A' G(\mathcal{A}_F)^1/H(\mathcal{A}_F) \longrightarrow A_G \backslash A_G G(\mathcal{A}_F)^1/H(\mathcal{A}_F) \longrightarrow A_G \backslash G(\mathcal{A}_F)/H(\mathcal{A}_F).$$

The first is a homeomorphism and the second is a closed embedding. Moreover the composite map sends $G(F)H(\mathcal{A}_F)$ to $A_G G(F)H(\mathcal{A}_F)$. Since $G(F)H(\mathcal{A}_F)$ is discrete and closed in $A' G(\mathcal{A}_F)^1/H(\mathcal{A}_F)$ we deduce that $A_G G(F)H(\mathcal{A}_F)$ is discrete and closed in $A_G \backslash G(\mathcal{A}_F)/H(\mathcal{A}_F)$. □

Theorem 17.6.1 gives us useful topological information about a single relative class. To proceed further, we must analyze all of the $H(\mathcal{A}_F)$-orbits of relatively semisimple elements of $X(F)$ in $X(\mathcal{A}_F)$. In the function field case, there are serious difficulties having to do with the fact that stabilizers may be nonsmooth and $H^1(F, I)$ can be infinite for nonreductive $I$. This motivates the following definition:
Definition 17.13. An element $\gamma \in X(F)$ is gcf (Galois cohomologically finite) if $H_\gamma$ is smooth and

$$|H^1(F_v, H_\gamma/H_\gamma)| < \infty$$

for all places $v$ of $F$.

We emphasize that in the number field case, all $\gamma$ are gcf by theorems 1.5.2 and 17.3.4. Clearly $\gamma$ is gcf if and only if every element of its class is gcf so it makes sense to speak of a class being gcf.

In the local setting, we analyzed classes in a geometric class using the class map of Lemma 17.3.1. We now provide the adelic analogue.

Let $S_0 \subseteq S$ be two finite sets of places of $F$. We allow $S_0$ to be empty and we require $S$ to contain the infinite places. Let $I \leq H$ be a smooth algebraic subgroup of the smooth algebraic group $H$ over $F$. We do not assume that $H$ is reductive. Let $X = I \setminus H$ and let $I, H$ and $X$ be smooth models of $I, H$ and $X$ over $O_F$ as in the beginning of this section.

Recall the cohomology sets of (17.39). For $v \notin S$ set

$$D_{\text{fppf}}(O_{F_v}, I, H) := \ker(H^1_{\text{fppf}}(O_{F_v}, I) \to H^1_{\text{fppf}}(O_{F_v}, H)),$$

where the map is given as in (17.40). We have a diagram

$$\begin{array}{ccc}
\tilde{H}^1_{\text{fppf}}(O_{F_v}, I) & \to & \tilde{H}^1_{\text{fppf}}(O_{F_v}, H) \\
\downarrow & & \downarrow \\
H^1(F_v, I) & \to & H^1(F_v, H),
\end{array}$$

(17.52)

where the vertical arrows are given by functoriality (17.40) and (17.44), the top arrow is given by (17.43), and the bottom arrow corresponds to the obvious morphism given by observing that a 1- with coefficients in $I(F_v^{\text{sep}})$ tautologically has coefficients in $H(F_v^{\text{sep}})$. The commutativity of the diagram follows from Lemma 17.5.1. Recall from (17.32) that

$$D(F, I, H) := \text{Im}(H^1(F, I) \to H^1(F, H)),$$

$$D(F_v, I, H) := \text{Im}(H^1(F_v, I) \to H^1(F_v, H)).$$

We let

$$D(O_{F_v}, I, H) := \text{Im}(D_{\text{fppf}}(O_{F_v}, I, H) \to D(F_v, I, H))$$

(17.53)

and set

$$D(\mathcal{A}_{F_v}^{S_0}, I, H) := \prod_{v \notin S_0} D(F_v, I, H),$$

(17.54)
where the prime indicates that we take the restricted direct product with respect to $\mathcal{D}(\mathcal{O}_F, \mathcal{I}, \mathcal{H}) \subseteq \mathcal{D}(F_v, I, H)$ for all $v \notin S$. We note that if $S' \supseteq S$ is finite then the natural map

$$\mathcal{D}(\mathcal{A}_{\mathcal{S}^0}^{\mathcal{F}}, \mathcal{I}, \mathcal{H}) \to \mathcal{D}(\mathcal{A}_{\mathcal{S}^0}^{\mathcal{F}}, \mathcal{I}_{\mathcal{O}_{\mathcal{S}^0}^{\mathcal{F}'}}, \mathcal{H}_{\mathcal{O}_{\mathcal{S}^0}^{\mathcal{F}'}})$$

is an isomorphism.

Assume now that $\mathcal{D}(F_v, I, H)$ is finite for all $v$. Consider the sets of the form

$$\Omega_{S'} \times \prod_{v \notin S'} \mathcal{D}(\mathcal{O}_{F_v}, \mathcal{I}, \mathcal{H})$$

for some finite set $S'$ of places of $F$ containing $S$, where $\Omega_{S'}$ is any subset of $\prod_{v \notin S'} \mathcal{D}(F_v, I, H)$. They form a base of open sets for a topology on $\mathcal{D}(\mathcal{A}_{\mathcal{S}^0}^{\mathcal{F}}, \mathcal{I}, \mathcal{H})$. With respect to this topology, every set of the form (17.55) is a compact open set.

The most important and nontrivial properties of $\mathcal{D}(\mathcal{A}_{\mathcal{S}^0}^{\mathcal{F}}, \mathcal{I}, \mathcal{H})$ are contained in the following theorem:

**Theorem 17.6.5** The image of the diagonal map $\mathcal{D}(F, I, H) \to \prod_v \mathcal{D}(F_v, I, H)$ is contained in $\mathcal{D}(\mathcal{A}_{\mathcal{S}^0}^{\mathcal{F}}, \mathcal{I}, \mathcal{H})$. If $H^1(F_v, I/\mathcal{I}^0)$ is finite for all $v$, then

$$\mathcal{D}(F, I, H) \to \mathcal{D}(\mathcal{A}_{\mathcal{S}^0}^{\mathcal{F}}, \mathcal{I}, \mathcal{H})$$

is proper if we give $\mathcal{D}(F, I, H)$ the discrete topology.

**Proof.** Using Lemma 17.5.1 we may regard an element of $\mathcal{D}(F, I, H)$ as the isomorphism class of an $I$-torsor $Y$ such that $Y \wedge^I H$ is isomorphic to the trivial $H$-torsor. By spreading out [Poo17, Theorem 3.2.1], for a sufficiently large finite set of places $S' \supseteq S$, the $I$-torsor $Y$ is isomorphic as an $I$-torsor to the generic fiber of an $\mathcal{I}_{\mathcal{O}_{\mathcal{S}^0}^{\mathcal{F}'}}$-torsor $\mathcal{Y}$ such that $\mathcal{Y} \wedge^{\mathcal{I}_{\mathcal{O}_{\mathcal{S}^0}^{\mathcal{F}'}}} \mathcal{H}_{\mathcal{O}_{\mathcal{S}^0}^{\mathcal{F}'}}$ is isomorphic to a trivial $\mathcal{H}_{\mathcal{O}_{\mathcal{S}^0}^{\mathcal{F}'}}$-torsor. This implies the first assertion of the theorem.

We claim that the inverse image of

$$\prod_{v \in S' - \mathcal{S}_0} \mathcal{D}(F_v, I, H) \times \prod_{v \notin S'} \mathcal{D}(\mathcal{O}_{F_v}, \mathcal{I}, \mathcal{H})$$

in $\mathcal{D}(F, I, H)$ is finite for all $S' \supseteq S$. This is enough to deduce the second assertion of the theorem. It suffices to show that the inverse image of

$$\prod_{v \in S' - \mathcal{S}_0} H^1(F_v, I) \times \prod_{v \notin S'} \text{Im}(\hat{H}^1_{\text{fppf}}(\mathcal{O}_{F_v}, \mathcal{I}) \to H^1(F_v, I))$$

in $H^1(F, I)$ is finite. This set is denoted by $H^1_{\mathcal{S}_0}(F, \mathcal{I})$ in [Poo17, Theorem 6.5.13] and we conclude by part (a) of that theorem. □
For every nonarchimedean place \( v \) of \( F \), let \( \mathcal{O}^\nr_F \) be the ring of integers of the maximal unramified extension \( F^\nr_v \) of \( F_v \) in some algebraic closure. We will use the following lemma several times below:

**Lemma 17.6.6** Let \( v \) be a nonarchimedean place of \( F \). If \( Y \) is a smooth scheme over \( \mathcal{O}_F \), then

\[
Y(\mathcal{O}^\nr_F) \neq \emptyset.
\]

**Proof.** Let \( \mathbb{F} \) be the residue field of \( \mathcal{O}_F \), and let \( \overline{\mathbb{F}} \) be its algebraic closure. It is the residue field of \( \mathcal{O}^\nr_F \). The surjection \( \mathcal{O}^\nr_F \rightarrow \mathbb{F} \) induces a surjection \( Y(\mathcal{O}^\nr_F) \rightarrow Y(\overline{\mathbb{F}}) \) by Hensel’s lemma [BLR90, §2.3, Proposition 5]. On the other hand, since \( Y \) is smooth, \( Y(\mathbb{F}) \) is nonempty by Lemma 17.1.7. \( \square \)

**Lemma 17.6.7** A class in \( H^1(F_v, H) \) is in the image of the map from \( \check{H}^1\text{fppf}(\mathcal{O}_F, H) \) if and only if it is represented by a 1- with values in \( H(\mathcal{O}_E) \) for some finite unramified Galois extension \( E/F_v \).

**Proof.** Let \( Y \) be an \( \mathcal{H}_{O_E} \)-torsor. It is smooth over \( \mathcal{O}_F \), and hence there is a finite unramified Galois extension \( E/F_v \) such that \( Y(\mathcal{O}_E) \) is nonempty by Lemma 17.6.6. The 1- defined using this point \( x \) as in the definition of (17.44) will take values in \( \check{H}(\mathcal{O}_E) \). This implies the “only if” implication.

Conversely, let \( c : \text{Gal}_v \rightarrow \mathcal{H}(\mathcal{O}_E) \) be a 1- where \( E/F_v \) is an unramified Galois extension. This defines a descent datum on \( \mathcal{H}_{O_E} \) [Poo17, Proposition 4.4.4(i), Remark 4.4.4] and [BLR90, §6.2.B]. Since \( \mathcal{H}_{O_E} \) is affine, the descent datum is effective [BLR90, §6.2.B]. In particular the descent datum is obtained from an \( \mathcal{H} \)-torsor \( Y \) by pullback along \( \text{Spec}(\mathcal{O}_E) \rightarrow \text{Spec}(\mathcal{O}_F) \). The image of this torsor under the map \( \check{H}^1\text{fppf}(\mathcal{O}_F, H) \rightarrow H^1(F_v, H) \) is cohomologous to \( c \), so we deduce the converse assertion. \( \square \)

For each place \( v \), we have a class map \( \text{cl} : (I/H)(F_v) 
\rightarrow D(F_v, I, H) \) as in the proof of Lemma 17.3.1.

**Lemma 17.6.8** The local class maps induce a map

\[
\text{cl} : (I/H)(k_F^\nr) \rightarrow D(k_F^\nr, I, H).
\]

If \( H^1(F_v, I/I^c) \) is finite for all \( v \), then \( \text{cl} \) sends compact sets to compact sets.

We do not know if \( \text{cl} \) is continuous.

**Proof.** Let \( \mathcal{O}^\nr_F \) be the ring of integers of the maximal unramified extension \( F^\nr_v \) in \( F^\sep \). By spreading out, for a finite set of places \( S' \supset S \) that is large enough, the map \( \mathcal{H}_{\mathcal{O}_F} \rightarrow \mathcal{X}_{\mathcal{O}_F} \) is smooth [Poo17, Theorem 3.2.1]. Hence the map \( \mathcal{H}(\mathcal{O}^\nr_F) \rightarrow \mathcal{X}(\mathcal{O}^\nr_F) \) is surjective for \( v \not\in S' \) by Lemma 17.6.6 applied to the fibers of the morphism \( \mathcal{H}_{\mathcal{O}_F} \rightarrow \mathcal{X}_{\mathcal{O}_F} \) for \( v \not\in S' \). Thus for \( v \not\in S' \) and...
given $x \in \mathcal{X}(O_{F_v})$, we can choose $h \in \mathcal{H}(O_{F_v})$ such that $I(F_{v}^{\operatorname{sep}})h = x$. This implies $\text{cl}(x)$ is contained in $\mathcal{D}(O_{F_v}, \mathcal{I}, \mathcal{H})$ by Lemma 17.6.7. We deduce the first assertion of the lemma.

For the second assertion, in view of Lemma 17.3.3 it suffices to observe that by the argument above $\text{cl}(x) \in \mathcal{D}(O_{F_v}, \mathcal{I}, \mathcal{H})$ for $v \notin S'$.

Proposition 17.6.9 One has a commutative diagram

$$
\begin{array}{cccc}
1 & \longrightarrow & I(F) \backslash H(F) & \longrightarrow & (I \backslash H)(F) \\
\downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & I(A_S^0) \backslash H(A_S^0) & \longrightarrow & (I \backslash H)(A_S^0) \\
\end{array}
$$

The top row is an exact sequences of pointed sets, and the map $\beta$ is proper if we give $\mathcal{D}(F, I, H)$ the discrete topology and $H^1(F, I/I^\circ)$ is finite for all $v$.

Proof. The commutativity of the diagram is clear, the top line is exact since (17.33) is exact, and $\beta$ is proper by Theorem 17.6.5.

We now prove that suitable subsets of the set of relative classes inside a geometric class are finite:

Theorem 17.6.10 Assume that $H^1(F_v, I/I^\circ)$ is finite for all $v$. Let $\Omega \subseteq (I \backslash H)(A_F^0)$ be a compact set. There are only finitely many $H(F)$-orbits $xH(F) \subseteq (I \backslash H)(F)$ such that $xH(A_F^0)$ intersects $\Omega$.

Proof. We use the diagram in Proposition 17.6.9. Let $x \in (I \backslash H)(F)$. The bottom $\text{cl}$ map is constant on $H(A_F^0)$-orbits. Thus if $xH(A_F^0) \cap \Omega$ is nonempty then

$$
\text{cl} \circ \alpha(x) \in \text{cl}(\Omega).
$$

By commutativity of the diagram this implies

$$
\beta \circ \text{cl}(x) \in \text{cl}(\Omega)
$$

and hence

$$
\text{cl}(x) \in \beta^{-1}(\text{cl}(\Omega)).
$$

Thus by Lemma 17.3.1 the set of $H(F)$-orbits $xH(F)$ in $(I \backslash H)(F)$ such that $xH(A_F^0)$ intersects $\Omega$ is mapped injectively via $\text{cl}$ into $\beta^{-1}(\text{cl}(\Omega))$. The set $\text{cl}(\Omega)$ is compact by Lemma 17.6.8. Since $\beta$ is proper, $\beta^{-1}(\text{cl}(\Omega))$ is finite.

In §17.7, we will use Theorem 17.6.10 and the following finiteness result for geometric classes:

Lemma 17.6.11 Assume $H$ is reductive. Let $\Omega \subseteq \mathcal{X}(A_F)$ be a compact set. There are only finitely many closed $H$-orbits $O \in (X/H)(F)$ such that $\gamma H(A_F) \cap \Omega$ is nonempty for some $\gamma \in O(F)$. 

17.7 Global relative orbital integrals

Proof. Let 

\[ p : X(\mathbb{A}_F) \longrightarrow (X/H)(\mathbb{A}_F) \]

be the canonical map. Let \( O \in (X/H)(F) \) be a closed \( H \)-orbit. If \( \gamma H(\mathbb{A}_F) \cap \Omega \neq \emptyset \) for some \( \gamma \in O(F) \), then \( O \in p(\Omega) \). Thus the set of closed \( H \)-orbits \( O \in (X/H)(F) \) such that \( \gamma H(\mathbb{A}_F) \cap \Omega \) is nonempty for some \( \gamma \in O(F) \) injects into \( (X/H)(F) \cap p(\Omega). \) (17.57)

But \( p(\Omega) \) is compact and \( (X/H)(F) \) is discrete and closed in \( (X/H)(\mathbb{A}_F) \) by Lemma 17.6.3, so (17.57) is finite. \( \square \)

We observe that if \( S_0 \) is nonempty and \( \Omega \subset X(\mathbb{A}_F^{S_0}) \) is compact then there may very well be infinitely many closed \( H \)-orbits \( O \in (X/H)(F) \) such that \( \gamma H(\mathbb{A}_F) \cap \Omega \) is nonempty for some \( \gamma \in O(F) \). Indeed, this is already true when \( H \) is the trivial group and \( X = \mathbb{G}_a \) by Theorem 2.1.4, the strong approximation theorem. This is in marked contrast to the situation in Theorem 17.6.10.

17.7 Global relative orbital integrals

For use in the relative trace formula, we require global analogues of the local orbital integrals (17.37). Before this, we must discuss what we mean by a smooth function in this setting.

For this section we continue to assume that \( F \) is a global field. Fix a model of \( X \) over \( \mathcal{O}_F \) in the sense of \( \S 2.4 \) and denote it again by \( X \) by abuse of notation. Here \( S \) is a sufficiently large set of places of \( F \) including all infinite places. Then for all \( v \notin S \), the characteristic function \( \mathbb{1}_{X(\mathcal{O}_F_v)} \) of \( X(\mathcal{O}_F_v) \) is an element of \( C^\infty_c(X(\mathcal{O}_F_v)) \). For \( S' \) a set of places of \( F \) including the infinite places we define \( C^\infty_c(X(\mathbb{A}_F^{S'})) = \mathbb{1}_{\cup S'} \otimes C^\infty_c(X(F_v)) \),

where the restricted tensor product is taken with respect to the functions \( \mathbb{1}_{X(\mathcal{O}_F_v)} \). It turns out that \( C^\infty_c(X(\mathbb{A}_F^{S'})) \), defined in this manner, is equal to the \( \mathbb{C} \)-vector space of compactly supported functions locally constant functions on \( X(\mathbb{A}_F^{S'}) \) (see Exercise 17.14).

Now assume \( S' \) is a set of infinite places of \( F \). If \( F \) is a function field, we let \( C^\infty_c(X(F_{S'})) \) be the space of locally constant compactly supported functions on \( X(F_{S'}) \). If \( F \) is a number field, we let \( C^\infty_c(X(F_{S'})) \) be the usual space of compactly supported smooth functions on \( X(F_{S'}) \) viewed as a real manifold. If \( S' \) is an arbitrary set of places of \( F \), we then set

\[ C^\infty_c(X(\mathbb{A}_F^{S'})) := C^\infty_c(X(F_\infty - S')) \otimes C^\infty_c(X(\mathbb{A}_F^{S'\cup \infty})). \]
Assume that
\[ \chi : H(\mathbb{A}_F) \rightarrow \mathbb{C}^\times \]
is a quasi-character trivial on \( H(F) \). We continue to denote by \( \chi \) the quasi-character
\[ H(\mathbb{A}_F) \hookrightarrow H(\mathbb{A}_F) \xrightarrow{\chi} \mathbb{C}^\times. \]
Here the first arrow is the canonical injection. For \( f^{S'} \in C_c^\infty(X(\mathbb{A}_F^{S'})) \), we consider the \textbf{global relative orbital integral}
\[ \text{RO}_\gamma(f^{S'}) := \text{RO}_0^\chi(f^{S'}) = \int_{H_\gamma(\mathbb{A}_F^{S'}) \cap H(\mathbb{A}_F^{S'})} f^{S'}(\gamma h) \chi(h) \frac{d_r h}{d_r h_\gamma}, \quad (17.58) \]
Here \( d_r h \) is a right Haar measure on \( H(\mathbb{A}_F^{S'}) \) and \( d_r h_\gamma \) is a right Haar measure on \( H_\gamma(\mathbb{A}_F^{S'}) \).

\textbf{Definition 17.14.} A relatively semisimple element \( \gamma \in X(F) \) is \textbf{\( \chi \)-relevant} if \( \chi \) is trivial on \( H_\gamma(\mathbb{A}_F) \).

\textbf{Proposition 17.7.1} If \( \gamma \in X(F) \) is relevant, relatively semisimple and relatively unimodular then the integral defining \( \text{RO}_\gamma(f^{S'}) \) is absolutely convergent.

\textbf{Proof.} As in the proof of Theorem 17.4.1, it suffices to show that the map
\[ H_\gamma(\mathbb{A}_F^{S'}) \cap H(\mathbb{A}_F^{S'}) \rightarrow X(\mathbb{A}_F^{S'}) \]
is proper. This is implied by Theorem 17.6.1. \( \square \)

We observe that orbital integrals factor into local orbital integrals. In more detail, suppose that \( f = \otimes_{v \notin S'} f_v \in C_c^\infty(X(\mathbb{A}_F^{S'})) \) where \( f_v \in C_c^\infty(X(F_v)) \) for all \( v \notin S' \). Thus \( f_v = \mathbb{1}_X(\mathbb{O}_{F_v}) \) for all but finitely many places \( v \). Assume that \( \gamma \in X(F) \) is relevant, relatively semisimple and relatively unimodular. Normalize the Haar measures \( d_r h_\gamma, d_r h_{\gamma,v}, d_r h, \) and \( d_r h_v \) on \( H_\gamma(\mathbb{A}_F^{S'}), H_\gamma(F_v), H(\mathbb{A}_F^{S'}), \) and \( H(F_v) \), respectively, so that
\[ d_r h_\gamma = \prod_{v \notin S'} d_r h_{\gamma,v} \quad \text{and} \quad d_r h = \prod_{v \notin S'} d_r h_v. \]
We moreover assume without loss of generality that there are compact open subgroups
\[ K_\gamma^\infty = \prod_{v \notin S'} K_{\gamma,v} \leq H_\gamma(\mathbb{A}_F^\infty) \quad \text{and} \quad K_v^\infty = \prod_{v \notin S'} K_v \leq H(\mathbb{A}_F^\infty) \]
such that \( \text{meas}_{d_r h_{\gamma,v}}(K_{\gamma,v}) = \text{meas}_{d_r h_v}(K_v) = 1 \) for \( v \nmid \infty \). By Exercise 17.15 one has \( K_{\gamma,v} = K_v \cap H_\gamma(F_v) \) for all but finitely many \( v \).

\textbf{Lemma 17.7.2} One has that
where for each \( v \) the local orbital integral \( \text{RO}_\gamma(f_v) \) is defined as in (17.37). In fact, for all but finitely many \( v \), \( \text{RO}_\gamma(1_{X(O_{F_v})}) = 1 \).

**Proof.** Let \( S'' \supseteq S' \) be a set of places of \( F \) such that \( S'' - S' \) is finite and let \( f^{S''} = \otimes_{v \in S''} f_v \). Then by Theorem 17.4.1, Proposition 17.7.1, and the Fubini-Tonelli theorem we have

\[
\text{RO}_\gamma(f^{S'}) = \text{RO}_\gamma(f^{S''}) \prod_{v \in S'' - S'} \text{RO}_\gamma(f_v). 
\]  

(17.60)

Thus to prove the lemma we are free to replace \( S' \) by \( S'' \supseteq S' \) where \( S'' - S' \) is finite.

It follows from Theorem 17.6.1 that the map \( H_\gamma(A_F^{S''}) \setminus H(A_F^{S'}) \to X(A_F^{S''}) \) is proper. This implies that the inverse image \( \Omega \) of \( X(O_f^{S''}) \) in \( H_\gamma(A_F^{S''}) \setminus H(A_F^{S'}) \) is compact and open. One has a canonical homeomorphism

\[
H_\gamma(A_F^{S''}) \setminus H(A_F^{S'}) \to \prod_{v \notin S'} H_\gamma(F_v) \setminus H(F_v) 
\]  

(17.61)

where the restricted direct product is with respect to the sets \( H_\gamma(F_v)K_v \) for all \( v \notin S' \). Applying Exercise 17.15 and replacing \( S' \) by a larger set of places if necessary we may assume that the image of \( \Omega \) under (17.61) is

\[
\prod_{v \notin S'} H_\gamma(F_v)H(O_{F_v}). 
\]

We can moreover assume that \( S' \) is chosen large enough that \( \chi \) is trivial on \( H(O_f^{S'}) \). Then

\[
\text{RO}_\gamma^\chi(1_{X(O_{F_v})}) = 1 
\]

for all \( v \notin S' \). The lemma follows. \( \square \)

In practice, one requires more than Proposition 17.7.1. More precisely, one needs conditions under which the integral

\[
\int_{H(F) \setminus H(A_F)} \sum_\gamma f(\gamma h)\chi(h)d_r h 
\]  

(17.62)

converges. This is a very delicate question in general and we will only address it in special cases. We first make the assumption that

\[
H^\circ := H_u \times H_r 
\]  

(17.63)
where $H_u$ is unipotent and $H_r$ has reductive neutral component. We also restrict to relatively unimodular elements so that our orbital integrals are well-defined, and to elements whose $H_r$-orbit is closed, because otherwise even the contribution of a single orbital integral can diverge [Art05, p. 21].

Let $C_X$ be the stabilizer of $X$ under $H_r$, and let

$$A := A_{C_X}$$

where $A_{C_X}$ is defined as in (2.17). Clearly if $A$ is nontrivial then (17.62) is not absolutely convergent. Therefore we consider instead

$$
\int_{AH(F) \setminus H(\mathbb{A}_F)} \sum_{\gamma} f(\gamma h) \chi(h) \, d_r h,
$$

(17.65)

where $f \in C_c^\infty(X(\mathbb{A}_F))$. There is an obvious obstruction for the contribution of a given $\gamma$ to converge. Letting $H_\gamma$ denote the stabilizer of $\gamma$ in $H$, it is possible that $AH_\gamma^2$ is not contained in $A$, which implies that the contribution of the class of (17.65) is not absolutely convergent by the unfolding argument in the proof of Theorem 17.7.4 below. We exclude this possibility using the notion of an elliptic element. Let $d_r h_\gamma$ be a right Haar measure on $A \setminus H_r(\mathbb{A}_F)$. We abuse notation and denoted again by $d_r h_\gamma$ the right $A \setminus H_r(\mathbb{A}_F)$-invariant Radon measure on $AH_\gamma(F) \setminus H_\gamma(\mathbb{A}_F)$ induced by this right Haar measure.

**Definition 17.15.** A $\gamma \in X(F)$ is **relatively elliptic** if

$$
\tau(H_\gamma) := \text{meas}_{d_r h_\gamma}(AH_\gamma(F) \setminus H_\gamma(\mathbb{A}_F))
$$

(17.66)

is finite.

If $H_\gamma$ has reductive neutral component then $\gamma \in X(F)$ is relatively elliptic if and only if $A_{H_\gamma} \leq A$ by Theorem 2.6.3.

The following lemma will be used to show that only relevant $\gamma$ contribute to (17.65):

**Lemma 17.7.3** If $\gamma$ is relatively elliptic then

$$
\int_{AH_\gamma(F) \setminus H_\gamma(\mathbb{A}_F)} \chi(h_\gamma) \, d_r h_\gamma
$$

converges absolutely. It is equal to $\tau(H_\gamma)$ if $\chi|_{H_\gamma(\mathbb{A}_F)}$ is trivial and equal to 0 otherwise.

We leave the proof as Exercise 17.18. We emphasize that we do not assume that $\chi : H(\mathbb{A}_F) \to \mathbb{C}^\times$ is unitary in Lemma 17.7.3.

We say that $\gamma \in H(F)$ is **relatively unimodular** if its image in $H(F_v)$ is relatively unimodular for all (or equivalently any) place $v$ of $F$. This assumption is convenient for making sense of local orbital integrals. Since we have assumed (17.63), the group $H(\mathbb{A}_F)$ is unimodular. Hence the assumption that
\( \gamma \) is relatively unimodular is equivalent to the assertion that \( H_\gamma(\mathbb{A}_F) \) is unimodular. If \( H^\circ = H_r \) and the \( H_r \)-orbit of \( \gamma \) is closed then the group \( H_\gamma(\mathbb{A}_F) \) is unimodular by Theorem 17.1.5 and Lemma 3.6.4.

Finally, in the function field case, we need to assume that our classes are gcf in the sense of Definition 17.13 above. Upon restricting our attention to the summands in (17.65) that satisfy the various desiderata above, the corresponding integral does indeed converge absolutely:

**Theorem 17.7.4** Assume (17.63) and that \( \chi \) is trivial on \( A \). The expression

\[
\int_{AH(F) \backslash H(\mathbb{A}_F)} \sum_\gamma |f(\gamma h)\chi(h)|dh
\]

(17.67)

is convergent. Here the sum is over \( \gamma \in X(F) \) that are gcf, relatively unimodular and relatively elliptic, and such that the \( H_r \)-orbit of \( \gamma \) is closed. Moreover

\[
\int_{AH(F) \backslash H(\mathbb{A}_F)} \sum_\gamma f(\gamma h)\chi(h)dh = \sum_{[\gamma]} \tau(H_\gamma) \text{RO}_\gamma(f)
\]

(17.68)

where the sum over \( \gamma \) on the left is as before and the sum on the right is over all classes that are relevant, gcf, relatively unimodular and elliptic and such that the \( H_r \)-orbit of \( \gamma \) is closed.

Several remarks are in order. First, the assumption that the \( H_r \)-orbit of \( \gamma \) is closed implies that \( \gamma \) is relatively semisimple with respect to \( H \) (see Lemma 17.1.10). We will use this fact without further mention below. In the definition of \( \text{RO}_\gamma(f) \), we use the measure \( \frac{dh}{dh_\gamma} \) on

\[
H_\gamma(\mathbb{A}_F) \backslash H(\mathbb{A}_F) = (A \backslash H_\gamma(\mathbb{A}_F)) \backslash (A \backslash H(\mathbb{A}_F))
\]

where \( dh_\gamma \) is a Haar measure on \( A \backslash H_\gamma(\mathbb{A}_F) \). We use the same measure \( dh_\gamma \) in the definition of \( \tau(H_\gamma) \). Finally, the assumption that \( \gamma \) is gcf is automatically satisfied if \( H_\gamma \) is smooth and connected or \( F \) is a number field (see Theorem 17.3.4).

We now prove the convergence claim in Theorem 17.7.4:

**Proposition 17.7.5** The expression (17.67) is convergent.

**Proof.** We first reduce to the case \( H = H^\circ \). Consider the set of right \( H^\circ(\mathbb{A}_F) \)-orbits in \( AH(F) \backslash H(\mathbb{A}_F) \). This set admits a natural right \( H^\circ(\mathbb{A}_F) \backslash H(\mathbb{A}_F) \)-action. The \( H^\circ(\mathbb{A}_F) \backslash H(\mathbb{A}_F) \)-orbit of \( AH^\circ(F) \backslash H^\circ(\mathbb{A}_F) \subseteq AH(F) \backslash H(\mathbb{A}_F) \) is the whole set, and its stabilizer is \( H(F)H^\circ(\mathbb{A}_F) \). Thus one has that

\[
\int_{AH(F) \backslash H(\mathbb{A}_F)} \sum_\gamma |f(\gamma h)\chi(h)|dh
\]
where \( d\gamma \) and \( dh \) are appropriately normalized Haar measures on \( H^s(\mathbb{A}_F) \) and \( H(\mathbb{A}_F) \), respectively. The set \( H^s(\mathbb{A}_F) \setminus H(\mathbb{A}_F) \) is compact \([Con12a, Proposition 3.2.1] \), so to prove that (17.67) is convergent, we can and do assume \( H^s = H \).

A relatively semisimple \( \gamma \in X(F) \) has closed \( H^s \)-orbit if and only if any element of its \( H \)-orbit \( O(\gamma) \) has closed \( H^s \)-orbit. Moreover, by Exercise 17.17, a relatively semisimple \( \gamma \in X(F) \) is gcf (resp. relatively unimodular, relatively elliptic) if and only if any element of \( O(\gamma) \) is gcf (resp. relatively unimodular, relatively elliptic). Thus we can talk about a geometric class having closed \( H^s \)-orbit, etc. With this in mind, one has that

\[
\int_{AH(F) \setminus H(\mathbb{A}_F)} |f(\gamma h)\chi(h)| \, dh = \sum_{[\gamma]} \int_{AH_s(F) \setminus H(\mathbb{A}_F)} |f(\gamma h)\chi(h)| \, dh, \tag{17.69}
\]

Here the outer sum is over closed \( H \)-orbits \( O \subseteq X \) such that any \( \gamma \in O(F) \) has closed \( H^s \)-orbit and is gcf, relatively unimodular and relatively elliptic. Unfolding the inner integral for a fixed \( O \), we obtain

\[
\int_{AH(F) \setminus H(\mathbb{A}_F)} \sum_{\gamma \in O(F)} |f(\gamma h)\chi(h)| \, dh = \sum_{[\gamma]} \int_{AH_s(F) \setminus H(\mathbb{A}_F)} |f(\gamma h)\chi(h)| \, dh,
\]

where the bottom sum is over the relative classes \([\gamma]\) in \( O(F) \). By Lemma 17.7.3 we have that

\[
\sum_{[\gamma]} \text{meas}_{dh_\gamma}(AH_\gamma(F) \setminus H_\gamma(\mathbb{A}_F)) \int_{H_\gamma(\mathbb{A}_F) \setminus H(\mathbb{A}_F)} |f(\gamma h)\chi(h)| \, dh \frac{dh}{dh_\gamma}, \tag{17.70}
\]

where for each relatively unimodular \( \gamma \), the measure \( dh_\gamma \) is a Haar measure on \( A \setminus H_\gamma(\mathbb{A}_F) \). The measure

\[
\tau(H_\gamma) = \text{meas}_{dh_\gamma}(AH_\gamma(F) \setminus H_\gamma(\mathbb{A}_F))
\]
is finite for all \( \gamma \) in (17.70) since all such \( \gamma \) are relatively elliptic. Thus (17.70) is finite by Proposition 17.7.1 and Theorem 17.6.10 with \( S_0 = \infty \).

It therefore suffices to show that there are only finitely many closed \( H \)-orbits \( O \subseteq X \) that can contribute a nonzero summand to (17.69). The sum (17.69) is equal to
\begin{equation}
\sum_{O} \int_{AH_r(F) \setminus H_r(A_F) \times [H_u]} \sum_{\gamma \in O(F)} f(\gamma h_r h_u) \chi(h_r h_u) dh_r dh_u
= \sum_{O} \int_{AH_r(F) \setminus H_r(A_F)} \sum_{\gamma \in O(F)} |\tilde{f}(\gamma h_r)\chi(h_r)| dh_r
\tag{17.71}
\end{equation}

where \( \tilde{f}(x) := \int_{F} f(x h_u) \chi(h_u) dh_u \in C_0^\infty(X(\mathbb{A}_F)) \) with \( F \) a compact fundamental domain for \( H_u(F) \) acting on \( H_u(\mathbb{A}_F) \) (see Theorem 2.6.3). Here the sum over \( O \) is as in (17.69). Let \( \Omega \subset X(\mathbb{A}_F) \) be the support of \( \tilde{f} \); it is a compact set. If the summand corresponding to \( O \) in (17.71) is nonzero, then for some \( H_r \)-orbit
\[ O_{H_r} \in (O/H_r)(F) \subset (X/H_r)(F) \]
and some \( \gamma \in O_{H_r}(F) \) one has \( \gamma H_r(\mathbb{A}_F) \cap \Omega \neq \emptyset \). But there are only finitely many such \( H_r \)-orbits \( O_{H_r} \) by Lemma 17.6.11. Thus there are only finitely many \( H \)-orbits \( O \) contributing a nonzero summand to (17.71), which is equal to (17.69).

\textbf{Proof of Theorem 17.7.4:} The convergence assertion of the theorem is the content of Proposition 17.7.5. We now prove the identity. All of the manipulations with integrals occurring in this proof are justified by Proposition 17.7.5.

Consider
\begin{equation}
\int_{AH(F) \setminus H(\mathbb{A}_F)} \sum_{\gamma} f(\gamma h) \chi(h) dh
\tag{17.72}
\end{equation}

where the sum is over \( \gamma \) with closed \( H_r \)-orbit that are relatively elliptic, relatively unimodular and gcf. We unfold the integral to obtain
\begin{align*}
\sum_{[\gamma]} \int_{AH_r(F) \setminus H(\mathbb{A}_F)} f(\gamma h) \chi(h) dh
&= \sum_{[\gamma]} \int_{AH_r(F) \setminus H_r(\mathbb{A}_F)} \chi(h_\gamma) dh_\gamma \tau(H_\gamma)(f)
&= \sum_{[\gamma]} \tau(H_\gamma)(f)
\end{align*}

where the first and second sums over \([\gamma]\) are over \( H(F) \)-orbits in \( X(F) \) with closed \( H_r \)-orbit that are gcf, relatively elliptic, and relatively unimodular. Here we have used Lemma 17.7.3. As in the statement of the theorem, the third sum is over relevant, gcf, relatively elliptic, relatively unimodular classes in \( X(F) \) with closed \( H_r \)-orbit. \qed
Exercises

17.1. Let $X \times H \rightarrow X$ be an action of an algebraic group $H$ on an affine scheme $X$ over a field $k$. Let $\gamma \in X(k)$. Prove that the stabilizer $H_\gamma \leq H$ is a closed subgroup scheme and deduce that it is again an affine algebraic group.

17.2. Give an example of a reductive group $H$ acting on an affine scheme $X$ over a field $k$ and a $\gamma \in X(k)$ such that $O(\gamma)$ is not closed but $H_\gamma$ is reductive.

17.3. Let $\sigma : H \rightarrow H$ be an automorphism of the smooth algebraic group $H$ over a field $k$. Prove that $H^\sigma$, defined as in (17.25), is an algebraic subgroup of $H$.

17.4. Give examples to show that the map $\Gamma_{ss}(k) \rightarrow H(k) \backslash X(k)$ need not be surjective and that the map

$$H(k) \backslash X(k) \rightarrow \Gamma_{geo, ss}(k)$$

need not be injective.

17.5. Give an example to show that if $H$ is a reductive group with simply connected derived group and $\sigma : H \rightarrow H$ is a semisimple automorphism then $H^\sigma$ need not be connected.

17.6. Let $k$ be an algebraically closed field. Assume that $\gamma \in GL_n(k)$ has finite order and that either $k$ is of characteristic zero or that the order of $\gamma$ is coprime to the characteristic of $k$. Prove that $\gamma$ is semisimple. Give a counterexample to show that if the characteristic of $k$ is positive and divides the order of $\gamma \in GL_n(k)$ then $\gamma$ need not be semisimple.

17.7. Let $k$ be a field with algebraic closure $\overline{k}$. Prove that two elements of $GL_n(k)$ are $GL_n(k)$-conjugate if and only if they are $GL_n(\overline{k})$-conjugate.

17.8. Let $k$ be a field with separable closure $k_{sep}$. Prove that a conjugacy class in $GL_n(k_{sep})$ that is stable under $\text{Gal}(k_{sep}/k)$ contains an element in $GL_n(k)$.

17.9. Let $G$ be a reductive group over a field $k$ and let $H \leq G \times G$ be the diagonal copy of $G$. Construct a bijection between $G(k)$-conjugacy classes in $G(k)$ and $H(k) \backslash G(k)/H(k)$.

17.10. Prove that (17.23) is an isomorphism.

17.11. Prove Lemma 17.2.9.

17.12. Using Proposition 17.1.8, prove Lemma 17.3.1.
17.13. Prove that the groups $I_h$ appearing in (17.34) are all inner forms of $I$.

17.14. Let $S$ be a set of places of the global field $F$ including the archimedean places. Prove that the space $C_c^\infty(X(\mathbb{A}_F^S))$, defined as a restricted direct product as in §17.7, is equal to the space of locally constant compactly supported functions on $X(\mathbb{A}_F^S)$.

17.15. Let $\{X_\alpha\}_{\alpha \in A}$ be a set of locally compact topological spaces indexed by a countable set $A$, and for all $\alpha$ outside a finite subset $S_0$ of $A$, let $K_\alpha \subseteq X_\alpha$ be a compact open subset of $X_\alpha$. Let

$$X := \prod_{\alpha \in A} X_\alpha$$

be the restricted direct product with respect to the $K_\alpha$. For all nonempty compact open subsets $\Omega \subset X$, prove that there is a finite subset $A'$ of $A$ such that

$$\Omega = \Omega_{A'} \times \prod_{\alpha \in A - A'} K_\alpha,$$

where $\Omega_{A'} \subseteq \prod_{\alpha \in A}, X_\alpha$ is a compact open set.

17.16. Prove that if $G$ is a commutative smooth algebraic group over a field $k$ and $c_1, c_2$ are 1-cocycles, then

$$\sigma \mapsto c_1(\sigma)c_2(\sigma)$$

is a 1-cocycle. This defines a binary operation on $H^1(k, G)$. Deduce that this binary operation endows $H^1(k, G)$ with an abelian group structure with the class of the neutral element as the identity element.

17.17. Let $F$ be a global field and let $H$ be a smooth algebraic group over $F$ acting on an affine scheme $X$ of finite type over $F$ on the right. Prove that $\gamma \in X(F)$ is relatively semisimple if and only if any element of $O(\gamma)(F)$ is relatively semisimple. Prove that the following conditions depend only on the relative geometric class of a relatively semisimple $\gamma \in X(F)$:

(a) $\gamma$ is relatively unimodular.
(b) $\gamma$ is relatively elliptic.
(c) $\gamma$ is gcf.

17.18. Prove Lemma 17.7.3.
Chapter 18
Simple Trace Formulae

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Abstract In this chapter, we prove a simple relative trace formula and overview its various specializations. This recovers (simple versions of) essentially all trace formulae that have appeared in the literature. Due to its continuing importance in analytic number theory, we work out the case of the Petersson-Bruggeman-Kuznetsov trace formula in some detail.

18.1 A brief history of trace formulae

The trace formula is a powerful tool for understanding automorphic representations on reductive groups. It is an expression for the integral operator $R(f)$ of §16.1 on the space of cusp forms in $L^2([G])$ in terms of orbital integrals. Relative trace formulae, which include the trace formula as a special case, are formulae that describe period integrals of automorphic forms in terms of the relative orbital integrals of Chapter 17.

The original trace formula was first introduced by Selberg in his seminal 1956 paper [Sel56]. He worked with an algebraic group $G$ over the real numbers and discrete subgroup $\Gamma \leq G(\mathbb{R})$. Selberg gave a general formula in the special case where $\Gamma \backslash G(\mathbb{R})$ is compact. Secondly he treated in detail certain special noncompact quotients such as $\text{SL}_2(\mathbb{Z}) \backslash \text{SL}_2(\mathbb{R})$ using the the-
ory of Eisenstein series. He succeeded in proving the existence of cusp forms on \( \text{SL}_2(\mathbb{Z}) \backslash \text{SL}_2(\mathbb{R}) \) invariant under \( \text{SO}_2(\mathbb{R}) \) on the right. In fact, he proved an asymptotic for the number of such forms as a function of their Laplacian eigenvalue. Results of this type are known as Weyl laws since the corresponding asymptotics were proven by Weyl for general compact manifolds [Wey12]. This is discussed in §19.2 in the next chapter.

A key difficulty in extending Selberg’s work to arbitrary groups was proving the analytic continuation of Eisenstein series in general and using them to decompose \( L^2([G]) \) as summarized in Chapter 10. Langlands achieved this in the groundbreaking work [Lan76]. Langlands’ study of Eisenstein series in turn led him to the formulation of his functoriality conjecture. All of this earned him an Abel prize in 2018.

Giving a decomposition of \( L^2([G]) \) in terms of Eisenstein series is a first step to generalizing Selberg’s trace formula, but it is not enough by itself. Over a lifetime of work, Arthur used the theory of Eisenstein series as the basis for a completely general trace formula for an arbitrary reductive group. The technical difficulties he overcame were so immense that the trace formula is often called the **Arthur-Selberg trace formula** in honor of his contributions.

Let \( f \in C_c^\infty(A_G \backslash G(A_F)) \). The basic idea behind the proof of the trace formula is to integrate the automorphic kernel function \( K_f(x, y) \) of (16.5) along the diagonal copy of \([G]\) and evaluate the expression in two different manners. Later, in [Jac86] (see also [Jac05a]), Jacquet realized that the trace formula could be generalized in an extremely useful manner. The idea behind Jacquet’s **relative trace formula** is to replace the diagonal integration in the usual trace formula by integration along a pair of subgroups \( H_1 \times H_2 \leq G \times G \).

Our first goal in this chapter is to prove Theorem 18.2.4, a simple version of the relative trace formula. This is a matter of collecting our preparatory work in Chapter 16 and Chapter 17. In the proof, we impose assumptions on the test function \( f \) to ensure that various analytic difficulties disappear. This is the reason we call the trace formula **simple**. The disadvantage is that in most cases, these assumptions restrict the class of automorphic representations one can treat. In the context of the usual trace formula, Arthur’s work mentioned above removes these assumptions. In the relative setting, this is the subject of ongoing work. See [Lap06, Zyd20, Zyd19].

After proving Theorem 18.2.4 we devote the remainder of the chapter to illustrating it via examples, recovering various (simple) trace formulae in the literature.
18.2 A general simple relative trace formula

We assume for the remainder of this chapter that $F$ is a number field. This is to avoid certain technicalities involving subgroups of $A_G$. We have already treated the most technical aspects of the theory in chapters 16 and 17. Thus the interested reader should be able to deduce a function field analogue of the simple relative trace formula (see Theorem 18.2.4) if desired.

Let $G$ be a reductive group over $F$ and let

$$H \leq G \times G$$

be a subgroup. We always assume that $H^\circ$ is the direct product of a reductive and a unipotent group. This implies in particular that $H(\mathbb{A}_F)$ is unimodular. In the setting of §17.1, we let $X = G$ equipped with the right action of $H$ given on points in an $F$-algebra $R$ by

$$a : G(R) \times H(R) \to G(R)$$

$$(g, (h_1, h_2)) \mapsto h_1^{-1}gh_2.$$  \hfill (18.1)

Let

$$A_{G,H} := H(\mathbb{A}_F) \cap (A_G \times A_G),$$

$$A := H(\mathbb{A}_F) \cap \Delta(A_G)$$

where

$$\Delta : G \to G \times G$$

is the diagonal embedding. To ease notation we sometimes write

$$G^2 = G \times G,$$

so $A_{G^2} = A_G \times A_G$ and

$$G^2(\mathbb{A}_F) = A_{G^2}(\mathbb{A}_F).$$

Since $H^\circ$ is the direct product of a unipotent and a reductive group, there is a unique maximal reductive subgroup $H_r$ of $H$. It moreover contains every reductive subgroup of $H$. We also assume

$$H_r(\mathbb{A}_F) = (H_r(\mathbb{A}_F) \cap A_{G^2}) (H_r(\mathbb{A}_F) \cap G^2(\mathbb{A}_F)).$$  \hfill (18.3)

This is not automatic by Exercise 18.4. However, it is no essential loss of generality for our purposes as we will explain after stating Theorem 18.2.4 below.

**Lemma 18.2.1** The assumption (18.3) implies that

$$H(\mathbb{A}_F) = (H(\mathbb{A}_F) \cap A_{G^2}) (H(\mathbb{A}_F) \cap G^2(\mathbb{A}_F)).$$
Proof. Let \( p : G^2(\mathbb{A}_F) \to A_{G^2} \to \mathbb{R}^n_{>0} \) be the composite of the canonical quotient map and a choice of isomorphism. It suffices to show that
\[
p(H(\mathbb{A}_F)) = p(H(\mathbb{A}_F) \cap A_{G^2}).
\]
It is clear that \( p(H(\mathbb{A}_F)) \supseteq p(H(\mathbb{A}_F) \cap A_{G^2}) \) so we are left with the reverse inclusion.

We claim that in fact
\[
p(H(\mathbb{A}_F)) = p(H_r(\mathbb{A}_F) \cap A_{G^2}).
\]
(18.4)
The decomposition (18.3) and Lemma 14.3.1 imply that
\[
p(H_r(\mathbb{A}_F))/p(H^o(\mathbb{A}_F)) = A_{H_r} \cap A_{G^2} \to \mathbb{R}^m_{>0}
\]
for some \( n \geq m \geq 0). Here the isomorphism is the restriction of the isomorphism in the definition of \( p \). Clearly \( p(H^o(\mathbb{A}_F)) = p(H_r(\mathbb{A}_F)) \). Thus there is a continuous injection
\[
p(H(\mathbb{A}_F))/p(H^o(\mathbb{A}_F)) \to \mathbb{R}^{n-m}_{>0}.
\]
The group \( p(H(\mathbb{A}_F))/p(H^o(\mathbb{A}_F)) \) is a continuous image of the group \( H(\mathbb{A}_F)/H^o(\mathbb{A}_F) \). Now by the proof of Theorem 17.6.1 there is a proper map
\[
H(\mathbb{A}_F)/H^o(\mathbb{A}_F) \to H/H^o(\mathbb{A}_F).
\]
The group \( H/H^o \) is finite étale by Lemma 17.1.3, hence proper. Applying [Con12b, Proposition 4.4] \( H/H^o(\mathbb{A}_F) \) is compact, and we deduce that \( H(\mathbb{A}_F)/H^o(\mathbb{A}_F) \) is as well. Thus \( p(H(\mathbb{A}_F))/p(H^o(\mathbb{A}_F)) \) is a compact group admitting a continuous injection into \( \mathbb{R}^{n-m}_{>0} \). Such a group is necessarily trivial. Hence
\[
p(H(\mathbb{A}_F)) = p(H^o(\mathbb{A}_F)) = p(H_r(\mathbb{A}_F)) = p(H_r(\mathbb{A}_F) \cap (A_{G^2}))
\]
by (18.5), proving (18.4). \( \square \)

Let \( C_G \leq H_r \) be the stabilizer of \( G \) under \( H_r \).

Lemma 18.2.2 One has that \( A = A_{C_G} \).
Thus the definition of \( A \) in this section is consistent with the definition (17.64).

Proof. By applying Weil restriction of scalars we can and do assume \( F = \mathbb{Q} \). Using Lemma 14.3.1 we have
\[
A = H(\mathbb{A}_F) \cap \Delta(A_G) = (H \cap \Delta(Z_G))(\mathbb{A}_Q) \cap A_{\Delta(G)} = A_{H \cap \Delta(Z_G)} \cap A_{\Delta(G)}.
\]
Since \( A_{H \cap \Delta(Z_G)} \) is contained in the real points of the greatest split torus in \( \Delta(Z_G) \) and \( A_{H \cap \Delta(Z_G)} \) is connected we deduce that \( A_{H \cap \Delta(Z_G)} \leq A_{\Delta(G)} \) and hence

\[
A = A_{H \cap \Delta(Z_G)}.
\]

On the other hand

\[
C_G(R) = \{(z, z) \in H(R) : z \in Z_G(R)\} = H(R) \cap \Delta(Z_G)(R).
\]

Thus \( A_{C_G} = A_{H \cap \Delta(Z_G)} \).

Let

\[
\chi : A_{G,H} \backslash H(A_F) \rightarrow \mathbb{C}^{	imes}
\]

be a quasi-character trivial on \( H(F) \) and let \( f \in C_c^\infty(A_G \backslash G(A_F)) \). As in (16.15), we define the relative trace of \( \pi(f) \) with respect to \( H \) and \( \chi \) to be

\[
\text{rtr} \pi(f) := \int_{A_{G,H} \backslash H(A_F) \backslash H(A_F)} K_{\pi(f)}(h_1, h_2) \chi(h_1, h_2) d(h_1, h_2). \quad (18.6)
\]

This depends on the choice of Haar measure \( d(h_1, h_2) \) on \( A_{G,H} \backslash H(A_F) \).

We now define relative orbital integrals in this context. For all \( \gamma \in G(F) \) one has

\[
A_{G,H} \cap H_\gamma(A_F) = A. \quad (18.7)
\]

Hence

\[
A_{G,H} H_\gamma(A_F) \backslash H(A_F) = (A \backslash H_\gamma(A_F)) \backslash (A_{G,H} \backslash H(A_F)).
\]

Assume that \( \gamma \) is relevant (i.e. \( \chi|_{H_\gamma(A_F)} = 1 \)) relatively semisimple (i.e. has closed \( H \)-orbit) and relatively unimodular (i.e. \( H_\gamma(A_F) \) is unimodular). We can then define an orbital integral

\[
\text{RO}_\gamma(f) := \int_{(A \backslash H_\gamma(A_F)) \backslash (A_{G,H} \backslash H(A_F))} f(h_1^{-1} \gamma h_2) \chi(h_1, h_2) \frac{d(h_1, h_2)}{dh_\gamma}. \quad (18.8)
\]

Note that this is slightly different from the definition in (17.58). The root of this is that it is more convenient for the spectral side of the trace formula to work with functions on \( A_{G,H} \backslash H(A_F) \) instead of \( G(A_F) \). In (18.8) \( dh_\gamma \) is a Haar measure on \( A_{G,H} \backslash H_\gamma(A_F) \). Thus \( \frac{d(h_1, h_2)}{dh_\gamma} \) is a right \( A_{G,H} \backslash H(A_F) \)-invariant Radon measure on

\[
(A \backslash H_\gamma(A_F)) \backslash (A_{G,H} \backslash H(A_F)).
\]

In order for Theorem 18.2.4 below to be valid, it is important that we use the same Haar measure \( d(h_1, h_2) \) on \( A_{G,H} \backslash H(A_F) \) used in the definition of the relative trace.
We now check that (18.8) is well-defined by slightly modifying the proof of Proposition 17.7.1:

**Proposition 18.2.3** If \( \gamma \) is relatively unimodular and relatively semisimple then

\[
\int_{(A \setminus H_\gamma(\mathbb{A}_F)) \setminus (A_{G,H} \setminus H(\mathbb{A}_F))} \left| f(h_1^{-1} \gamma h_2) \chi(h_1, h_2) \right| \frac{d(h_1, h_2)}{d h_\gamma} < \infty.
\]

**Proof.** It follows from Theorem 17.6.1 that

\[
H_\gamma(\mathbb{A}_F^\infty) \setminus H(\mathbb{A}_F^\infty) \xrightarrow{\leftarrow} G(\mathbb{A}_F^\infty)
\]

\[
(h_1, h_2) \mapsto h_1^{-1} \gamma h_2
\]

(18.9) is proper.

Using Lemma 18.2.1 one checks that the stabilizer of \( A_G \gamma \in A_G \setminus G(F_\infty) \) in \( H(F_\infty) \) is \( A_{G,H} H(\mathbb{A}_F^\infty) \). Thus we have a commutative diagram

\[
\begin{array}{ccc}
H_\gamma(F_\infty) \setminus H(F_\infty) & \xrightarrow{\leftarrow} & G(F_\infty) \\
\downarrow & & \downarrow \\
(A \setminus H_\gamma(F_\infty)) \setminus (A_{G,H} \setminus H(F_\infty)) & \xrightarrow{\leftarrow} & A_G \setminus G(F_\infty)
\end{array}
\]

(18.10)

where the vertical arrows are the quotient maps and the horizontal arrows are given by the obvious analogues of (18.9). The image of the top horizontal arrow is closed by Lemma 17.3.3 and hence the image of the bottom horizontal map is closed. We deduce that the bottom vertical map is proper.

As in the proof of Theorem 17.4.1 and Proposition 17.7.1 the properness of (18.9) and the bottom arrow of (18.10) suffices to prove the proposition. \( \square \)

Since \( A = A_{G,G} \) by Lemma 18.2.2 the measure

\[
\tau(H_\gamma) := \text{meas}_{d h_\gamma}((A \setminus H_\gamma(\mathbb{A}_F)) \setminus H_\gamma(\mathbb{A}_F))
\]

(18.11)

of (17.66) is finite if \( \gamma \) is relatively elliptic in the sense of Definition 17.15. Here we have abused notation and denoted again by \( d h_\gamma \) the right \( A \setminus H_\gamma(\mathbb{A}_F) \)-invariant Radon measure induced by the Haar measure on \( A \setminus H_\gamma(\mathbb{A}_F) \) used in the definition of the relative orbital integral (18.8).

Recall that we are assuming \( F \) is a number field and that \( H^\circ \) is the direct product of a reductive group \( H_r \) and a unipotent group \( H_u \). We moreover assume \( H_r \) satisfies (18.3).

The following is the general simple relative trace formula:

**Theorem 18.2.4** Under the assumptions above, let \( f \in C_c^\infty (A_{G,G} \setminus G(\mathbb{A}_F)) \) be a function such that \( R(f) \) has cuspidal image and such that if the \( H(\mathbb{A}_F) \)-orbit of \( \gamma \) intersects the support of \( f \) then \( \gamma \) is relatively elliptic, relatively unimodular, and the \( H_r \)-orbit of \( \gamma \) is closed. Then
\[ \sum_{\pi} \text{tr} \pi(f) = \sum_{[\gamma]} \tau(H_{\gamma}) \text{RO}_{\gamma}(f) \]

where the sum on the left is over \((H, \chi)\)-distinguished cuspidal automorphic representations \(\pi\) and the sum on the right is over relevant relative classes \([\gamma]\) that are relatively unimodular, relatively elliptic, and such that the \(H_{\gamma}\)-orbit of \(\gamma\) is closed. Both sums are absolutely convergent.

We recall that if the \(H_{\gamma}\)-orbit of \(\gamma\) is closed, then \(\gamma\) is relatively semisimple by Lemma 17.1.10. Following Langlands, the left hand side of the trace formula is called the \textit{spectral side} and the right hand side of the trace formula is called the \textit{geometric side}. We call the trace formula of Theorem 18.2.4 general because it includes essentially all trace formulae studied in the literature as special cases. Indeed the formula does not hold for a general connected algebraic subgroup \(H \subset G \times G\) without serious modification, so in some sense it is as general as possible. It is simple because we have imposed assumptions on \(f\) to eliminate analytic difficulties on the spectral and geometric sides.

We now explain why assumption (18.3) is no essential loss of generality. If we replace an arbitrary subgroup \(H \subset G \times G\) with \(H(\mathbb{A}_F)\) then (18.3) is satisfied. This has the effect of isolating cuspidal automorphic representations that have a particular central character depending on \(\chi\). We chose not to assume that \(Z_G \times Z_G \subset H\) for aesthetic reasons. In particular, in the settings of §18.4, §18.5 and Theorem 18.9.2 assuming \(Z_G \times Z_G\) is a subgroup of \(H\) complicates the formula for local orbital integrals at nonarchimedean places.

We require the following analogue of Lemma 17.6.11:

**Lemma 18.2.5** Assume that \(H \subset G \times G\) is reductive and (18.3) holds, that is,
\[ H(\mathbb{A}_F) = (H(\mathbb{A}_F) \cap A_{G^2})(H(\mathbb{A}_F) \cap G^2(\mathbb{A}_F)^1). \]
Let \(\Omega \subset A_G \setminus G(\mathbb{A}_F)\) be a compact set. The set of relatively semisimple geometric classes in \(G(F)\) whose \(H(\mathbb{A}_F)\)-orbit intersects \(\Omega\) is finite.

**Proof.** Let \(H(\mathbb{A}_F)^0 = H(\mathbb{A}_F) \cap G^2(\mathbb{A}_F)^1\) and \(A_{G,H} = H(\mathbb{A}_F) \cap A_{G^2}\) as above. Then
\[ A_{G,H} H(\mathbb{A}_F)^0 = H(\mathbb{A}_F), \]
by assumption and \(A_{G,H} \cap H(\mathbb{A}_F)^0 = 1\). Using the fact that \(G(\mathbb{A}_F)\) is the direct product of \(A_G\) and \(G(\mathbb{A}_F)^1\) and \(H(\mathbb{A}_F)\) is the direct product of \(A_{G,H}\) and \(H(\mathbb{A}_F)^0\) we deduce that there is a homeomorphism
\[ A_G / a(1, A_{G,H}) \times G(\mathbb{A}_F)^1 / H(\mathbb{A}_F)^0 \rightarrow G(\mathbb{A}_F) / H(\mathbb{A}_F) \]
\[ (z, g) \mapsto zg \]
where \(a(\cdot, \cdot)\) is the action map of (18.1). If we let \(A_G\) act via its canonical left actions on \(A_G / a(1, A_{G,H})\) in the domain and on \(G(\mathbb{A}_F)\) in the codomain, the
homeomorphism (18.13) is $A_G$-equivariant. Thus (18.13) induces a homeomorphism
\[
G(\mathfrak{a}_F) / H(\mathfrak{a}_F) \rightarrow A_G \setminus G(\mathfrak{a}_F) / H(\mathfrak{a}_F).
\] (18.14)

We observe that $H(\mathfrak{a}_F) \geq H(\mathfrak{a}_F)^1 > H(F)$ and $G(\mathfrak{a}_F)^1 > G(F)$. We have a commutative diagram
\[
\begin{array}{ccc}
G(F) / H(F) & \rightarrow & a(1, A_{G,H}) \setminus G(\mathfrak{a}_F)^1 / H(\mathfrak{a}_F) \\
\downarrow & & \downarrow \\
A_G \setminus G(\mathfrak{a}_F) / H(\mathfrak{a}_F) & \sim & G(\mathfrak{a}_F) / H(\mathfrak{a}_F) \sim a(1, A_{G,H}) \setminus G(\mathfrak{a}_F)^1 / H(\mathfrak{a}_F)
\end{array}
\]

where all arrows are the canonical maps and the horizontal arrows are homeomorphisms. Here to construct the right homeomorphism we have used (18.13).

Thus it suffices to show that given a compact subset
\[
\Omega \subseteq a(1, A_{G, H}) \setminus G(\mathfrak{a}_F)^1 / H(\mathfrak{a}_F),
\]
there are only finitely many relatively semisimple geometric classes in $G(F)$ whose image in $a(1, A_{G, H}) \setminus G(\mathfrak{a}_F)^1 / H(\mathfrak{a}_F)$ intersects $\Omega$. Let
\[
p : a(1, A_{G, H}) \setminus G(\mathfrak{a}_F)^1 / H(\mathfrak{a}_F) \rightarrow (G/H) (\mathfrak{a}_F)
\]
denote the restriction of the natural map $G(\mathfrak{a}_F) / H(\mathfrak{a}_F) \rightarrow (G/H) (\mathfrak{a}_F)$. In view of Proposition 17.1.6, it suffices to observe that $p(\Omega)$ is compact and $(G/H)(F)$ is discrete and closed by Lemma 17.6.3, hence
\[
p(\Omega) \cap (G/H)(F)
\]
is finite.

\[\square\]

**Proof of Theorem 18.2.4:** Let
\[
K_f(g, h) = \sum_{\gamma \in G(F)} f(g^{-1} \gamma h).
\]

Since we have assumed that $R(f)$ has cuspidal image, one has that
\[
K_f(g, h) = K_{f, \text{cusp}}(g, h)
\]
in the notation of (16.6). By Theorem 16.2.6 we have that
\[
\int_{A_{G,H} H(F) \setminus H(\mathfrak{a}_F)} K_f(h_1, h_2) \chi(h_1, h_2) d(h_1, h_2) = \sum_\pi \text{tr} \pi(f).
\] (18.15)
At this point, if $A_{G,H} = A$ then we can invoke Theorem 17.7.4 to finish the proof. In general we can still follow the argument of the proof of Theorem 17.7.4. We now give the details. We will check below that it is permissible to unfold the integral in (18.15). Assuming this for the moment, we have that it is equal to

$$
\sum_{[\gamma]} \int f(h_1^{-1} \gamma h_2) \chi(h_1, h_2) \frac{d(h_1, h_2)}{dh_2} \int_{AH_r(F) \setminus H_r(\A_F)} \chi(h_\gamma) dh_\gamma,
$$

where the outer integral is over

$$(A \setminus H_r(\A_F)) \setminus (A_{G,H} \setminus H(\A_F))$$

and the sum on $[\gamma]$ is over classes that are relatively unimodular, relatively elliptic, and such that the $H_r$-orbit of $\gamma$ is closed. Applying Lemma 17.7.3 the sum above is equal to

$$
\sum_{[\gamma]} \tau(H_\gamma) \int_{(A \setminus H_r(\A_F)) \setminus (A_{G,H} \setminus H(\A_F))} f(h_1^{-1} \gamma h_2) \chi(h_1, h_2) \frac{d(h_1, h_2)}{dh_2} = \sum_{[\gamma]} \tau(H_\gamma) RO_{\gamma}(f),
$$

(18.16)

where the sum is over classes $[\gamma]$ as in the statement of the theorem.

To justify unfolding, using Lemma 17.7.3 it suffices to check that

$$
\sum_{[\gamma]} \tau(H_\gamma) \int_{(A \setminus H_r(\A_F)) \setminus (A_{G,H} \setminus H(\A_F))} |f(h_1^{-1} \gamma h_2)\chi(h_1, h_2)| \frac{d(h_1, h_2)}{dh_2} \tag{18.17}
$$

is finite, where the sum is over classes $[\gamma]$ that are relatively unimodular, relatively elliptic, and such that the $H_r$-orbit of $\gamma$ is closed. The integral in (18.17) converges absolutely by Proposition 18.2.3. It therefore suffices to prove that only finitely many classes $[\gamma]$ can contribute a nonzero summand to (18.17).

Consider the contribution of a particular relatively semisimple geometric class $[\gamma_0]$. By Theorem 17.6.10, for any compact set $\Omega \subset (H_{\gamma_0} \setminus H(\A_F^\infty))$, there are only finitely many $H(F)$-orbits $xH(F) \subset (H_\gamma \setminus H(F))$ such that $xH(\A_F^\infty)$ intersects $\Omega$. Here we are using that $F$ is a number field, so the condition that $H^1(F_v, H_\gamma/H_\gamma^\infty)$ is finite is automatic by Theorem 17.3.4. This implies that there are only finitely many relative classes in the geometric class of $[\gamma_0]$ that can contribute to the sum.

Lastly we show that there are only finitely many geometric classes that can contribute to the sum. By the same argument used in the proof of Proposition 17.7.5, we are reduced to showing that for any compact set $\Omega \subset A_G \setminus G(\A_F)$, there are only finitely many geometric classes in $G(F)$ with closed $H_r$-orbit whose $H_r(\A_F)$-orbit intersects $\Omega$. This is implied by Lemma 18.2.5. \hfill \Box
We now make some comments on the assumptions on $f$ in Theorem 18.2.4. For more information on the assumption on $f$ involving the geometric side of the trace formula, we refer to exercises 18.1, 18.2 and 18.3.

In practice, in order to ensure that $f$ has cuspidal image, one chooses a finite place $v$ and assumes $f(g) = f_v(g_v) f^v(g^v)$ where $f^v \in C_c^\infty(A_Q \setminus G(A_F^v))$ and $f_v \in C_c^\infty(G(F_v))$ is supercuspidal. This suffices to ensure $R(f)$ has cuspidal image by Lemma 16.4.1. Unfortunately, assuming that $f_v$ is supercuspidal limits the set of $\pi$ that can be studied using the simple relative trace formula:

**Lemma 18.2.6** Let $v$ be a finite place of $F$. Assume that $f \in C_c^\infty(G(F_v))$ is supercuspidal. If $\pi_v$ is an irreducible admissible representation of $G(F_v)$ and $\pi_v(f) \neq 0$, then $\pi_v$ is supercuspidal.

**Proof.** To ease notation, we drop the subscript $v$, writing $F := F_v$, etc. The current lemma was proved under the assumption that $Z_G(F) \subset G$ is compact in Exercise 16.5. We reduce to this case. This provides an opportunity to illustrate some techniques useful for considering restrictions of representations to derived subgroups. For the facts recalled in the remainder of this paragraph see [GK82, Lemma 2.1]. The restriction of $\pi$ to $Z_G(F)G^\text{der}(F)$ decomposes as a finite direct sum of irreducible representations, all with the same multiplicity. Since $Z_G(F)$ acts via scalars on the space of $\pi$, the restriction of $\pi$ to $G^\text{der}(F)$ also decomposes into a finite direct sum of irreducible representations, all with the same multiplicity:

$$\pi|_{G^\text{der}(F)} = \bigoplus_{i=1}^n \pi_i^{\oplus m_i}.$$  

Here $\pi_i \cong \pi_j$ if and only if $i = j$. The group $G(F)$ acts via conjugation on the set of isomorphism classes of the $\pi_i$. If we let $G(F)_{\pi_1} \leq G(F)$ be the subgroup fixing the isomorphism class of $\pi_1$, the group $G(F)/G(F)_{\pi_1}$ permutes the isomorphism classes of the $\pi_i$'s transitively. Finally, the $\pi_i$ are admissible.

Since supercuspidal representations are representations whose matrix coefficients are square integrable modulo the center, $\pi$ is supercuspidal if and only if $\pi_i$ is supercuspidal for all $i$, which is equivalent to $\pi_i$ being supercuspidal for a single $i$ due to the transitive action of $G(F)/G(F)_{\pi_1}$.

Since $\pi(f) \neq 0$, there is a $\varphi$ in the space of $\pi$ such that

$$\int_{G(F)} f(g) \pi(g) \varphi \neq 0$$

which implies that there exists an $a \in G(F)$ such that

$$\int_{G^\text{der}(F)} f(ga) \pi(ga) \varphi \neq 0.$$  

Let $f_a(g) := f(ga)$. For some $i$, the operator $\pi_i(f_a)$ is nonzero. Since $f$ is supercuspidal so is $f_a$ (as a function on $G^\text{der}(F)$) so we deduce that $\pi_i$ is
supercuspidal by Exercise 16.5. As already mentioned, this implies that \( \pi \) is supercuspidal.

Thus if we assume that \( f_v \) is supercuspidal, we can only study cuspidal automorphic representations \( \pi \) such that \( \pi_v \) is supercuspidal. When \( F = \mathbb{Q} \) and \( G \) is split and adjoint, Lindenstrauss and Venkatesh [LV07] found another method to construct functions \( f \in C_\infty^c(A_G \backslash G(\mathbb{A}_F)) \) that have cuspidal image. Their idea was applied again in [GH15]. A more flexible method for isolating the cuspidal spectrum was introduced in [BPLZZ19]. Under mild assumptions on a cuspidal representation \( \pi \), they prove that one can construct a test function \( f \) in an enlargement of \( C_\infty^c(A_G \cap G(\mathbb{A}_F)) \) with cuspidal image such that \( \pi(f) \neq 0 \). The assumptions are mild enough that they are satisfied for every cuspidal automorphic representation of \( \text{GL}_n(\mathbb{A}_F) \). Interestingly, the method requires test functions that are not compactly supported at infinity, but merely Schwartz in an appropriate sense.

18.3 Products of subgroups

Assume that \( H = H_1 \times H_2 \leq G \times G \), where both \( H_1 \) and \( H_2 \) are subgroups of \( G \), and for each \( i \) the neutral component \( H_i^o \) is either unipotent or reductive. Then

\[
G(F)/H(F) = H_1(F) \backslash G(F)/H_2(F). \tag{18.18}
\]

Assume moreover that \( H \) satisfies (18.3), and that \( \chi = \chi_1 \otimes \chi_2 \), where \( \chi_i \) is a character of \( H_i(\mathbb{A}_F) \). In this case, the formula in Theorem 18.2.4 simplifies somewhat. Let \( K_1^{\infty} \) be a maximal compact subgroup of \( G(F_{\infty}) \).

**Corollary 18.3.1** Under the assumptions above, let \( f \in C_\infty^c(A_G \backslash G(\mathbb{A}_F)) \) be a \( K_\infty^{\infty} \)-finite function. Assume that \( R(f) \) has cuspidal image and that if the \( H(\mathbb{A}_F) \)-orbit of \( \gamma \) intersects the support of \( f \) then \( \gamma \) is relatively elliptic and the \( H_\gamma \)-orbit of \( \gamma \) is closed. Then

\[
\sum_\pi \sum_{\varphi \in \mathcal{B}(\pi)} P_{\chi_1}(\pi(f)(\varphi)) \overline{P_{\chi_2}(\varphi)} = \sum_{[\gamma] \in H_1(F) \backslash G(F)/H_2(F)} \tau(H_\gamma) R_\gamma(f)
\]

where \( \mathcal{B}(\pi) \) is an orthonormal basis of \( L^2_{\text{cusp}}(\pi) \) consisting of \( K_\infty^{\infty} \)-finite (and hence smooth) vectors. Here the sum on the right is over relevant \([\gamma]\) that are relatively elliptic and such that the \( H_\gamma \)-orbit of \( \gamma \) is closed.

Here we are using the notation (14.3) for the period integrals occurring in the corollary. It is important to point out that the only \( \pi \) that contribute a nonzero summand to the identity in Corollary 18.3.1 are \( \pi \) that are both \((H_1, \chi_1)\) and \((H_2, \chi_2)\)-distinguished. Moreover the assumption on relative unimodularity in Theorem 18.2.4 can be (and is) dropped in the case at
hand. This sort of relative trace formula first appeared in work of Jacquet [Jac86], and has been applied many times in the intervening years.

**Proof.** Let \( \pi \) be a cuspidal automorphic representation of \( \mathcal{A}_G \). By Proposition 4.4.3, \( K_\infty \)-finite vectors are dense in \( L^2_{\text{cusp}}(\pi) \). Thus we can choose an orthonormal basis \( \mathcal{B}(\pi) \) of this space consisting of \( K_\infty \)-finite vectors. Since \( f \) is \( K_\infty \)-finite we deduce that in the expansion

\[
K_{\pi(f)}(x, y) = \sum_{\varphi \in \mathcal{B}(\pi)} \pi(f) \varphi(x) \overline{\varphi(y)}
\]  

(18.19)

of (16.12) only finitely many \( \varphi \) can contribute a nonzero summand. Since \( K_\infty \)-finite vectors are smooth by Proposition 4.4.3 we deduce that

\[
\text{rtr} \pi(f) = \sum_{\varphi \in \mathcal{B}(\pi)} \mathcal{P}_{\chi_1}(\pi(f) \varphi) \overline{\mathcal{P}_{\chi_2}(\varphi)}.
\]

To complete the proof, we show that a relatively semisimple element of \( G(F) \) is automatically relatively unimodular. If \( H_1^\gamma \) and \( H_2^\gamma \) are both reductive and \( \gamma \) is relatively semisimple then \( H_\gamma \) has reductive neutral component by Theorem 17.1.5. Thus by Lemma 3.6.4, \( H_\gamma(\mathcal{A}_F) \) is unimodular.

Now assume that \( H_1^\gamma \) is unipotent. For \( F \)-algebras \( R \), one has that

\[
H_\gamma(R) = \{ (h, \gamma^{-1} h \gamma) \in H_1(R) \times H_2(R) \}.
\]

The neutral component of this group is isomorphic to a subgroup of \( H_1^\gamma \). Hence it is unipotent, which implies that \( H_\gamma(\mathcal{A}_F) \) is unimodular by Exercise 18.7.

\[\square\]

**18.4 The simple trace formula**

For \( \gamma \in G(F) \) let

\[
C_\gamma(R) := \{ g \in G(R) : g \gamma g^{-1} = \gamma \}
\]

(18.20)

be the centralizer of \( \gamma \) and let

\[
\text{O}_\gamma(f) = \int_{C_\gamma(\mathcal{A}_F) \backslash G(\mathcal{A}_F)} f(g^{-1} \gamma g) \frac{dg}{dg_\gamma}
\]

(18.21)

whenever this integral is well-defined. This is the usual **orbital integral** of \( f \) along the conjugacy class of \( \gamma \). We say an element \( \gamma \in G(F) \) is **elliptic** if the measure of \( A_G C_\gamma(F) \backslash C_\gamma(\mathcal{A}_F) \) is finite with respect to a right \( A_G \backslash C_\gamma(\mathcal{A}_F) \)-invariant Radon measure.
The simple trace formula of Deligne and Kazhdan [DKV84, Rog83] is essentially the following special case of the simple relative trace formula of Theorem 18.2.4:

**Corollary 18.4.1** Let $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$. Assume that $R(f)$ has cuspidal image and that any element of $G(F)$ that is $G(\mathbb{A}_F)$-conjugate to the support of $f$ is elliptic and semisimple. Then

$$
\sum_\pi m(\pi)\text{tr}(f) = \sum_{[\gamma]} \tau(C_\gamma)O_\gamma(f),
$$

where the sum on the left is over isomorphism classes of cuspidal automorphic representations of $A_G \backslash G(\mathbb{A}_F)$, $m(\pi)$ is the multiplicity of $\pi$ in the cuspidal spectrum of $L^2([G])$ and the sum on the right is over elliptic, semisimple conjugacy classes in $G(F)$.

As in the statement of Theorem 18.2.4, some care must be taken with measures in order for the identity above to be correct. We choose a Haar measure $dg$ on $A_G \cap G(\mathbb{A}_F)$ and for each semisimple $\gamma$, a Haar measure $dg_\gamma$ on $A_G \backslash C_\gamma(\mathbb{A}_F)$. Then we use $dg'$ to define the operator $\pi(f)$ (and hence its trace) and use $dg'_\gamma$ to define $\tau(C_\gamma)$. Finally, we normalize the Radon measure $\frac{dg}{dg_\gamma}$ on $C_\gamma(\mathbb{A}_F) \backslash G(\mathbb{A}_F)$ used to define $O_\gamma(f)$ so that it corresponds to $\frac{dg'}{dg'_\gamma}$ under the natural isomorphism

$$C_\gamma(\mathbb{A}_F) \backslash G(\mathbb{A}_F) \cong (A_G \backslash C_\gamma(\mathbb{A}_F)) \backslash (A_G \backslash G(\mathbb{A}_F)).$$

**Proof.** In Theorem 18.2.4, we take $\chi$ to be trivial and take $H$ to be the diagonal copy of $G$ inside $G \times G$. Then (18.3) is satisfied, $A = A_{G,H} = \Delta(A_G)$, and a class is just a conjugacy classes in $G(F)$. Thus $RO_\gamma(f) = O_\gamma(f)$.

All classes are trivially relevant, and $\gamma \in G(F)$ is semisimple in the usual sense if and only if it is relatively semisimple by [Ste65, Corollary 6.13]. Thus $C_\gamma$ is reductive for all semisimple $\gamma$ by Theorem 17.1.5. This implies all semisimple classes are relatively unimodular by Lemma 3.6.4. Finally, it is clear that $\gamma$ is relatively elliptic if and only if it is elliptic. Thus the geometric side of the trace formula of Theorem 18.2.4 is equal to the right hand of (18.22).

As for the left hand side,

$$
\text{rtr} \pi(f) := \int_{[G]} K_{\pi(f)}(g,g)dg = m(\pi)\text{tr}(f)
$$

by the discussion around (16.9). Thus the spectral side of the trace formula of Theorem 18.2.4 is equal to the left hand side of (18.22). □

To understand Corollary 18.4.1 it is helpful to have some qualitative understanding of orbital integrals. Since orbital integrals factor as explained in Lemma 17.7.2, it suffices to consider local orbital integrals. Thus let $v$ be a
nonarchimedean place of $F$ which we omit from notation, writing $F := F_v$.
Let $K \leq G(F)$ be a compact open subgroup and let $\mathbb{1}_K$ be its characteristic function. Then for semisimple $\gamma$ one has that

$$O_\gamma(\mathbb{1}_K) := \int_{C_\gamma(F)\backslash G(F)} \mathbb{1}_K(g^{-1} \gamma g) \frac{dg}{dg_\gamma}$$

$$= \sum_{C_\gamma(F)gK \in C_\gamma(F) \backslash G(F)/K} \text{meas}_{dg_\gamma}(C_\gamma(F) \backslash C_\gamma(F)gK)$$

$$= \sum_{C_\gamma(F)gK \in C_\gamma(F) \backslash G(F)/K} \frac{\text{meas}_{dg_\gamma}(K)}{\text{meas}_{dg_\gamma}(C_\gamma(F) \cap gKg^{-1})}. \quad (18.24)$$

For general linear groups when $K$ is hyperspecial this can be made much more explicit. We refer to [Kot05, §5] for the computation and many similar computations when $G = \text{GL}_2$ and [Yun13] for the case when $G = \text{GL}_n$. In these cases orbital integrals can be evaluated in fairly elementary terms. However, the situation is wild when one ventures beyond the general linear group [Hal94].

### 18.5 The simple twisted trace formula

Let $\theta$ be a finite order, hence semisimple automorphism of a reductive group $G$ over number field $F$. One then has a right action of $G$ on itself by $\theta$-conjugation, given on points in an $F$-algebra $R$ by

$$G(R) \times G(R) \rightarrow G(R)$$

$$(x, g) \mapsto g^{-1}x\theta(g). \quad (18.25)$$

When $\theta$ is the identity, this is simply conjugation. Following the lead of §18.4, we let

$$C^\theta_\gamma(R) := \{ g \in G(R) : g^{-1}\gamma \theta(g) = \gamma \} \quad (18.26)$$

be the $\theta$-centralizer of $\gamma$. Write

$$A^\theta_G := \{ a \in A_G : \theta(a) = a \}.$$ 

We define the twisted orbital integral

$$\text{TO}_\gamma(f) := \int_{A^\theta_G \backslash C^\theta_\gamma(A_F) \backslash (A_G \backslash G(A_F))} f(g^{-1} \gamma \theta(g)) \frac{dg}{dg_\gamma}. \quad (18.27)$$
We say \( \gamma \) is \( \theta \)-semisimple if its \( G \)-orbit under \( \theta \)-conjugation is closed and \( \theta \)-elliptic if
\[
\tau(C_\gamma^\theta) := \text{meas}_{G_\gamma}(A_G C_\gamma^\theta(F) \backslash C_\gamma^\theta(A_F))
\]
is finite.

We have an operator
\[
I_\theta : L^2([G]) \to L^2([G])
\]
that preserves \( L^2_{\text{cusp}}([G]) \). For each \( f \in C_c^\infty(A_G \backslash G(A_F)) \) the composite
\[
R(f) \circ I_\theta : L^2([G]) \to L^2([G])
\]
has kernel function \( K_f(g, \theta(h)) \). Since \( R_{\text{cusp}}(f) := R(f)|_{L^2_{\text{cusp}}([G])} \) is of trace class by Theorem 9.1.1, the same is true of \( R_{\text{cusp}}(f) \circ I_\theta \).

For each cuspidal automorphic representation \( \pi \), the operator \( R(f) \circ I_\theta \) restricts to induce an operator
\[
\pi(f) \circ I_\theta : L^2(\pi) \to L^2(\pi^\theta),
\]
where
\[
\pi^\theta(g) := \pi(\theta(g)).
\]

The spectral counterpart of a twisted orbital integral is the twisted trace of \( \pi(f) \) with respect to \( \theta \):
\[
\text{tr}(\pi(f) \circ I_\theta) := \int_{[G]} K_{\pi(f)}(g, \theta(g)) dg.
\]

We observe that if \( \pi \) occurs with multiplicity greater than 1 then this is incorporated into the twisted trace. In contrast, we separated the trace and the multiplicity in the statement of Corollary 18.4.1. The twisted trace \( \text{tr}(\pi(f) \circ I_\theta) \) vanishes unless \( \pi \cong \pi^\theta \) by Exercise 14.4. When \( \pi \cong \pi^\theta \), so \( \pi(f) \circ I_\theta \) is an endomorphism of \( L^2(\pi) \), then \( \int_{[G]} K_{\pi(f)}(g, \theta(g)) dg \) is the trace of the trace class operator (18.29) by [Bri91, Corollary 3.2].

The following is a simple version of the twisted trace formula:

**Corollary 18.5.1** Let \( f \in C_c^\infty(A_G \backslash G(A_F)) \). Assume that \( R(f) \) has cuspidal image and that any element of \( G(F) \) that is \( G(A_F) \)-conjugate to the support of \( f \) is \( \theta \)-elliptic and \( \theta \)-semisimple. Then
\[
\sum_\pi \text{tr}(\pi(f) \circ I_\theta) = \sum_{[\gamma]} \tau(C_\gamma^\theta) \text{TO}_\gamma(f),
\]
(18.31)
where the sum over \( \pi \) is over isomorphism classes of cuspidal automorphic representations of \( \mathcal{A}_G \backslash G(\mathbb{A}_F) \) such that \( \pi \cong \pi^\theta \) and the sum on the right is over \( \theta \)-conjugacy classes in \( G(F) \) that are \( \theta \)-elliptic and \( \theta \)-semisimple.

**Proof.** In Theorem 18.2.4, we take \( \chi \) to be trivial and take \( H \) to be the twisted diagonal copy of \( G \) in \( G \times G \); that is, for \( F \)-algebras \( R \), one has that

\[
H(R) = \{(g, \theta(g)) : g \in G(R)\}.
\]

In this setting, classes are just \( \theta \)-conjugacy classes and

\[
\mathcal{A}_{G,H} = \{(a, \theta(a)) : a \in \mathcal{A}_G\} \quad \text{and} \quad \mathcal{A} = \{(a, a) : a = \theta(a), a \in \mathcal{A}_G\}.
\]

In particular (18.3) is valid and the relative orbital integral \( \text{RO}_{\mathcal{X}}(f) \) is \( \text{TO}_{\mathcal{X}}(f) \).

All classes are trivially relevant, all \( \theta \)-semisimple (resp. \( \theta \)-elliptic) elements are relatively semisimple (resp. elliptic), and all \( \theta \)-semisimple elements are relatively unimodular by Theorem 17.1.5. \( \square \)

The fact that the twisted trace formula isolates representations satisfying \( \pi \cong \pi^\theta \) was the primary motivation for its development. We explain why this fact is useful in \( \S \)19.4 below. The twisted trace formula was first introduced by Saito \[\text{Sai75}\] and Shintani \[\text{Shi79}\]. For a completely general twisted trace formula, we refer to \[\text{LW13}\]. A relative version of the simple twisted trace formula (which is another special case of Theorem 18.2.4) was introduced in \[\text{Hah09}\].

### 18.6 A variant

It has proven to be beneficial to work with a variant of the simple relative trace formula of Theorem 18.2.4. In this section, we discuss the variant in impressionistic terms to aid the reader when it is encountered in the literature.

Let \( X \) be a smooth affine scheme over the global field \( F \) equipped with a (right) action of the reductive \( F \)-group \( G \). We assume for simplicity that \( \mathcal{A}_G = 1 \). Since \( X \) is smooth, the Schwartz space \( \mathcal{S}(X(\mathbb{A}_F)) \) is defined (see \( \S \)17.7).

Let \( H \leq G \) be a subgroup such that \( H^\circ \) is the direct product of a reductive group and a unipotent group and let \( \chi : H(\mathbb{A}_F) \to \mathbb{C}^\times \) be a character trivial on \( H(F) \). For each \( x \in X(F) \) with closed orbit under \( G \), the stabilizer \( G_x \) has reductive neutral component by Theorem 17.1.5 and we can consider the subgroup

\[
H_x := G_x \times H \leq G \times G.
\]
18.6 A variant

The character $\chi$ extends to a character

$$G_x(\mathbb{A}_F) \times H(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$$

$$(g, h) \mapsto \chi(h)$$

which we will continue to denote by $\chi$. We are now in the setting of §18.2
and we can form the relative trace

$$\text{rtt}_{H_x, \chi} \pi(f)$$

for any $f \in C_c^\infty(G(\mathbb{A}_F))$.

Now let $\phi \in C_c^\infty(X(\mathbb{A}_F))$. We assume that

$$\int_{H(F) \setminus H(\mathbb{A}_F)} \sum_{x \in X(F)} |\phi(xh)\chi(h)|dh$$

converges. For conditions ensuring convergence, see Theorem 17.7.4. Thus
the integral

$$\int_{H(F) \setminus H(\mathbb{A}_F)} \sum_{x \in X(F)} \phi(xh)\chi(h)dh$$

(18.33)

is well-defined.

Assume that for each $x \in X(F)$, we can choose $f_x \in C_c^\infty(G(\mathbb{A}_F))$ such that

$$\phi(xy) = \int_{G_x(\mathbb{A}_F)} f_x(g^{-1}y)dg$$

(18.34)

for $y \in G(\mathbb{A}_F)$ (this is not always possible). Then

$$\int_{H(F) \setminus H(\mathbb{A}_F)} \sum_{x \in X(F)} \phi(xh)\chi(h)dh$$

$$= \sum_{x \in X(F)/G(F)} \int_{H_x(F) \setminus H_x(\mathbb{A}_F)} \sum_{g \in G(F)} f_x(g^{-1}h_1)\chi(h_2)dh_1dh_2. \quad (18.35)$$

Thus (18.33) is a sum over $x \in X(F)/G(F)$ of geometric sides (or spectral sides) of relative trace formula for the various $H_x$. Studying (18.33) amounts

This method of packaging distinction problems together has proven crucial,

尤其是工作上的Gan-Gross-Prasad conjecture（见§14.7）。我们指出，即使在(18.34)中不可能选择$f_x$，仍然

可能将函数
\[
\sum_{x \in X(F)} \phi(xy) \in C^\infty([G])
\]
spectrally. More or less it will give something like the right hand side of (18.35), but the \( f_x \) involved will not necessarily be compactly supported.

In some sense this is the key point. Replacing \( G \) by \( X \) has the effect of changing the function space \( S(X(\mathbb{A}_F)) \). As an example, let \( E/F \) be a quadratic extension and let \( G = \text{Res}_{E/F} \text{GL}_n \). Of course \( AG \neq 1 \), but the discussion above can be modified to treat this setting. Let \( X \) be the space of Hermitian matrices with respect to \( E/F \). Thus \( X(F) \) is the space of \( n \times n \) matrices \( x \) with coefficients in \( E \) such that \( \theta(x) = x^t \), where \( \theta \) is a generator of \( \text{Gal}(E/F) \). Then there is a natural right action of \( G \) on \( X \) given on points in an \( F \)-algebra \( R \) by

\[
X(R) \times G(R) \rightarrow X(R) \\
(x, g) \mapsto g^t x \theta(g).
\]

The stabilizer of any invertible element of \( X(F) \) is a unitary group. Jacquet took \( H \) to be the unipotent radical of a Borel subgroup of \( G \) and \( \chi \) to be a generic character and then studied (18.33) in order to prove Theorem 14.5.2 [Jac05b, Jac10] (see also [Ngô99b, Ngô99a]). The fact that \( X \) is a linear space, and hence admits a Fourier transform, turns out to be crucial.

One can also ask what happens when \( X \) is no longer smooth. This is the subject of important ongoing research. We refer to [BK00, BK02, BNS16, Get20, GL19, GH20, GHL21, JLZ20, Ngô14, Ngô20, Sak12, SW20].

### 18.7 The Petersson-Bruggeman-Kuznetsov formula

Given two nonzero integers \( m \) and \( n \) and a positive integer \( c \), the classical Kloosterman sum is defined as

\[
K(m, n; c) := \sum_{1 \leq a \leq c \atop (a, c) = 1} e^{2\pi i(am+bn)/c},
\]

where \( a\bar{a} \equiv 1 \pmod{c} \). Petersson [Pet32] expressed a certain sum of Fourier coefficients of holomorphic weight \( k \) cusp forms in terms of the Bessel function \( J_{k-1} \) and Kloosterman sums. Historically, this was the first relative trace formula, although cosmetically Petersson’s proof appears different. It involved constructing a Poincaré series and computing its coefficients in two different manners rather than integrating an automorphic kernel function. Bruggeman [Bru78] and Kuznetsov [Kuz80] later showed how to incorporate Maass forms into the formula. Traditionally, the formula is called the **Kuznetsov formula**.
Using the formula, Kuznetsov was able to prove estimates for the sum
\[ \sum_{c=1}^{\infty} \frac{K(m, n; c)}{c} \phi \left( \frac{2\pi \sqrt{mn}}{c} \right) \]
for suitable \( \phi \in C^\infty(\mathbb{R}_{>0}) \). We refer to [IK04, Chapter 16] for more details.

There are many other analytic number theory papers that use the formula, for example [Iwa02, GS83]. It has also played a role in work on Langlands’ beyond endoscopy proposal [Her11, Her12, Her16, Sar, Ven04].

Following Jacquet, we will derive the Kuznetsov formula as a relative trace formula (see [KL08, KL13] for a nice exposition for \( \text{GL}_2 \), and [Ye00] for a slightly different take on the higher rank case). The beautiful fact about this trace formula is that the general simple relative trace formula (Theorem 18.2.4) together with a small additional argument for the spectral side can be used to derive it in complete generality, with no additional assumptions on the test functions involved. We state this precisely in Theorem 18.9.2 below.

In this sense, it is analytically the simplest trace formula. The Petersson-Kuznetsov formula remains an important tool in analytic number theory and has not been fully exploited in higher rank (see [BB20] for example for recent results for \( \text{GL}_3 \)). We have therefore structured our treatment of the formula with applications to analytic number theory in mind.

We now set notation. Let \( N := N_r \leq \text{GL}_r \) be the unipotent radical of the Borel \( B := B_r \) of upper triangular matrices and let \( T := T_r \leq B \) be the maximal torus of diagonal matrices. For this trace formula, we take \( H := N \times N \leq \text{GL}_r \times \text{GL}_r \).

Thus we have a right action
\[ \text{GL}_r \times N \times N \rightarrow \text{GL}_r \]
given on points by \( (n_1, n_2) \cdot \gamma = n_1^{-1} \gamma n_2 \). For a global or local field \( F \), we denote the stabilizer of a \( \gamma \in \text{GL}_r(F) \) under this action by \( (N \times N)_\gamma := H_\gamma \leq N \times N \).

Thus for \( F \)-algebras \( R \), one has that
\[ (N \times N)_\gamma(R) := \{ (n, \gamma^{-1} n \gamma) : n \in N(R) \cap \gamma N(R) \gamma^{-1} \} \] (18.36)

**Lemma 18.7.1** Every element of \( \text{GL}_r(F) \) is relatively semisimple, relatively unimodular, and relatively elliptic.

**Proof.** For every \( \gamma \in \text{GL}_r(F) \), the stabilizer \( (N \times N)_\gamma \) is unipotent, hence unimodular by Exercise 18.7. Moreover the fact that \( (N \times N)_\gamma \) is unipotent implies that \( [(N \times N)_\gamma] \) is compact by Theorem 2.6.3; we deduce that \( \gamma \) is
relatively elliptic. Finally [Mil17, Theorem 17.64] implies that every element of $\text{GL}_r(F)$ is relatively semisimple.

The partial determinants
\[
\det_i : \text{GL}_r \to \mathbb{G}_a
\]
given on points by
\[
\det_i (A_i) = \det(A_i)
\]
are invariant under the action of $N \times N$ for $1 \leq i \leq r$. Here $A_i$ is an $i \times i$ matrix.

For every representative $w \in \text{GL}_r(F)$ of the Weyl group, we define a morphism (of schemes, not group schemes)
\[
\mathcal{T}_w : G_m^r \to \text{GL}_r
\]
by
\[
\mathcal{T}_w(c_1, \ldots, c_r) = \begin{pmatrix} 
\frac{c_r}{c_{r-1}} & \frac{c_{r-1}}{c_{r-2}} & \cdots & \frac{c_2}{c_1} \\
1 & & & \\
& & \ddots & \\
& & & 1
\end{pmatrix} w. \tag{18.38}
\]

By the Bruhat decomposition [Mil17, Example 21.46], one has
\[
\text{GL}_r(F) = \prod_{w \in W(\text{GL}_r, T)(F)} N(F)T(F)wN(F)
\]
and thus every element of $N(F)\backslash \text{GL}_r(F)/N(F)$ has a representative of the form $\mathcal{T}_w(c)$ for some $c \in (F^\times)^r$. The orbit of a given $w \in W(\text{GL}_r, T)(F)$ under $B \times B$ is called a Bruhat cell. Let
\[
w_0 = \begin{pmatrix} 1 \\
1 & \ddots \\
& & 1
\end{pmatrix}
\]
be (the standard representative of) the long Weyl element. The orbit of $w_0$ is the unique open Bruhat cell [Mil17, Theorem 21.73].

**Lemma 18.7.2** The morphism $\mathcal{T}_{w_0}$ induces a bijection
\[
\mathcal{T}_{w_0} : (F^\times)^r \to N(F)\backslash B(F)w_0B(F)/N(F).
\]
Moreover, the stabilizer $(N \times N)_\gamma$ of any element of $B(F)w_0B(F)$ is trivial.
This lemma implies in particular that all elements of the open Bruhat cell are relevant with respect to any quasi-character in both the local and the global setting.

Proof. It is clear that any element of $N(F)\backslash B(F)w_0B(F)/N(F)$ is in the image of the map $\mathcal{T}_{w_0}$. On the other hand, the partial determinants $\det_i$ are all $N \times N$-invariant and

$$\det_i(\mathcal{T}_{w_0}(c)) = \pm c_i,$$

where $c_i$ is the $i$th entry of $c$ and the sign $\pm$ depends only on $i$ and $r$. Thus $\mathcal{T}_{w_0}$ is injective and induces a bijection as claimed.

For the claim on triviality of stabilizers, we can assume $\gamma = tw_0$ for $t \in T(F)$ by what we have already proven. Then if $(n_1, n_2) \in (N \times N)_{\gamma}(F)$, one has

$$w_0^{-1}t^{-1}n_1tw_0 = n_2. \tag{18.39}$$

The left hand side is in the unipotent radical of the Borel opposite to $B$, so both sides must be equal to $I_r$. \qed

### 18.8 Kloosterman integrals

The geometric side of the Kuznetsov formula is a sum of Kloosterman integrals. We define Kloosterman integrals in this section and discuss their basic properties. The reader unfamiliar with integrals over totally disconnected groups may want to consult Appendix B and complete the exercises of that appendix before reading this section.

We define a morphism

$$\nu : \mathbb{G}_m^{r-1} \longrightarrow T \tag{18.40}$$

via

$$\nu(m) := \begin{pmatrix} \prod_{i=1}^{r-1} m_i \\ \vdots \\ m_{r-2}m_{r-1} \\ m_{r-1} \\ 1 \end{pmatrix}. \tag{18.41}$$

We note that
Let $F$ be a global field and fix a nontrivial character

$$\psi : F \backslash \mathbb{A}_F \rightarrow \mathbb{C}^\times.$$ 

It factors as $\psi = \prod_v \psi_v$, where $\psi_v : F_v \rightarrow \mathbb{C}^\times$ is a nontrivial character. For $m = (m_i) \in F^{r-1}$, we set

$$\psi_m \left( \begin{array}{ccc} 1 & x_1 & \ast \\ \vdots & \vdots & \vdots \\ 1 & x_{r-1} & 1 \end{array} \right) := \psi \left( \sum_{i=1}^{r-1} m_i x_i \right).$$  (18.43)

We also use the obvious local analogue of this notation.

We thus obtain a character

$$\psi_{m_1} \otimes \overline{\psi}_{m_2} : N(\mathbb{A}_F) \times N(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$$

trivial on $N(F) \times N(F)$. We now discuss which elements of $N(F) \backslash GL_r(F)/N(F)$ are $\psi_{m_1} \otimes \overline{\psi}_{m_2}$-relevant. One has that

$$S_r \rightarrow W(GL_r, T)(F).$$

The map sends a permutation $\sigma \in S_r$ to the class of the permutation matrix $w_\sigma$ such that $w_\sigma$ has nonzero entries in the $(i, \sigma(i))$ positions.

The following lemma was attributed to Piatetski-Shapiro in [Fri87, p. 175].

**Lemma 18.8.1** Assume $m_1, m_2 \in (F^\times)^{r-1}$. A class in $N(F) \backslash GL_r(F)/N(F)$ is $\psi_{m_1} \otimes \overline{\psi}_{m_2}$-relevant only if it is a subset of $N(F)T(F)w_\sigma N(F)$ where $\sigma^{-1}(i) < \sigma^{-1}(i + 1)$ implies $\sigma^{-1}(i + 1) - \sigma^{-1}(i) = 1$ for $1 \leq i < r$. In other words, a class contained in $N(F)T(F)w_\sigma N(F)$ is relevant only if

$$w_\sigma = \left( \begin{array}{ccc} I_{r_1} \\ I_{r_2} \\ \vdots \\ I_{r_k} \end{array} \right)$$

where $\sum_{i=1}^k r_i = r$.

**Proof.** Consider $u := I + \sum_{i=1}^{r-1} x_i E_{i,i+1}$ where $x_i \in \mathbb{A}_F$ and $E_{i,j}$ is the elementary matrix with a 1 in the $(i, j)$ position and zeros elsewhere. One has that
where convergent and Proposition 18.8.2 analogous notation and terminology in the local setting.

Suppose that \( \sigma^{-1}(i) < \sigma^{-1}(i+1) \). Then

\[
\{ (I + x_i E_{\sigma^{-1}(i), \sigma^{-1}(i+1)}, I + x_i E_{i,i+1}) : x_i \in \mathbb{A}_F \} \leq (N \times N)_w(e_\mathbb{A}_F).
\]

In order for \( \psi_{m_1} \otimes \overline{\psi}_{m_2} \) to be trivial on this subgroup of \( (N \times N)_w(e_\mathbb{A}_F) \), it is necessary that \( \sigma^{-1}(i+1) - \sigma^{-1}(i) = 1 \). This proves that \( N(F)w_\sigma N(F) \) is \( \psi_{m_1} \otimes \overline{\psi}_{m_2} \)-relevant only if \( \sigma^{-1}(i) < \sigma^{-1}(i+1) \) implies \( \sigma^{-1}(i+1) - \sigma^{-1}(i) = 1 \) for \( 1 \leq i < r \). On the other hand, for \( c \in (F^\times)^{r-1} \) and \( a \in F^\times \) the element \( au(c)w_\sigma \) is \( \psi_{m_1} \otimes \overline{\psi}_{m_2} \)-relevant only if \( w_\sigma \) is \( \psi_{m_1c} \otimes \overline{\psi}_{m_2} \)-relevant by a change of variables. Here

\[
m_1c := (m_{11}c_1, \ldots, m_{1(r-1)c_{r-1}}),
\]

where \( m_1 = (m_{11}, \ldots, m_{1(r-1)}) \) and \( c = (c_1, \ldots, c_{r-1}) \). Thus we deduce that a general element of \( N(F)T(F)w_\sigma N(F) \) is \( \psi_{m_1} \otimes \overline{\psi}_{m_2} \)-relevant only if \( \sigma^{-1}(i) < \sigma^{-1}(i+1) \) implies \( \sigma^{-1}(i+1) - \sigma^{-1}(i) = 1 \) for \( 1 \leq i < r \).

For \( f \in C_c^\infty(A_{GL_r} \backslash GL_r(\mathbb{A}_F)) \) and \( \psi_{m_1} \otimes \overline{\psi}_{m_2} \)-relevant \( \gamma \in GL_r(F) \), we define the orbital integral

\[
\text{RO}_\gamma(f, m_1, m_2) := \int_{(N \times N)_\gamma(\mathbb{A}_F) \backslash N^2(\mathbb{A}_F)} f(n_1^{-1} \gamma n_2) \psi_{m_1}(n_1) \overline{\psi}_{m_2}(n_2) dn_1 dn_2. \tag{18.45}
\]

By Proposition 18.2.3 and Lemma 18.7.1, the integral in (18.45) is absolutely convergent. It is sometimes called a Kloosterman integral. We use the analogous notation and terminology in the local setting.

**Proposition 18.8.2** The sum

\[
\sum_{[\gamma] \in (N \times N)_\gamma(\mathbb{F}F) \backslash N(F)} \tau((N \times N)_\gamma) |\text{RO}_\gamma(f, m_1, m_2)|
\]

is convergent and

\[
\int_{[N]^2} K_f(n_1, n_2) \psi_{m_1}(n_1) \overline{\psi}_{m_2}(n_2) dn_1 dn_2
\]

\[
= \sum_{[\gamma] \in (N \times N)_\gamma(\mathbb{F}F) \backslash N(F)} \tau((N \times N)_\gamma) \text{RO}_\gamma(f, m_1, m_2)
\]

where \( [N] := N(F) \backslash N(\mathbb{A}_F) \) and the sum is over \( \psi_{m_1} \otimes \overline{\psi}_{m_2} \)-relevant classes.
Proof. By Lemma 18.7.1, every $\gamma$ is relatively unimodular, relatively elliptic, and relatively semisimple. Thus the proposition follows from the proof of Theorem 18.2.4. \qed

In the remainder of this section, we assume that $F$ is nonarchimedean with ring of integers $\mathcal{O}_F$. For $\gamma$ in the open Bruhat cell, we will relate \( \mathrm{RO}_\gamma(\mathbb{I}_{\GL_r}(\mathcal{O}_F), m_1, m_2) \) to classical Kloosterman sums, justifying calling these orbital integrals Kloosterman integrals.

We start with some easy observations (compare [Ste87, Fri87]):

Lemma 18.8.3 The Kloosterman integral $\mathrm{RO}_\gamma(\mathbb{I}_{\GL_r}(\mathcal{O}_F), m_1, m_2)$ is nonzero only if $\det \gamma \in \mathcal{O}_F^\times$ and $\det_i(\gamma) \in \mathcal{O}_F$ for $1 \leq i \leq r$.

Proof. This follows from the fact that the partial determinants $\det_i$ are invariant under the action of $N \times N$ on $\GL_r$. \qed

Lemma 18.8.4 Let $f'(g) := f(g^{-1})$. Then one has that

\[
\mathrm{RO}_{\gamma}(f, m_1, m_2) = \mathrm{RO}_{\gamma}^{-1}(f', -m_2, -m_1).
\]

Proof. We compute that

\[
\begin{align*}
\mathrm{RO}_{\gamma}(f, m_1, m_2) \\
= & \int f(n_1^{-1} \gamma(n_2) \psi_{m_1}(n_1) \overline{\psi}_{m_2}(n_2) dn_1 dn_2 \\
= & \int f'(n_1^{-1} \gamma(n_2) \psi_{m_1}(n_1) \overline{\psi}_{m_2}(n_2) dn_1 dn_2 \\
= & \int f'(n_2^{-1} \gamma(n_1) \psi_{m_1}(n_1) \overline{\psi}_{m_2}(n_2) dn_1 dn_2,
\end{align*}
\]

where the integrals are over $(N \times N)\gamma(F) \backslash N(F) \times N(F)$. We have a bijection

\[
(N \times N)\gamma(F) \backslash N(F) \times N(F) \rightarrow (N \times N)\gamma^{-1}(F) \backslash N(F) \times N(F) \\
(n_1, n_2) \mapsto (n_2, n_1).
\]

Changing variables using this bijection we see that (18.46) is equal to

\[
\int f'(n_1^{-1} \gamma(n_1) \psi_{m_1}(n_1) \overline{\psi}_{m_2}(n_2) dn_1 dn_2 \\
= \mathrm{RO}_{\gamma}^{-1}(f', -m_2, -m_1)
\]

where the integral is now over $(N \times N)\gamma^{-1}(F) \backslash N(F) \times N(F)$. \qed

It turns out that for suitable $c$ the integral $\mathrm{RO}_{\gamma}(f, m_1, m_2)$ can be expressed as the product of Kloosterman sums, see Proposition 18.8.6 below. Before stating and proving Proposition 18.8.6 we require a preparatory lemma:
Lemma 18.8.5 Assume that $c = (c_1, c_2, \ldots, c_r) \in (\mathcal{O}_F \cap F^\times)^r$ and that $c_1, c_r \in \mathcal{O}_F^\times$. Then one has that

$$
\text{RO}_{\mathcal{O}_F} (\mathbf{1}_{\mathcal{O}_F} m_1, m_2) = \mathbf{1}_{\mathcal{O}_F} (m_{1(r-1)} \mathbf{1}_{\mathcal{O}_F} m_2) \text{RO}_{\mathcal{O}_F} (\mathbf{1}_{\mathcal{O}_F} m_1, m_2),
$$

where $\tilde{w}_0 \in \text{GL}_{r-1}(F)$ is the standard representative for the long Weyl element,

$$
\tilde{c} := (c_2/c_1, c_3/c_1, \ldots, c_r/c_1) \in (\mathcal{O}_F \cap F^\times)^{r-1},
$$

and

$$
\tilde{m}_1 := (m_{11}, m_{12}, \ldots, m_{1(r-2)}), \tilde{m}_2 := (m_{22}, m_{22}, \ldots, m_{2(r-1)}) \in F^{r-2}.
$$

In other words, under suitable assumptions, Kloosterman integrals on $\text{GL}_r$ can be written in terms of Kloosterman integrals on $\text{GL}_{r-1}$.

Proof. To ease notation, let $C$ be the $(r-1) \times (r-1)$ matrix given by

$$
C := \begin{pmatrix}
  c_r/c_{r-1} \\
  \vdots \\
  c_2/c_1
\end{pmatrix}.
$$

For $n_1, n_2 \in N_{r-1}(F)$, $\alpha_1, \alpha_2 \in F^{r-1}$ (the latter viewed as column vectors) we compute

$$
\begin{pmatrix}
  n_1 \\
  1
\end{pmatrix}
\begin{pmatrix}
  C \\
  c_1
\end{pmatrix}
\begin{pmatrix}
  1 \alpha_1' \\
  n_2
\end{pmatrix} =
\begin{pmatrix}
  c_1 \alpha_1 \\
  c_1
\end{pmatrix}
\begin{pmatrix}
  1 \alpha_1' \\
  n_2
\end{pmatrix} =
\begin{pmatrix}
  c_1 \alpha_1 / c_2 \\
  c_1 \alpha_2
\end{pmatrix}.
$$

(18.47)

Here

$$
\tilde{C} = c_1 \alpha_2 \alpha_1' + n_1 C n_2
$$

is an $(r-1) \times (r-1)$ matrix, $c_1 \alpha_1$ is a column vector, and $c_1 \alpha_1'$ is a row vector. Since we assumed that $c_1 \in \mathcal{O}_F^\times$ and $c_r \in \mathcal{O}_F^\times$, (18.47) is in $\text{GL}_r(\mathcal{O}_F)$ if and only if $\alpha_1, \alpha_2 \in \mathcal{O}_F^{r-1}$ and $n_1 C n_2 \in \text{GL}_{r-1}(\mathcal{O}_F)$. The lemma follows. \qed\\

Proposition 18.8.6 Assume that $c = (c_1, \ldots, c_r) \in (\mathcal{O}_F \cap F^\times)^r$ where $c_r \in \mathcal{O}_F^\times$, $c_j \notin \mathcal{O}_F^\times$, and $c_i \in \mathcal{O}_F^\times$ for all $i \neq j$. Then

$$
\text{RO}_{\mathcal{O}_F} (\mathbf{1}_{\mathcal{O}_F} m_1, m_2) = 0
$$

unless $m_1, m_2 \in \mathcal{O}_F^{r-1}$, in which case it is equal to

$$
\sum_{x \in (\mathcal{O}_F/c_j)^\times} \psi \left( \frac{-m_{2j} x + m_{1(r-j)} c_{j+1} c_{j-1} \overline{c}_j}{c_j} \right).
$$

(18.48)
Here $\pi \in \mathcal{O}_F$ is chosen so that $x\pi \equiv 1 \pmod{c}$ and we set $c_{j-1} = 1$ if $j = 1$.

When $F = \mathbb{Q}_p$, (18.48) is a classical Kloosterman sum. This explains why the relative orbital integral $\text{RO}_{\alpha_0}(c)(\mathbb{I}_{\text{GL}_r(\mathcal{O}_F)}, m_1, m_2)$ is called a Kloosterman integral.

**Proof.** By Lemma 18.8.5 iterated $j - 1$ times, we have that

$$\text{RO}_{\alpha_0}(c)(\mathbb{I}_{\text{GL}_r(\mathcal{O}_F)}, m_1, m_2) = 0$$

unless $m_1(r-1), \ldots, m_1(r-j+1), m_21, \ldots, m_2(j-1) \in \mathcal{O}_F$. Here $m_1$ and $m_2$ are the $i$th entry of $m_1$ and $m_2$ respectively. Assuming this is the case, the same lemma implies that

$$\text{RO}_{\alpha_0}(c)(\mathbb{I}_{\text{GL}_r(\mathcal{O}_F)}, m_1, m_2) = \text{RO}_{\alpha_0}(c)(\mathbb{I}_{\text{GL}_r(-(j-1))(\mathcal{O}_F)}, \bar{m}_1, \bar{m}_2).$$

Here $\bar{w} \in \text{GL}_{r-(j-1)}(F)$ is the standard representative for the long Weyl element and

$$\bar{c} := (c_j/c_{j-1}, c_j/c_{j-1}, \ldots, c_j/c_{j-1}),$$
$$\bar{m}_1 := (m_{11}, \ldots, m_{1(r-j)}),$$
$$\bar{m}_2 := (m_{2j}, \ldots, m_{2(r-1)}).$$

We now apply Lemma 18.8.4 to write this as

$$\text{RO}_{\alpha_0}(c-1)(\mathbb{I}_{\text{GL}_r(-(j-1))(\mathcal{O}_F)}, -\bar{m}_2, -\bar{m}_1)$$

$$= \text{RO}_{\alpha_0}(c/c_{r-1}c/c_{r-1}c/c_{r-1}c/c_{r-1})(\mathbb{I}_{\text{GL}_r(-(j-1))(\mathcal{O}_F)}, -\bar{m}_2, -\bar{m}_1).$$

Applying Lemma 18.8.5 again, this vanishes unless

$$m_{11}, \ldots, m_{1(r-j-1)}; m_{2(j+1)}, \ldots, m_{2(r-1)} \in \mathcal{O}_F$$

in which case it is

$$\text{RO}_{\alpha_0}(c)(\mathbb{I}_{\text{GL}_2(\mathcal{O}_F)}, -m_{2j}, -m_{1(r-j)}).$$

We compute this directly. It is equal to

$$\int \mathbb{I}_{\text{GL}_2(\mathcal{O}_F)} \left( \begin{array}{cc} 1 - x & 1 \\ 1 & 1 \\ \end{array} \right) \left( \begin{array}{cc} c_{j-1}/c_j \\ c_j/c_{j+1} \\ \end{array} \right) \psi(m_{2j}x)\psi(m_{1(r-j)}y) \ dx \ dy$$

$$= \int \mathbb{I}_{\text{GL}_2(\mathcal{O}_F)} \left( \begin{array}{cc} -c_jx/c_{j+1} - c_jxy/c_{j+1} + c_{j-1}/c_j \\ c_j/c_{j-1} \\ c_jy/c_{j-1} \\ \end{array} \right) \psi(m_{2j}x)\psi(m_{1(r-j)}y) \ dx \ dy$$

$$= \frac{1}{|c_j|^2} \int \mathbb{I}_{\text{GL}_2(\mathcal{O}_F)} \left( \begin{array}{cc} -x/c_{j-1} - xy/c_{j-1} + c_{j+1}/c_j \\ c_j/c_{j-1} \\ c_jy/c_{j-1} \\ \end{array} \right) \psi \left( \frac{-m_{2j}x + m_{1(r-j)}y}{c_j} \right) \ dx \ dy.$$
which completes the proof. □

18.9 Sums of Whittaker coefficients

The spectral side of the Kuznetsov formula is a sum of Whittaker coefficients. We make this precise in the current section and then prove the Kuznetsov formula in Theorem 18.9.2 below.

Let \( f \in C_c^\infty(A_{GL_r} \backslash GL_r(\mathbb{A}_F)) \). If \( \pi \) is a cuspidal automorphic representation of \( A_{GL_r} \backslash GL_r(\mathbb{A}_F) \) then we can form the relative trace

\[
\text{rtr}(\pi(f), m_1, m_2) := \int_{[N]^2} K_{\pi(f)}(n_1, n_2) \psi_{m_1}(n_1) \overline{\psi}_{m_2}(n_2) dn_1 dn_2. \tag{18.49}
\]

Since \([N]\) is compact, this is absolutely convergent. We then have the following theorem:

**Theorem 18.9.1** Assume that \( R(f) \) has cuspidal image. Then one has an equality of absolutely convergent sums

\[
\sum_{\pi} \text{rtr}(\pi(f), m_1, m_2) = \sum_{[\gamma] \in N(F) \backslash GL_r(F)/N(F)} \tau(N_\gamma) \text{RO}_\gamma(f, m_1, m_2), \tag{18.50}
\]

where the sum on the left is over isomorphism classes of cuspidal automorphic representations of \( A_{GL_r} \backslash GL_r(\mathbb{A}_F) \) and the sum on the right is over \( \psi_{m_1} \otimes \overline{\psi}_{m_2} \)-relevant classes.

**Proof.** This is immediate from Theorem 16.2.6 and Proposition 18.8.2. □

We now use Langlands’ spectral expansion of the automorphic kernel function \( K_f(x, y) \) from §16.3 to give a general version of Theorem 18.9.1 that requires no assumption on \( f \). For unexplained notation below we refer to Chapter 10.

Let \( B \leq GL_r \) be the Borel subgroup of upper triangular matrices. Call a parabolic subgroup \( P \leq GL_r \) standard if it contains \( B \). We let \( M \leq P \) be the unique Levi subgroup containing the maximal torus of diagonal matrices. We let \( A_P \leq \text{Res}_{F/\mathbb{Q}} M(\mathbb{R}) \) be the connected component of the identity (in the real topology) and let

\[
a_P := \text{Lie } A_P.
\]

Then Langlands’ spectral decomposition of \( L^2([GL_r]) \) implies that
where the sum on $P$ is over standard parabolic subgroups of $GL_r$, $n_P$ is the order of the association class of $P$ and the sum on $\sigma$ is over isomorphism classes of irreducible $A_M \cap M(F)$-subrepresentations of $L^2_{\text{disc}}([M])$. Here

$$K_I(x, y) := \sum_{\varphi \in B(\sigma)} E(x, I(\sigma, \lambda)(f) \varphi, \lambda) E(y, \varphi, \lambda),$$

(18.52)

where $B(\sigma)$ is an orthonormal basis of the $\sigma$-isotypic subspace of

$$\text{Ind}_{P}^{GL_r}(L^2_{\text{disc}}([M])(\sigma))$$

consisting of vectors in

$$\text{Ind}_{P}^{GL_r}(L^2_{\text{disc}}([M])(\sigma))^0.$$

This expression is only an $L^2$-expansion of the kernel. If $f$ is finite under a maximal compact subgroup of $GL_r(F)$ then the sum can be taken to be finite and hence the identity (18.52) is valid pointwise. Even if this is not the case, the sum is still represented by a smooth function of $x$ and $y$. We refer to [Art78, §4] for proofs of these assertions. In fact they can be proven by an easy analogue of the argument proving Theorem 16.2.3

We set

$$rtr(I(\sigma, \lambda)(f), m_1, m_2) := \int_{[N]^2} K_I(\sigma, \lambda)(f)(n_1, n_2) \psi_{m_1}(n_1) \overline{\psi}_{m_2}(n_2) dn_1 dn_2.$$

Since $[N]$ is compact, this is absolutely convergent.

The following is the full version of the Petersson-Bruggemann-Kuznetsov formula:

**Theorem 18.9.2** One has that

$$\sum_P n_P^{-1} \sum_{\sigma} \int_{\text{iap}} |rtr(I(\sigma, \lambda)(f), m_1, m_2)| d\lambda < \infty.$$

Moreover

$$\sum_P n_P^{-1} \sum_{\sigma} \int_{\text{iap}} rtr(I(\sigma, \lambda)(f), m_1, m_2) d\lambda$$

$$= \sum_{[\gamma] \in N(F) \setminus GL_r(F) / N(F)} \tau((N \times N)_{\gamma}) \text{RO}_{\gamma}(f, m_1, m_2),$$

where the sum on the right is over $\psi_{m_1} \otimes \overline{\psi}_{m_2}$-relevant classes.
Proof. The absolute convergence statement follows from the fact that \([N]\) is compact together with the estimates on the terms in the spectral expansion (18.51) contained in [Art78, §4]. Thus by (18.51), we conclude that

\[
\int_{[N]^2} K_f(n_1, n_2)\psi_{m_1}(n_1)\overline{\psi}_{m_2}(n_2)dn_1dn_2 = \sum_P n_P^{-1} \sum_\sigma \int_{ia_P} \text{rtr}(I(\sigma, \lambda))(f, m_1, m_2)d\lambda.
\]

The theorem now follows from Proposition 18.8.2. □

It is important to observe that for \(m_1, m_2 \in (F^\times)^{v-1}\) only generic representations can contribute to the spectral side of Theorem 18.9.2. In particular, for \(P = G\), only cuspidal representations contribute by the following theorem:

**Theorem 18.9.3** If \(\pi\) is a generic discrete automorphic representation of \(A_{GL_r} \backslash GL_r(\mathbb{A}_F)\), then \(\pi\) is cuspidal.

**Proof.** This is a combination of the main theorem of [MgW89] and [JL13, Theorem 5.5]. □

In analytic number theory, it is useful to understand the dependence of the quantities \(\text{rtr}(I(\sigma, \lambda))(f, m_1, m_2)\) on \(m_1\) and \(m_2\). We now make this explicit. Assume \(I(\sigma, \lambda)\) is unramified outside a finite set \(S\) of places of \(F\) including the infinite places. Assume moreover that \(I(\sigma_v, \lambda)\) admits a nonzero \(\psi_v\)-Whittaker functional for each place \(v\).

Let \(\psi := \psi_{(1, \ldots, 1)}\), where \(\psi_m\) is defined in (18.43). Using the notation of (11.5) for Whittaker models, we have the following lemma:

**Lemma 18.9.4** Let \(\sigma\) be discrete automorphic representation of \(A_M \backslash M(\mathbb{A}_F)\) and let \(\lambda \in ia_P\). If \(I(\sigma, \lambda)\) is generic then there is a unique vector

\[ W^S \in W(I(\sigma, \lambda)^S, \overline{\psi}^S)_{GL_r(O_F^S)} : = \prod_{v \in S} W(I(\sigma_v, \lambda), \overline{\psi}_v)_{GL_r(O_{F_v})} \]

such that \(W^S(I) = 1\).

**Proof.** We first prove that \(I(\sigma, \lambda)\) is irreducible. This is equivalent to the assertion that \(I(\sigma_v, \lambda)\) is irreducible for all \(v\). Since \(I(\sigma_v, \lambda)\) is generic, it is irreducible for \(v|\infty\) (see below [Jac09, Lemma 2.5]). For nonarchimedean places \(v\), the representation \(I(\sigma, \lambda)_v\) is known to be irreducible even if we do not assume it is generic [Ber84].

Since \(I(\sigma, \lambda)_v\) is irreducible and admits a nonzero \(\psi_v\)-Whittaker functional it is generic. Hence we conclude by Theorem 11.4.1. □

For \(f \in C_c^\infty(A_{GL_r} \backslash GL_r(\mathbb{A}_F))\) and \(m_{1S}, m_{2S} \in (F^\times_S)^{v-1}\), let

\[
f_{m_{1S}, m_{2S}}(g) := f(m_{1S}g)g(m_{2S})^{-1}) \in C_c^\infty(A_{GL_r} \backslash GL_r(\mathbb{A}_F)), \quad (18.53)
\]
where \( \nu \) is defined as in (18.41).

**Lemma 18.9.5** Assume that \( f \) and \( I(\sigma, \lambda) \) are unramified outside \( S \). One has that

\[
\text{rtr}(I(\sigma, \lambda)(f_{m_1, S}, m_1, m_2)) = W^S(\nu(m_1)) W^S(\nu(m_2)) \text{rtr}(\pi(\lambda), 1, 1),
\]

where \( W^S \) is as in Lemma 18.9.4.

Here the 1 refers to the vector in \((F^\times)^{r-1}\) that is identically 1 in all entries.

The quantity \( W^S(\nu(m_1)) \) can be computed in terms of the Satake parameters of \( I(\sigma, \lambda)^S \) using Corollary 11.4.2.

**Proof.** To ease notation in the proof, write

\[
\tilde{f} := f_{m_1, S, m_2}.
\]

We change variables

\[
(n_1, n_2) \mapsto (\nu(m_1)^{-1} n_1 \nu(m_1), \nu(m_2)^{-1} n_2 \nu(m_2))
\]

and use (18.42) to see that

\[
\text{rtr}(\pi(\tilde{f}), m_1, m_2)
= \int_{[N]^2} K_{I(\sigma, \lambda)(\tilde{f})} (\nu(m_1)^{-1} n_1 \nu(m_1), \nu(m_2)^{-1} n_2 \nu(m_2)) \psi_1(n_1) \overline{\psi}_1(n_2) dn_1 dn_2
= \int_{[N]^2} K_{I(\sigma, \lambda)(\tilde{f})} (n_1 \nu(m_1), n_2 \nu(m_2)) \psi(n_1) \overline{\psi}(n_2) dn_1 dn_2.
\]

Here we have used the fact that \( K_{I(\sigma, \lambda)(\tilde{f})}(g_1, g_2) \) is left invariant under \( G(F) \times G(F) \). This in turn is equal to

\[
\int_{[N]^2} K_{I(\sigma, \lambda)(\tilde{f})}(n_1 \nu(m_1^S), n_2 \nu(m_2^S)) \psi(n_1) \overline{\psi}(n_2) dn_1 dn_2.
\]

The function

\[
(g_1, g_2) \mapsto \int_{[N]^2} K_{I(\sigma, \lambda)(\tilde{f})}(n_1 g_1, n_2 g_2) \psi(n_1) \overline{\psi}(n_2) dn_1 dn_2
\]

is a global Whittaker function for \( I(\sigma, \lambda) \otimes I(\sigma, \lambda)^\nu \) for the character \((n_1, n_2) \mapsto \psi(n_1) \overline{\psi}(n_2)\) that is unramified outside \( S \). Thus the lemma follows from uniqueness of Whittaker models (see Theorem 11.3.1). \( \square \)
Exercises

For the first three problems we assume that we are in the setting of §18.2, although we allow \( F \) to be a global function field. Let \( v \) be a finite place of the global field \( F \) and let

\[
f^v \in C^\infty_c(A_G \backslash G(\mathbb{A}_F)), \quad f_v \in C^\infty_c(G(F_v)), \quad f := f_v f^v.
\]

Let \( \gamma \in G(F) \) and let \( \Omega \subset A_G \backslash G(\mathbb{A}_F) \) be the \( H(\mathbb{A}_F) \)-orbit of \( \gamma \).

18.1. Prove that if \( f_v \) is supported in the set of elements of \( G(F_v) \) with closed \( H_{r'} \)-orbit and \( f(x) \neq 0 \) for some \( x \in \Omega \) then \( \gamma \) is relatively semisimple.

18.2. Prove that if \( f_v \) is supported in

\[ \{ \gamma' \in G(F_v) : H_{r'} = H_{r'}^\gamma \text{ is smooth} \} \]

and \( f(x) \neq 0 \) for some \( x \in \Omega \) then \( \gamma \) is gcf.

18.3. Prove that if \( f_v \) is supported in the set of elements of

\[ \{ \gamma' \in G(F_v) : X^*((Z_H \cap \Delta(Z_G)) \backslash H_{r'}) = 0 \} \]

and \( f(x) \neq 0 \) for some \( x \in \Omega \) then \( \gamma \) is relatively elliptic. Here, as before, \( \Delta : G \to G \times G \) is the diagonal embedding.

18.4. Give an example of a pair of reductive groups \( H \leq G \) such that

\[ H(\mathbb{A}_F) \neq (A_G \cap H(\mathbb{A}_F))(G(\mathbb{A}_F)^1 \cap H(\mathbb{A}_F)). \]

18.5. Under suitable assumptions on \( K \) if necessary, state and prove an analogue of (18.24) for twisted orbital integrals.

18.6. For \( f \in C^\infty_c(A_{GL_r} \backslash GL_r(F_v)) \) and \( m_1, m_2 \in (F_\infty^\times)^{r-1} \), let

\[
f_{m_1, m_2}(g) := f(\nu(m_1)g\nu(m_2)^{-1}).
\]

For \( c := (c_1, \ldots, c_r) \in (F_\infty^\times)^r \), prove that

\[
RO_{T_w}(f_{m_1, m_2}, m_1, m_2) \neq 0
\]

only if

\[
|c_r \det(\nu(m_1m_2^{-1}))|_v \asymp f Y
\]

for some \( Y \in \mathbb{R}_{>0} \) independent of \( v|\infty \) and \( c \) satisfies

\[
|\det_i(\nu(m_1)T_{w}(c)\nu(m_2)^{-1})|_v \ll f |c_r \det(\nu(m_1m_2^{-1}))|_v^{i/r}
\]

for all \( v|\infty \) and \( 1 \leq i \leq r \). In particular, if \( w = w_0 \), then
\[ RO_{w_0}(c) \left( f_{m_1, m_2}, m_1, m_2 \right) = 0 \]

unless
\[ |c_i|_v \ll_f |c_r \det \nu(m_1m_2^{-1})|_v^{1/r} |\det_i(\nu(m_1)w_0\nu(m_2^{-1}))|_v^{-1} \]

for all \( v|\infty \) and \( 1 \leq i \leq r \).

**18.7.** Let \( H \) be a unipotent algebraic group over a global field \( F \). Prove that \( H(\mathbb{A}_F) \) is unimodular.

**18.8.** Let \( X \) be the space of \( n \times n \) matrices, equipped with the natural action of \( \text{GL}_n \) on the right. Let \( \phi \in C_c(\mathbb{A}_F/\text{GL}_n) \). Prove that the restriction of \( \phi \) to \( \text{GL}_n(\mathbb{A}_F)/\text{GL}_n \) is smooth but not compactly supported.
Chapter 19
Applications of Trace Formulae

Abstract In this chapter, we discuss applications of relative trace formulae. Our goal is to provide an introduction to the key motifs of existence and comparison underlying these applications.

19.1 Existence and comparison

Selberg’s motivation for proving the trace formula for $\text{SL}_2$ over $\mathbb{Q}$ was to prove the existence of cusp forms that are fixed by $\text{SO}_2(\mathbb{R})$. In the process, he actually estimated the number of such cusp forms in a suitable family and showed that they exist in abundance. This was the first application of the trace formula, and it can be viewed as a (quantifiable) existence result. We discuss this in §19.2.

Motivated by the Langlands functoriality conjecture, Jacquet and Langlands [JL70] gave a very different type of application of the trace formula. They compared two different trace formulae on different groups in order to deduce a relation between automorphic representations on the two groups. This idea has been extended to what is known as the theory of twisted endoscopy and continues to play a key role in automorphic representation theory. We discuss this in the second part of the chapter. All sections of the chapter are more or less brief surveys. Our main aim is merely to whet the appetite of the reader and give them some guidance on further reading. We assume for this chapter that $F$ is a number field because most of the results we will cite below are proven in this setting.
19.2 The Weyl law

Let $M$ be a smooth compact Riemannian manifold. Associated to $M$ is a Laplace-Beltrami operator $\Delta$. Call a function $\varphi \in C^\infty(M) \cap L^2(M)$ a Laplacian eigenform with eigenvalue $\lambda$ if

$$\Delta \varphi = \lambda \varphi.$$ 

It is known that each eigenvalue $\lambda$ is a positive real number, that the set of eigenvalues is discrete, and that $L^2(M)$ is the Hilbert space direct sum of the corresponding eigenspaces $L^2(M)(\lambda)$. Moreover $\dim L^2(M)(\lambda) < \infty$ for each $\lambda$. For $x > 0$, consider the counting function

$$N(x) := \{ \dim L^2(M)(\lambda) : \sqrt{\lambda} \leq x \}.$$

The following is the Weyl law (see [Wey12] and [MP49]):

**Theorem 19.2.1 (The Weyl law)** One has that

$$N(x) = \frac{\text{vol}(M)}{(4\pi)^{\frac{\dim M}{2}}} \frac{\Gamma\left(\frac{\dim M}{2} + 1\right)}{\Gamma\left(\frac{\dim M}{2}\right)} x^{\dim M} + o(x^{\dim M}) \quad (19.1)$$

as $x \to \infty$. □

Consider the Shimura manifolds $\text{Sh}(G, X)^K$ of §15.2. These are finite unions of Riemannian manifolds. When they are compact, the Weyl law implies the existence of an abundance of automorphic forms on them. However they are often noncompact (see Theorem 2.6.3 for a precise statement). Even if one compactifies $\text{Sh}(G, X)^K$, it is still unclear in the noncompact case whether the eigenforms produced by the Weyl law correspond to the cuspidal or noncuspidal spectrum. Selberg developed the trace formula for $\text{SL}_2$ over $\mathbb{Q}$ to address precisely this question [Sel56], and he proved an analogue of Theorem 19.2.1 in this setting.

Müller [Mö7] established the analogue of the Weyl law for $G = \text{SL}_n$ over $\mathbb{Q}$. His proof makes use of the full noninvariant trace formula of Arthur, and can treat cusp forms that are not necessarily fixed by $\text{O}_n(\mathbb{R})$. About the same time, Lindenstrauss and Venkatesh [LV07] proved an analogous result for all split semisimple adjoint groups over $\mathbb{Q}$. Notably, their method is soft in that it does not require the full trace formula, only a simple analogue. It is based on a novel construction of test functions with purely cuspidal image. Their method currently only treats cusp forms that are fixed by a maximal compact subgroup at infinity, but it is likely that this can be removed.

Using the same technique, the authors [GH15] established a relative analogue of the Weyl law under some assumptions via the general simple relative trace formula (Theorem 18.2.4). This relative Weyl law gives an asymptotic formula for the number of Laplacian eigenfunctions on $\text{Sh}(G, X)^K$ weighted
by the \( L^2 \)-restriction norm over a locally symmetric subspace associated to a subgroup \( H \leq G \).

The approach to the Weyl law via the full Arthur-Selberg trace formula is more complicated, but it still has its merits. The results of Lindenstrauss and Venkatesh mentioned above give no control on the error term in the Weyl law. Using the trace formula, a Weyl law for \( SL_n \) over \( \mathbb{Q} \) with remainder was established by Lapid and Müller in \([LM09]\) using the approach of \([DKV79b, DKV79a]\) combined with the Arthur-Selberg trace formula.

### 19.3 The comparison strategy

The Langlands functoriality conjecture predicts that for two reductive groups \( G_1 \) and \( G_1 \) defined over the number field \( F \) and a \( L \)-map

\[
r : L^{G_1} \rightarrow L^{G_2},
\]

there should be a transfer from \( L \)-packets of automorphic representations of \( G_1(\mathbb{A}_F) \) to \( L \)-packets of automorphic representations of \( G_2(\mathbb{A}_F) \). The question becomes how one should prove such a transfer. One option which has proven to be effective is to study the geometric sides of suitable trace formulae for \( G_1 \) and \( G_2 \).

In more detail, suppose that we are given subgroups \( H_i \leq G_i \times G_i \) for \( 1 \leq i \leq 2 \) and characters

\[
\chi_i : (H_i(\mathbb{A}_F) \cap (A_{G} \times A_{G}))/H_i(\mathbb{A}_F) \rightarrow \mathbb{C}^\times.
\]

For \( F \)-algebras \( R \), the groups \( H_i \) act on \( G_i \) via

\[
G_i(R) \times H_i(R) \rightarrow G_i(R)
\]

\[
(g, (h_r, h_r)) \mapsto h_r^{-1} g h_r
\]
as in (18.1). We assume that the neutral component of \( H_i \) is the direct product of a unipotent and a reductive group. We write \( H_{ir} \) for the unique maximal reductive subgroup of \( H_i^0 \).

Suppose that we can construct a correspondence

\[
\{\chi_2 \text{-relevant } \delta \in G_2(F)/H_2(F)\} \leftrightarrow \{\chi_1 \text{-relevant } \gamma \in G_1(F)/H_1(F)\}.
\]

For example, one might have a bijection, but often the situation is more complicated. We refer to this as a geometric correspondence between relevant relative classes. We say that the class of \( \delta \) matches the class of \( \gamma \) if the two classes correspond. Suppose that for matching \( \delta \) and \( \gamma \), one can relate the orbital integrals \( \tau(G_{2\delta})\text{RO}_\delta(f) \) and \( \tau(G_{1\gamma})\text{RO}_\gamma(f^{G_1}) \) for suitable test functions \( f \in C_c^\infty(A_{G_2} \setminus G_2(\mathbb{A}_F)) \) and \( f^{G_1} \in C_c^\infty(A_{G_1} \setminus G_1(\mathbb{A}_F)) \). Then
one can hope to prove an identity of the form

\[
\sum_{\delta \in G_2(F)/H_2(F)} \tau(G_{2\delta}) \text{RO}_\delta(f) = \sum_{\gamma \in G_1(F)/H_1(F)} \tau(G_{1\gamma}) \text{RO}_\gamma(f^{G_1}). \quad (19.2)
\]

Here the sums over \(\gamma\) and \(\delta\) are over \(\chi_1\) and \(\chi_2\)-relevant elements, respectively, that are additionally relatively regular, relatively elliptic, relatively unimodular and such that the \(H_1\)-orbit of \(\gamma\) (resp. the \(H_2\)-orbit of \(\delta\)) is closed. We are assuming these things so that the sums are convergent by Theorem 18.2.4. Assuming that an identity like (19.2) can be established for a sufficiently rich set of test functions, one can obtain a relationship between \((H_1 \hookrightarrow G_1)\)-distinguished representations of \(G_1(F_v) \rtimes G_1(F_v)\) and \((H_2 \hookrightarrow G_2)\)-distinguished representations of \(G_2(F_v) \rtimes G_2(F_v)\) and thereby deduce an instance of Langlands functoriality (or its relative analogue due to Jacquet, Sakellaridis, and Venkatesh). We will explain this in more detail momentarily.

In many cases one has to incorporate families of \((G_i \hookrightarrow H_i)\) on both sides of the formula. We ignore this complication because we are just speaking in rough terms at the moment.

Let us elaborate what we mean by a relation between \(\tau(G_{2\delta}) \text{RO}_\delta(f)\) and \(\tau(G_{1\gamma}) \text{RO}_\gamma(f^{G_1})\). We assume for simplicity that the geometric correspondence is a bijection, though this is rarely the case. One needs to be able to relate the measures \(\tau(G_{1\gamma})\) and \(\tau(G_{2\delta})\) for matching \(\gamma\) and \(\delta\), but this is usually not the key issue.

For almost all finite places \(v\), there exists a hyperspecial subgroup \(K_{1v} \leq G_1(F_v)\) by Proposition 2.4.5. The map

\[ r : \text{LG}_1 \rightarrow \text{LG}_2 \]

induces an algebra homomorphism

\[ \tilde{r} : C_c^\infty(G_2(F_v) \parallel K_{2v}) \rightarrow C_c^\infty(G_1(F_v) \parallel K_{1v}) \quad (19.3) \]

via Theorem 7.5.1 (see also Exercise 19.2). Dual to this algebra homomorphism we have a transfer

\[ \pi_v \mapsto r(\pi_v) \quad (19.4) \]

from the set of equivalence classes of \(K_{1v}\)-unramified representations of \(G_1(F_v)\) to the set of equivalence classes of \(K_{2v}\)-unramified representations of \(G_2(F_v)\). Here we are abusing notation and using the same notation for a representation and its equivalence class. The transfer (19.4) is given by stipulating that the Langlands class of \(r(\pi_v)\) is the image under \(r\) of the Langlands class of \(\pi_v\). See below Corollary 7.5.2 for the definition of the Langlands class of an unramified representation. By Exercise 19.2 for \(f \in C_c^\infty(G_2(F_v) \parallel K_{2v})\) one has
\[ \text{tr } \pi_v(\tilde{f}(f)) = \text{tr } r(\pi_v)(f). \tag{19.5} \]

Here we normalize the Haar measures implicit in the traces so that the \( K_i \) are given measure 1. Globally, assume \( S \) is a finite set of places of \( F \) including the infinite places and for each \( i \) we are given compact open subgroups \( K_i^S \leq G_i(\mathbb{A}_F^S) \) is with \( K_{i_0} \) hyperspecial for all \( v \). Assume moreover that \( \pi_v^S \) is an admissible irreducible representation of \( G_1(\mathbb{A}_F^S) \) with a nonzero \( K_{i_0}^S \)-fixed vector. We then define

\[ r(\pi) := \otimes'_v r(\pi_v). \]

It is an irreducible admissible representation of \( G_2(\mathbb{A}_F^S) \) with a nonzero \( K_2^S \)-fixed vector.

One wants to prove that if \( \gamma \) and \( \delta \) match then for almost all finite places \( v \), one has the equality

\[ \text{RO}_{\delta_v}(f_v) = \text{RO}_{\gamma_v}(\tilde{f}(f_v)) \tag{19.6} \]

for all \( f_v \in C_c^\infty(G_2(F_v) \backslash K_{2v}) \). This is often too naive. The equality will only hold up to an explicit function of \( \delta_v \) and \( \gamma_v \) called a transfer factor, but we suppress this difficulty. Formulae of the form (19.6) are known as the fundamental lemma for the Hecke algebra. The identity (19.6) for \( f_v = 1_{K_{2v}} \) is known as the fundamental lemma, or the fundamental lemma for the unit element of the Hecke algebra. In practice, one deduces the fundamental lemma for the Hecke algebra from the fundamental lemma. Finally, one needs to know that for all places \( v \), there is a sufficiently large supply of functions \( f_v \in C_c^\infty(G_2(F_v)) \) and \( f_v^{G_1} \in C_c^\infty(G_1(F_v)) \) that match, meaning that when \( \delta_v \) and \( \gamma_v \) match one has that

\[ \text{RO}_{\delta_v}(f_v) = \text{RO}_{\gamma_v}(f_v^{G_1}). \tag{19.7} \]

Statements of this type are known as transfer. Of course, for an arbitrary \( f \in C_c^\infty(A_{G_1} \backslash G_2(\mathbb{A}_F)) \), we will have \( f_v \in C_c^\infty(G(F_{2v}) \backslash K_{2v}) \) with \( K_{2v} \) hyperspecial for almost all \( v \), and in this case, one can take \( f_v^{G_1} = \tilde{f}(f_v) \). As in the statement of the fundamental lemma, hoping for an equality as simple as (19.7) is often too naive; it will only hold up to an explicit transfer factor.

Given the fundamental lemma for the Hecke algebra and transfer, the identity (16.8) implies an identity

\[ \sum_{\pi'} \text{tr} r_{H_2, \chi_2} \pi'(f) = \sum_{\pi} \text{tr} r_{H_1, \chi_1} \pi(f^{G_1}), \tag{19.8} \]

at least for \( f \) and \( f^{G_1} \) satisfying the assumptions of Theorem 18.2.4. Here the sum on the left is over isomorphism classes of cuspidal automorphic representations \( \pi' \) of \( A_{G_1} \backslash G_2(\mathbb{A}_F) \) such that \( \pi' \otimes \pi^{\vee} \) is \((H_2, \chi_2)\)-distinguished and the sum on the right is over isomorphism classes of cuspidal automorphic representations of \( A_{G_1} \backslash G_1(\mathbb{A}_F) \) such that \( \pi \otimes \pi^{\vee} \) is \((H_1, \chi_1)\)-distinguished. Let \( S \) be a finite set of places of \( F \) including all the infinite places. If \( f \)
(resp. $f^{G_1}$) is fixed on the left and right under $K^S_2$ (resp. $K^S_1$) where $K_{iv}$ is hyperspecial for $v \notin S$ then this identity implies that
\[
\sum_{\pi'} \text{tr} \pi'^S(f^S) \text{rtr}_{H_2, \chi_2} \pi'(f^S \mathbb{1}_{K^S_2}) = \sum_{\pi} \text{tr} \pi^S(f^{G_1}S) \text{rtr}_{H_1, \chi_1} \pi(f^G_1 \mathbb{1}_{K^S_1})
\]
(see Exercise 19.4). Though the set of representations on each side of this equation is in general infinite, in practice one can use a version of linear independence of characters to refine it to an identity
\[
\sum_{\pi': \pi'^S \equiv r^S(\pi^S_0)} \text{tr} \pi'^S(f^S) \text{rtr}_{H_2, \chi_2}(\pi'(f^S \mathbb{1}_{K^S_2})) = \sum_{\pi: \pi^S \equiv \pi^S_0} \text{tr} \pi^S(f^{G_1}S) \text{rtr}_{H_1, \chi_1}(\pi(f^G_1 \mathbb{1}_{K^S_1}))
\]  
(19.9)
for each cuspidal automorphic representation $\pi_0$ of $A_{G_1} \backslash G_1(\mathbb{A}_F)$. Of course, both sums can be zero, but if one sum is nonzero, the other is also nonzero. If one can establish (19.9), then one can use it to establish a transfer from (weak) $L$-packets of automorphic representations of $A_{G_1} \backslash G_1(\mathbb{A}_F)$ to (weak) $L$-packets of automorphic representations of $A_{G_2} \backslash G_2(\mathbb{A}_F)$ and characterize the image of the transfer. Often one can refine this comparison to the level of $L$-packets (or even use it to define $L$-packets). We observe that in certain circumstances, it is better to start with a map of relative $L$-groups as explained by Sakellaridis and Venkatesh [SV17].

To recap, we started with a geometric assumption of matching. Assuming this, we further assumed the fundamental lemma and transfer. Given this pile of assumptions, we asserted that one could prove instances of Langlands functoriality. Amazingly, it is possible to make this heuristic method rigorous and prove fantastic results using it.

### 19.4 Jacquet-Langlands transfer and base change

In this section, we describe some settings where the general strategy in §19.3 has been successfully applied. They are related to the trace formula and the twisted trace formula of §18.4 and §18.5.

We start by taking $G_1$ to be an inner form of the quasi-split reductive group $G_2$ over $F$. Then there is an isomorphism
\[
r : L^{G_1} \longrightarrow L^{G_2}
\]
by Exercise 19.3. We assume that $G_2^{\text{der}}$ is simply connected so that we can apply Theorem 19.4.1 below.
We take $H_i \leq G_i \times G_i$ to be the diagonal copy of $G_i$ and we take $\chi$ to be trivial. Then in view of Proposition 17.1.6 and Example 17.2, $(G_i/H_i)(F)$ is the set of semisimple conjugacy classes in $G_i(F)$ that are $\text{Gal}(F/F)$-invariant. Using this and the definition of an inner form, one can construct an isomorphism of affine $F$-schemes

$$G_1/H_1 \sim \rightarrow G_2/H_2. \quad (19.11)$$

We give the construction in a special case in Lemma 19.4.2 below; the construction in general is essentially the same. This is almost the first step of the procedure explained in the previous section. There is a slight wrinkle. Consider the map

$$\{\text{semisimple } \gamma \in G_i(F)\} \longrightarrow (G_i/H_i)(F) \quad (19.12)$$

sending $\gamma$ to its geometric class. Because $G_1$ need not be quasi-split, for $i = 1$, the map (19.12) need not be surjective. However, for $i = 2$, the group $G_2$ is quasi-split and hence the map (19.12) is surjective by the following theorem: [Kot82]:

**Theorem 19.4.1 (Kottwitz and Steinberg)** Let $G$ be a quasi-split reductive group over a characteristic zero field $k$ with simply connected derived group. A semisimple $G(k)$-conjugacy class fixed under $\text{Gal}_k$ contains an element of $G(k)$. \hfill $\square$

In the current setting, the fundamental lemma for the Hecke algebra is trivial because $G_1/F_v$ and $G_2/F_v$ are isomorphic for almost all place $v$ (see Exercise 19.6). Thus proving transfer of automorphic representations is reduced to understanding transfer of functions. Jacquet and Langlands [JL70] were the first to introduce an argument of this type and they used it to give a complete treatment of the special case where $G_1$ is the unit group of a quaternion algebra and $G_2 = \text{GL}_2$. Because of this, transfers attached to the maps (19.10) are known as Jacquet-Langlands transfers. Though the full theory of Jacquet-Langlands transfers is still not known, many results exist in the literature. In particular, the theory for $\text{GL}_n$ is fairly complete. Let $D$ be a simple algebra of dimension $d^2$ over $F$. For $F$-algebras $R$, let

$$\text{GL}_{n,D}(R) := (D \otimes_F M_n(R))^\times.$$ 

Let $\text{GL}_{n,d}$ act on itself by conjugation and denote by $\text{GL}_{n,d}/\sim$ the GIT quotient. We define $\text{GL}_{n,D}/\sim$ similarly.

**Lemma 19.4.2** There is an isomorphism

$$t: \text{GL}_{n,d}/\sim \sim \rightarrow \text{GL}_{n,D}/\sim.$$ 

**Proof.** Let $M_{n,D}$ be the affine $F$-scheme whose points in an $F$-algebra $R$ are given by
Let $E/F$ be a finite Galois extension such that there is an isomorphism of (noncommutative) $E$-algebras $M_d(E)\to D \otimes_F E$. It induces an isomorphism

$$\tilde{t} : (M_{nd})_E \xrightarrow{\sim} (M_{nD})_E$$

which in turn induces an isomorphism of affine $E$-schemes

$$t : (M_{nd})_E/GL_{ndE} \xrightarrow{\sim} (M_{nD})_E/(GL_{nD})_E,$$

where the quotients are the GIT quotients with respect to the conjugation action. We claim that this isomorphism is the base change to $E$ of an isomorphism

$$M_{nd}/GL_{nd} \xrightarrow{\sim} M_{nD}/GL_{nD}.$$  \hfill (19.15)

In other words, $t$ descends to $F$. This suffices to complete the proof of the lemma, as the restriction of (19.15) to $GL_{nd}$ yields the isomorphism in the lemma.

To check this, by Galois descent it suffices to check that $t$ is $\text{Gal}(E/F)$-invariant. In more detail, for any affine scheme $X$ of finite type over $F$, there is an action of $\text{Gal}(E/F)$ (viewed as a constant group scheme over $E$, say) on $\text{Spec}(E)$ and on $X_E$ such that the structure map $X_E \to \text{Spec}(E)$ is equivariant. Concretely, at the level of points, for an $E$-algebra $R$, the set $X_E(R)$ is a system of solutions in $R$ to a set of polynomials defined over $F$ and so inherits a $\text{Gal}(E/F)$-action. If $Y$ is another affine $F$-scheme of finite type, to check that a morphism $\varphi : X_E \to Y_E$ the base change of a morphism $X \to Y$, it suffices to check that $\varphi$ is $\text{Gal}(E/F)$-invariant. For a more sophisticated perspective on Galois descent, [Poo17, §4.4] is a nice place to begin. Now for $\sigma \in \text{Gal}(E/F)$ consider

$$\tilde{t}^{-1} \circ \sigma \circ \tilde{t} : (M_{nd})_E \longrightarrow (M_{nd})_E.$$  

This is an automorphism of $(M_{nd})_E$. It follows from the Skolem-Noether theorem that $\tilde{t}^{-1} \circ \sigma \circ \tilde{t}$ is given by conjugation by some $g_\sigma \in GL_{nd}(E)$. Thus $\tilde{t}^{-1} \circ \sigma \circ \tilde{t}$ induces the identity automorphism of $(M_{nd})_E/(GL_{nd})_E$ for all $\sigma$, which is to say that $t$ is $\text{Gal}(E/F)$-invariant. \hfill $\square$

For every place $v$, we have a diagram

$$\begin{array}{ccc}
\{\text{semisimple } \delta_v \in \text{GL}_{nd}(F_v)\} & \{\text{semisimple } \gamma_v \in \text{GL}_{nD}(F_v)\} & \\
\downarrow \rho & \downarrow \rho D & \\
(GL_{nd}/\sim)(F_v) & \xrightarrow{t} & (GL_{nD}/\sim)(F_v)
\end{array}$$
where the quotients on the bottom are the GIT quotients of \(GL_{nd}\) and \(GL_{nD}\) by the conjugation action of \(GL_{nd}\) and \(GL_{nD}\), respectively, and the vertical maps are the projections. The map \(p\) is surjective (see Exercise 19.7), but \(p_D\) is not. We say that \(\delta_v\) and \(\gamma_v\) match if \(t(p(\delta_v)) = p_D(\gamma_v)\). An irreducible admissible representation \(\pi_v\) of \(GL_{nd}(F_v)\) is \(D_v\)-compatible if the character \(\Theta_{\pi_v}(\gamma_v)\) is nonzero for some regular semisimple \(\delta_v \in GL_{nd}(F_v)\) that matches a \(\gamma_v \in GL_{nD}(F_v)\). See Theorem 8.5.1 for the definition of the character \(\Theta_{\pi_v}\) in the nonarchimedean case. The archimedean case is similar [Kna86, Theorem 10.25]. An automorphic representation \(\pi\) of \(A_{GL_{nd}} \backslash GL_{nd}(\mathbb{A}_F)\) is \(D\)-compatible if \(\pi_v\) is \(D_v\)-compatible for all \(v\). Let \(S\) be the finite set of places \(v\) of \(F\) where \(D_v \not\subseteq M_d(F_v)\). Here \(M_n(F_v)\) is the space of \(n \times n\) matrices with entries in \(F_v\). Then \(\pi\) is \(D\)-compatible if and only if \(\pi_v\) is \(D_v\)-compatible for all \(v \in S\).

The following theorem is due to Badulescu and Grbac [Bad08] following several previous works [Rog83, BDKV84], going back to [JL70], where the \(n = 2\) case was treated.

**Theorem 19.4.3 (Jacquet-Langlands correspondence)** There exists a unique injection \(r\) from the set of isomorphism classes of discrete automorphic representations of \(A_{GL_{nd}} \backslash GL_{nd}(\mathbb{A}_F)\) to the set of isomorphism classes of discrete automorphic representations of \(A_{GL_{nd}} \backslash GL_{nd}(\mathbb{A}_F)\) such that \(r(\pi)_v \cong \pi_v\) for \(v \not\in S\). The image consists of discrete automorphic representations that are \(D\)-compatible.

Here we are using the isomorphism \(GL_{nd}(F_v) \cong GL_{nD}(F_v)\) for \(v \not\in S\) to make sense of the assertion that \(r(\pi)_v \cong \pi_v\).

In the setting above, the groups \(G_1\) and \(G_2\) were not that different from each other; indeed, they become isomorphic over the algebraic closure. Saito and Shintani [Sai75, Shi79] discovered the first geometric correspondence involving groups that are not isomorphic over the algebraic closure. It was fully developed for \(GL_2\) by Langlands in [Lan80]. Take \(G_1 = G\) to be reductive with simply connected derived group, let \(E/F\) be a cyclic field extension, and let \(\theta\) be a generator of \(\text{Gal}(E/F)\). We then take \(G_2 = \text{Res}_{E/F} G\), and let

\[
r : L^1 G \longrightarrow L^1 \text{Res}_{E/F} G
\]

be the map given by the diagonal embedding on the neutral component and the identity on the Galois factor. To set up our geometric correspondence, we let the \(\chi_i\) be trivial, let \(H_1\) be the diagonal copy of \(G\) in \(G \times G\), and let \(H_2\) be the \(\theta\)-twisted diagonal defined for \(F\)-algebras \(R\) by

\[
H_2(R) := \{(g, \theta(g)) : g \in \text{Res}_{E/F} G(R)\}.
\]

Thus \(G(F)/H_1(F)\) is the space of conjugacy classes and \(\text{Res}_{E/F} G(F)/H_2(F)\) is the space of \(\theta\)-conjugacy classes in the sense of §18.5.

The geometric correspondence is defined as follows. A reference is [Kot82]. Let \(\delta \in \text{Res}_{E/F} G(F)\) and let
N(δ) := θ^{[E:F]−1}(δ) · θ^{[E:F]−2}(δ) · · · δ.

View N(δ) as an element of G(E). Then

θ(N(δ)) = δN(δ)δ^{-1}

which implies that the G(E)-conjugacy class of N(δ) is fixed under Gal(E/F).

Thus there is an element of G(F) in the conjugacy class by Theorem 19.4.1. We call such an element a norm of δ. This gives us a map

\{θ-semisimple conjugacy classes in Res_{E/F}G(F)\} \rightarrow \{semisimple conjugacy classes in G(F)\}. \tag{19.17}

We say that a θ-semisimple δ and a semisimple γ match if γ is a norm of δ (see §18.5 for the notion of a θ-semisimple class).

This geometric correspondence forms the basis for a comparison of the trace formula of §18.4 and the twisted trace formula of §18.5. We point out that one might anticipate such a comparison by consideration of the spectral sides of the two trace formulae. Indeed, the image of the functorial transfer induced by r is contained in the set of representations π' that satisfy π' ≅ π^θ, and the spectral side of the twisted trace formula is given in terms of such representations.

Unlike in the Jacquet-Langlands case, the fundamental lemma is nontrivial in this setting. However Kottwitz found a beautiful, direct, and simple argument establishing it [Kot86a]. The fundamental lemma for the whole Hecke algebra was then deduced by Clozel [Clo90a] and Labesse [Lab90]. This was the basis for the work of Arthur and Clozel on cyclic base change for GL_n stated in §13.5.

19.5 Twisted endoscopy

The twisted trace formula was formulated for an arbitrary finite order automorphism θ of an arbitrary reductive group G, not just an inner form of GL_n. However there are serious new difficulties involved in general. The key point is that for essentially every reductive group other than GL_n, two semisimple elements of G(F) that are G(F)-conjugate need not be G(F)-conjugate. In particular, the cohomology sets D(F, I, G) of (17.32) need not be singletons.

In the settings described in §19.4, we did not really relate elements of G_1(F)/H_1(F) and G_2(F)/H_2(F). We really related elements of (G_1/H_1)(F) and (G_2/H_2)(F). For general G_i and H_i, these really are different, and the difference can be measured in terms of the Galois cohomology sets of (17.32). Be-
cause of this subtlety, to understand the quotient \((G_2/H_2)(F)\), it is appropriate to place \(G_1\) into a whole family of so-called \textbf{twisted endoscopic groups}. This subject is known as the \textbf{theory of twisted endoscopy}. It would take at least another book to discuss this fully and fortunately such books are already available. The reader might start with \([\text{Kot84, Kot86b}]\) which are based on \([\text{Lan83, Lan79b}]\). More recent references include \([\text{KS99, Lab99, CHLN11}]\). Perhaps the definitive treatment is \([\text{MW16a, MW16b}]\).

For arbitrary \(\theta\), the geometric comparison was made precise fairly early in the references mentioned above, though it is far more complicated than the geometric comparisons mentioned in \S19.4 for the general linear group. Originally, it was expected that the fundamental lemma in this context would be a routine combinatorial exercise as it was in the case of \(\text{GL}_2\), and this is why it was described as a lemma. In fact, proving the fundamental lemma, transfer, and then applying them became a struggle of Homeric proportions involving at least two generations of mathematicians over a span of almost 30 years.

While the fundamental lemma was isolated and made precise by Shelstad and Langlands \([\text{LS87}]\), Arthur began developing the trace formula in a form that was designed to apply the lemma. It would take too much space to record his work and accomplishments here; fortunately there is a freely available archive of his work online. Some of the fruits of his work are described in \S13.5 and \S13.8. We will also not give an overview of the “long march” in the words of Ngo, \([\text{Ngô10a}]\) to the fundamental lemma, preferring to leave this to the survey \([\text{Ngô10a}]\) written by the person who completed the proof in his Fields medal winning work.

We would be remiss not to point out that despite the amazing success of twisted endoscopy, it has obvious drawbacks. As currently understood, it can only hope to establish cases of functoriality where the image of the functorial lift consists of representations that are isomorphic to their conjugates under a single automorphism. A generalization to two automorphisms is proposed in \([\text{Get12, Get20}]\). These papers are based on interconnected ideas of Braverman, Kazhdan, L. Lafforgue, Langlands, Ngô, and Sakellaridis \([\text{BK00, Laf14, Lan04, Ngô14, Sak12}]\).

\section*{19.6 The interplay of distinction and twisted endoscopy}

The theory of twisted endoscopy discussed in the previous section has reached a degree of maturity. One understands how to compare the twisted trace formulae and the trace formula. Since Jacquet has given us the new tool of the relative trace formula, we now have a much broader arena to apply the experience gained in the theory of twisted endoscopy and prove new results. We mention a few geometric correspondences here and connect them to the results on distinction reviewed in Chapter 14.
We start by explaining the comparison underlying Theorem 14.5.2, which characterizes the cuspidal automorphic representations of Res$_{E/F}$GL$_n(\mathbb{A}_F)$ that are distinguished by a quasi-split unitary group. Here $E/F$ is a quadratic extension. Let $G_1 = \text{GL}_n$, $G_2 = \text{Res}_{E/F}\text{GL}_n$, and let

$$r : L\text{GL}_n \rightarrow L\text{Res}_{E/F}\text{GL}_n$$

be the base change map. It is given by the diagonal embedding on the neutral component and the identity on the Galois factor.

In this case, our basic outline from §19.3 must be modified along the lines of §19.6. Thus for $F$-algebras $R$, we let

$$X(R) := \{ x \in M_n(E \otimes_F R) : \theta(x) = x^t \},$$

where $\theta$ is the generator of $\text{Gal}(E/F)$. This admits an action of $\text{Res}_{E/F}\text{GL}_n$:

$$X(R) \times \text{Res}_{E/F}\text{GL}_n(R) \rightarrow X(R)$$

$$(x, g) \mapsto \theta(g)^t x g.$$ 

The stabilizer of any $x \in X(F) \cap \text{Res}_{E/F}\text{GL}_n(F)$ is a unitary group. For $f \in C^\infty_c(X(\mathbb{A}_F))$, the function

$$\sum_{x \in X(F)} f(\theta(g)^t x g)$$

is left $G(F)$-invariant. As in §18.6, it can be expanded spectrally in terms of automorphic representations of $\text{Res}_{E/F}\text{GL}_n(\mathbb{A}_F)$ that are distinguished by some unitary group.

We take $H_2 = \text{Res}_{E/F}N_n$ where $N_n$ is the unipotent radical of the Borel subgroup of upper triangular matrices in $\text{GL}_n$. It acts on $X$ via restriction of the action of $\text{Res}_{E/F}\text{GL}_n$. We then let

$$\chi_2 : \text{Res}_{E/F}N_n(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$$

$$u \mapsto \psi(u \theta(u)),$$

where $\psi : N_n(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$ is a generic character trivial on $N_n(F)$. We do not need to modify the geometric setting for $G_1$. We can simply take

$$H_1(R) := \{(u_1^{-t}, u_2) : u_1, u_2 \in N_n(R)\}$$

for an $F$-algebra $R$ and let $\chi_1(u_1, u_2) = \psi(u_1^t u_2)$. Note that this is precisely the geometric setting of the Kuznetsov formula of §18.8, apart from slightly modifying the action of the unipotent groups involved. Let $T \leq \text{GL}_n$ be the maximal torus of diagonal matrices. We have a diagram.
The interplay of distinction and twisted endoscopy

19.6

\[ \begin{array}{c}
\xymatrix{
T(F) \\
\ar[ur]^{p_2} \ar[dr]_{p_1} & \ar[d] \ar[dr] & \\
X(F)/\text{Res}_{E/F}N_n(F) & & G_1(F)/H_1(F)
}\end{array} \]

where the two arrows send an element of \( T(F) \) to its class. We say that a \( \chi_2 \)-relevant \( \delta \) and \( \chi_1 \)-relevant \( \gamma \) match if there is a \( t \in T(F) \) such that \( \delta \in p_2(t) \) and \( \gamma \in p_1(t) \). In this setting, the fundamental lemma was conjectured by Jacquet and Ye [JY96] and proven in two different manners by Ngô [Ngô99a, Ngô99b] and Jacquet [Jac04, Jac05b].

Jacquet and Rallis [JR11] later proposed a related comparison that eventually led to the proof of the results on the Gan-Gross-Prasad conjecture discussed in §14.7. A useful reference is [Cha19]. Let \( E/F \) be a quadratic extension of number fields and let \( U_n \) be the quasi-split unitary group of §13.7. We take

\[ (G_1 = U_{n+1} \times U_n) \quad \text{and} \quad (G_2 = \text{Res}_{E/F}GL_{n+1} \times \text{Res}_{E/F}GL_n) \]

and let

\[ r : L(U_{n+1} \times U_n) \longrightarrow L(\text{Res}_{E/F}GL_{n+1} \times \text{Res}_{E/F}GL_n) \]

be induced by the standard representations \( L(U_{n+1}) \to L(\text{Res}_{E/F}GL_{n+1}) \) and \( L(U_n) \to \text{Res}_{E/F}GL_n \) (see §13.7). We take \( H_i \leq G_i \times G_i \) to be the subgroups with points in an \( F \)-algebra \( R \) given by

\[ H_2(R) : = \{(g_1, g) : g \in \text{Res}_{E/F}GL_n(R)\} \times GL_{n+1}(R) \times GL_n(R), \]

\[ H_1(R) : = \iota(U_n)(R) \times \iota(U_n)(R), \]

where \( \iota : U_n \to U_{n+1} \times U_n \) is an appropriate “diagonal” embedding. Thus \( H_2 \) is isomorphic to \( \text{Res}_{E/F}GL_n \times GL_{n+1} \times GL_n \) and \( H_1 \) is isomorphic to \( U_n \times U_n \). Letting

\[ \eta : F^\times \backslash \mathcal{A}_E^\times /\text{N}_{E/F}(\mathcal{A}_E^\times) \longrightarrow \{\pm 1\} \]

be the character attached to \( E/F \) by class field theory, we set

\[ \chi_2((g_1, g), (g_1, g_2)) = \eta(\det g_1)^{n+1} \eta(\det g_2)^n. \]

We take \( \chi_1 \) to be the trivial character. In this setting, one has an isomorphism of GIT quotients

\[ G_2/H_2 \cong G_1/H_1 \]

and we say that relatively regular semisimple \( \delta \in G_2(F) \) and \( \gamma \in G_1(F) \) match if their geometric classes correspond under this isomorphism. It turns
out that regular semisimple elements have trivial stabilizers so the question of relevance is irrelevant here.

In this setting, the fundamental lemma was proven by Z. Yun [Yun11]. W. Zhang [Zha14] proved transfer and used it to deduce the global Gan-Gross-Prasad conjecture for unitary groups under certain local restrictions. These were later removed by various authors as explained in §14.7.

Both of the comparisons mentioned above are in some sense much simpler than the comparisons encountered in the theory of twisted endoscopy. The origin of this is easy to pinpoint. It is that stabilizers of relatively regular semisimple elements are trivial. As the reader can see from Chapter 17, the Galois cohomology of centralizers creates a great deal of complication in understanding orbital integrals. There is no reason to expect that one will always be able to avoid nontrivial centralizers, and it is important to develop relative analogues of the theory of twisted endoscopy discussed in §19.5.

As an example where it does not seem possible to avoid centralizers, we discuss the comparison underlying Theorem 14.6.1. As discussed in §14.6, this setup and its variants are fertile ground for future research.

For a quadratic extension $E/F$, we let $U_{n_{E/F}} \leq \text{Res}_{E/F} \text{GL}_n$ be the quasisplit unitary group defined in §13.7. We give ourselves a biquadratic extension

Thus $\sigma$ is a generator for $\text{Gal}(E/F)$, etc.

Let $G_1 = \text{Res}_{E/F} U_{n_{M/F}}$ and $G_2 = \text{Res}_{E/M} \text{GL}_n$. The map

$$\tau : L \text{Res}_{E/F}(U_{n_{M/F}})_E \rightarrow L \text{Res}_{E/M} \text{GL}_n$$

under consideration is induced by the standard representation $L U_{n_{M/F}} \rightarrow L \text{Res}_{M/F} \text{GL}_n$.

The automorphism $\sigma$ induces an automorphism of $\text{Res}_{E/F} U_n$ with $U_n$ as its fixed point group and an automorphism (still denoted $\sigma$) of $\text{Res}_{E/M} \text{GL}_n$ with $\text{Res}_{M/F} \text{GL}_n$ as its fixed point group. Likewise, the automorphism $\tau$ induces an automorphism of $\text{Res}_{E/M} \text{GL}_n$, with $\text{Res}_{E/F} U_n$ as its fixed point group. We set $\theta = \sigma \circ \tau$. Its fixed point group is $\text{Res}_{L/F} U_{n_{EM/L}}$. For $F$-algebras $R$, we take

$$H_2(R) = \text{Res}_{L/F U_{n_{EM/L}}}(R) \times \text{Res}_{M/F} \text{GL}_n(R),$$
$$H_1(R) = U_{n_{M/F}}(R) \times U_{n_{M/F}}(R).$$
To relate these two quotients, we use the constructions related to symmetric subgroups recalled in §17.2. We have morphisms (of affine $F$-schemes, not groups)

$$B_\theta : \text{Res}_{E/F}GL_n \rightarrow \text{Res}_{E/F}GL_n,$$

$$B_\sigma : \text{Res}_{E/F}(U_{nM/F})_E \rightarrow \text{Res}_{E/F}(U_{nM/F})_E$$

defined as in (17.27). Let $S$ be the scheme theoretic image of $B_\theta$ and $Q$ the scheme theoretic image of $B_\sigma$. We then have isomorphisms

$$G_2/H_2 \rightarrow S/\text{Res}_{M/F}GL_n,$$

$$G_1/H_1 \rightarrow Q/U_{nM/F},$$

where $\text{Res}_{M/F}GL_n$ acts via $\tau$-conjugation on $S$ and $U_n$ acts via conjugation on $Q$. This is formally similar to the geometric setting underlying base change discussed in §19.4: one has a set of $\tau$-conjugacy classes and a set of conjugacy classes that one wishes to relate. For a relatively regular semisimple $\delta \in \text{Res}_{E/M}GL_n(F)$, we say that $\gamma$ is a norm of $\delta$ if there is an $h \in H_2(F)$ such that

$$h\gamma\gamma^{-\sigma}h^{-1} = \delta\delta^{-\theta}(\delta\delta^{-\theta}).$$

One can show that every relatively regular semisimple $\delta$ admits a norm, and we say that

$$\delta \in G_2(F) = \text{Res}_{E/M}GL_n(F) = GL_n(EM)$$

and

$$\gamma \in G_1(F) = \text{Res}_{E/F}(U_{nM/F})_E(F) = U_{nEM/E}(E)$$

match if $\gamma$ is a norm of $\delta$. In this context, the fundamental lemma can be obtained from the fundamental lemma for base change and transfer for a sufficiently rich supply of test functions was proven in [GW14]. This provides enough information to establish Theorem 14.6.1. However there is a relative analogue of twisted endoscopy lurking here that has yet to be developed. This theory of twisted relative endoscopy will probably be necessary to obtain a version of Theorem 14.6.1 valid for arbitrary cuspidal automorphic representations of $U_{nEM/E}(A_E)$.

**Exercises**

**19.1.** Relate the Laplace-Beltrami operator on a Shimura manifold $\text{Sh}(G, X)^K$ and the Casimir operator of Exercise 4.7.

**19.2.** For $i = 1, 2$, let $G_i$ be a reductive group over the nonarchimedean field $F$ admitting a hyperspecial subgroup $K_i \leq G_i(F)$. If
is an $L$-map, prove that there is a unique algebra homomorphism

$$r : C_c^\infty(G_2(F) \backslash K_2) \longrightarrow C_c^\infty(G_1(F) \backslash K_1)$$  \hspace{1cm} (19.18)$$

such that if $\pi$ is an irreducible admissible unramified representation of $G_1(F)$ with transfer $r(\pi)$ to $G_2(F)$ then

$$\text{tr} \ r(\pi)(f) = \text{tr} \ \pi(\widehat{f}(f))$$

for all $f \in C_c^\infty(G_2(F) \backslash K_2)$. Here we normalize the Haar measures implicit in the traces so that the $K_i$ are given measure 1.

19.3. If $G_1$ and $G_2$ are two reductive groups over a local or global field $F$ that are inner forms of each other, prove that $^L G_1$ is isomorphic to $^L G_2$.

19.4. Let $G$ be a reductive group over the global field $F$ and let $H \leq G \times G$ and $\chi : A_{G,H} \backslash H(\mathbb{A}_F) \to \mathbb{C}^\times$ be as in §18.2. Let $S$ be a finite set of places of $F$ including the infinite places and let $K^S < G(\mathbb{A}_F^S)$ be a maximal compact open subgroup such that $K_v$ is hyperspecial for $v \not\in S$. Let

$$f = f_S f^S \in C_c^\infty(A_G \backslash G(\mathbb{A}_F) \backslash K^S).$$

Prove that

$$\text{rt} \ r(\pi(f)) = \text{tr} \ \pi(f^S) \text{rt} \ r(\pi(f_S 1_{K^S})).$$

19.5. Construct the isomorphism (19.11).

19.6. Let $G_1$ and $G_2$ be reductive groups over a global field $F$ that are inner forms of each other. Prove that $G_1F_v \cong G_2F_v$ for all but finitely many places $v$ of $F$.

19.7. Let $k$ be a field. Prove that every semisimple conjugacy class in $\text{GL}_n(k_{\text{sep}})$ that is fixed by $\text{Gal}_k$ has an element in $\text{GL}_n(k)$. Prove moreover that two elements $\gamma_1, \gamma_2 \in \text{GL}_n(k)$ that are $\text{GL}_n(k_{\text{sep}})$-conjugate are $\text{GL}_n(k)$-conjugate.

19.8. Let $T \leq \text{GL}_n$ be a maximal torus over a field $k$. Prove that $D(k, T, \text{GL}_n)$, defined as in (17.32), has only one element.
Appendix A
The Iwasawa Decomposition

Abstract The Iwasawa decomposition plays an important role in the theory of automorphic representations. We prove the existence of the decomposition using theorems of Bruhat and Tits to treat the ramified case. We also discuss the compatibility of the Iwasawa decomposition and the Levi decomposition of parabolic subgroups.

A.1 Introduction

Let $F$ be a local or global field and let $P$ be a parabolic subgroup of a reductive $F$-group $G$. Fix a Levi subgroup $M \leq P$ and let $N \leq P$ be the unipotent radical. Thus $P = MN$. In the local case, we say that a maximal compact subgroup $K \subseteq G(F)$ is in good position with respect to $(P \hookrightarrow M)$ if the Iwasawa decomposition

$$G(F) = P(F)K$$

(A.1)

holds and

$$P(F) \cap K = (M(F) \cap K)(N(F) \cap K).$$

(A.2)

Similarly, if $F$ is a global field we say that a maximal compact subgroup $K \subseteq G(\mathbb{A}_F)$ is in good position with respect to $(P, M)$ if it is good position with respect to $(P_{F_v}, M_{F_v})$ for all places $v$ of $F$.

Our goal in this appendix is to prove the following theorem:

**Theorem A.1.1** Let $G$ be a reductive group over a global or local field $F$ and let $P \leq G$ be a parabolic subgroup with Levi subgroup $M$. There exists a maximal compact subgroup $K$ of $G(\mathbb{A}_F)$ or $G(F)$, respectively, in good position with respect to $(P, M)$. We can even assume that $M(\mathbb{A}_F) \cap K$ and $M(F) \cap K$ are maximal compact subgroups of $M(\mathbb{A}_F)$ and $M(F)$, respectively.
The proof of this theorem, in general, invokes the formidable work of Bruhat-Tits. However, at least outside of a finite set $S$ of places of $F$ including the infinite places that in practice one can control, one can obtain the decomposition using much less. Our aim in this appendix is to explain this to readers who have some background in algebraic geometry. The proof will also explain precisely how one chooses $K$, at least outside of finite set $S$ of places of $F$. The reason for including this material is twofold. First, in most applications one needs at least a basic understanding of the relationship between the $K$ and $P(k_F)$ occurring in the Iwasawa decomposition. Second, the proof gives us the opportunity to introduce concepts and techniques that are useful in a broad context.

### A.2 Some group schemes

In order to explain how one can choose $K$ given $P$, it is useful to discuss notions of reductive and parabolic subgroups over the base ring $\mathcal{O}_F$. Our primary reference for this is [Con14]. Before beginning, we recall some terminology that is ubiquitous in algebraic geometry (specialized to the case of schemes over an affine base).

Let $k$ be a commutative Noetherian ring with identity. A **geometric point** of the affine scheme $\text{Spec}(k)$ is the spectrum $\bar{s} := \text{Spec}(\bar{L})$ of an algebraic closure of a field $L$ equipped with a morphism

$$\overline{s} \rightarrow \text{Spec}(k).$$

Recall from §1.2 that to give such a morphism is equivalent to giving a morphism of commutative rings with identity $k \rightarrow L$. A **geometric fiber** of a $k$-scheme $X$ is the base change $X_{\overline{s}}$ where $\overline{s}$ is a geometric point of $\text{Spec}(k)$.

**Definition A.1.** A reductive group scheme $G$ over $k$ is an affine smooth group scheme such that its geometric fibers are reductive groups.

This (trivially) recovers our original definition of reductive group when $k$ is a field. Now suppose $G$ is a reductive group scheme over $k$.

**Definition A.2.** A parabolic subgroup scheme $P$ of $G$ is a smooth affine $k$-group scheme equipped with a monic homomorphism $P \rightarrow G$ of group schemes such that for all geometric points $\overline{s} \rightarrow \text{Spec}(k)$ the fiber $P_{\overline{s}}$ is a parabolic subgroup of $G_{\overline{s}}$. A **Borel subgroup scheme** is a parabolic subgroup scheme $B$ such that $B_{\overline{s}}$ is a Borel subgroup of $G_{\overline{s}}$ for all geometric points $\overline{s} \rightarrow \text{Spec}(k)$.

If $P$ is a parabolic subgroup of $G$ then the map $P \rightarrow G$ is a closed immersion by [Con14, Proposition 5.2.3].
A.2 Some group schemes

Assume we are given a reductive group scheme \( G \) and a parabolic subgroup scheme \( P \leq G \). Consider the functor which assigns to any \( k \)-algebra \( R \) the set of closed subgroup schemes of \( G \) that are étale locally conjugate to \( P \). Then this functor is representable by a scheme called \( G/P \) over \( k \). This scheme is not affine, but it is smooth and proper and equipped with a canonical \( k \)-ample line bundle (and hence admits a canonical embedding into a projective space over \( k \)) [Con14, Corollary 5.2.8]. We note that there is a natural transformation of functors \( G \to G/P \) given on points in a \( k \)-algebra \( R \) by

\[
G(R) \to (G/P)(R) \quad\quad g \mapsto gP(R)g^{-1}.
\] (A.3)

If \( R \) is a field the map (A.3) is surjective:

**Lemma A.2.1** If \( k \) is a field then the map

\[
G(k) \to (G/P)(k)
\]

is surjective and induces a bijection \( G(k)/P(k) \cong (G/P)(k) \).

**Proof.** Let \( \overline{k} \) be an algebraic closure of \( k \). By the generalities on parabolic subgroups over fields recalled after Theorem 1.9.1, every parabolic subgroup of \( G(k) \) that is \( G(k) \)-conjugate to \( P \) is in fact \( G(\overline{k}) \)-conjugate to \( P \). \( \square \)

The following lemma is extremely useful:

**Lemma A.2.2** Let \( G \) be a smooth affine group scheme of finite type over a discrete valuation ring \( O \) with finite residue field \( \kappa \) and let \( X \) be a \( G \)-torsor. If the special fiber \( G_{\kappa} \) is connected then

\[
X(O) \neq \emptyset.
\]

**Proof.** Since \( G \) is smooth, so is \( X \) [BLR90, §6.4]. Therefore, by Hensel’s lemma [BLR90, §2.3, Proposition 5], the map \( X(O) \to X(\kappa) \) is surjective. It therefore suffices to show that \( X(\kappa) \neq \emptyset \). But \( X_{\kappa} \) is a torsor under the connected affine algebraic group \( G_{\kappa} \), so \( X(\kappa) \neq \emptyset \) by a theorem of Lang [Lan56]. \( \square \)

We can now prove a version of the Iwasawa decomposition if we assume the existence of the parabolic subgroup scheme \( P \):

**Theorem A.2.3** Suppose that \( S \) is a finite set of places of a global field \( F \) containing all infinite places, that \( G \) is a reductive group scheme over \( O_F^S \) and that \( P \leq G \) is a parabolic subgroup scheme over \( O_F^S \). Then

\[
\]

**Proof.** Suppose that \( g \in G(A_F^S) \). Then \( g \in G(O_{F_v}) \) for all but finitely many places \( v \), so it suffices to show that for every \( v \notin S \), one has that
\[ G(F_v) = \mathcal{P}(F_v)G(O_{F_v}) \].

Fix \( v \not\in S \) and let \( \varpi \in O_{F_v} \) be a uniformizer.

One has a commutative diagram

\[
\begin{array}{ccc}
G(O_{F_v}) & \longrightarrow & G(F_v) \\
\downarrow^{a} & & \downarrow \\
(G/P)(O_{F_v}) & \longrightarrow & (G/P)(F_v).
\end{array}
\]

The scheme \( G/P \) is projective over \( O_{F_v} \), hence proper, and thus the arrow \( b \) is a bijection by the valuative criterion of properness. In view of Lemma A.2.1 it therefore suffices to show that the arrow \( a \) is surjective. The fiber of \( a \) over any point of \( (G/P)(O_{F_v}) \) is a \( P \)-torsor, and hence has a point by Lemma A.2.2. \( \square \)

### A.3 The dynamic method

Now in the statement of Theorem A.1.1, there is no mention of the group scheme \( G \) or \( P \); one has only the reductive group \( G \) and its parabolic subgroup \( P \). To apply Theorem A.2.3 in the setting of Theorem A.1.1 we must construct \( G \) and \( P \) given \( G \) and \( P \). One starts by choosing a faithful representation \( G \rightarrow \text{GL}_n \). For a sufficiently large finite set \( S \) of places of \( F \) containing the infinite ones, the schematic closure \( \bar{G} \) of \( G \) in \( \text{GL}_nO_S \) is then reductive by Proposition 2.4.5. Thus we can construct \( \bar{G} \). We now construct \( P \).

**Proposition A.3.1** Let \( S \) be a finite set of places of a global field \( F \) containing all infinite places and let \( \bar{G} \) be a reductive group scheme over \( O_F^S \). Let \( P \) be a parabolic subgroup of \( \bar{G}_F \) with Levi decomposition \( P = MN \). After possibly adding a finite set of places to \( S \), the schematic closure \( \bar{P} \) of \( P \) in \( \bar{G} \) is a parabolic subgroup scheme of \( \bar{G} \) admitting a decomposition

\[
\bar{P} = M \ltimes N, \tag{A.4}
\]

where \( M \) is a reductive group scheme, \( N \) is a connected smooth group scheme with unipotent fibers, \( M_F = M \), and \( N_F = N \).

In the setting of the lemma, we have \( \mathcal{P}(O_{F_v}) = \mathcal{M}(O_{F_v}) \ltimes N(O_{F_v}) \) for \( v \not\in S \). This implies that

\[
P(F_v) \cap G(O_{F_v}) = (M(F_v) \cap G(O_{F_v}))(N(F_v) \cap G(O_{F_v})),
\]

the condition required in (A.2). We also note that \( \mathcal{M}(O_{F_v}) \leq M(F_v) \) is a hyperspecial subgroup (and in particular is maximal, see Theorem 2.4.1).
To prove Proposition A.3.1 we use the so-called dynamic description of parabolic subgroups [Con14]. Suppose that we are given an affine group scheme \( \mathcal{G} \) of finite type over a ring \( k \) and a cocharacter, i.e. a morphism of group schemes

\[ \lambda : \mathbb{G}_m \rightarrow \mathcal{G}. \]

This defines a left action of \( \mathbb{G}_m \) on \( \mathcal{G} \)

\[ \lambda : \mathbb{G}_m \times \mathcal{G} \rightarrow \mathcal{G} \]

given on points by

\[ \lambda(t, g) := \lambda(t)g\lambda(t)^{-1}. \]

We say that \( \lim_{t \to 0} \lambda(t, g) \) exists if the morphism \( \lambda \) extends to a morphism

\[ \lambda : \mathbb{G}_a \times \mathcal{G} \rightarrow \mathcal{G}. \]

\textbf{Example A.1.} Let \( \lambda : \mathbb{G}_m \rightarrow \text{GL}_3 \) be the cocharacter given as

\[ \lambda(t) = \begin{pmatrix} t & t & t^{-2} \\ t^{-1} & t & t^2 \\ t^{-3} & t^{-1} & t^3 \end{pmatrix}. \]

Then

\[ \lambda(t) \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \lambda(t)^{-1} = \begin{pmatrix} a_{11} & t^2a_{12} & t^3a_{13} \\ t^{-2}a_{12} & a_{22} & ta_{23} \\ t^{-3}a_{13} & t^{-1}a_{23} & a_{33} \end{pmatrix}. \]

Thus \( \lim_{t \to 0} \lambda(t, g) \) exists if and only if \( g \) is a point of the Borel subgroup of upper triangular matrices. Moreover \( \lim_{t \to 0} \lambda(t, g) = 1 \), the identity in \( \text{GL}_3 \), if and only if \( g \) lies in the group of strictly upper triangular matrices, and \( g \) commutes with the action of \( \lambda \) if and only if \( g \) is a diagonal matrix.

Let \( P(\lambda) \) be the group functor on \( k \)-algebras given on a \( k \)-algebra \( R \) by

\[ P(\lambda)(R) := \{ g \in \mathcal{G}(R) : \lim_{t \to 0} \lambda(t, g) \text{ exists} \}. \]

We also define

\[ N(\lambda)(R) = \{ g \in \mathcal{G}(R) : \lim_{t \to 0} \lambda(t, g) = 1 \} \]

and let \( Z(\lambda)(R) \leq \mathcal{G}(R) \) be the subgroup that commutes with the action \( \lambda \). We then have

\[ P(\lambda)(R) = Z(\lambda)(R) \ltimes N(\lambda)(R). \quad (A.5) \]

In the example above, \( P(\lambda) \) turned out to be a parabolic subgroup of the ambient group scheme. Moreover, the geometric fibers of \( N(\lambda) \) (resp. \( Z(\lambda) \))
turned out to be the unipotent radical (resp. Levi subgroup) of the corresponding fiber of $P(\lambda)$. This is a general phenomenon, and is the basis of the dynamic description of parabolic subgroups, which is summarized in Theorem A.3.2 and Theorem A.3.3 below. For the proof of Theorem A.3.2 see [Con14, Theorem 4.1.7, Example 5.2.2].

**Theorem A.3.2** Assume that $\mathcal{G}$ is smooth. The functor $P(\lambda)$ is represented by a closed smooth subgroup of $\mathcal{G}$. If $\mathcal{G}$ has connected fibers then so does $P(\lambda)$. If $\mathcal{G}$ is connected and reductive, $T \leq \mathcal{G}$ is a maximal torus, and $\lambda$ has image in $T$ then $P(\lambda)$ is a parabolic subgroup scheme of $\mathcal{G}$.

Here a torus $T$ over $\text{Spec}(k)$ is a group scheme over $k$ that is fpqc locally isomorphic to $\mathbb{G}_m^n$ for some $n \geq 0$. In other words, for each point $x \in \text{Spec}(k)$ there exists a neighborhood $U \subseteq \text{Spec}(k)$ of $x$ and a faithfully flat quasi-compact morphism $U' \to U$ such that $T_{U'} \cong \mathbb{G}_m^{n_{U'}}$. A torus $T$ is a maximal torus of $\mathcal{G}$ if it is a closed subgroup scheme of $\mathcal{G}$ such that for all geometric points $\overline{x}$ of $\text{Spec}(k)$, the fiber $T_{\overline{x}}$ is a maximal torus of $\mathcal{G}_{\overline{x}}$.

**Theorem A.3.3** Assume that $k$ is a field and $G$ is a reductive group over $k$. Let $P \leq G$ be a parabolic subgroup with Levi decomposition $P = MN$ and let $T \leq M$ be a maximal torus of $G$. There is a cocharacter $\lambda : \mathbb{G}_m \to T$ such that $P = P(\lambda)$, $M = Z(\lambda)$, and $N = N(\lambda)$.

*Proof.* The fact that every $P$ is a $P(\lambda)$ is [Mil17, Theorem 25.1] and $N = N(\lambda)$ for any cocharacter $\lambda$ such that $P = P(\lambda)$. It follows from [Mil17, Theorem 25.6] that all Levi subgroups of $P$ are conjugate under $P(k)$, we deduce that upon conjugating $\lambda$ by an element of $P(k)$, we may assume that $M = M(\lambda)$.

Using these theorems, we can now prove Proposition A.3.1:

*Proof of Proposition A.3.1:* Let $P$ be a parabolic subgroup of $G$ containing a maximal torus $T$ of $G$. Then there is a character $\lambda : T \to \mathbb{G}_m$ such that $P = P(\lambda)$ by Theorem A.3.3. Choose a faithful representation $G \to \text{GL}_n$. Upon choosing a sufficiently large finite set $S$ of places of $F$, we can assume that the schematic closure $\mathcal{G}$ of $G$ in $\text{GL}_n_{\overline{\mathcal{O}_F}}$ is reductive (see Proposition 2.4.5), that the schematic closure $T$ of $T$ in $\mathcal{G}$ is a maximal torus of $\mathcal{G}$ [Con14, Corollary 3.1.8 and Proposition 3.1.9], and that the cocharacter $\lambda : \mathbb{G}_m \to T$ extends to a cocharacter $A : \mathbb{G}_m \to T$. Here we are using some basic facts on spreading out that are summarized in [Poo17, §3.2]. Let

$$\mathcal{P} := P(A).$$

Then by Theorem A.3.2, $\mathcal{P}$ is a parabolic subgroup of $\mathcal{G}$ and we deduce the proposition. \qed
A.4 Proof of Theorem A.1.1

We now complete the proof of Theorem A.1.1. In view of Theorem A.2.3 and Proposition A.3.1, it suffices to prove Theorem A.1.1 in the local case. Thus we assume for the remainder of the section that $F$ is a local field.

We will actually prove something more precise than what is required by Theorem A.1.1. Fix a minimal parabolic subgroup $P_0 \leq G$ and a Levi subgroup $M_0 \leq P_0$. We write $N_0$ for the unipotent radical of $P_0$. We call a pair $(P,M)$ consisting of a parabolic subgroup $P \leq G$ and a Levi subgroup $M \leq P$ a standard parabolic pair if $P_0 \leq P$ and $M_0 \leq M$.

**Theorem A.4.1** Assume $F$ is archimedean. There is a maximal compact subgroup $K \leq G(F)$ such that for any standard parabolic pair $(P,M)$, one has that $G(F) = P(F)K$, $M(F) \cap K$ is a maximal compact subgroup of $M(F)$, and $P(F) \cap K = M(F) \cap K$.

**Proof.** By [Kna02, Proposition 7.31], the Iwasawa decomposition $G(F) = P(F)K$ holds for some maximal compact subgroup $K \leq G(F)$. In the following, we use a $+$ to denote the neutral component of a real Lie group. All maximal compact subgroups of $G^{\text{der}}(F)^+$ are conjugate by [Hel01, §VI.2]. It follows that all maximal compact subgroups of $G(F)$ are conjugate. Thus $G(F) = P(F)K$ holds for all maximal compact subgroups $K$.

The assertion that $K$ is a maximal compact subgroup of $G(F)$ is equivalent to the statement that $K \cap G(F)^+$ is maximal in $G(F)^+$ and $K$ meets all components of $G(F)$. In view of this, the assertion that $M(F) \cap K$ is a maximal compact subgroup of $M(F)$ is part of [HC75, Lemma 8].

Consider the image of $K$ in $P(F)/N(F) = M(F)$. It is a compact subgroup containing $M(F) \cap K$, hence equal to $M(F) \cap K$. If $mn \in K$ with $(m,n) \in M(F) \times N(F)$, we deduce that $m \in M(F) \cap K$, and hence $n \in N(F) \cap K = I$. This completes the proof of the last assertion. □

**Theorem A.4.2** Assume $F$ is nonarchimedean. There is a maximal compact subgroup $K \leq G(F)$ such that for any standard parabolic pair $(P,M)$, one has that $G(F) = P(F)K$, $M(F) \cap K$ is a maximal compact subgroup of $M(F)$, and $P(F) \cap K = (M(F) \cap K)(N(F) \cap K)$.

We do not know of a proof of this result that does not use Bruhat-Tits theory. We will have to assume some of this in the proof. A survey is given in [Tit79] and the proofs are contained in [BT72, BT84]. The paper [Lan00] is also useful; there one can find the definition of the extended Bruhat-Tits building. Finally the book [KP] is a wonderful resource.

Let $B^e(G)$ denote the extended Bruhat-Tits building of $G$ over $F$. It is a topological space equipped with an action of $G(F)$ and is a union of subsets called (extended) apartments $A^e(G,T)$ indexed by the maximally split tori $T$ of $G$:

$$B^e(G) := \bigcup gA^e(G,T).$$

(A.6)
Usually $B^e(G)$ is denoted by $B^e(G, F)$, etc, but since $G$ is an $F$-group, we have omitted this from notation. The extended apartment $A^e(G, T)$ is a $X_*(T)_{\mathbb{R}}$-torsor (affine $X_*(T)_{\mathbb{R}}$-space) equipped with a homomorphism

$$N_G(T)(F) \rightarrow \text{Aff}(A^e(G, T))$$

where $\text{Aff}(A^e(G, T))$ is the group of affine transformations of $A^e(G, T)$.

**Proof.** Assume that $T \leq M_0$ is a maximal split torus. Let $x \in A^e(G, T)$ be a special point [Lan00, §2.2] and let $K$ be the stabilizer of $x$. Then $K$ is a maximal compact subgroup of $G(F)$ and one has $P_0(F)K = G(F)$ [BT72, 4.4.6(ii)]. It follows immediately that $P(F)K = G(F)$ for all standard parabolic subgroups $P$ of $G$.

We now claim that we can choose $x$ so that for each standard parabolic pair, $M(F) \cap K$ is a maximal compact subgroup of $M(F)$. Assuming this, the rest of the theorem follows as in the proof of Theorem A.4.1. The inclusion $M \leq G$ induces an inclusion

$$B^e(M) \rightarrow B^e(G)$$

that is equivariant with respect to the action of $M(F)$. Moreover, the inclusion sends $A^e(M, T)$ bijectively to $A^e(G, T)$ [Lan00, Proposition 2.1.5]. Let $x \in A^e(G, T)$ be a special point. One checks directly from the definition of a special point that the inverse image of $x$ in $A^e(M, T)$ is again a special point. The claim follows. \[\Box\]

### A.5 An addendum in the hyperspecial case

Suppose that $G$ is a reductive group over a local field $F$ that is unramified (quasi-split and split over an unramified extension). Thus there is a smooth model $\mathcal{G}$ of $G$ over $\mathcal{O}_F$ with reductive fibers by Theorem 2.4.1. The group $\mathcal{G}(\mathcal{O}_F)$ is a hyperspecial subgroup of $G(F)$.

**Lemma A.5.1** There exists a Borel subgroup scheme $\mathcal{B}$ of $\mathcal{G}$ and a maximal torus $\mathcal{T}$ of $\mathcal{G}$ satisfying $\mathcal{T} \leq \mathcal{B}$.

**Proof.** We first check that a Borel subgroup exists. The functor assigning to an $\mathcal{O}_F$-scheme $S$ the set of Borel subgroups of $\mathcal{G}_S$ is representable by a smooth proper $\mathcal{O}_F$-scheme, say $\mathcal{X}$ [DG74, XXII, Corollaire 5.8.3(i)]. We have $\mathcal{X}(F) = \mathcal{X}(\mathcal{O}_F)$ by the valuative criterion of properness, so there is a Borel subgroup $\mathcal{B} \leq \mathcal{G}$. The existence of $\mathcal{T}$ is [DG74, XXII, Corollaire 5.9.7]. \[\Box\]

**Lemma A.5.2** For $\mathcal{B}$ and $\mathcal{T}$ as in Lemma A.5.1, the subgroup $\mathcal{T}(\mathcal{O}_F) = \mathcal{T}(F) \cap \mathcal{G}(\mathcal{O}_F)$ is a maximal compact subgroup of $\mathcal{T}(F)$ and $\mathcal{G}(\mathcal{O}_F)$ is in good position with respect to $(\mathcal{B}_F, \mathcal{T}_F)$. 
Proof. The assertion that $\mathcal{T}(\mathcal{O}_F) \leq \mathcal{T}(F)$ is a maximal compact subgroup is [Mac17, Proposition 4.6]. One has a decomposition

$$\mathcal{B} = \mathcal{T} \ltimes \mathcal{N}$$

where $\mathcal{N}$ is a connected smooth group scheme with unipotent fibers [DG74, XXVI, Proposition 1.6]. In particular, $\mathcal{N}_F$ is the unipotent radical of $\mathcal{B}_F$. This implies that

$$\mathcal{B}(F) \cap \mathcal{G}(\mathcal{O}_F) = \mathcal{B}(\mathcal{O}_F) = \mathcal{T}(\mathcal{O}_F)\mathcal{N}(\mathcal{O}_F) = (\mathcal{T}(F) \cap \mathcal{G}(\mathcal{O}_F))(\mathcal{N}(F) \cap \mathcal{G}(\mathcal{O}_F)).$$

Moreover, $\mathcal{B}(F)\mathcal{G}(\mathcal{O}_F) = \mathcal{G}(F)$ by the argument proving Theorem A.2.3 \qed
Appendix B
Poisson Summation

Abstract We define the adelic version of the Fourier transform on a vector space and state the corresponding version of Poisson summation.

B.1 The standard additive characters

All of the material in this appendix is either implicit or explicit in Tate’s thesis \cite{Tat67}. A leisurely account is given in \cite[Chapter 7]{RV99}.

The goal of this section is to fix a standard additive character on every local field and an additive character of $F \backslash \mathbb{A}_F$ for every global field. We first consider the case of completions of $\mathbb{Q}$ and $\mathbb{F}_p(t)$ for primes $p$. We let

$$\psi_{\mathbb{R}} : \mathbb{R} \to \mathbb{C} \times \xrightarrow{a \mapsto e^{-2\pi i a}}$$

and

$$\psi_{\mathbb{Q}_p} : \mathbb{Q}_p \to \mathbb{C} \times \xrightarrow{a \mapsto e^{2\pi i pr(a)}}.$$

Here the principal part $pr(a) \in \mathbb{Z}[p^{-1}]$ is any element such that

$$a \equiv pr(a) \pmod{\mathbb{Z}_p}.$$

If $F$ is a characteristic zero local field then it is an extension of a completion $F_0$ of $\mathbb{Q}$, and we let

$$\psi_F := \psi \circ tr_{F/F_0} : F \to \mathbb{C} \times.$$

We now treat the characteristic $p$ case. We define the additive character
\( \psi_{F_p((t^{-1}))} : F_p((t^{-1})) \to \mathbb{C}^\times \)

\[
\sum_{n=-\infty}^{r} a_n t^n \mapsto e^{-2\pi i a_1/p}.
\]

Here the \(a_i\) are representatives in \(\mathbb{Z}\) for \(a_i \in F_p\). Let \(\varpi\) be an irreducible polynomial in \(F_p[t]\) and let \(F_p(t)_{\varpi}\) is the completion of \(F_p(t)\) at the corresponding place. If \(k = F_p[t]/\varpi\) then we define

\[
\psi_{F_p(t)_{\varpi}} : F_p(t)_{\varpi} \to \mathbb{C}^\times
\]

\[
\sum_{n=r}^{\infty} a_n \varpi^n \mapsto e^{2\pi i r_k/F_p(a_{-1})/p}.
\]

Here the \(a_i\) are in \(k\) (or rather a fixed set of representatives for \(k = F_p[t]/\varpi\) in \(F_p[t]\)). On the right, we are implicitly identifying \(tr_{k/F_p}(a_{-1})\) with an element of \(\mathbb{Z}\) mapping to it under the quotient map \(\mathbb{Z} \to F_p\). If \(F\) is a characteristic \(p\) local field then it is an extension of a completion \(F_0\) of \(F_p(t)\) and we let

\[
\psi_F := \psi \circ tr_{F/F_0} : F \to \mathbb{C}^\times.
\]

One checks that all of the \(\psi_F\) we have defined are indeed nontrivial characters.

For an abelian topological group \(A\) let \(\hat{A}\) be the unitary dual of \(A\). It is the set of all continuous homomorphisms \(A \to \mathbb{C}^\times\). For the following lemma see Exercise B.2:

**Lemma B.1.1** If \(F\) is a local field and \(\psi : F \to \mathbb{C}^\times\) is any nontrivial character then there is a bijection

\[
F \to \hat{F},
\]

\[
\alpha \mapsto (x \mapsto \psi(\alpha x)).
\]

\(\square\)

If \(F\) is a global field we let

\[
\psi_F := \prod_v \psi_{F_v}.
\]

(B.1)

This defines a nontrivial character of \(\mathbb{A}_F\) that is trivial on \(F\). The global analogue of Lemma B.1.1 is the following lemma:

**Lemma B.1.2** If \(F\) is a global field and \(\psi : F\backslash \mathbb{A}_F \to \mathbb{C}^\times\) is any nontrivial character then there is a bijection

\[
F \to \hat{F}\backslash \mathbb{A}_F
\]

\[
\alpha \mapsto (x \mapsto \psi(\alpha x)).
\]

\(\square\)
B.2 Local Schwartz spaces and Fourier transforms

Let $F$ be a local field. If $F$ is nonarchimedean we define $S(F) = C^c_c(F)$, and if $F$ is archimedean we let $S(F)$ denote the usual Schwartz space. We refer to $S(F)$ as the Schwartz space of $F$.

If $W$ is a finite dimensional $F$-vector space, we define $S(W)$ analogously.

Let $h : W \times W \rightarrow F$ be a perfect pairing, let $\psi : F \rightarrow \mathbb{C}^\times$ be a character, and let $dy$ be a Haar measure on $W$. In the case where $W = F$, the pairing is usually taken to be $\langle x, y \rangle := xy$. For $f \in S(W)$, one has a Fourier transform

$$\hat{f}(x) := \int_W f(y) \psi(\langle x, y \rangle) dy$$

which is also an element of $S(W)$. In the archimedean case the assertion that $\hat{f} \in S(W)$ is standard, and in the nonarchimedean case it is Exercise B.5.

The Fourier transform depends on $\psi$, the pairing $\langle , \rangle$ and the Haar measure $dy$. Given $\psi$ and $\langle , \rangle$, there is a unique way to normalize $dy$ so that for all $f \in S(W)$ the Fourier inversion formula holds:

$$f(x) = \hat{\hat{f}}(-x).$$

This Haar measure $dy$ is known as the self-dual Haar measure (with respect to the pairing and additive character).

It follows from the Fourier inversion formula that the Fourier transform defines a $\mathbb{C}$-linear isomorphism

$$\hat{\cdot} : S(W) \rightarrow S(W).$$

B.3 Global Schwartz spaces and Poisson summation

Let $F$ be a global field and let $W$ be a finite dimensional $F$-vector space. For an $F$-algebra $R$, let $W(R) := W \otimes_F R$. We define

$$S(W(A_F)) := S(W(F_\infty)) \otimes_{\mathbb{C}} C_c^\infty(W(A_F^\infty)).$$

When $F$ is a number field, we view $W(F_\infty)$ as an $\mathbb{R}$-vector space to make sense of $S(W(F_\infty))$.

Let $h : W \times W \rightarrow F$
be a perfect pairing, let $\psi : F \backslash \mathbb{A}_F \to \mathbb{C}^\times$ be a nontrivial character, and let $dy$ be a Haar measure on $W(\mathbb{A}_F)$. For $f \in \mathcal{S}(W(\mathbb{A}_F))$, we then have an adelic Fourier transform

$$\hat{f}(x) := \int_{W(\mathbb{A}_F)} f(y)\psi(\langle x, y \rangle) \, dy. \quad (B.5)$$

Lemma B.3.1 One has that $\hat{f} \in \mathcal{S}(W(\mathbb{A}_F))$.

Proof. It suffices to verify the lemma in the special case where $W = F^r$ and $\langle , \rangle$ is the standard product

$$\langle (x_1, \ldots, x_r), (y_1, \ldots, y_r) \rangle \mapsto \sum_{i=1}^r x_i y_i.$$ 

Let $S$ be a finite set of places of $F$ including the infinite places and all places where $F_v$ is ramified. If $S$ is large enough then

$$f = f_S \mathbb{1}_{(\hat{\mathbb{A}}^r_F)^*}.$$ 

It also follows from Lemma B.1.2 that $\psi = \psi_S \psi^S$ where $\psi^S = \prod_{v \not\in S} \psi_{F_v}$; that is, outside of $S$ the character $\psi^S$ is the standard character. Finally we can assume that $dy = dy_S dy^S$ where $dy^S = \prod_{v \not\in S} dy_v$ and $dy_v(\mathcal{O}_{F_v}) = 1$ for $v \not\in S$. Thus

$$\hat{f} = \hat{f}_S \mathbb{1}_{(\hat{\mathbb{A}}^r_F)^*}.$$ 

We already know that $\hat{f}_S \in \mathcal{S}(W(F_{\infty})) \otimes \mathcal{S}(W(F_S^S))$, so it suffices to check that $\mathbb{1}_{(\hat{\mathbb{A}}^r_F)^*} = \mathbb{1}_{(\hat{\mathbb{A}}^r_F)^*}$. This is the content of Exercise B.7. \qed

Given $\langle , \rangle$ and $\psi$, there is a unique choice of Haar measure $dy$ on $W(\mathbb{A}_F)$ such that the Fourier inversion formula holds:

$$f(x) = \hat{f}(-x). \quad (B.6)$$

This is the self-dual Haar measure (with respect to $\langle , \rangle$ and $\psi$).

As in the local setting, the Fourier inversion formula implies that the Fourier transform induces an isomorphism

$$\begin{array}{c}
\mathcal{S}(W(\mathbb{A}_F)) \rightarrow \mathcal{S}(W(\mathbb{A}_F)).
\end{array} \quad (B.7)$$

We can now state an adelic version of Poisson summation:

Theorem B.3.2 (Poisson summation) For $f \in \mathcal{S}(W(\mathbb{A}_F))$ one has that

$$\sum_{\gamma \in W(F)} f(\gamma) = \sum_{\gamma \in W(F)} \hat{f}(\gamma).$$

\qed
Exercises

B.1. Let $F$ be a local field. Prove that the maps $\psi_F$ defined in §B.1 are characters.


B.3. Let $F$ be a global field. Prove that $\psi_F$ is a character of $\mathbb{A}_F$ trivial on $F$.


B.5. Let $F$ be a nonarchimedean local field and let $f \in C_c^\infty(F)$. Prove that $\hat{f} \in C_c^\infty(F)$. Here $\hat{f}$ can be defined with respect to any Haar measure on $F$ and nontrivial character $\psi : F \to \mathbb{C}^\times$.

B.6. Prove that the Fourier inversion formula (B.3) holds when $F$ is a nonarchimedean local field.

B.7. Let $F$ be a nonarchimedean local field that is unramified over its prime field. Show that

$$\hat{1}_{\mathcal{O}_F} = 1_{\mathcal{O}_F}$$

where the Fourier transform is defined with respect to the standard additive character $\psi_F$ and the Haar measure on $F$ such that $d\gamma(\mathcal{O}_F) = 1$. Conclude that this Haar measure is self-dual with respect to $\psi_F$ and the pairing $(x, y) \mapsto xy$. 

Hints to selected exercises

Chapter 1

1.12 Use the fact that parabolic subgroups would have unipotent elements. This would imply that $D$ has zero divisors, contradicting the assumption that it is a division algebra.

Chapter 2

2.6 One approach uses the alternate definition of group schemes described in Exercise 1.7.

2.7 See [CF86, Chapter II.6].

2.10 Note that $\text{GL}_n(F_S)$ is an open subset of $\mathfrak{gl}_n(F_S)$ for any finite set of places $S$ and use weak approximation for $\mathbb{A}_F$ (see Theorem 2.1.4).

2.14 In the archimedean case, use the Gram-Schmidt process. In the nonarchimedean case, one can proceed in an elementary manner using induction on $n$. See [Bum97, Proposition 4.5.2].

2.15 Let $H_G : G(\mathbb{A}_F) \to \text{Hom}(X^*(G), \mathbb{R})$ be defined by

$$\langle H_G(g), \lambda \rangle = \log |\lambda(g)|$$

for $\lambda \in X^*(G)$. Then $H_G$ is injective when restricted to $A_G$ and has kernel $G(\mathbb{A}_F)^1$. The subgroup index $[H_G(A_G) : H_G(G(\mathbb{A}_F))]$ is finite. When $F$ is a number field $H_G(A_G) = H_G(G(\mathbb{A}_F))$.

2.18 Using the description of $\text{GL}_2(\mathbb{Q}) \backslash \text{GL}_2(\mathbb{A}_Q)/\text{GL}_2(\widehat{\mathbb{Z}})$ given at the end of §2.6, the exercise can be deduced from the following well-known fact: Every
element of the complex upper half plane is $\text{SL}_2(\mathbb{Z})$-equivalent (under Möbius transformations) to an element $z$ satisfying $-\frac{1}{2} \leq \text{Re}(z) \leq \frac{1}{2}$ and $|z| > 1$.

Chapter 3

3.1 See [Fol95, Proposition 2.6].

3.2 Average a left Haar measure.

3.3 See [Fol95, §2.4], bearing in mind that $\delta_G(g) = \Delta(g)^{-1}$ in the notation of loc. cit.

3.15 Prove first that the complex vector space admits a $G$-invariant inner product.

Chapter 4

4.2 See [Bum97, Theorem 2.4.1].

4.4 See [Bum97, Proposition 2.4.4].

4.10 The proof is an extension of the well-known construction of all the finite dimensional irreducible representations of the Lie algebra $\mathfrak{sl}_2$.

Chapter 5

5.6 Show that $V$ has a countable basis and invoke Lemma 4.5.1.

5.11 See [Car79].

5.12 Mimic the proof of Proposition 4.2.5.

Chapter 7

7.5 See [Lau96, Lemma D.1.5].
Chapter 8

8.5 Right exactness is the only point that is not obvious. Suppose that one has a surjection \( a : V \to W \) of smooth representations of \( M(F) \). Let \( \varphi : G \to W \) be an element of \( \text{Ind}_P^G(W) \), and choose a compact open subgroup \( K \leq G(F) \) fixing \( \varphi \). By Lemma 8.2.1 for all \( g \in G(F) \), one has

\[
\varphi(g) \in V^{M(F)\cap gKg^{-1}}.
\]

Since \( V^{M(F)\cap gKg^{-1}} \to W^{M(F)\cap gKg^{-1}} \) is surjective by Exercise 7.5 there is a \( v_g \in V^{M(F)\cap gKg^{-1}} \) such that \( a(v_g) = \varphi(g) \). Fix a minimal set of representatives \( X \) for \( P(F) \backslash G(F) / K \). We then define

\[
\tilde{\varphi} : G \to V,
\]

\[
mnzk \mapsto \delta^{1/2}(m)m.v_x
\]

for \( (n, m, x, k) \in N(F) \times M(F) \times X \times K \). The function \( \tilde{\varphi} \) is fixed by \( K \) and is therefore smooth, and by construction \( a(\tilde{\varphi}) = \varphi \). Thus the map \( \text{Ind}_P^G(V) \to \text{Ind}_P^G(W) \) is surjective.

8.6 Additivity is clear. Assume that

\[
0 \to V_1 \to V_2 \to V_3 \to 0
\]

is exact. It is easy to verify that \( V_1N \to V_2N \to V_3N \to 0 \) is exact. On the other hand

\[
V_1(N) = V_1 \cap V_2(N),
\]

and it follows that \( V_1N \to V_2N \) is injective.

8.7 If \( \varphi \in V \) and \( n' \in N(F) \) then there is a compact open subgroup \( K_N \leq N(F) \) such that \( n' \in K_N \). Choosing \( K_N \) in this manner we see that

\[
\int_{K_N} \pi(n)(\pi(n')\varphi - \varphi)dn = 0.
\]

Conversely, suppose that \( \varphi \in V \) is such that

\[
\int_{K_N} \pi(n)\varphi dn = 0
\]

for a compact open subgroup \( K_N \leq N(F) \). Choose a compact open subgroup \( K'_N \leq K_N \) such that \( \varphi \in V^{K'_N} \). Then

\[
0 = \int_{K_N} \pi(n)\varphi dn = \sum_{n \in K_N/K_N'} \frac{1}{[K_N : K_N']} \pi(n)\varphi
\]
which implies
\[ \varphi = \frac{1}{[K_N : K'_N]} \sum_{n \in K_N / K'_N} (\pi(n)\varphi - \varphi). \]

8.9 For each \( i \) let \( m_i \leq C^\infty_c(G(F) \parallel K) \) be the annihilator of \( V_i \). It is a maximal ideal, and \( m_i = m_j \) if and only if \( i = j \). Choose \( f_1, \ldots, f_n \in C^\infty_c(G(F) \parallel K) \) such that \( f_i \in m_i \) and \( \sum_{i=1}^n f_i = 1_K \), where we have normalized the Haar measure so that \( K \) has measure 1. Then \( 1_K - f_1 \) is the identity on \( V_1 \) and zero on \( V_i \) for \( i \neq 1 \).

8.15 Show that if \( Q(\sigma^a, \lambda_a) \) and \( Q(\sigma^{a'}, \lambda_{a'}) \) are linked then one of their central characters is not unitary.

Chapter 9

9.4 See [DE09, Proposition 5.3.2].

Chapter 10

10.5 Use Proposition 8.3.1.

Chapter 11

11.5 Consider the action of an unramified Hecke operator on \( J(\lambda) \). For \( \mu \) dominant, relate the value of the Whittaker function at \( \mu(\pi) \) to the action of the Hecke operator \( 1_{GL_2(O_F)} \mu(\pi) GL_2(O_F) \). Then use that the eigenvalues of such an operator can be computed in terms of \( \lambda \) by Proposition 7.6.8.

11.6 See [JS81b, §2.5].

Chapter 12

12.1 Use Lemma 5.1.2.

12.11 Assume first that the \( \pi_i \) are all square integrable. Then since all the \( \pi_i \) are tempered, none of them are linked in the sense of §8.4. Hence we conclude by Theorem 8.4.4. The general case can be reduced to this case by Theorem 8.4.5 and the transitivity of induction.
Chapter 14

14.1 Let $H \leq \mathbb{R}^n$ be a closed real Lie subgroup with finitely many connected components. Then by the correspondence between connected Lie groups and Lie algebras we deduce that the neutral component $H^0 \leq H$ (in the real topology) is isomorphic to $\mathbb{R}^k$ for some $k \leq n$. Thus the quotient

$$H/H^0 \leq \mathbb{R}^n/\mathbb{R}^k$$

is a finite subgroup of $\mathbb{R}^{n-k}$ and hence is trivial by Lemma 5.1.2.

Chapter 15

15.4 Let

$$K(N) := \{ g \in GL_n(\mathbb{Z}) : g \equiv I_n \mod N\mathbb{Z}\}.$$ 

Then $GL_n(\mathbb{Q}) \cap K(N)$ is the group in the proof of Lemma 15.2.1 and hence is neat. The set of eigenvalues of matrices in $GL_n(\mathbb{Q}) \cap g^{-1}K(N)g$ is independent of $g \in GL_n(A^\mathbb{Q})$.

Chapter 17

17.2 Consider the natural action of $\mathbb{G}_m$ on $X = \mathbb{G}_a$.

Chapter 18

18.4 Let $G = \mathbb{G}_m \times SL_2$ and let $H < G$ be the subgroup whose points in an $F$-algebra $R$ are given by

$$H(R) := \{ (t, \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}) : t \in R^\times \}.$$ 

18.5 See [Kot86a].

Appendix B

B.2 See [RV99, Exercise 7.1].
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