CEO Horizon, Optimal Pay Duration, and the Escalation of Short-termism

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Abstract

This paper studies optimal CEO contracts when managers manipulate their performance measure, sometimes at the expense of firm value. Optimal contracts defer compensation. The manager’s incentives vest over time at an increasing rate, and compensation becomes increasingly sensitive to short-term performance. This process generates an endogenous CEO horizon problem whereby managers intensify performance manipulation in their final years in office. Contracts are designed to foster effort while minimizing the adverse effects of manipulation. We characterize the optimal mix of short- and long-term compensation along the manager’s tenure, the optimal vesting period of incentive pay, and the resulting dynamics of managerial short-termism over the CEO’s tenure. Our paper provides a rationale for issuing stock awards with performance-based vesting provisions, a practice increasingly adopted by U.S. firms.

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I have nothing to disclose.
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1 Introduction

Short-termism is prevalent among managers. Graham, Harvey, and Rajgopal (2005) find that 78% of U.S. CEOs are willing to sacrifice long-term value to beat market expectations. For example, Dechow and Sloan (1991) argue that, by the end of tenure, CEOs tend to cut R&D investments which, though profitable, have negative implications for the firm’s reported earnings. Managerial short-termism has been an important concern for many years, but it has assumed a particularly prominent role in recent years following the Enron scandal and the financial crisis in 2008.

To understand this phenomenon, the theoretical literature has adopted two approaches. One approach studies CEO behavior, taking managerial incentives as given, and thus it is silent about optimal incentives (see e.g., Stein (1989)). However, the complexity of CEO contracts in the real world (which include accounting-based bonuses, stock options, restricted stock, deferred compensation, clawbacks, etc.) suggest that shareholders are aware of CEO’s potential manipulations and design compensation to mitigate the consequences of such manipulations. An alternative approach studies optimal compensation contracts that are designed to fully remove CEO manipulations. In this class of models, manipulation is not observed on the equilibrium path (Edmans, Gabaix, Sadzik, and Sannikov (2012)). This approach is particularly helpful in settings in which CEO manipulations are too costly to the firm or easy to rule out, but it cannot explain why manipulation seems so frequent in practice or why real world contracts may tolerate them or even induce manipulation (see Bergstresser and Philippon (2006)).

We study optimal compensation contracts when CEOs exert hidden effort but can also manipulate the firm’s performance to increase their compensation, sometimes at the expense of firm value. Building on Holmstrom and Milgrom (1987), we consider a setting with a risk-averse CEO, who can save privately and consume continuously, and who exerts two costly actions: effort and manipulation. Both actions increase the CEO’s performance in the short run, but manipulation also has negative consequences for firm value. As in Stein (1989), we assume these consequences are not perfectly/immediately captured by the performance measure but rather take time to be verified, potentially creating an externality when the CEO tenure is shorter than the firm’s life span.

Our paper makes two contributions to the literature. First, on the normative side, we
study the contract that maximizes firm value in the presence of manipulation. We characterize the optimal mix of long- and short-term incentives, the duration of CEO pay throughout CEO tenure, and the ideal design of clawbacks and post-retirement compensation. Second, on the positive side, we are able to make predictions about the evolution of CEO manipulations along CEO tenure, and we establish the existence of an endogenous CEO horizon problem.

We study the timing of manipulation: how it evolves over the CEO tenure and whether optimal contracts may generate a horizon effect in which the CEO distorts performance at the end of his tenure. Previous literature has shown that in dynamic settings one can implement positive effort and zero manipulation at the same time (unlike in static settings) by appropriately balancing the mix of short- and long-run incentives. However, in our setting, inducing zero manipulation is not optimal; rather, tolerating some manipulation is desirable because, in return, this allows the firm to elicit higher levels of effort than a manipulation-free contract. Furthermore, to fully discourage manipulations, the firm would have to provide the CEO with a large post-retirement compensation package that ties his wealth to the firm’s post-retirement performance. Such post-retirement compensation is costly to the firm, as it imposes risk on the CEO during a period of time when effort does not need to be incentivized, and the CEO must be compensated for bearing this extra risk (see, e.g., Dehaan et al. (2013)).

In the absence of manipulation, short-term incentives, measured as the contract’s pay-performance sensitivity ($PPS$), are constant over time, as in Holmstrom and Milgrom (1987). Unfortunately, the simplicity of this contract vanishes under the possibility of manipulation. A constant $PPS$ contract is no longer optimal because it induces excessive manipulation, particularly in the final years in office. Indeed, offering the CEO a stationary contract would lead him to aggressively shift performance across periods, boosting current performance at the expense of firm value. To mitigate this behavior, an optimal contract implements lower levels of short-term compensation and higher levels of long-term compensation, measured, roughly, as the present value of the contract’s future $slopes$.

Also, in the absence of manipulation, the CEO incentives vest deterministically, but, under the possibility of manipulation, vesting is contingent upon firm performance. This is empirically relevant. Bettis, Bizjak, Coles, and Kalpathy (2010) assert that, even though restricted stock awards with time-vesting provisions historically account for the majority of
performance-based pay in U.S. companies, shareholder advocacy groups and proxy research services have expressed concern that these provisions do not provide sufficiently strong incentives and have suggested that compensation contracts should include performance-based vesting conditions. In fact, since the mid-1990s, U.S. firms have increasingly issued option and stock awards with sophisticated performance-based vesting conditions. Our paper provides a rationale for this phenomenon. Under the possibility of manipulation, the optimal contract defers compensation and includes performance-based vesting provisions. In the absence of manipulation, the vesting date of incentives is known at the start of the CEO’s tenure, and it is independent of the firm’s performance. When the CEO can manipulate performance, the optimal contract includes performance-based vesting. Thus, the duration of incentives is random: vesting accelerates with positive shocks and is delayed with negative shocks. Random vesting is helpful in the presence of manipulation because it allows the principal to change the level of long-term incentives without having to simultaneously distort the short-term incentives to avoid creating an imbalance, which would trigger extra manipulation. Hence, performance-based vesting provides the principal with an additional degree of freedom to reduce the CEO’s long term incentives without having to distort effort to contain manipulation.

The optimal contract also includes a post-retirement package that ties the manager’s wealth value to the performance of the firm, observed for some time after his retirement. This contracting tool is helpful but has limited power when the CEO is risk averse: even when the firm has the ability to tie forever – and to any degree – the manager’s wealth to the firm’s post-retirement performance, the contract generally induces some manipulation. Although it would be possible to defer compensation long enough to deter manipulation altogether, firms might not do so, given the cost. A key insight in this paper is that firms find it more beneficial to defer compensation while the CEO is still on the job rather than after he retires. This result implies that long-term incentives are larger at the beginning of the CEO’s tenure and decay toward the end.

Under the possibility of manipulation, optimal CEO contracts are non-linear, unlike in Holmstrom and Milgrom (1987). Following Edmans et al. (2012), we first characterize the optimal contract within the subclass of contracts that implement deterministic sequences of effort and manipulation. Under such deterministic contracts, long-term incentives and effort are intertwined. Long-term incentives can be reduced only via increasing the current slope
of short-term compensation, which necessarily distorts the level of effort. This is why, in general, the optimal contract implements incentives that are path dependent. The benefit of providing stochastic incentives which are history dependent and lead to stochastic effort and manipulation, resides precisely in allowing the principal to control the evolution of long-term incentives independently of the CEO’s effort. We find that at the beginning of the CEO’s tenure, the performance sensitivity of long-term incentives is close to zero. However, towards the end of the tenure, such sensitivity becomes negative; positive shocks accelerate vesting thereby reducing long-term incentives. Generally, we find that long-term incentives are mean reverting and follow a target level. If, due to their stochastic nature, the long-term incentives increase relative to its target, the sensitivity of long-term incentives with respect to shocks becomes negative, in order to drive the long-term incentives back down.

**Related Literature**  Beginning with Narayanan (1985), Dye (1988), and Stein (1989), a large number of studies in accounting, economics, and finance have examined the causes and consequences of performance manipulation in corporate settings. Most of the literature studying managerial short-termism has either taken incentives as exogenously given (Stein (1989); Fischer and Verrecchia (2000); Guttman et al. (2006), Kartik, Ottaviani, and Squintani (2007)) or has restricted attention to static or two-period settings with linear contracts, which are unsuited for studying the dynamics of short-termism and its relation to optimal long-term incentives (Baker, 1992; Goldman and Slezak, 2006; Dutta and Fan, 2014; Peng and Röell, 2014). A related strand of the literature examines optimal compensation contracts in the presence of moral hazard and adverse selection (Beyer et al. (2014); Maggi and Rodriguez-Clare (1995); Crocker and Slemrod (2007)).

A more recent literature stream studies dynamic contracts under the possibility of manipulation (Edmans, Gabaix, Sadzik, and Sannikov, 2012; Varas, forthcoming; Zhu, 2018). These studies restricts their attention to contracts that prevent manipulation altogether. Because we are interested in making predictions about the evolution of short-termism, we consider more general contracts that implement optimal levels of manipulation. Sabac (2008) studies CEO horizon effects in a multi-period model with renegotiation where effort has long-term consequences. He finds that effort can decrease while incentive rates increase as managers approach retirement. DeMarzo et al. (2013) study a dynamic agency model where the CEO can take on “tail risk,” thereby gambling with the firm’s money. The optimal con-
tract must strike a balance between providing effort incentives and controlling the manager’s risk-taking behavior. The authors show that when the manager’s continuation value reaches low levels, due to poor performance, the manager engages in excessive risk taking.

On the technical side, we borrow heavily from Holmstrom and Milgrom (1987), Williams (2011), He, Wei, and Yu (2014), and Sannikov (2014). The problem of managerial short-termism is closely related to the long-run moral hazard problem in Sannikov (2014), and we model the inter-temporal effect of the CEO’s action in a similar way as Sannikov (2014). In Sannikov (2014), the future CEO’s productivity is determined by today’s effort, so long-term incentives are required to incentivize effort today. He et al. (2014) studies the design of long-term contracts when the manager’s ability is unknown and learned over time and shows that a combination of private saving and CARA utility provides great tractability to analyze dynamic contracting problems with persistent private information.

2 Model

We study a dynamic agency problem where the agent (hereafter, CEO) can manipulate the firm’s performance to boost his own compensation. The CEO exerts two costly actions, effort \( a_t \) and manipulation \( m_t \). The principal observes neither action but a noisy measure of firm performance.

Let \( \{B_t\}_{t \geq 0} \) be a standard Brownian motion in a probability space with measure \( P \), and let \( \{\mathcal{F}_t\}_{t \geq 0} \) be the filtration generated by \( B \). For any \( \mathcal{F}_t \)-adapted effort, \( a_t \), and manipulation, \( m_t \), processes, the firm’s cash flow process is given by

\[
dX_t = (a_t + m_t - \theta M_t)dt + \sigma dB_t
\]

\[
M_t = \int_0^t e^{-\kappa(t-s)} m_s ds,
\]

when \( M_t \) is the stock of manipulation accumulated through time \( t \). The stock of manipulation \( M_t \) reduces the firm cash flows; at each point, the marginal effect of manipulation on the firm cash flows is \(-\theta\). At the same time, the stock \( M_t \) depreciates at rate \( \kappa \); this means that the consequences of manipulation are more persistent when \( \kappa \) is smaller. Also, as \( \theta \) vanishes, manipulation ceases to have future cash-flow consequences as it is qualitatively equivalent to effort.
This representation of short-termism goes back to Stein (1989) and captures the idea that manipulation may increase current cash flows but eventually destroys firm value. The accounting literature refers to this behavior as real earnings management. Some examples of this behavior arise when managers cut advertising expenditures or R&D to boost reported earnings, offer excessive discounts to meet earnings expectations; or risk the firm’s reputation by lowering product quality/safety. More generally, we can think of manipulation as any potentially unproductive actions aimed at boosting the firm’s short-run profits. Notice that unlike effort, manipulation is inherently dynamic: it increases today’s performance but decreases the firm’s cash flows in future periods. In a nutshell, manipulation is a bad investment, or a mechanism the manager might use to borrow performance from the future to boost current performance.\footnote{Prior literature (see Dutta and Fan (2014)) has modeled the reversal of manipulation as taking place in the second period of a two-period setting. Although not fundamental, one of the benefits of our specification is its flexibility to accommodate different reversal speeds. In our model, the effect of manipulation vanishes gradually based on $\kappa$. As we shall see, this parameter is a key determinant of the manager manipulation patterns.}

Following Holmstrom and Milgrom (1987), we assume that the CEO has exponential preferences given by

$$u(c, a, m) = -e^{-\gamma (c-h(a)-g(m))}/\gamma,$$

where $h(a) \equiv a^2/2$ and $g(m) \equiv gm^2/2$. By assuming that manipulation is costly to the manager, we thus follow the literature on costly state falsification (Dye, 1988; Fischer and Verrecchia, 2000; Lacker and Weinberg, 1989; Crocker and Morgan, 1998; Kartik, 2009). The cost of manipulation $g(m)$ captures the various personal costs the CEO bears from manipulating the firm’s performance. These costs include the effort required to find ways of distorting cash-flows, litigation risk, fines imposed by the SEC, or the natural distaste from behaving unethically.\footnote{This last interpretation is consistent with introspection and existent experimental evidence (Gneezy, 2005).} We allow the cost parameter $g$ to be zero, which captures the case when manipulation is costless.

The CEO is infinitely-lived but works for the firm for a finite time period, $t \in [0, T]$. We refer to $T$ as the manager’s retirement date, and to $T - t$ as his horizon at time $t$. We assume the contract can stipulate compensation beyond retirement, until time $T + \tau$ for $\tau \geq 0$. In other words, the manager’s compensation can be made contingent on outcomes.
observed after his retirement. This possibility captures the principal’s ability to implement a clawback: a longer $\tau$ represents an environment where clawbacks can be enforced for a longer period of time.

In practice, $T$ is random. For tractability reasons, we assume that the retirement date $T$ is known. One can think of $T$ as an approximation to some predictable separation date. For example, Cziraki and Xu (2014) find that CEO terminations are mostly concentrated around the end date of their contracts. The CEO’s expected utility given a consumption flow $\{c_t\}_{t \geq 0}$ is

$$U(c, a, m) = E^{(a,m)} \left[ \int_0^T e^{-rt} u(c_t, a_t, m_t) dt + \int_T^{T+\tau} e^{-rt} u^R(c_t) dt + e^{-r(T+\tau)} u^R(c_{T+\tau}) \right],$$

where $u^R(c) \equiv u(c, 0, 0)$ is the flow utility accrued to the CEO after retirement. Notice again that the CEO tenure is finite but the principal effectively controls the CEO’s compensation over his entire life, even after $T + \tau$. As will become clear, this assumption make it possible to study the effect of changes in the contracting environment while holding the CEO’s life expectancy constant. By doing so, we remove dynamic effects on the structure of the CEO’s compensation driven merely by the shortening of the period length available to pay the CEO as he grows old. Edmans et al. (2012) show that with a finite life, pay-performance sensitivity rises over time as the number of periods to pay the agent his promised compensation is reduced. The same would be true in our setting if we assumed the CEO’s life ends at $T + \tau$.

Following He et al. (2014), we assume that the CEO can borrow and save privately – that is, a hidden action – at the common interest rate $r$. This possibility allows the CEO to smooth consumption intertemporally and hence restricts the ability of the principal to implement compensation schemes that lead to steep expected consumption patterns. Being able to privately save and borrow, the CEO can smooth consumption over time.

The principal designs the contract to maximize firm value. A contract is a consumption process $\{c_t\}_{t \geq 0}$ and an effort-manipulation pair $\{(a_t, m_t)\}_{t \in [0,T]}$ adapted to the filtration generated by the performance measure $X_t$. Formally, the principal chooses the contract to maximize the present value of the firm’s discounted cash flow net of the CEO’s compensation, as given by

$$V(c, a, m) = E^{(a,m)} \left[ \int_0^\infty e^{-rt}(dX_t - c_t 1_{\{t \leq T+\tau\}}) dt - e^{-r(T+\tau)} \frac{c_{T+\tau}}{r} \right].$$
The firm value is thus equal to the discounted net stream of cash flows, which includes a terminal bonus granting the manager a consumption flow \( c_{T+\tau} \) from time \( T+\tau \) onward (these are the CEO pension benefits).

We assume that the negative long-run effect of manipulation dominates the instant benefit, so manipulation destroys value. In other words, manipulation is a negative NPV project. Using integration by parts we find that

\[
E \left[ \int_0^\infty e^{-rt} dX_t \right] = E \left[ \int_0^T e^{-rt} (a_t - \lambda m_t) dt \right],
\]

where \( \lambda \equiv \frac{\theta}{r+\kappa} - 1 \) captures the value-destroying effect of manipulation. When \( \theta \) is large relative to \( r + \kappa \), manipulation is detrimental to firm value – yet potentially attractive to the manager. In other words, manipulation destroys value when either the stock of manipulation \( M_t \) is highly persistent (i.e., \( r + \kappa \) is small) or the marginal effect of \( M_t \) on the firm current cash flows is large (i.e., \( \theta \) is large). Throughout the paper, we make the following parametric assumption.

**Assumption 1.** \( \theta \geq r + \kappa \).

If \( \theta = r + \kappa \), that is if \( \lambda = 0 \), then manipulation has no cash flow effect; it only shifts income across periods, being akin to accrual earnings management. In contrast, if \( \theta > r + \kappa \), \( (\lambda > 0) \), then manipulation not only shifts cash flows across periods but also destroys firm value, being akin to real earnings management.

The principal’s expected payoff given \( (c_t, a_t, m_t) \) is

\[
V(c, a, m) = E \left[ \int_0^T e^{-rt} (a_t - \lambda m_t) dt - \int_0^{T+\tau} e^{-rt} c_t dt - e^{-r(T+\tau)} \frac{c_{T+\tau}}{r} \right].
\]

### 3 The CEO’s Problem

In order to solve for the optimal contract, we first characterize the CEO’s behavior given an arbitrary contract. We state the CEO’s problem given an arbitrary contract, state the necessary incentive compatibility constraints, and provide an informal discussion of these conditions. After describing the necessary conditions for incentive compatibility in Proposition 1, we provide a formal analysis of the CEO’s optimization problem and derive these
conditions using the stochastic maximum principle.

The CEO can save and borrow privately. We denote by $S_t$ the balance in the CEO’s savings account, so given an arbitrary contract prescribing actions $(c_t, a_t, m_t)$ the CEO solves the following problem:

$$
\begin{align*}
\sup_{\hat{c}, \hat{a}, \hat{m}} U(\hat{c}, \hat{a}, \hat{m}) \\
\text{subject to} \\
\begin{aligned}
    dX_t &= (\hat{a}_t + \hat{m}_t - \theta \hat{M}_t)dt + \sigma dB_t \\
    dS_t &= (rS_t - \hat{c}_t + c_t)dt, \quad S_0 = 0.
\end{aligned}
\end{align*}
$$

We rely on the first-order approach to characterize the necessary incentive compatibility constraints: that is, we characterize incentive compatibility by looking at the first order conditions of the CEO’s problem. We verify the sufficiency of the first-order conditions later on in Section 7.

As is common in dynamic contracting problems, we use the CEO’s continuation utility, $W_t$ as a state variable in the recursive formulation of the contract. The continuation utility can be written as

$$
W_t = E_t \left[ \int_t^{T+\tau} e^{-r(s-t)} u(c_s, a_s, m_s) ds + e^{-r(T+\tau-t)} \frac{u^R(c_{T+\tau})}{r} \right].
$$

The contract stipulates the sensitivity of $W_t$ to cash flow shocks, $dW_t/dX_t$, as captured by the slope of incentives, which we denote by $-W_t \beta_t$. We refer to $\beta_t$ as the contract pay-performance sensitivity.

Often in models of dynamic contracting, the continuation value is a sufficient statistic; however, in our setting, manipulation has a persistent effect on the firm’s performance. Thus, the CEO’s incentive to manipulate is tied to the manager’s expectation of the contract’s future pay-performance sensitivities, $\beta_t$. Because of the persistent effect of manipulation, we need to include an additional state variable that measures the impact of manipulation on future outcomes. This inter-temporal effect is captured by the state variable $p_t$ given by

$$
p_t = E_t \left[ \int_t^{T+\tau} e^{-(r+\kappa)(s-t)} \frac{dW_s}{dX_s} ds \right].
$$
When choosing manipulation, the CEO faces a trade-off between boosting current compensation and lowering future compensation. The latter effect is discounted by the interest rate $r$ and the “depreciation rate” of the manipulation stock, $\kappa$.

The presence of long term-incentives is crucial to attenuate manipulation. When the CEO inflates $dX_t$ by an extra dollar, he expects compensation to go down in the future, as the manipulation reverses, more so the larger the CEO’s long-term incentives $p_t$ become. The speed $\kappa$ and intensity $\theta$ of the manipulation reversal determines how much effort the CEO will exert before resorting to manipulation.

The contract also specifies how long-term incentives $p_t$ react to shocks, namely, the performance sensitivity of long-term incentives, which we denote by $\sigma p_t$.

We can now state the necessary incentive compatibility constraint. In Section 7 we return to the issue of sufficiency.

**Proposition 1.** A necessary condition for the contract $(c_t, a_t, m_t)$ to be incentive compatible is that for any $t \in [0, T]$:

\[ r \gamma h'(a_t) = \beta_t \]  
\[ g'(m_t) = \phi \frac{p_t}{W_t} + \frac{\beta_t}{r \gamma} \text{ if } m_t > 0 \]  
\[ g'(m_t) \geq \phi \frac{p_t}{W_t} + \frac{\beta_t}{r \gamma} \text{ if } m_t = 0, \]

where $\phi \equiv \frac{\theta}{r \gamma}$ and $(W_t, p_t, \beta_t, \sigma p_t)_{t \geq 0}$ solves the backward stochastic differential equation

\[ dW_t = -\sigma \beta_t W_t dB_t, \quad W_{T+\tau} = \frac{u_R(c_{T+\tau})}{r} \]  
\[ dp_t = [(r + \kappa)p_t + \beta_t W_t]dt + \sigma \sigma p_t dB_t, \quad p_{T+\tau} = 0. \]

Finally, the private savings condition requires that for any $t \in [0, T+\tau]$ consumption satisfies

\[ r W_t = u(c_t, a_t, m_t). \]
CEO’s marginal cost of effort must equal the sensitivity of his continuation utility to performance $\beta_t$, which captures the CEO’s marginal benefit of effort. This condition is analogous to the incentive compatibility constraint in a static setting with linear contracts, which states that the marginal cost of effort is equal to the slope of incentives. Equations (4b)–(4c) are analogous incentive constraints for manipulation: The marginal cost of manipulation equals the marginal benefit of manipulation. The latter has two components: 

1. A positive component capturing the extra compensation the manager gets by manipulating the performance measure today $\beta_t$, and
2. A negative component capturing the decrease in future compensation arising from the reversal of future cash flows triggered by manipulations, which is proportional to $p_t$ in equation (3). This represents the negative effect in present value that today’s manipulation has on future payoffs. Equations (5a)–(5b) provide the evolution of the state variables $(W_t, p_t)$, which is derived in the appendix. Equation (6) is a consumption Euler equation arising as a consequence of the private savings assumption and the absence of wealth effects under CARA preferences. The Euler equation for consumption implies that the marginal utility of consumption is a martingale while CARA utility implies that the marginal utility is proportional to the current utility. Hence, which means that the current utility $u_t$ equals $rW_t$, so the continuation value $W_t$ is a martingale (its drift is given by $rW_t - u_t = 0$). As He et al. (2014) point out, for general utility functions, allowing for private savings introduces an additional state variable, but this is not the case in a CARA setting, where the marginal value of savings is proportional to the level of current utility. Introducing private savings greatly simplifies our problem and the characterization of the optimal contract. When the agent cannot smooth consumption, the principal chooses the agent’s consumption to control the ratio between the agent’s current utility and his continuation value to provide incentives.

4 Principal’s Problem

Having characterized the CEO incentive compatibility conditions, we can study the principal’s optimization problem. We need to solve a two-dimensional stochastic control problem because manipulation generates a persistent state variable, the stock of manipulation $M_t$. This means we must keep track of the contract’s long-term incentives $p$ in addition to the CEO’s continuation utility $W$. Fortunately, the absence of wealth effects in the manager’s
CARA preference, along with the possibility of private savings, allows us to work with a single state variable, as we shall demonstrate.

The principal’s original optimization problem can be written as follows:

\[
\begin{align*}
V(W_0, p_0) &= \sup_{c, a, m, \beta, \sigma} V(c, a, m) \\
\text{subject to} \quad &dW_t = -\sigma \beta_t W_t dB_t, \quad W_{T+\tau} = \frac{u(R(c_T+\tau))}{r} \\
&dp_t = [(r + \kappa)p_t + \beta_t W_t] dt + \sigma \sigma_p W_t dB_t, \quad p_{T+\tau} = 0 \\
rW_t = u(c_t, a_t, m_t) \\
\text{IC constraints (4a)-(4c)}.
\end{align*}
\]

In principle, we need to keep track of two state variables, \( W \) and \( p \). However, using the properties of CARA preferences under private savings, we can rewrite the principal’s optimization problem as a function of a single state variable: \( z_t \equiv -p_t/W_t \). This variable represents the contract’s long-term incentives scaled by the agent’s continuation utility \( W \). We can think of \( z \) as measuring the importance of deferred compensation (e.g., the number of equity grants that will vest in the future, scaled by the total continuation value of the CEO). From now on, we refer to \( z_t \) as long-term incentives.

Before reformulating the principal’s problem, notice that the private savings condition immediately pins down the manager’s consumption process as, given by:

\[
c_t = h(a_t) + g(m_t) - \frac{\log(-r \gamma W_t)}{\gamma}.
\]

Given private savings, consumption depends on the CEO’s continuation value, effort, and manipulation, so we do not need to consider it as a separate control variable. This further simplifies the formulation of the problem.

**Lemma 1.** The principal value function can be written as

\[
V(W, p) = \text{constant} + \frac{\log(-r \gamma W)}{r \gamma} + F\left(-\frac{p}{W}\right),
\]

where, defining \( z \equiv -p/W \), \( F(z) \) solves the maximization problem.
subject to the incentive compatibility constraints (4a)-(4b), and the law of motion for $z_t$

$$dz_t = [(r + \kappa)z_t + \beta_t(\sigma\sigma_{zt} - 1)]dt + \sigma_{zt}dB_t, \ z_{T+\tau} = 0,$$

where $\sigma_{zt} \equiv \sigma(\beta_tz_t - \sigma_{pt})$.

The optimal contract boils down to the following problem. The principal chooses both short-term incentives $\beta_t$ and the sensitivity of long-term incentives $\sigma_{zt}$ to maximize the firm value. We interpret changes in long-term incentives $dz_t$ as capturing vesting of the CEO’s incentives. PPS can be zero today, but if the CEO has equity grants that vest in the future, then $z_t$ will be positive. When $z_t$ goes down, the duration of the CEO incentives decreases, as if the incentives vested earlier. In contrast, an increase in $z_t$ amounts to delaying vesting and is thus equivalent to increasing the duration of the CEO incentives.

The principal also chooses $\sigma_{zt}$, which captures the sensitivity of incentives $z_t$ to performance shocks, $dB_t$. The principal can implement history-dependent incentives by setting $\sigma_{zt} \neq 0$, effectively implementing performance vesting. As is apparent from equation (9), $\sigma_{zt}$ increases the drift of incentives $z_t$. So, by choosing $\sigma_{zt}$, the principal indirectly controls the rate of vesting. By contrast, when $\sigma_{zt} = 0$ incentives evolve in a deterministic fashion over time.

We make the following technical assumptions. First, we restrict attention to contracts with bounded incentive slopes. That is, for some arbitrarily large constant $\bar{\beta}$, we consider contracts with sensitivity $\beta_t$ bounded by $\bar{\beta}$ for all time $t$. This assumption is similar to the restriction in He et al. (2014)\(^3\), and it implies that $z_t$ remains bounded for all feasible contracts. Notice that we can always choose a value of $\bar{\beta}$ large enough so that the constraint is binding with very small probability. The upper bound $\bar{\beta}$ also implies an upper bound $\bar{\alpha} \equiv (r\gamma)^{-1}\bar{\beta}$ on the implemented effort. We also assume that the agent can freely dispose the output. The assumption of free disposal is standard in the static contracting literature (for example, see Innes (1990)), and more recently, it has been used in a dynamic setting by

\(^3\)As in He et al. (2014), this assumption serves the role of a transversality condition.
Zhu (2018). The free disposal imposes a non-negativity constraint on $\beta_t$. This means that we restrict attention to contracts that have a sensitivity $\beta_t \in [0, \hat{\beta}]$ before retirement.

5 Infinite Tenure and Irrelevance of Manipulation

What is the role of the retirement date $T$ on the structure of CEO pay? As a benchmark, here we study the case with $T = \infty$. We start with a variation of the moral-hazard-without-manipulation problem studied by Holmstrom and Milgrom (1987) but allow for both intermediate consumption and private savings. This problem serves as a benchmark for evaluating the effect of short-termism on the optimal contract.

In the absence of manipulation (namely when the manager cannot manipulate the performance measure), deferring compensation beyond $T$ plays no role; hence, $z_T = 0$ and the principal’s problem boils down to

$$\max_{a_t} \mathbb{E} \left[ \int_0^T e^{-rt} \left( a_t - h(a_t) - \frac{\sigma^2 r \gamma h'(a_t)^2}{2} \right) dt \right].$$

The principal can optimize the agent’s effort point-wise, which yields

$$a_{HM} \equiv \frac{1}{1 + r \gamma \sigma^2}.$$

$a_{HM}$ is the optimal effort arising in the absence of manipulation concerns, as Holmstrom and Milgrom (1987) have shown (this is also the level of effort arising in the case without learning studied by He, Wei, and Yu (2014)). We have thus confirmed that under CARA preferences and Brownian shocks the optimal contract is linear and implements constant effort over time.

In practice, the benefits of non-linear contracts are controversial. The executive compensation literature often argues that firms should get rid of non-linearities in compensation schemes to prevent performance manipulation. For example, Jensen (2001, 2003) argues that non-linearities induce managers to manipulate compensation over time, distracting them from the optimal long-term policies. Consistent with this view, we show that even when manipulation is possible, the above linear contract remains optimal as long as the CEO works forever. More precisely, we show that when $T = \infty$, the optimal contract is linear.
and induces no manipulation. In fact, a perpetual linear contract aligns the incentives of the principal and the CEO, eliminating his incentive to manipulate performance over time. Later on we show that a no manipulation contract is no longer optimal when $T < \infty$, even though it might still be feasible.

The no manipulation constraint can be written as follows:

$$\beta_t - \theta E_t \left[ \int_t^\infty e^{-(r+\kappa)(s-t)} \frac{\beta_s W_s}{W_t} ds \right] \leq 0.$$

If pay-performance-sensitivity, $\beta_t$, is constant, then the no manipulation constraint can be reduced to

$$\beta \left(1 - \theta \int_t^\infty e^{-(r+\kappa)(s-t)} ds \right) \leq 0$$

because $W_t$ is a martingale. In turn, this is always satisfied under the condition that manipulation has negative NPV, as stated in Assumption 1 (i.e., $\theta \geq r+\kappa$). Hence, we have verified that, under infinite tenure, the optimal contract in the relaxed problem that ignores the possibility of manipulation continues to be feasible when the manager is allowed to manipulate. When the manager’s tenure is infinite, the possibility of manipulation is thus irrelevant and the optimal contract implements no manipulation, being thus identical to that in Holmstrom and Milgrom (1987). We summarize the previous discussion in the following proposition.

**Proposition 2.** When $\theta \geq r+\kappa$ and $T = \infty$ the optimal contract entails no manipulation. The optimal contract is linear and stationary implementing $a_t = a^{HM}$.

## 6 Finite Tenure

In the sequel, we solve the principal’s problem for finite $T$, using backward induction. First, we characterize the optimal contract after retirement, in $t \in (T, T + \tau]$, and then solve for the contract before retirement, in $t \in [0, T]$, taking as given the optimal post-retirement compensation.

### 6.1 Post-Retirement Compensation

An important aspect of a CEO contract is post-retirement compensation and the extent to which it is tied to the firm’s performance. By linking the CEO’s wealth to the firm’s post-
retirement performance the firm can mitigate the CEO’s manipulation in the final years in office. In this section, we fix the contract’s promised post-retirement incentives, \( z_T \), and study how the contract optimally allocates incentives over time after retirement.

We focus on contracts in which \( z_t \) is deterministic after retirement (that is, on \( (T, T + \tau) \)). This means we set \( \sigma_{zt} = 0 \) over the interval \( (T, T + \tau] \), so vesting is indeed deterministic once the CEO is out of office. We make this assumption for tractability, as accommodating the terminal condition \( z_{T+\tau} = 0 \) is extremely difficult in the case of stochastic contracts.\(^4\) Moreover, the restriction to contracts with deterministic vesting makes it possible to solve for the value function at time \( T \) in closed form, which simplifies the analysis of the HJB equation (13). However, the restriction to deterministic contracts is not fundamental for the qualitative aspects of the contract during the employment period \( [0, T] \), which is the main focus of our paper. The main qualitative results follow from the fact that the cost of providing long-term incentives \( z_T \) is convex in \( z_T \), which should hold for stochastic contracts. In fact, the structure of the stochastic problem for the post-retirement compensation in the particular case where \( \tau = \infty \) is similar to the one in He et al. (2014), who show in their setting that the cost of providing long-term incentives is convex.

For a fixed promise \( z_T \), the principal must spread payments to the manager over the period \( (T, T + \tau] \) to minimize the overall cost of providing incentives. As previously mentioned, we focus on contracts that implement deterministic vesting after retirement. Hence, we can formulate the problem as the following cost minimization problem:

\[
\begin{align*}
\min_{\beta} & \int_T^{T+\tau} e^{-(t-T)} \frac{\sigma^2 \beta^2}{2r\gamma} dt \\
\text{subject to} & \\
z_T & = \int_T^{T+\tau} e^{-(r+\kappa)(t-T)} \beta_t dt.
\end{align*}
\]

This formulation shows that providing post-retirement incentives is particularly costly when cash flows are noisy or the manager is more risk-averse. Risk aversion explains why the flow cost to the firm is convex (quadratic) in \( \beta_t \). The principal prefers to smooth the stream of incentives rather than cluster them in a short time period.

\(^4\)We are not aware of a treatment of this terminal value problem in the stochastic control literature.
The Lagrangian of this minimization problem is

\[ \mathcal{L} = \int_T^{T+\tau} e^{-r(t-T)} e^{-r(\kappa)(t-T)} \beta_t dt + \ell \left( z_T - \int_T^{T+\tau} e^{-r(\kappa)(t-T)} \beta_t dt \right), \]

where \( \ell \) is the Lagrange multiplier. We can minimize the objective with respect to \( \beta_t \) pointwise, which yields

\[ \beta_t = \ell \frac{2r\gamma}{\sigma^2} e^{-\kappa(t-T)}. \]

We find the value of the multiplier \( \ell \) by replacing \( \beta_t \) in the constraint. The multiplier is proportional to the level of deferred compensation at time \( T \), represented by \( z_T \). Thus, the level of the post-retirement PPS is proportional to \( z_T \) and decreases exponentially over time at speed \( \kappa \), which is the depreciation rate of the stock of manipulation.

The cost of post-retirement incentives is given by \( \frac{1}{2}C z_T^2 \), where

\[ C \equiv \frac{\sigma^2 (r + 2\kappa)}{r\gamma (1 - e^{-(r+2\kappa)T})}. \]

Hence, the cost increases, in a convex fashion, in the magnitude of incentives \( z_T \). This convexity implies that the principal dislikes volatility in \( z_T \). On the surface, this suggests that implementing (random) history-dependent incentives is suboptimal because it may lead to a large post-retirement compensation package \( z_T \), but – as we shall discuss later – having the possibility to implement history-dependent incentives also has benefits.

Post-retirement incentives (to deter manipulation) are more costly (on net) than pre-retirement incentives because they do not also incentivize effort. In general, post-retirement incentives are more costly when the performance measure is more volatile \( (\sigma) \), the manager is more risk averse \( (\gamma) \), the stock of manipulation depreciates faster \( (\kappa) \), and \( \tau \) is smaller. Notice, that the principal is always better off as \( \tau \) increases. However, the cost of post-retirement incentives does not go to zero as \( \tau \to \infty \).

7 Optimal Contract

Next, we analyze the compensation contract through the CEO’s active life. We investigate how the mix of long- versus short-term incentives and the duration of incentives respond to
shocks throughout the CEO’s tenure.

Given the optimal post-retirement scheme derived previously, we can specialize the principal problem in Lemma 1. Hereafter, it is convenient to eliminate manipulation as one of the controls by noting that manipulation is given by $m_t = (a_t - \phi z_t)^+/g$ and defining the payoff function as

$$\pi(a, z) \equiv a - \frac{\lambda}{g} (a - \phi z)^+ - \frac{1}{2g} [(a - \phi z)^+]^2 - \frac{(1 + r\gamma \sigma^2)a^2}{2}.$$ 

We can rewrite the principal problem in Lemma 1 as

$$\begin{align*}
F(z, 0) &= \sup_{z_0, a, \sigma} \mathbb{E} \left[ \int_0^T e^{-rt} \pi(a_t, z_t) dt - e^{-rT} \frac{1}{2} C z_T^2 \right] \\
\text{subject to} \\
dz_t &= [(r + \kappa)z_t + r\gamma a_t (\sigma \sigma z_t - 1)] dt + \sigma z_t dB_t
\end{align*}$$

The Hamilton–Jacobi–Bellman equation associated with problem (12) is as follows:

$$\begin{align*}
r F &= \max_{a, \sigma} \pi(a, z) + F_t + [(r + \kappa)z + r\gamma a(\sigma \sigma z - 1)] F_z + \frac{1}{2} \sigma^2 F_{zz} \\
F(z, T) &= -\frac{1}{2} C z^2.
\end{align*}$$

The terminal condition captures the cost of providing post-retirement incentives $z_t$, as derived in Section 6.1. Once we solve for the value function for arbitrary $z_0$, we initialize the contract at $z_0 = \arg \max_z F(z, 0)$.

Optimizing with respect to $\sigma_z$ requires that the value function is concave. We can always ensure that the value function is weakly concave by introducing public randomization. However, we have not been able to prove that the solution is strictly concave. Hereafter, we assume that the value function is concave so that the following characterization applies to regions of the state space where such randomization is not required. We have not found instances in our examples where the use of randomization is required. Moreover, $F(z, t)$ is strictly concave in $z$ at time $T$ (because it satisfies the terminal condition), so $F(z, t)$ is also

---

$^5$We use the usual notation $x^+ = \max \{x, 0\}$.

$^6$We omit the specification of the value function at the upper bound on $z_t$ as this is not needed for the following analysis. We provide the specification of the boundary condition at the upper bound to the online appendix when we provide the numerical algorithm used for the solution of the HJB equation.
strictly concave for $t$ close to $T$.\footnote{The main challenge in establishing concavity on $[0,T]$, for an arbitrary time $T$, comes from the interaction term $a_t \sigma_{zt}$ in the drift of $z_t$.} \footnote{The HJB equation presents the difficulty that the diffusion coefficient $\sigma_{zt}$ is not bounded away from zero, which means that the HJB equation is a degenerate parabolic PDE. Because a classical solution may fail to exist, we resort to the theory of viscosity solutions for the analysis. The fact that the value function is a viscosity solution of the HJB equations follows from the principle of dynamic programming. The uniqueness of the solution follows from the comparison principle in Fleming and Soner (2006).} The contract entails history-dependent incentives ($\sigma_{zt} \neq 0$). Why is this so? Exposing the CEO to risk after retirement ($t > T$) by setting $z_T > 0$ may deter some manipulation, but it is costly; it is inefficient from a risk-sharing point of view, and does not stimulate effort. Hence, the principal would like to reduce long-term incentives just before the CEO’s retirement, at time $T$. Now, if we look at the SDE for $z_t$ in (12), we see that the drift of $z_t$ depends both on the level of effort $a_t$ and the sensitivity of incentives, $\sigma_{zt}$. The principal can reduce $z_t$ by either increasing effort or by implementing a negative sensitivity, $\sigma_{zt}$. The latter amounts to implementing a negative correlation between incentives $p_t$ and the CEO continuation value $W_t$.

By setting $\sigma_{zt} \neq 0$, the principal effectively implements performance vesting. The optimality of (random) performance vesting arises in this model because, by changing $\sigma_{zt}$, the principal can control the evolution (i.e., drift) of incentives $z$. This is useful because it allows the principal to accelerate vesting toward retirement and reduce the level of incentives $z_t$ without having to increase the short-term incentives $\beta$, which would trigger more manipulation. In other words, by controlling the sensitivity of incentives $\sigma_{zt}$, the principal can partially decouple incentives for effort provision (which are driven by $\beta_t$) from manipulation incentives.

As previously mentioned, implementing history-dependent incentives is costly. A high sensitivity $\sigma_{zt}$ leads to volatile incentives $z_t$, and as discussed in Section 6.1, the cost of giving the CEO post-retirement incentives increases, in a convex manner, in the size of those incentives, $z_T$, given the agent’s risk aversion. This explains why the absolute magnitude of $\sigma_{zt}$ is relatively small, as seen in Figure 1.

Given the value function $F$, we find the optimal effort, manipulation, and sensitivity by
solving the optimization problem in the HJB equation. The optimal policy is then given by

\[ a(z, t) = \min \{ \tilde{a}(z, t), \bar{a} \}^+ \]  
(14a)

\[ \tilde{a}(z, t) = \begin{cases} 
\frac{a - \lambda + \phi z - r \gamma F_z}{1 + g H(F_z, F_{zz})} & \text{if } 1 - r \gamma F_z \geq \phi z H(F_z, F_{zz}) + \frac{\lambda}{g} \\
\frac{1 - r \gamma F_z}{H(F_z, F_{zz})} & \text{if } 1 - r \gamma F_z < \phi z H(F_z, F_{zz}) \\
\phi z & \text{otherwise} 
\end{cases} \]

\[ m(z, t) = \frac{1}{g} (a(z, t) - \phi z)^+ \]  
(14b)

\[ \sigma_z(z, t) = -r \gamma \sigma a(z, t) \frac{F_z}{F_{zz}}, \]  
(14c)

where \( H(F_z, F_{zz}) \equiv 1 + r \gamma \sigma^2 + r^2 \gamma^2 \sigma^2 (F_{zz})^{-1} F_z^2 \). The second-order condition requires that \( 1 + g H(F_z, F_{zz}) \geq 0 \). \( \sigma_z(z, t) \) remains determined by (14c) if this condition is not satisfied. However, the optimal effort now is either \( \phi z \) or \( \bar{a} \). In the particular case where \( g > \lambda \), effort is given by \( \min(\phi z, \bar{a}) \).

Due to its non-linearity, it is difficult to obtain analytical results by analyzing the HJB equation (13) directly. However, we can derive some insights about the optimal contract indirectly by analyzing the sample paths of the dual variable \( \psi_t = F_z(z_t, t) \), which captures the Principal’s marginal value of providing long-term incentives to the CEO. The approach of analyzing the optimal contract by studying the sample paths of \( \psi_t \) follows the approach used by Farhi and Werning (2013) and Sannikov (2014), and it is similar to the analysis of deterministic contracts using optimal control developed in Section 8 (and the analogous stochastic maximum principle). In fact, \( \psi_t \) is the stochastic analogue to the traditional co-state variable in optimal control problems.

Because the payoff function \( \pi(a, z) \) fails to be differentiable at \( a = \phi z \) when \( \lambda > 0 \), we assume that \( \lambda = 0 \) throughout the rest of this section. This assumption is required purely for technical considerations and is not required when we solve the model numerically. Using Ito’s Lemma and the Envelope Theorem, we derive the following representation for the dual variable \( \psi_t \), which is key in the characterization of the optimal contract. We find that the
pair \((\psi_t, z_t)\) solves the forward–backward stochastic differential equation

\[
d\psi_t = - (\kappa \psi_t + \phi m_t) dt - r \gamma \sigma a_t \psi_t dB_t, \quad \psi_0 = 0,
\]

\[
dz_t = [(r + \kappa) z_t + r \gamma a_t (\sigma \sigma_{zt} - 1)] dt + \sigma_{zt} dB_t, \quad z_T = -C^{-1} \psi_T.
\]

This means that, for any \(t \in [0, T]\), the value of \(\psi_t\) is given by

\[
\psi_t = -\phi \int_0^t e^{-\kappa (t-s)} \mathcal{E}_{s,t} m_s ds
\]

\[
\mathcal{E}_{s,t} \equiv \exp \left\{ - \int_s^t r \gamma \sigma a_u dB_u - \frac{1}{2} \int_s^t r^2 \gamma^2 \sigma^2 a_u^2 du \right\},
\]

and hence \(\psi_t \leq 0\) for all \(t \in [0, T]\). Notice that \(\psi_t = 0\) if and only if \(m_s = 0\) for all \(s < t\). This implies that \(\sigma_{zt}\) is zero (so incentives are deterministic) if there has been no manipulation before time \(t\).\(^9\) Equation (15) makes it possible to derive the qualitative properties of the optimal contract and has the following implication: The marginal value of incentives has an upper boundary at zero, and this implies that if the value function is concave, then there is a lower bound \(\bar{z}(t)\) for the long-term incentives; that is, \(z_t \geq \bar{z}(t)\) where \(F_z(\bar{z}(t), t) = 0\). We provide a qualitative characterization of the lower boundary \(\bar{z}(t)\) in Proposition 3 and show that this boundary decreases over time. This is consistent with the notion that the principal wishes to reduce long-term incentives over time to avoid leaving the manager with a large post-retirement package.

Furthermore, combining Equations (14c) and (15), we find that the sensitivity of incentives is non-positive: positive shocks reduce the long-term incentive, reducing the duration of incentives. This establishes the optimality of performance vesting discussed at the beginning of this section. The following proposition records these results.

**Proposition 3.** If \(\theta = r + \kappa\), then the optimal contract has the following properties:

1. Lower bound on long-term incentives: There is a decreasing function \(\bar{z}(t) \geq 0\) such that \(z_t \geq \bar{z}(t)\) for all \(t \in [0, T]\), where \(\bar{z}(T) = 0\) and \(\bar{z}(t) > 0\) for all \(t < T\).

2. Long-term incentives and performance are negatively correlated: \(\sigma_{zt} \leq 0\) for all \(t \in [0, T]\).

\(^9\)We can show that this never happens if \(\lambda = 0\), but the analysis of deterministic contracts suggests that it could be the case if \(\lambda > 0\).
3. Whenever \( m_t > 0 \) the drift of long term incentives is negative, that is, \( E_t(dz_t) < 0 \).

Performance vesting is optimal (i.e., \( \sigma_{zt} < 0 \)) because it provides an extra degree of freedom to control the level of incentives \( z_t \) without triggering excessive manipulation \( m_t \). In contrast, under deterministic vesting, the only way to reduce the long-term incentives is by increasing short-term incentives \( \beta_t \), which exacerbates manipulation and lowers the level of effort the principal can implement in the future. This is precisely where performance vesting helps: Long-term incentives can be reduced over-time without necessarily distorting the level of effort. By adjusting the sensitivity of incentives \( \sigma_{zt} \), the principal can control the drift of \( z_t \) while holding the trajectory of effort constant.

It is precisely the possibility of manipulation that justifies performance vesting in our setting. On the surface, one might think that performance vesting exacerbates the CEO’s incentive to manipulate since, by inflating performance, the CEO can accelerate vesting. This logic is flawed. The manager’s manipulation incentive at a given point depends upon the sensitivity of his continuation value to performance \( \beta_t \) and duration \( z_t \), not upon the sensitivity of duration \( \sigma_{zt} \). Making vesting more or less sensitive to performance at time \( t \), by modifying \( \sigma_{zt} \), does not affect the CEO manipulation incentives at time \( t \). For example, consider the case when \( \beta_t = 0 \). The manager has no incentive to manipulate performance, and this is true independent of the sensitivity of incentives \( \sigma_{zt} \). In other words, as long as \( \beta_t \) does not change, the choice of \( \sigma_{zt} \) will not affect manipulation incentives at time \( t \). Of course, \( \sigma_{zt} \) has an indirect effect on incentives to manipulate in future periods due to its effect on the duration of incentives \( (z_t) \).

By setting a negative sensitivity \( \sigma_{zt} \), the principal effectively implements a negative correlation between \( p_t \) and \( W_t \). Hence, positive shocks that boost the agent’s continuation value \( W_t \) reduce the duration of incentives \( p_t \). In brief, good performance accelerates vesting.

The evolution of \( z_t \) resembles a mean-reverting process that follows a time-varying target \( \tilde{z}(t) \) converging to zero as \( T \) becomes closer. Figure 2 shows the evolution of the lower boundary \( \tilde{z}(t) \) together with the drift of \( z_t \). The lower bound decreases over time towards zero, and the drift of \( z \) is negative above the lower bound on incentives, \( \tilde{z}(t) \). Moreover, we show that whenever the optimal contract implements positive manipulation, the drift of long-term incentives is negative. In particular, we have shown that the drift is negative when \( z_t \) is close to the lower boundary. \(^{10}\) Long-term incentives revert toward the target \( \tilde{z}(t) \)

\(^{10}\) We have not been able to sign the drift for values of \( z_t \) such that \( m_t = 0 \). However, we show that in the
over time, and the magnitude of the negative drift of $z_t$ increases when we are close to the retirement date $T$. The relative importance of short-term incentives increases as the CEO gets closer to retirement, explaining the CEO horizon effect. Figure 3 shows the evolution of expected long-term incentives, effort, manipulation and sensitivity. Long-term incentives and effort decrease, and manipulation increases over time. The volatility of incentives is low at the beginning of tenure – so the contract’s evolution is close to deterministic; so that it decreases over time (its absolute value increases). This means that the contract becomes more sensitive to performance over time.

Consider the effect of enforcement on the optimal contract. Our model includes an upper bound $\tau$ for the length of the clawback period $[T, T + \tau]$. This parameter captures the fact that the principal cannot impose risk on the manager’s wealth forever, as this would be impossible to enforce. It is not surprising then that a lower $\tau$ reduces the level of long-term incentives. When the clawback period is shorter, providing long-term incentives towards the end of the CEO’s tenure is more costly because it makes the CEO compensation more risky. Moreover, a lower $\tau$ reduces the overall duration of CEO incentives. In fact, a lower $\tau$ reduces the importance of long-term incentives at the beginning of the CEO’s tenure and the lower bound on long-term incentives $\bar{z}(t)$, as the following proposition proves.

**Proposition 4.** If $\theta = \tau + \kappa$, then

1. The lower boundary on long-term incentives, $\bar{z}(t)$, is increasing in $\tau$.
2. Initial long-term incentive $z_0$ are increasing in $\tau$ and $T$.
3. This implies that the initial effort is increasing in $\tau$ and $T$ while the initial manipulation is decreasing in both $\tau$ and $T$.

We conclude this section by revisiting the CEO’s problem and establishing sufficient conditions for the validity of the first-order approach. This approach makes it possible to find a recursive formulation for the principal problem and analyze the relaxed problem in which we only consider the first-order conditions. Solving for the optimal contract is not possible if the first-order approach is not valid because one lacks a recursive formulation that case of contracts with deterministic vesting, that is $\sigma_{zt} = 0$, the drift of $z_t$ is always negative on the optimal path.
Figure 1: Solution to principal’s optimization problem. Parameters: \( r = 0.1, \gamma = 1, g = 1, \theta = 0.4, \kappa = 0.3, \sigma = 2, T = 10, \tau = 5 \). This plot shows the solution of the optimal contract in the \( t, z \) space. The bottom left panel reveals that manipulation is zero when both \( t \) and \( z \) are low. As \( t \) approaches retirement date \( T \), manipulation escalates. In general, incentives are stochastic, as reflected by \( \sigma_{zt} \neq 0 \). However, the sign of \( \sigma_{zt} \) depends both on \( t \) and \( z_t \). Early on in the CEO’s career, the contract is virtually deterministic, but as the manager approaches retirement, the contract implements performance contingent vesting, which leads to vesting being positively correlated with performance.

can be analyzed using the tools of stochastic control theory. The next proposition provides sufficient conditions for the validity of the first-order approach.

**Proposition 5.** Assume that \( r\gamma\sigma^2 > 1 \) and that the cost of manipulation satisfies

\[
g \geq \frac{1}{r\gamma\sigma^2 - 1}. \tag{16}
\]

Then, given the optimal contract characterized in Proposition 3, the necessary incentive compatibility constraint is also sufficient. If either \( r\gamma\sigma^2 \leq 1 \) or (16) are not satisfied, then there is a bound \( L_v > 0 \) such that the necessary incentive compatibility constraint is also
Figure 2: Lower bound ($\bar{z}(t)$) and drift ($E_t(dz_t)$) of incentives. Parameters: $r = 0.1$, $\gamma = 1$, $g = 1$, $\theta = 0.4$, $\kappa = 0.3$, $\sigma = 2$, $T = 10$, $\tau = 5$. This plot shows the evolution of $\bar{z}(t)$ together with the drift of the continuation value. The lower bound $\bar{z}(t)$ decreases over time towards zero, and the drift of $z_t$ is negative, which means that long-term incentives revert toward the target $\bar{z}(t)$. The black curve represents $\bar{z}(t)$ while the contour lines represent the value of drift, $E_t(dz_t)$, for different pairs $(t, z)$. The relevant state space on-path corresponds to the pairs $(t, z)$ above the curve $\bar{z}(t)$, and a darker background represents a lower (more negative) drift.

As in previous literature (He et al., 2014; Sannikov, 2014), the sufficiency of the first-order approach requires that the sensitivity of long term incentives be bounded. However, because in our setting this sensitivity is never positive, we only need to bound the sensitivity from below. Moreover, when $r\gamma\sigma^2$ and $g$ are high enough, the first-order approach is valid for any non-positive sensitivity, and there is no need to impose any additional bound on the sensitivity of incentives. Finally, notice that the first-order approach is always valid if $\sigma_{zt} = 0$; so, the first order approach is always valid for the deterministic contracts considered later in Section 8.

8 Deterministic Incentives

In general, effort and manipulation are history dependent; however, one can gain further insights into the dynamics of compensation and CEO behavior by following Edmans et al. (2012) and He et al. (2014) and looking at the subclass of contracts that implement de-
terministic sequences of effort and manipulation. In this section, we characterize the best contract among the class of contracts that implement deterministic incentives. Hence, in this section, by optimal contract we mean the “best deterministic contract.” With deterministic incentives, $\sigma_{zt} = 0$, the evolution of long-term incentives specializes to

$$
\dot{z}_t = (r + \kappa)z_t - r\gamma a_t.
$$

Equation (17) reveals a fundamental limitation of a deterministic contract: effort and long-term incentives are intertwined: to reduce long-term incentives, $z_t$, the contract must increase current effort $a_t$ and vice versa. The following proposition characterizes the path of incentives and CEO actions induced by the optimal deterministic contract.

**Proposition 6.** Based on the incentives to manipulate, the manager’s tenure $T$ can be divided into three regions characterized by thresholds $t^* \leq t^{**}$:

- In the first region, $[0, t^*)$, there is no manipulation, and the effort level is the same as that arising when manipulation is impossible (that is, $a_t = a^{HM}$).
- In the second region, $(t^*, t^{**}]$, there is zero manipulation, but the level of effort is
bounded by the magnitude of long-term incentives \(a_t = \phi z_t\).

- In the third region, \((t^{**}, T]\), manipulation is positive and increasing over time.

- Depending on parameters, the regions \((t^*, t^{**}]\) and \((t^{**}, T]\) can be empty.

Over time, long-term incentives \(z_t\) are weakly decreasing while manipulation \(m_t\) is weakly increasing.

There are three distinct regions. In the first region, manipulation is not a concern. In the second region, there is no manipulation, but preventing manipulation forces the principal to lower the level of effort implemented. In the third region, preventing manipulation is too costly: both effort and manipulation go up over time. The region \((t^{**}, T]\) is empty when the no-manipulation contract identified in the previous section is optimal. Figure 4 shows a numerical example in which the three regions identified above are present. Surprisingly, effort is non-monotone: it decreases at the beginning and increases toward the end (this is not true in general, and depending on the parameters, effort can be either increasing or decreasing in the final region \((t^{**}, T]\)).

Why is effort increasing in the final region? The reason is simple: vesting of long-term incentives accelerates toward the end of the CEO’s tenure, to avoid leaving the manager with a large post-retirement package, thus boosting short-term incentives. Edmans et al. (2012) find a similar result, but driven by a different mechanism. In their model, the CEO is finitely lived, so vesting accelerates by the end of his tenure because fewer periods are available to compensate the CEO. Hence, payments have to be spread within a shorter time period to keep the manager from shirking. In our setting, the CEO is infinitely lived so there is no need to accelerate vesting to satisfy the promise-keeping constraint. In our setting, vesting accelerates because deferring compensation after retirement is more costly than deferring compensation while the CEO is active. Having excessive deferred compensation after the CEO retires is costly; hence, vesting accelerates towards the end of his tenure to lower the level of post-retirement incentives. In turn, this means that \(PPS\) increases by the end of tenure, thereby increasing both effort and manipulation.
The optimal contract induces manipulation sometimes, but not necessarily in every instant of the manager’s tenure. As mentioned above, the CEO’s tenure consists of three phases, ranked by the intensity of manipulation. During the first phase, manipulation incentives are weak because the CEO horizon is long, which means the principal has enough time to “detect” and penalize the manager’s manipulation. As a consequence, short-run incentives are strong and the manager exerts high effort and zero manipulation. During the second phase, the manager’s manipulation incentives are binding, but the contract still implements zero manipulation. However, to prevent manipulation, the principal is forced to distort the contract \( PPS \) downward, which leads to a pattern of decreasing effort. During the third phase, vesting speeds up, manipulation incentives become stronger, and manipulation escalates.

Long-term incentives have to mature over time, as the manager approaches retirement, and this process tilts incentives toward the short run. In turn, this triggers manipulation, but may also boost effort in the final years. We can think of these two effects as mirror images: providing high post-retirement compensation is costly. To reduce it, some of the contract’s long-term incentives must mature, which in turn increases short-term incentives. The relative length of the three phases in the manager’s tenure depends on the severity of the
manipulation problem. Thus, for instance, when the reversal of manipulation is slow (low $\theta$), enforcement is weak (low $\tau$), or manipulation is easy (low $g$), the relative importance of the third phase grows at the expense of the other two phases, especially the first one.

We find that the optimal contract follows similar patterns to the ones in the contract with deterministic vesting. In the next section, we discuss the predictions of the model and provide some numerical examples and comparative statics. In most of our examples, the long-term incentive sensitivity implemented by the stochastic contract – which is itself random – is very small on average. Hence, the deterministic contract seems like a good approximation of the state-contingent contract, and it captures the evolution of CEO behavior and incentive-pay very accurately, particularly at the beginning of CEO’s tenure.

9 The Model at Work: Numerical Examples and Empirical Implications

In this section, we discuss the empirical implications of the model and relate them to existing evidence.

Vesting and short-termism With regard to short-termism and vesting, Edmans et al. (2013) show that CEO manipulations increase during years with significant amounts of shares and option vesting. The authors find that, in years in which the CEOs experience significant equity vesting, they cut investments in R&D, advertising, and capital expenditures. Seemingly, vesting induces CEOs to act myopically in order to meet short-term targets.

Horizon, Short-Termism, and Pay Duration. The executive compensation literature hypothesizes the existence of a “CEO Horizon problem” whereby CEO short-termism would be particularly severe in the final years of CEO office, in so far as the manager is unable to internalize the consequences of his actions. Gibbons and Murphy (1992) indeed hypothesize that existing compensation policies induce executives to reduce investments during their last years of office but do not find conclusive evidence of greater manipulation. Gonzalez-Uribe and Groen-Xu (2015) find that “CEOs with more years remaining in their contract pursue more influential, broad and varied innovations.” Dechow and Sloan (1991) document that
managers tend to reduce R&D expenditures as they approach retirement, and the reductions in R&D are mitigated by CEO stock ownership.

Although intuitive, the CEO horizon hypothesis seems to ignore the fact that manager incentives are endogenous. If shareholders anticipate the CEO horizon problem, arguably they will adjust compensation contracts accordingly. This could explain why the empirical evidence regarding the relation between manipulation and tenure is ambivalent (Gibbons and Murphy (1992)). Cheng (2004), for instance finds that compensation contracts become particularly insensitive to accounting performance measures that are easily manipulable by the end of the manager’s tenure, suggesting that compensation committees are able to anticipate the manager’s incentives. In this paper, we show that a CEO horizon problem exists even in the presence of endogenous incentives. The finite nature of CEO’s tenure and the fact that deferring compensation after retirement is costly, explain why optimal contracts implement manipulation in our setting. In Section 5, we show that when the manager horizon grows large \((T \to \infty)\), the possibility of manipulation is irrelevant. Linear contracts such as the one analyzed by Holmstrom and Milgrom (1987) suffice to eliminate manipulation. This result is consistent with Jensen (2001, 2003), who recommends linear contracts to prevent managers from gaming compensation systems. In sum, our analysis suggests that when the CEO has a limited horizon, linear contracts are unable to prevent short-termism and may even induce too much.\(^{11}\)

From a contracting perspective, two tools are effective at dealing with the possibility of manipulations: \(i\) deferred compensation and \(ii\) clawbacks. Both tools are used in practice. Some empirical evidence suggests that after SOX the average duration of CEO compensation increased, and firms started to rely more on restricted stock to compensate managers. Gopalan et al. (2014) provide evidence that the duration of stock-based compensation is about three to five years. They document a negative association between the duration of incentives and measures of manipulation such as discretionary accruals. In particular, they find that this duration is shorter for older executives and those with longer tenures. The second instrument is clawbacks. A clawback is a contractual clause included in employment contracts whereby the manager is obliged to return previously awarded compensation due to special circumstances, which are described in the contract, for example a fraud or re-

\(^{11}\)Kothari and Sloan (1992) provides evidence that accounting earnings commonly take up to three years to reflect changes in firm value
statement. The growing popularity of clawback provisions is due, at least in part, to the Sarbanes–Oxley Act of 2002, which requires the U.S. Securities and Exchange Commission (SEC) to pursue the repayment of incentive compensation from senior executives who are involved in a fraud or a restatement.\(^\text{12}\) Although we do not incorporate clawbacks – as a discrete event triggered by a restatement – in our model, the fact that the manager’s income depends on post-retirement performance captures the essence of clawbacks as an incentive mechanism.

**Pay-for-Performance** The executive compensation literature has documented at least two puzzles regarding pay-performance sensitivity. First, pay for performance evolves with CEO tenure (Brickley et al. (1999)). Unlike in Holmstrom and Milgrom (1987), a constant \(PPS\) is not optimal in our setting. Indeed, a constant \(PPS\) would lead to excessive manipulation, especially around the retirement date. Our model predicts a profile of increasing manipulation along with a relatively low but potentially increasing \(PPS\). Some evidence suggests that the \(PPS\) of CEO compensation increases over time, as manager’s stock ownership grows (Gibbons and Murphy (1992)). At first blush, this fact seems to contradict the predictions of our model. In our setting, \(PPS\) may increase over time; however, it is never higher than at the start of tenure, and it is non-monotonic in time; it only increases at the end. A time profile of increasing \(PPS\) is consistent with an extended version of the model in which the performance measure is a distorted version of the firm’s cash flows (for example, the firm earnings). A second empirical puzzle that was identified in the 1990s is the low \(PPS\) in CEO contracts (see e.g., Jensen (2001)). Our model predicts that such low \(PPS\) could be the result of the possibility of manipulation, as already suggested by Goldman and Slezak (2006).

**Corporate Governance and Short-Termism** The CEO horizon problem is ultimately a corporate governance weakness reflecting the inability of the firm to monitor the CEO’s actions. If we understand corporate governance as a set of mechanisms (some of which are exogenous to the firm) that make it more costly for the manager to manipulate performance (for example, by increasing the cost of manipulation \(g\)), then our model predicts that better

\(^{12}\)The prevalence of clawback provisions among Fortune 100 companies increased from less than 3% prior to 2005 to 82% in 2010.
corporate governance would result in higher short-term compensation (lower duration) and greater firm value. Maybe paradoxically, it does not predict that the levels of manipulation will be lower. If better corporate governance makes short-run incentives relatively more effective at stimulating effort, vis-a-vis manipulation, then the firm may find it optimal to offer stronger short-term incentives, even at the expense of tolerating greater manipulations. This effect is present in previous static models of costly state falsification. For example, Lacker and Weinberg (1989) show that no manipulation is optimal when the cost of manipulation is not overly convex. In our setting, with a quadratic falsification cost, this condition translates into a low value of $g$. From the IC constraint, we find that the sensitivity of manipulation to changes in effort (for a fixed $z$) is $1/r\gamma g$. This means that when the marginal cost of manipulation $g$ is low, the trade-off between higher effort and higher manipulation is too high. A small increment in effort generates so much manipulation that it makes the no-manipulation optimal. In fact, when $g = 0$, the optimal contract implements no manipulation. Hence, in this case, the optimal contract implements no manipulation. Of course one needs to be careful when interpreting this observation as evidence that short-run incentives cause manipulation (Bergstresser and Philippon (2006)). As the cost of manipulation $g$ grows large, the manager’s manipulation incentives are vanishingly low. The contract then becomes stationary – with constant $PPS$ – because short-term incentives suffice to induce effort.

Another parameter that relates to corporate governance is $\tau$. Recall that $\tau$ captures the length of the clawback period; namely, how long after retirement principal the principal can tie the CEO’s wealth to the firm’s performance. Figure 5a shows that a longer clawback period allows the principal to induce higher effort and less manipulation. The contract tends to rely more on performance-contingent vesting, and long-term incentives tend to be higher. As mentioned previously, extending $\tau$ is not a panacea. The consequences of manipulation are present even as $\tau$ grows large because providing a large post-retirement package over a long clawback period can eliminate manipulation, although, as a downside, this would impose excessive risk on the manager.
10 Conclusion

This paper studies optimal CEO contracts when managers can increase short-term performance at the expense of firm value. Our model is flexible, nesting both the case when the CEO can manipulate performance by distorting the timing of cash flows and that when the manager can manipulate accruals. We consider a setting in which the manager horizon is finite. We find that long-term incentives decrease over time, managerial short-termism increases, and effort may be non-monotonic in time, increasing at the end of the CEO’s career. The optimal compensation scheme includes deferred compensation. Vesting of the manager incentives accelerates at the end of tenure, thus shifting the balance of incentives towards short-term compensation. This process gives rise to a CEO horizon problem – as an inherent feature of optimal contracts – whereby managers intensify performance manipulation in their final years in office. We characterize the optimal mix of short- and long-term incentives and the optimal duration and vesting of incentives along the manager’s tenure.

We explore the optimality of deferred compensation as a contracting tool for alleviating the effects of CEO manipulation. Though potentially effective, post-retirement compensation may impose significant risk on the CEO during a time when incentives are not needed to stimulate effort. This makes it costly from the firm’s perspective, limiting the effectiveness
of such compensation.

Unlike in Holmstrom and Milgrom (1987), the optimal contract is non-linear in performance. Moreover, it implements path-dependent effort and manipulation, effectively making the firm’s performance more noisy. Under a deterministic contract, the firm can modify the long-term incentives only by distorting the CEO’s effort (for the contract to preserve incentive compatibility). History-dependent incentives help because they allow the firm to control the evolution of long-term incentives without exacerbating manipulations. The optimal contract is one in which the sensitivity of long-term incentives to firm performance at the beginning of the CEO’s tenure and at the end are qualitatively different. At the beginning, positive performance shocks increase the use of long-term incentives; in other words, the duration of incentives increases when the firm is performing well. On the contrary, at the end of the CEO’s tenure, long-term incentives are negatively correlated to the firm’s performance. Positive performance shocks lead to an acceleration of incentive vesting.

We conclude by noting that the design of monetary incentives alone is not enough to eliminate managerial short-termism. In practice, other corporate governance tools may complement the disciplining role of compensation. For example, we can presume that CEOs’ discretion to make short-term investments or cut long-term ones evolves over time, being a function of the manager’s horizon. There are different ways in which this could be addressed. For example, the level of discretion a CEO receives affects the freedom he has to manipulate performance, but it also makes him less productive. In other words, the CEO might not be able to manipulate performance as freely as before, and the associated lack of flexibility could also reduce his productivity. Specifically, assume that under low discretion the manager’s effort only produces a fraction of what it produces otherwise (that is, the marginal productivity of effort is $\alpha a_t$ for some $\alpha < 1$). Our analysis suggests that CEOs should be given more discretion at the beginning of their tenures, with an increment in board oversight taking place as he gets close to retirement. Of course, this policy recommendation must be taken with a grain of salt. There are additional factors that we have ignored in the model. One of these factors is learning; a young and inexperienced CEO’s talent may be uncertain, and the board may want to monitor his actions more closely.
References


Appendix

A Necessary Conditions Incentive Compatibility

First we analyze the CEO’s optimization problem in (19) and derive the conditions stated in Proposition 1. Given a contract prescribing actions \((c_t, a_t, m_t)\), and any CEO strategy \((\hat{c}, \hat{a}, \hat{m})\), we denote a deviation from the contract’s recommended actions by \(\Delta c \equiv \hat{c} - c\), \(\Delta a \equiv \hat{a} - a\) and \(\Delta m \equiv \hat{m} - m\). We simplify the notation by denoting the CEO’s utility flow if he follows the contract’s recommendation by \(u_t \equiv u(c_t, a_t, m_t)\) and by \(u_t^\Delta \equiv u(c_t + \Delta c_t, a_t + \Delta a_t, m_t + \Delta m_t)\) if he deviates.

A contract is a function of the entire performance path \(X_t\), which makes the analysis of the CEO’s problem involved. To overcome this challenge, we follow the approach proposed by Williams (2011) and introduce the following change of measure. Let \(P\) be the probability measure under recommendation \(\{(a, m)\}_{t \in [0,T]}\) and let \(P^\Delta\) be the probability measure induced by the deviation \(\{(\hat{a}, \hat{m})\}_{t \in [0,T]}\). For any such deviation, we define the exponential martingale:

\[
\xi_t = \exp\left( -\frac{1}{2} \int_0^t \eta_s^2 ds + \int_0^t \eta_s dB_s \right)
\]

\[
\eta_t = \frac{1}{\sigma} (\Delta a_t + \Delta m_t - \theta \Delta M_t).
\]

By Girsanov’s theorem, the Radon-Nikodym derivative between \(P^\Delta\) and \(P\) is given by \(dP^\Delta/dP = \xi_{T+\tau}\). Using the fact that \(E(\xi_{T+\tau} | \mathcal{F}_t) = \xi_t\) and the law of iterated expectations, we can write the CEO’s expected payoff given a deviation as

\[
U(\hat{c}, \hat{a}, \hat{m}) = \mathbb{E}^{(a, m)} \left[ \int_0^{T+\tau} e^{-r\xi_t} u_t^\Delta dt + e^{-r(T+\tau)} \xi_{T+\tau} \frac{u_{T+\tau}^\Delta}{r} \right].
\]

The change of variables in equation (18) is useful because it allows to fix the expectation operator by introducing the new state variable \(\xi_t\) in the CEO’s optimization problem. Under this change of variables the agent problem is a stochastic control problem with random coefficients (Williams (2011)).

Notice that, without loss of generality, we can take the recommendation \((c_t, a_t, m_t)\) as a reference point and consider the optimization with respect to \((\Delta c_t, \Delta a_t, \Delta m_t)\). A contract is
incentive compatible if and only if $\Delta c = \Delta a = \Delta m = 0$ is the CEO’s optimal choice. Using equation (18), we can write the CEO’s problem as

\[
\begin{align*}
\sup_{\Delta c, \Delta a, \Delta m} U(c + \Delta c, a + \Delta a, m + \Delta m) \\
\text{subject to} \\
d\xi_t &= -\frac{\xi_t}{\sigma}(\Delta a_t + \Delta m_t - \theta \Delta M_t)dB_t \\
d\Delta M_t &= (\Delta m_t - \kappa \Delta M_t)dt \\
dS_t &= (rS_t - \Delta c_t)dt.
\end{align*}
\] (19)

As mentioned above, this is a stochastic control problem which can be analyzed using the stochastic maximum principle – a generalization of Pontryagin’s maximum principle to stochastic control problems (Yong and Zhou, 1999).

We begin by defining the (current value) Hamiltonian function $\mathcal{H}$ as follows:

\[
\mathcal{H} = \xi u^\Delta + q^M(\Delta m - \kappa \Delta M) + q^S(rS - \Delta c) - \nu^\xi \frac{\xi^2}{\sigma}(\Delta a + \Delta m - \theta \Delta M),
\] (20)

There are three control variables ($\Delta a, \Delta m, \Delta c$) three state variables ($\Delta M, S, \xi$) and their associated adjoint variables ($q^M, q^S, q^\xi$). The first two state variables ($\Delta M, S$) have drift but are not (directly) sensitive to cash flow shocks. The third state variable $\xi$ is a martingale; its sensitivity to shocks is $\nu^\xi$.

We maximize the Hamiltonian with respect to the control variables. Because the Hamiltonian is jointly concave in ($\Delta c, \Delta a, \Delta m$), it suffices to consider the first order conditions evaluated at $\Delta a = \Delta m = \Delta c = 0$. This yields the following first order conditions:

\[
u^\xi \frac{\xi^2}{\sigma}
\]

\[
u^\xi
\]

\[
u^\xi
\]

\[
u^\xi
\]

along with the complementary slackness conditions for the non-negativity constraint of $m_t$. The three adjoint variables ($q^S_t, q^M_t, q^\xi_t$) follow stochastic differential equations which are the
stochastic analogue of the differential equations in optimal control theory:

\[ dq^S_t = r q^S_t dt - \frac{\partial H_t}{\partial S_t} dt + \nu^S_t dB_t = \nu^S_t dB_t \]

\[ dq^M_t = r q^M_t dt - \frac{\partial H_t}{\partial M_t} dt + \nu^M_t dB_t = \left( (r + \kappa) q^M_t + \frac{\theta \nu^\xi_t}{\sigma} \right) dt + \nu^M_t dB_t \]

\[ dq^\xi_t = r q^\xi_t dt - \frac{\partial H_t}{\partial \xi_t} dt + \nu^\xi_t dB_t = \left( r q^\xi_t - u_t \right) dt + \nu^\xi_t dB_t. \]

The adjoint equations must satisfy the transversality conditions

\[ q^\xi_T + \tau = u^T + \frac{\tau r}{\sigma}, \quad q^M_T + \tau = 0 \text{ and } q^S_T + \tau = u^c_T + \frac{\tau r}{\sigma}. \]

Theses equations are standard in stochastic control theory, but their economic meaning will become clear later.

First, we solve for \( q^\xi_t \) by integrating its SDE and using the corresponding transversality condition, which yields

\[ q^\xi_t = E_t \left[ \int_t^T e^{-r(s-t)} u_s ds + e^{-r(T-t)} \frac{u_T + \tau r}{r} \right]. \]

It’s now apparent that the adjoint variable \( q^\xi_t \) captures the evolution of the CEO’s continuation value, so we follow the standard notation in the contracting literature and denote \( W_t = q^\xi_t \). It is also convenient to write the sensitivity of the continuation value to cash flow shocks as \( \nu^\xi_t \equiv -\beta_t W_t \sigma \). The SDE of the continuation value can thus be rewritten as

\[ dW_t = (rW_t - u_t) dt - \beta_t W_t \sigma dB_t. \]

\( \beta_t \) thus captures the sensitivity of \( W_t \) to shocks \( dB_t \) (recall that \( W_t \) is negative, given the negative exponential utility). \( \beta_t \) is often referred to as the pay-performance-sensitivity.\(^{13}\)

Hereafter, we refer to \( \beta_t \) as the CEO’s short-run incentives or \( PPS \).

We arrive at equation (5a) by plugging the private savings condition (6) in the above equation. Equation (5b), on the other hand, is derived from the stochastic differential equation for the adjoint variable of manipulation, \( q^M_t \). Specifically, \( p_t \equiv q^M_t / \theta \) captures the contract’s long-term incentives and measures the incentive power of deferred compensation

\(^{13}\)In the case of contracts implementing a deterministic sequence of effort and manipulation, \( \beta_t \) is proportional to the sensitivity of consumption to cash-flows that corresponds to the traditional definition of PPS in the empirical literature adapted to our setting.
to deter manipulation.

Finally, we study the CEO’s saving strategy. Using the first order condition for consumption \( u_c = q^S \) and the stochastic differential equation for the adjoint variable \( q^S_t \), we conclude that the marginal utility of consumption follows the following stochastic differential equation:

\[
du_{ct} = \nu^S_t \sigma dB_t. \tag{22}
\]

Equation (22) is the continuous-time version of the classic Euler equation for consumption, which states that the marginal utility of consumption must be a martingale when the CEO’s discount rate is equal to the market’s interest rate, or else the CEO would save or borrow money to smooth-out his consumption path.

We find equation (22) by using a guess and verify approach. As in He et al. (2014), we conjecture that \( W_t = ru_t \), and given the CARA utility, we find that \( u_c = -\gamma u = -\gamma rW \). Replacing this relation into equation (22) and setting \( \nu^S_t = r\gamma \beta_t W_t \sigma \), we obtain that

\[
du_{ct} = -r\gamma dW_t = r\gamma \beta_t W_t \sigma dB_t. \tag{23}
\]

Dividing by \(-r\gamma\) we find that equation (23) coincides with equation (21) and verify that \( u_{ct} = -r\gamma W_t \) solves the adjoint equation for \( q^S_t \). Hence, the marginal utility of consumption \( u_c \) and the continuation utility \( W_t \) are martingales. This result, due to He et al. (2014), combines two observations. First, the CEO can smooth consumption inter-temporally, so his marginal utility of consumption is a martingale. Second, under CARA preferences, the CEO’s continuation value \( W_t \) is linear in flow utility \( u_t \).

The first order conditions along with the private savings condition, \( u = rW \), yield the necessary incentive conditions for effort and manipulation stated in Proposition 1:

\begin{align*}
r\gamma h'(a_t) &= \beta_t \quad \tag{24a} \\
r\gamma g'(m_t) &= \beta_t + \theta \frac{p_t}{W_t}. \quad \tag{24b}
\end{align*}

The complementary slackness condition equation (5a) is obtained by replacing \( u_t = rW_t \) into equation (21) and defining \( \nu^M_t \equiv \sigma_{pt} W_t \sigma \).
B Sufficiency Agent Incentive Compatibility

To prove sufficiency, we follow Sannikov (2014) and He et al. (2014) in the construction of an upper bound for the payoff after a deviation. In particular, we construct an upper bound of the form

$$\hat{W}_t = W_t e^{-r\gamma(S_t + L_1 \Delta M_t^2) + \theta z_t \Delta M_t}.$$ 

Let $P^\Delta$ be the measure induced by a deviation $(\Delta a_t, \Delta m_t, \Delta c_t)_{t \in [0,T]}$ and let

$$B_t^\Delta \equiv B_t - \int_0^t \frac{(\Delta a_s + \Delta m_s - \theta \Delta M_s)}{\sigma} ds.$$ 

By Girsanov’s theorem, $B_t^\Delta$ is a Brownian motion under $P^\Delta$, which means that under $P^\Delta$

$$dW_t = -\beta_t W_t (\Delta a_t + \Delta m_t - \theta \Delta M_t) dt - \beta_t W_t \sigma dB_t^\Delta$$
$$dz_t = [(r + \kappa) z_t - \beta_t (\sigma \sigma_t - 1) + \frac{2\sigma_t}{\sigma} (\Delta a_t + \Delta m_t - \theta \Delta M_t)] dt + \sigma_t dB_t^\Delta$$
$$W_{T+\tau} = u^{\phi(CT+\tau)}$$
$$z_{T+\tau} = 0$$

Using Ito’s Lemma we find that

$$d\hat{W}_t = \hat{W}_t (\mu_t^\hat{W} dt + \sigma_t^\hat{W} dB_t^\Delta)$$

where

$$\mu_t^\hat{W} = -r\gamma(rS_t - \Delta c_t) + 2r\gamma\kappa L_1 \Delta M_t^2 - r\gamma(-\phi z_t + 2L_1 \Delta M_t) \Delta m_t + \frac{1}{2} r^2 \gamma^2 \phi^2 \Delta M_t^2 \sigma_{zt}^2$$
$$+ r\gamma \phi \Delta M_t [r z_t + \beta_t (\sigma \sigma_t - 1)] + r\gamma \phi \Delta M_t \frac{\sigma_{zt}}{\sigma} (\Delta a_t + \Delta m_t - \theta \Delta M_t)$$
$$- r\gamma a_t (\Delta a_t + \Delta m_t - \theta \Delta M_t) - r\gamma \phi \sigma \beta_t \Delta M_t \sigma_{zt}.$$ 

Let $W_t^\Delta$ be the expected payoff given the deviation $(\Delta a_t, \Delta m_t, \Delta c_t)_{t \in [0,T+\tau]}$. For any fixed $t_0$ and $t \in [t_0, T + \tau]$, define the process

$$G_{t_0,t} \equiv \int_{t_0}^t e^{-r(s-t_0)} u_s^\Delta ds + e^{-r(t-t_0)} \hat{W}_t,$$
where

\[ u^\Delta_t \equiv u(c_t + \Delta c_t, (a_t + \Delta a_t) 1_{t \leq T}, (m_t + \Delta m_t) 1_{t \leq T}), \]

and notice that by definition \( E[G_{t0,T+\tau}] = W^\Delta_t \). Differentiating \( G_{t0,t} \) we get

\[
e^{rt} dG_{t0,t} = u^\Delta_t - r \hat{W}_t dt - r \gamma \mu_t \hat{W}_t dt + \sigma_t \hat{W}_t dB_t.
\]

Using the first order condition for the agent’s consumption, \( r W_t = u_t \equiv u(c_t, a_t 1_{t < T}, m_t 1_{t < T}) \), we can write the previous expression as

\[
e^{rt} dG_{t0,t} = r \hat{W}_t \left[ \left( \frac{u^\Delta_t}{u_t} e^{\gamma (S_t + L_1 \Delta M_t^2) - \theta \Delta z_t M_t} - 1 \right) dt - \gamma \mu_t \hat{W}_t \right] + \sigma_t \hat{W}_t dB_t^\Delta.
\]

Given that \( \hat{W}_t < 0 \), we need to show that

\[ Q_t \equiv \frac{u^\Delta_t}{u_t} e^{\gamma (S_t + L_1 \Delta M_t^2) - \theta \Delta z_t M_t} - 1 + \frac{\mu_t \hat{W}_t}{r} \geq 0 \]

Using the inequality \( e^x \geq 1 + x \), we find that that

\[
e^{\gamma (S_t + L_1 \Delta M_t^2) - \theta \Delta z_t M_t} \gamma \left( r (S_t + L_1 \Delta M_t^2 - \theta z_t \Delta M_t - \left( \Delta c_t - a_t \Delta a_t - \frac{\Delta a_t^2}{2} - gm_t \Delta m_t - g \frac{\Delta m_t^2}{2} \right) - 1 \right.
\]

\[ \geq \gamma \left[ r(S_t + L_1 \Delta M_t^2) - \theta z_t \Delta M_t - \left( \Delta c_t - a_t \Delta a_t - \frac{\Delta a_t^2}{2} - gm_t \Delta m_t - g \frac{\Delta m_t^2}{2} \right) \right].\]
Replacing $\mu_t \hat{W}$ we get

$$
\frac{Q_t}{\gamma} = \left( (r + 2\kappa) L_1 - \phi \frac{\sigma_{zt}}{\sigma} \right) \Delta M_t^2 + \left( \frac{\Delta a_t^2}{2} + g m_t \Delta m_t + g \frac{\Delta m_t^2}{2} \right) + (\phi z_t - 2L_1 \Delta M_t) \Delta m_t + \frac{1}{2} r \gamma \phi^2 \Delta M_t^2 \sigma_{zt}^2 + \theta a_t \Delta M_t (\sigma_{zt} - 1) + \phi \Delta M_t \frac{\sigma_{zt}}{\sigma} (\Delta a_t + \Delta m_t) - a_t (\Delta M_t - \theta \Delta M_t) - \theta \sigma a_t \Delta M_t \sigma_{zt}
$$

$$
= \left( (r + 2\kappa) L_1 - \phi \frac{\sigma_{zt}}{\sigma} + \frac{1}{2} r \gamma \phi^2 \sigma_{zt}^2 \right) \Delta M_t^2 + \frac{\Delta a_t^2}{2} + g \frac{\Delta m_t^2}{2} + (gm_t - (a_t - \phi z_t)) \Delta m_t - 2L_1 \Delta M_t \Delta m_t + \theta a_t \sigma \Delta M_t \sigma_{zt} + \phi \Delta M_t \frac{\sigma_{zt}}{\sigma} (\Delta a_t + \Delta m_t) - \theta \sigma a_t \Delta M_t \sigma_{zt}
$$

$$
\geq \left( (r + 2\kappa) L_1 - \phi \frac{\sigma_{zt}}{\sigma} + \frac{1}{2} r \gamma \phi^2 \sigma_{zt}^2 \right) \Delta M_t^2 + \frac{\Delta a_t^2}{2} + g \frac{\Delta m_t^2}{2} + \left( \phi \frac{\sigma_{zt}}{\sigma} - 2L_1 \right) \Delta m_t \Delta M_t + \phi \frac{\sigma_{zt}}{\sigma} \Delta a_t \Delta M_t,
$$

where in last line we have used that $(gm_t - (a - \phi z_t)) \Delta m_t \geq 0$. Completing squares

$$
\frac{Q_t}{\gamma} \geq \left[ (r + 2\kappa) L_1 - \phi \frac{\sigma_{zt}}{\sigma} + \frac{1}{2} r \gamma \phi^2 \sigma_{zt}^2 - \frac{\phi^2 \sigma_{zt}^2}{2 \sigma^2} - \frac{1}{2g} \left( \phi \frac{\sigma_{zt}}{\sigma} - 2L_1 \right)^2 \right] \Delta M_t^2 + \frac{1}{2} \left( \Delta a_t + \phi \frac{\sigma_{zt}}{\sigma} \Delta M_t \right)^2 + \frac{g}{2} \left( \Delta m_t + \frac{1}{g} \left( \phi \frac{\sigma_{zt}}{\sigma} - 2L_1 \right) \Delta M_t \right)^2
$$

Hence, if

$$
(r + 2\kappa) L_1 - \phi \frac{\sigma_{zt}}{\sigma} + \frac{1}{2} r \gamma \phi^2 \sigma_{zt}^2 - \frac{\phi^2 \sigma_{zt}^2}{2 \sigma^2} - \frac{1}{2g} \left( \phi \frac{\sigma_{zt}}{\sigma} - 2L_1 \right)^2 \geq 0,
$$

then $G_{t_0,t}$ has a negative drift so

$$
W^\Delta_{t_0} = E_{t_0}^\Delta [G_{t_0,T+r}] \leq G_{t_0,t_0} = \hat{W}_{t_0}.
$$

Accordingly, if inequality (25) is satisfied, the optimality of $\Delta a_t = 0$, $\Delta m_t = 0$, $\Delta c_t = 0$ follows directly from the fact that $W_0 = \hat{W}_0 \geq W^\Delta_0$, and for any $t$ such that $\Delta a_s = 0$, $\Delta m_s = 0$, $\Delta c_s = 0$ for all $s \leq t$ we have $W_t = \hat{W}_t \geq W^\Delta_t$.

Next, we find conditions such that (25) holds. We can write the LHS of equation (25) as
a function of $L_1$:

$$H(L_1) \equiv (r + 2\kappa) L_1 - \phi \left( \theta - \frac{L_1}{g} \right) \frac{\sigma_{zt}}{\sigma} + \left( \frac{r\gamma}{2} - \frac{1}{2\sigma^2} - \frac{1}{2\sigma^2 g} \right) \phi^2 \sigma_{zt}^2 - \frac{2L_1^2}{g}.$$ 

First, if $r\gamma\sigma^2 - 1 > 0$ and

$$g \geq \frac{1}{r\gamma\sigma^2 - 1},$$

then, because $\sigma_{zt} \leq 0$, we can simply consider $L_1 = 0$ and get

$$H(0) = -\phi \theta \frac{\sigma_{zt}}{\sigma} + \left( r\gamma - \frac{1}{\sigma^2} - \frac{1}{\sigma^2 g} \right) \frac{\phi^2 \sigma_{zt}^2}{2} \geq 0.$$ 

Second, if $r\gamma\sigma^2 - 1 > 0$ or (26) does not hold, then we can take $L_1$ to maximize

$$(r + 2\kappa) L_1 - \frac{2L_1^2}{g},$$

which yields

$$L_1 = \frac{(r + 2\kappa) g}{4}.$$ 

After replacing in $H(L_1)$ we get

$$H(L_1) = \frac{(r + 2\kappa) g}{8} - \frac{\phi}{4} \left( 4\theta - r - 2\kappa \right) \frac{\sigma_{zt}}{\sigma} + \left( \frac{r\gamma}{2} - \frac{1}{2\sigma^2} - \frac{1}{2\sigma^2 g} \right) \phi^2 \sigma_{zt}^2.$$ 

Because $\sigma_{zt} \leq 0$, we only need to consider the negative root, so there is $L_v > 0$ such that if $\sigma_{zt} \geq -L_v$ then $H(L_1) \geq 0$, where $L_v$ is the absolute value of the negative root and is given by

$$L_v = \frac{r\gamma \sigma \left( 4\theta - r - 2\kappa \right) + \sqrt{(4\theta - r - 2\kappa)^2 - 4(r + 2\kappa)^2 (r\gamma \sigma^2 - g - 1)}}{1 + g^{-1} - r\gamma \sigma^2}.$$
C Principal Problem

Proof Lemma 1

Proof. Plugging equation (7) in the objective function yields the principal’s expected payoff as a function of $a_t$, $m_t$, and $W_t$ alone:

$$E \left[ \int_0^T e^{-rt} (a_t - \lambda m_t - h(a_t) - g(m_t)) dt + \int_0^{T+\tau} e^{-rt} \left( \frac{\log(-W_t)}{\gamma} + \frac{\log(r \gamma)}{\gamma} \right) dt 
+ e^{-r(T+\tau)} \frac{\log(-r \gamma W_{T+\tau})}{r \gamma} \right].$$ (27)

Using Itô’s Lemma, we compute the expected value of $\log(-W_t)$, which is given by

$$E[\log(-W_t)] = \log(-W_0) - \frac{1}{2} E \left[ \int_0^t \sigma^2 \beta_s^2 ds \right].$$

If we change the order of integration we obtain that

$$E \left[ \int_0^{T+\tau} e^{-rt} \int_0^t \sigma^2 \beta_s^2 ds dt \right] = E \left[ \int_0^{T+\tau} (e^{-rt} - e^{-r(T+\tau)}) \frac{\sigma^2 \beta_t^2}{r} dt \right].$$

Replacing this expression in equation (27), and ignoring constant terms, we can write the principal’s expected payoff as:

$$E \left[ \int_0^T e^{-rt} (a_t - \lambda m_t - h(a_t) - g(m_t)) dt - \int_0^{T+\tau} e^{-rt} \frac{\sigma^2 \beta_t^2}{2r \gamma} dt \right].$$ (28)

The first term inside brackets captures the cash flow realized throughout the CEO’s tenure. The second term captures the monetary impact of the compensation risk borne by the manager till the end of the clawback period, at time $T + \tau$.

After removing the dependence of the principal’s payoff on the manager’s continuation utility $W_t$, we proceed to remove such dependence from the incentives constraints as well. As mentioned above we shall use $z_t \equiv -p_t/W_t$ as state variable. Hereafter, we refer to $z$ as the contract’s long-term incentives.

Using the law of motion of $W_t$ and $p_t$ in (5a) and (5b), along with Ito’s lemma, we find
that the law of motion of $z$ follows the following stochastic differential equation:

$$dz_t = [(r + \kappa)z_t + \beta_t(\sigma z - 1)]dt + \sigma z dB_t,$$

where $\sigma z \equiv \sigma(\beta z - \sigma_p)$. Also, the incentive compatibility constraint is defined by $g'(m_t) = h'(a_t) - \phi z_t = (a_t - \phi z_t)/g$. We have thus reduced the optimal contract to a finite horizon one dimensional stochastic control problem that can be written as follows:

$$F(z) = \sup_{a,m,\beta,v} E \left[ \int_0^T e^{-rt} (a_t - \lambda m_t - h(a_t) - g(m_t)) dt - \int_0^{T+\tau} e^{-rt} \frac{\sigma^2 \beta^2}{2r\gamma} dt \right]$$

subject to the law of motion of $z_t$ in (9) and the manager’s participation and incentive constraints (equations (4a) and (4b)).

\[\square\]

C.1 Optimal Contract

Maximization HJB Equation

We have the following HJB equation:

$$rF = \max_{a,m,\beta,v} \pi(a,m) + F_t + [(r + \kappa)z + ar\gamma(\sigma v - 1)]F_z + \frac{1}{2}\sigma^2 z F_{zz},$$

subject to

$$m \geq \frac{a - \phi z}{g}$$

$$m \geq 0.$$ 

with boundary conditions

$$F(z,T) = -\frac{1}{2}C z^2$$

$$F(0,t) = 0.$$
The Lagrangean for the optimization problem is

\[
L = a - \lambda m - \frac{gm^2}{2} - \frac{(1 + r\gamma\sigma^2)a^2}{2} + [(r + \kappa)z + ar\gamma(\sigma\sigma_z - 1)]F_z + \frac{1}{2}\sigma_z^2F_{zz} \\
+ \eta \left( m - \frac{a - \phi z}{g} \right) + \nu m.
\]

The first order condition with respect to \(a\) and \(m\) is

\[
1 - (1 + r\gamma\sigma^2)a - \frac{\eta}{g} + r\gamma(\sigma\sigma_z - 1)F_z = 0 \\
-\lambda - gm + \eta + \nu = 0 \\
ar\gamma\sigma F_z + \sigma_z F_{zz} = 0,
\]

where \(\nu = 0\) if \(m > 0\) and \(\eta = 0\) if \(m > (a - \phi z)/g\) (which means that \(m = 0\)). The volatility of the continuation value is

\[
v = -\frac{ar\gamma\sigma F_z}{F_{zz}}
\]

Suppose that \(m > 0\). If this is the case, we have that \(\nu = 0\) and

\[
\eta = \lambda + gm = \lambda + a - \phi z.
\]

Replacing in the first order condition for effort, we get

\[
g - \left( 1 + g \left( 1 + r\gamma\sigma^2 + \frac{r^2\gamma^2\sigma^2 F_z^2}{F_{zz}} \right) \right) a - \lambda + \phi z - r\gamma g F_z = 0,
\]

hence, we get

\[
a = \frac{g - \lambda + \phi z - r\gamma g F_z}{1 + g \left( 1 + r\gamma\sigma^2 + \frac{r^2\gamma^2\sigma^2 F_z^2}{F_{zz}} \right)}
\]

\[
m = \frac{a - \phi z}{g}
\]

\[
\sigma_z = -\frac{ar\gamma\sigma F_z}{F_{zz}}
\]

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This solution satisfies the constraints if and only if
\[
\frac{g - \lambda - r\gamma g F_z}{g} \geq \phi z \left(1 + r\gamma \sigma^2 + \frac{r^2\gamma^2 \sigma^2 F^2_z}{F_{zz}}\right).
\]

If this condition is violated, it must be the case that \( m_t = 0 \). Suppose that \( \eta = 0 \), which means that the constraint \( m \geq (a - \phi z)/g \) is slack. In this case, we find that
\[
a = \frac{1 - r\gamma F_z}{1 + r\gamma \sigma^2 + \frac{r^2\gamma^2 \sigma^2 F^2_z}{F_{zz}}}.
\]

Replacing in the constraints, we find that \( m \geq (a - \phi z)/g \) is slack if and only if
\[
1 - r\gamma g F_z < \phi z \left(1 + r\gamma \sigma^2 + \frac{r^2\gamma^2 \sigma^2 F^2_z}{F_{zz}}\right).
\]

Finally, if the two conditions above are violated, that is if
\[
1 - \frac{\lambda + r\gamma g F_z}{g} < \phi z \left(1 + r\gamma \sigma^2 + \frac{r^2\gamma^2 \sigma^2 F^2_z}{F_{zz}}\right) < 1 - r\gamma g F_z,
\]
then it must be the case that \( m_t = 0 \) and \( a_t = \phi z_t \). We find then that the solution to the maximization problem in the HJB equation is
\[
a(z, t) = \begin{cases} 
\frac{g - \lambda + \phi z - r\gamma g F_z}{1 + g H(z, t)} & \text{if } 1 - \frac{\lambda + r\gamma g F_z}{g} \geq \phi z H(z, t) \\
\frac{1 - r\gamma F_z}{H(z, t)} & \text{if } 1 - r\gamma F_z < \phi z H(z, t) \\
\phi z & \text{otherwise}
\end{cases}
\]
\[
m(z, t) = \frac{1}{g} (a(z, t) - \phi z)^+
\]
\[
\sigma_z(z, t) = -a(z, t) r\gamma \sigma \frac{F_z}{F_{zz}}.
\]

where
\[
H(z, t) = 1 + r\gamma \sigma^2 + \frac{r^2\gamma^2 \sigma^2 F^2_z}{F_{zz}}
\]

Finally, we need to show that the second order conditions are satisfied. Let’s consider the function
\[ G(a, m, \sigma_z) \equiv a - \lambda m - \frac{gm^2}{2} - \frac{(1 + r\gamma\sigma^2)a^2}{2} + ar\gamma(\sigma\sigma_z - 1)F_z + \frac{1}{2}\sigma_zF_{zz} \]

If we maximize with respect to \( \sigma_z \) we get that
\[
\sigma_z = -r\gamma\sigma a F_z F_{zz}
\]

The second order condition for this maximization is that \( F_{zz} < 0 \). Replacing in \( G \) we get the following optimization problem for \( a \) and \( m \)

\[
\max_{a,m} a - \lambda m - \frac{gm^2}{2} - \frac{(1 + r\gamma\sigma^2)a^2}{2} - r\gamma a F_z + \frac{1}{2}r^2\gamma^2\sigma^2 a^2 F_z^2 F_{zz}
\]

subject to
\[ gm - a + \phi z \geq 0 \]

Because, \( z_t \geq \bar{z}(t) \) (so \( F_z(z_t, t) \leq 0 \)), we can restrict attention to \( z \) such that \( F_z(z_t, t) \leq 0 \) when we check the second order condition. We consider three cases:

1. \( H(z, t) > 0 \)
2. \( H(z, t) < 0 \) and \( 1 + gh(z, t) > 0 \)
3. \( 1 + gh(z, t) < 0 \).

Let’s consider the case in which the IC constraint for \( m \) holds with equality. In this case, we can consider the bordered Hessian which is given by

\[
\begin{bmatrix}
0 & g & -1 \\
g & -g & 0 \\
-1 & 0 & -(1 + r\gamma\sigma^2) - r^2\gamma^2\sigma^2 F_z^2 F_{zz}
\end{bmatrix}
\]

Thus, the second order condition for the maximization problem for \( a, m \) is

\[
\begin{vmatrix}
0 & g & -1 \\
g & -g & 0 \\
-1 & 0 & -(1 + r\gamma\sigma^2) - r^2\gamma^2\sigma^2 F_z^2 F_{zz}
\end{vmatrix} > 0.
\]

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which corresponds to the condition

\[ 1 + g \left( 1 + r\gamma\sigma^2 + r^2\gamma^2\sigma^2 \frac{F_z^2}{F_{zz}} \right) = 1 + g H(z,t) > 0. \]

If this condition is not satisfied, then the solution to the optimization problem is an extreme point. We can notice this considering the optimization problem

\[ \max_a a - \frac{\lambda}{g} (a - \phi_z)^+ - \frac{[(a - \phi_z)^+]^2}{2g} - \frac{(1 + r\gamma\sigma^2) a^2}{2} - r\gamma a F_z - \frac{1}{2} r^2\gamma^2\sigma^2 a^2 \frac{F_z^2}{F_{zz}} \] \hspace{1cm} (30)

The optimal effort is an extreme point when \( 1 + g H(z,t) < 0 \), which means that there are two candidate for the solution \( \phi_z \) and \( \bar{a} \). Suppose that \( \phi_z < \bar{a} \). In this case we have that

\[ \bar{a} - \frac{\lambda}{g} (\bar{a} - \phi_z) - \frac{(\bar{a} - \phi_z)^2}{2g} - \frac{(1 + r\gamma\sigma^2) \bar{a}^2}{2} - r\gamma \bar{a} F_z - \frac{1}{2} r^2\gamma^2\sigma^2 \bar{a}^2 \frac{F_z^2}{F_{zz}} \geq \]

\[ \phi_z - \frac{(1 + r\gamma\sigma^2) (\phi_z)^2}{2} - r\gamma \phi_z F_z - \frac{1}{2} r^2\gamma^2\sigma^2 (\phi_z)^2 \frac{F_z^2}{F_{zz}} \]

or

\[ g - \lambda - r\gamma g F_z = \frac{1}{2} \left( 1 + g (1 + r\gamma\sigma^2) + gr^2\gamma^2\sigma^2 \frac{F_z^2}{F_{zz}} \right) (\bar{a} + \phi_z) > 0 \]

This condition is satisfied if \( g > \lambda \) and \( 1 + g H(z,t) < 0 \).

Finally, we consider the situation in which the constraint for \( m_t \) is slack. In this case, the second order condition for \( a \) in (30) is \( 1 + r\gamma\sigma^2 + r^2\gamma^2\sigma^2 \frac{F_z^2}{F_{zz}} = H(z,t) > 0 \) (which immediately implies that \( 1 + g H(z,t) > 0 \)). If this condition is not satisfied, then it must be the case that \( a \in [\phi_z, \bar{a}] \) because the solution can not be an interior point of \([0, \phi_z]\) and in this case

\[ 1 - r\gamma F_z - \frac{\phi_z}{2} H(z,t) > 0 \]

so the value of \( a = \phi_z \) is higher than the value of \( a = 0 \).
Internet Appendix

A Optimal Contract

Stochastic Representation $\psi_t$

First, we obtain the stochastic representation using the HJB equation together with the envelope theorem. This derivation is intuitive but requires that the value function is sufficiently smooth (which is not guaranteed in our setting). Later we provide an alternative proof that relies in the stochastic maximum principle and does not require the smoothness assumption (Yong and Zhou, 1999; Pham, 2009).

With some abuse of notation, let $F_z(t) \equiv F_z(z_t, t) = \psi_t$, $F_{tz}(t) \equiv F_{tz}(z_t, t)$, $F_{zz}(t) \equiv F_{zz}(z_t, t)$ and $F_{zzz}(t) \equiv F_{zzz}(z_t, t)$. Applying Ito’s Lemma we get

$$dF_z(t) = \left[ F_{zt}(t) + (r + \kappa)z_t + r\gamma a_t(\sigma \sigma_z - 1) \right] dt + \sigma z_t F_{zz}(t) dB_t$$

Applying the Envelope theorem to the HJB equation we get

$$rF_z = \pi_z(a, z) + F_{zt} + (r + \kappa)F_z + [(r + \kappa)z + r\gamma a(\sigma \sigma_z - 1)]F_{zz} + \frac{1}{2}\sigma_z^2 F_{zzz}$$

Replacing in the stochastic differential equation for $F_z$ we get

$$dF_z(t) = -[\kappa F_z(t) - \pi_z(a_t, z_t)] dt + \sigma z_t F_{zz}(t) dB_t$$

Finally, we arrive to the desired expression after replacing the first order condition for $\sigma_{zt}$ we get

$$dF_z(t) = -[\kappa F_z(t) - \pi_z(a_t, z_t)] dt - r\gamma \sigma a_t F_z(t) dB_t$$

The terminal condition is obtained by differentiating the terminal condition in the HJB equation.

The only step left is to solve the SDE for $\psi_t$ and show that $\psi_t \leq 0$. Let’s define the
following processes:

\[ Y_t = e^{\kappa t} \psi_t \]
\[ H_t = \int_0^t (-\phi m_s) ds \]
\[ U_t = \int_0^t (-r \gamma a_s) dB_s \]
\[ \mathcal{E}(U)_t = \exp \left\{ U_t - \frac{1}{2} [U, U]_t \right\}. \]

where \([U, U]_t\) is the quadratic variation of the martingale \(U\). Applying Theorem V.52 in Protter (1990) we get that

\[ Y_t = \int_0^t \mathcal{E}(U)_t (\phi m_s) ds. \]

The result follows from replacing the previous variables and using the fact that \([H, U]_t = 0\) and

\[ [U, U]_t = r^2 \gamma^2 \sigma^2 \int_0^t a_s^2 ds \]

**Stochastic Maximum Principle** As we already mentioned, we can also derive the representation for \(\psi_t\) using the stochastic maximum principle. The Hamiltonian for the stochastic control problem is

\[ \mathcal{H} = \psi[(r + \kappa)z + ar \gamma (\sigma \sigma_z - 1)] + \sigma z \sigma \psi + \pi(a, z) \]

where the co-state variable solves

\[ d\psi_t = -(\kappa \psi_t + \pi_z(a, z)) dt + \sigma_{\psi,t} dB_t, \quad \psi_T = -C z_T \]

Moreover, given that the initial state \(z_0\) is free, the co-state variable also satisfies the initial condition \(\psi_0 = 0\). A necessary condition of optimality is that \(\sigma_{zt}\) maximizes \(\mathcal{H}\), which yields \(\sigma_{zt} = -r \gamma \sigma a_t\) so

\[ d\psi_t = -(\kappa \psi_t + \pi_z(a_t, z_t)) dt - r \gamma \sigma a_t dB_t, \quad \psi_T = -C z_T. \]
The representation follows from the relation between the co-state variable and the derivative (more generally the superdifferential) in Yong and Zhou (1999). Note that this condition is necessary and not sufficient; however, for the purpose of the representation for \( \psi_t \) we only need its necessity as we are not using the stochastic maximum principle to solve for \( a_t \) but only using it to describe some qualitative aspects that an optimal contract must satisfy.

**Proof of Proposition 3**

The subsequent analysis relies on the notion of viscosity solution for parabolic PDEs. For completeness, we start introducing the notion of viscosity solution: A standard reference for the theory of viscosity solutions is Crandall, Ishii, and Lions (1992); Fleming and Soner (2006) presents the theory in the context of stochastic control of diffusion processes, while Katzourakis (2014) provides a more accessible introduction.

**Viscosity Solutions:** Let \( G(\psi, z, p, Q) \) be a continuous function satisfying

\[
G(\psi, z, p, Q') \leq G(\psi, z, p, Q'')
\]

whenever \( Q' \leq Q'' \). If \( G \) satisfies (31) we say that it is *degenerate elliptic* (Crandall et al., 1992). Consider the partial differential equation (PDE)

\[
-Z_t + G(\psi, Z, Z_\psi, Z_{\psi\psi}) = 0
\]

(32)

**Definition 1.** Let \( Z : \mathbb{R} \times (0, T) \to \mathbb{R} \) be a locally bounded continuous function.

1. \( Z \) is a viscosity subsolution of (32) on \( \mathbb{R} \times (0, T) \) if

\[
-Z_t(\bar{\psi}, \bar{t}) + G(\bar{\psi}, Z(\bar{\psi}, \bar{t}), \varphi_\psi(\bar{\psi}, \bar{t}), \varphi_{\psi\psi}(\bar{\psi}, \bar{t})) \leq 0,
\]

for all \( (\bar{\psi}, \bar{t}) \in \mathbb{R} \times (0, T) \) and for all \( \varphi \in C^{2,1}(\mathbb{R} \times (0, T)) \) such that \( Z(\bar{\psi}, \bar{t}) = \varphi(\bar{\psi}, \bar{t}) \) and \( (\bar{\psi}, \bar{t}) \) is a maximum of \( Z - \varphi \).

2. \( Z \) is a viscosity supersolution of (32) on \( \mathbb{R} \times (0, T) \) if

\[
-Z_t(\bar{\psi}, \bar{t}) + G(\bar{\psi}, Z(\bar{\psi}, \bar{t}), \varphi_\psi(\bar{\psi}, \bar{t}), \varphi_{\psi\psi}(\bar{\psi}, \bar{t})) \geq 0,
\]
for all $(\bar{\psi}, \bar{t}) \in \mathbb{R} \times (0, T)$ and for all $\varphi \in C^{2,1}(\mathbb{R} \times (0, T))$ such that $Z(\bar{\psi}, \bar{t}) = \varphi(\bar{\psi}, \bar{t})$ and $(\bar{\psi}, \bar{t})$ is a minimum of $Z - \varphi$.

3. We say that $Z$ is a viscosity solution of (32) on $\mathbb{R} \times (0, T)$ if it is both a subsolution and a supersolution of (32).

**Proof:** The strategy is to analyze the function $Z(\psi, t)$ defined as the inverse $\psi = F_z(Z(\psi, t), t)$ and note that the boundary $\bar{z}(t)$ corresponds by definition to the curve $Z(0, t)$; and finally, we use the fact that $\bar{z}(t) \geq 0$ to conclude that $a_t > 0$ so $\sigma z_t \leq 0$.

Before deriving a PDE for $Z$, we introduce some definitions.

\[
\Sigma_z(z, F_z, F_{zz}) = -r\gamma\sigma A(z, F_z, F_{zz}) \frac{F_z}{F_{zz}}
\]

Let

\[
A^*(\psi, z, F_{zz}) = \begin{cases}
\frac{g + \phi z - r\gamma \psi}{1 + gH(\psi, F_{zz})} & \text{if } 1 - r\gamma \psi \geq \phi z H(\psi, F_{zz}) \\
\frac{1 - r\gamma \psi}{H(\psi, F_{zz})} & \text{if } 1 - r\gamma \psi < \phi z H(\psi, F_{zz})
\end{cases}
\]

and

\[
A(z, F_z, F_{zz}) = \begin{cases}
\bar{a} & \text{if } A^*(z, F_z, F_{zz}) > \bar{a} \\
A^*(z, F_z, F_{zz}) & \text{if } A^*(z, F_z, F_{zz}) \in [0, \bar{a}] \\
0 & \text{if } A^*(z, F_z, F_{zz}) < 0
\end{cases}
\]

If the objective function fails to be concave, that is if \(1 + g(1 + r\gamma \sigma^2 + r^2 \gamma^2 \sigma^2 (F_{zz})^{-1} F_z^2) < 0\), then the solution belongs to the extreme set \(\{0, \bar{a}\}\), and it can be verified that the solution is $\bar{a}$. If we denote $F_{zz}(t) \equiv F_{zz}(z_t, t)$, then we can write the FBSDE for $(\psi_t, z_t)$ as

\[
\begin{align*}
d\psi_t &= -\left(\kappa \psi_t + \frac{\phi}{g}(A(z, \psi_t, F_{zz}(t)) - \phi z_t)^+\right) dt - r\gamma\sigma A(z_t, \psi_t, F_{zz}(t)) \psi_t dB_t \\
\psi_0 &= 0 \\
dz_t &= [(r + \kappa)z_t + r\gamma A(z_t, \psi_t, F_{zz}(t))(\sigma \Sigma_z(z_t, \psi_t, F_{zz}(t)) - 1)] dt + \Sigma_z(z_t, \psi_t, F_{zz}(t)) dB_t \\
\psi_T &= -C z_T
\end{align*}
\]

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Next, let $Z(\psi, t)$ be the inverse $\psi = F_z(\psi, t)$ so

$$Z_\psi = \frac{1}{F_{zz}(\psi, t)} < 0$$

We can write the optimal policy as a function of $Z$

$$\Sigma_z(\psi, Z_\psi) = -r\gamma \sigma A(\psi, Z_\psi) Z_\psi$$

$$A^*(\psi, Z, Z_\psi) = \begin{cases} \frac{\phi Z + g(1 - r\gamma \psi)}{1 + H(\psi, Z_\psi)} & \text{if } 1 - r\gamma \psi \geq \phi z H(\psi, Z_\psi) \\ \frac{1 - r\gamma \psi}{H(\psi, Z_\psi)} & \text{if } 1 - r\gamma \psi < \phi z H(\psi, Z_\psi) \end{cases}$$

Using Ito’s lemma, we find that the drift of $Z(\psi, t)$ is

$$Z_t = \left(\kappa \psi_t + \frac{\phi}{g} (A(z, \psi_t, Z_\psi) - \phi Z)\right) Z_\psi + \frac{1}{2} r^2 \gamma^2 A(z_t, \psi_t, Z_\psi(t))^2 \psi_t^2 Z_{\psi^2}$$

and the diffusion is

$$-r\gamma \sigma A(z, \psi_t, Z_\psi) Z_\psi \psi_t Z_\psi.$$

If we match the drift and the diffusion, above to the one in the FBSDE we get the system of equations

$$Z_t - \left(\kappa \psi_t + \frac{\phi}{g} (A(Z(\psi_t), \psi_t, Z_\psi) - \phi Z)\right) Z_\psi + \frac{1}{2} r^2 \gamma^2 A(z_t, \psi_t, Z_\psi(t))^2 \psi_t^2 Z_{\psi^2} =

\left(r + \kappa\right)Z(\psi_t) + r\gamma A(Z(\psi_t), \psi_t, Z_\psi)(\sigma \Sigma_z(\psi_t, Z_\psi) - 1)$$

where

$$\Sigma_z(\psi_t, Z_\psi) = -r\gamma \sigma A(z, \psi_t, Z_\psi) \psi_t Z_\psi.$$

Combining these two equations, we get the following quasilinear parabolic PDE

$$\begin{cases} Z_t + \frac{1}{2} r^2 \gamma^2 \sigma^2 A(\psi, Z, Z_\psi)^2 \psi^2 Z_{\psi^2} - \left(\kappa \psi + \frac{\phi}{g} m(\psi, Z, Z_\psi)\right) Z_\psi \\ -(r + \kappa)Z + r\gamma A(\psi, Z, Z_\psi)(r\gamma \sigma^2 A(\psi, Z, Z_\psi) \psi Z_\psi + 1) = 0 \quad (\psi, t) \in \mathbb{R}^- \times (0, T) \end{cases}$$

$$Z(\psi, T) = -C^{-1}\psi \quad (\psi, t) \in \mathbb{R}^- \times \{T\}$$
The previous equation is degenerated at $\psi = 0$; thus, there is no guarantee that a smooth solution exists and we need to consider viscosity solutions (Ma and Yong, 1999, Theorem 8.2.1). By definition, the free boundary $\bar{z}(t)$ is given by $Z(0,t)$; hence, it is enough to show that $Z(0,t)$ is decreasing to show that the free boundary $\bar{z}(t)$ is decreasing as well. Before proceeding with the analysis of $Z(0,t)$ we prove the following useful Lemma

**Lemma 2.** If $\lambda = 0$, then $\phi z_0 < a^{HM}$ and for any $t$ such that $\psi_t = 0$ we have that $a_t < a^{HM}$

*Proof.* We only need to prove the statement for $t = 0$. Suppose that $\phi z_0 \geq a^{HM}$, then at time zero we have that

$$a_0 = \frac{1 - r\gamma \psi_0}{H(\psi_0, F_{zz})} = \frac{1}{1 + r\gamma \sigma^2} = a^{HM}.$$ 

This means that $\sigma_{z0} = 0$ and

$$dz_0 = ((r + \kappa)z_0 - r\gamma a^{HM})dt \geq ((r + \kappa)z_0 - r\gamma \phi z_0)dt = 0,$$

where we have used the fact that $\phi \equiv \theta/r\gamma$ and $r + \kappa = \theta$. In particular, this means that $\phi z_t > a^{HM}$ in a small interval $(0, \epsilon)$. In such an interval, we have that $m_t = 0$ and accordingly $\psi_t = 0 \forall t \in (0, \epsilon)$. We can conclude that $\phi z_\epsilon > a^{HM}$, $\psi_\epsilon = 0$ and so $\sigma_{z\epsilon} = 0$. Using induction over $t$ and repeating the same argument for any arbitrary $t$ such that $\psi_t = 0$ and $\phi z_t \geq a^{HM}$, we can conclude that $\phi Z_T > a^{HM}$ and $\psi_T = 0$, which violates the transversality condition $Z_T = -C^{-1}\psi_T$.

Lemma 2 implies that $\phi Z(0,t) < a^{HM}$, and replacing in Equation (34) we conclude that

$$(r + \kappa)Z(0,t) - r\gamma A(0, Z, Z_{\psi}) < 0. \quad (36)$$

Now, we can proceed to complete the proof.

**Monotonicity of $\bar{z}(t)$:** In order to apply results of viscosity solutions, we reverse the sign in Equation (35) so the equation is degenerate elliptic as required by the theory of viscosity
solutions (Crandall et al., 1992). Defining the function

\[ G(\psi, z, p, Q) \equiv -\frac{1}{2} r^2 \gamma^2 \sigma^2 A(\psi, z, p)^2 \psi^2 Q + \left( \kappa \psi + \frac{\phi}{g} m(\psi, z, p) - r^2 \gamma^2 \sigma^2 A(\psi, z, p) \psi \right) p + (r + \kappa)z - r\gamma A(\psi, z, p), \]

we can write the PDE for \( Z \) as

\[ -Z_t + G(\psi, Z, Z_\psi, Z_{\psi\psi}) = 0. \]

Consider a time \( \hat{t} \) and smooth function \( \varphi(\psi, t) \) such that \( \varphi(0, \hat{t}) = Z(0, \hat{t}) \) and \( Z - \varphi \) has a minimum at \( (0, \hat{t}) \) [The set of points \((\psi, t)\) on which such a function exists is dense in \( \mathbb{R} \times [0, T] \), so we can assume that such a function exists at \( \hat{t} \) (Katzourakis, 2014, Lemma 2.8)\(^\text{14}\)]. Given that \( Z \) is a viscosity solution, and so a fortiori a viscosity supersolution, we get that

\[ -\varphi_t + G(0, \varphi, \varphi_\psi, \varphi_{\psi\psi}) \geq 0, \]

or replacing in \( G \)

\[ -\varphi_t + \frac{\phi}{g} m(0, \varphi, \varphi_\psi) \varphi_\psi + (r + \kappa)\varphi - r\gamma A(0, \varphi, \varphi_\psi) \geq 0. \]

Due to the concavity of \( F \), it is the case that \( Z \) is decreasing in \( \psi \) and so \( \varphi_\psi \leq 0 \). Moreover, the drift of \( z_t \) at \( \psi = 0 \) \([(r + \kappa)z_t - r\gamma a_t]\) is nonpositive (Equation (36)) so \( (r + \kappa)\varphi - r\gamma A(0, \varphi, \varphi_\psi) \leq 0 \). Hence, we get that

\[ -\varphi_t \geq -\frac{\phi}{g} m(0, \varphi, \varphi_\psi) \varphi_\psi - ((r + \kappa)\varphi - r\gamma A(0, \varphi, \varphi_\psi)) > 0. \]

This implies that \( \varphi_t < 0 \) so there is an interval \((\hat{t} - \epsilon, \hat{t})\) such that \( \varphi(0, t) \geq \varphi(0, \hat{t}) = Z(0, \hat{t}) \) for all \( t \in (\hat{t} - \epsilon, \hat{t}) \). Moreover, \( Z - \varphi \) has a local minimum at \((0, \hat{t})\) which means that \( Z(0, t) - \varphi(0, t) \geq 0 \) in \((\hat{t} - \epsilon, \hat{t})\), and so \( Z(0, t) \geq \varphi(0, t) \geq \varphi(0, \hat{t}) = Z(0, \hat{t}) \). We conclude that \( Z(0, t) \geq Z(0, \hat{t}) \) for all \( t \in (\hat{t} - \epsilon, \hat{t}) \). And, because \( \hat{t} \in [0, T] \) was arbitrary and \( Z(0, T) = 0 \), we can conclude that \( Z(0, t) \) is nonincreasing in \( t \) and that \( Z(0, t) \geq 0 \).

\(^{14}\)For any time \( \hat{t} \) in which such a test function \( \varphi \) does not exist, we can show that \( Z(0, t_n) \) is decreasing along a sequence \( t_n \to \hat{t} \), which by continuity implies that is decreasing at \( \hat{t} \).
Sign diffusion coefficient of $z_t$: Finally, we can verify that $\sigma_{zt} \leq 0$. The diffusion coefficient of $z_t$ is
\[
\Sigma_z(\psi_t, Z_\psi(\psi_t, t)) = -r\gamma A(\psi_t, Z_\psi(\psi_t, t))\psi Z_\psi(\psi_t, t),
\]
We know that $\psi_t \leq 0$, and given the hypothesis that $F(\cdot, t)$ is concave we get that $Z_\psi(\psi_t, t) \leq 0$. Hence, given that $A(\psi_t, Z_\psi(\psi_t, t)) \geq 0$ so $\Sigma_z(\psi_t, Z_\psi(\psi_t, t)) \leq 0$ (the free disposal assumption implies that $\beta_t$ is positive and so $a_t$). Moreover, because $Z$ is decreasing in $\psi$ and $Z(0, t) \geq 0$ we can conclude that $Z(\psi, t) \geq 0$ for all $\psi \leq 0$ so $A^*$ defined in (34) is positive.

Drift of $z_t$: We have that
\[
\frac{E_t (dz_t)}{dt} = (r + \kappa)z_t + r\gamma a_t(\sigma_{zt} - 1)
\]
Using the fact that $a_t \sigma_{zt} \leq 0$ we get that
\[
\frac{E_t (dz_t)}{dt} \leq (r + \kappa)z_t - r\gamma a_t
\]
Notice that when $m_t > 0$ we have that
\[
a_t = gm_t + \phi z_t
\]
which means that
\[
\frac{E_t (dz_t)}{dt} \leq (r + \kappa - \theta)z_t - r\gamma gm_t < 0,
\]
where the second inequality follows from the fact that $r + \kappa - \theta \leq 0$.

Proof of Proposition 4

Consider the family of stochastic control problems
\[
\begin{cases}
F(z_0, t|C) = \sup_{a,v} E_t \left[ \int_t^T e^{-r(s-t)} \tilde{\pi}(a_s, z_s) ds - e^{-r(T-t)} \frac{1}{2} C z_T^2 \right] \\
\text{subject to} \\
dz_s = [(r + \kappa)z_s + r\gamma a_s(\sigma\sigma_{zs} - 1)] ds + \sigma_{zs} dB_s, \ z_t = z_0
\end{cases}
\]
By the envelope theorem,
\[ \frac{\partial}{\partial C} F(z_0, t|C) = -e^{-r(T-t)} \frac{1}{2} E_t(z_T^2). \]

Let \( z_T(z_0) \) be the solution to \( z_t \) at time \( T \) given an initial condition \( z_0 \). If we take \( z_0^2 > z_0^1 \), then we get by the comparison theorem for SDEs in (Protter, 1990, Theorem V.54) that \( z_T(z_0^2) \geq z_T(z_0^1) \); hence, as \( z_T \geq 0 \), we conclude that
\[ \frac{\partial}{\partial C} (F(z_0^2, t|C) - F(z_0^1, t|C)) = -e^{-r(T-t)} \frac{1}{2} E_t \left[ (z_T(z_0^2))^2 - (z_T(z_0^1))^2 \right] \leq 0. \]
Hence, \( F(z, t|C) \) has decreasing differences in \((z, C)\) which means that \( z^*(t) = \arg \max_z F(z, t|C) \) is nonincreasing in \( C \). Moreover, noting that \( z^*(t) = \bar{z}(t) \) we find that \( \bar{z}(t) \) is nonincreasing in \( C \). The result follows as \( C \) is decreasing in \( \tau \).

## B Deterministic Contracts

### B.1 Best Deterministic No-Manipulation Contract

We start considering contracts that implement zero manipulation (Edmans, Gabaix, Sadzik, and Sannikov, 2012; Varas, forthcoming; Zhu, 2018), that is, we study the contract that induces zero manipulation \( m_t = 0 \) throughout the CEO tenure. Such a contract can be represented by the following optimization problem:

\[
\max_{z_0, a} \int_0^T e^{-rt} \left( a_t - \frac{(1 + r\gamma \sigma^2) a_t^2}{2} \right) dt - e^{-rT} C z_T^2
\]

s.t.
\[
\begin{align*}
\dot{z}_t &= (r + \kappa) z_t - r \gamma a_t \\
a_t &\leq \phi z_t.
\end{align*}
\]

We solve the problem using Pontryagin’s maximum principle for problems with mixed-state constraints. Because the objective function is concave, and the dynamics and mixed constraint are linear, the necessary conditions in the maximum principle are also sufficient. The
Hamiltonian of the optimal control problem is
\[ H(z_t, a_t, m_t, \psi_t) = \pi(a_t, m_t) + \psi_t((r + \kappa)z_t - r\gamma a_t). \]

Given the constraints of the optimization problem, we must consider the augmented Hamiltonian
\[ \mathcal{L}(z_t, a_t, m_t, \psi_t, \mu_t) = H(z_t, a_t, m_t, \psi_t) + \mu_t(\phi z_t - a_t) \]
where \( \mu_t \) is a Lagrange multiplier and the adjoint variable \( \psi_t \) solves the initial value problem
\[ \dot{\psi}_t = -\kappa \psi_t - \phi \mu_t, \quad \psi_0 = 0. \]
with the transversality condition \( \psi_T = -C z_T \). The optimal effort solves the first order condition
\[ a_t = \frac{1 - r\gamma \psi_t - \mu_t}{1 + r\gamma \sigma^2}. \]

If the no manipulation constraint is not binding before time \( t \), then we have that \( \psi_t = \mu_t = 0 \) and so \( a_t = a^{nm} \). If the no manipulation constraint is binding, increasing effort requires to increase long term incentives today. The multiplier \( \mu_t \) represents the shadow cost of such increment at \( t \) while \( \psi_t \) represents the effect of the associated increment of long term incentives in future periods. Integrating the equation for \( \psi_t \) and using the transversality condition we get
\[ \psi_t = -C z_T + \phi \int_t^T e^{\kappa(s-t)} \mu_s ds. \quad (37) \]

Equation (37) shows that the marginal effect of increasing long term incentives today has two effects. It relaxes the no manipulation constraint in all future periods with an associated shadow benefit of \( \mu_t \). However this increment in long term incentives also increases the cost of providing post-retirement compensation after time \( T \). The following proposition presents the optimal effort path implemented by the zero manipulation contract.

**Proposition 7.** In the best contract implementing zero manipulation there is \( t^* \leq T \) such that \( a_t = a^{nm} \), all \( t \in [0, t^*) \) and \( a_t = \phi z_t \) for all \( t \in [t^*, T] \).

In particular,
\[
\frac{\phi^2 2\kappa + r - 2\theta + (\theta - \kappa) e^{-(\theta - r - \kappa)T} + e^{(\theta - \kappa)T} (\theta - r - \kappa)}{C e^{-(\theta - r - \kappa)T} (2\theta - 2\kappa - r) (\theta - \kappa)} \leq a^{HM}
\]

Then the optimal effort is \( a_t = \phi z_t \), where \( z_t = e^{-(\theta - r - \kappa)t} z_0 \) and

\[
z_0 = \frac{\phi \left( 1 - e^{(\theta - \kappa)T} \right) \left( \phi - \kappa \right)}{C e^{(r + \kappa - \theta)T} - \phi^2 \left( 1 + r \gamma \sigma^2 \right) e^{(r + \kappa - \theta)T} - \phi e^{(\theta - \kappa)T}}
\]

Otherwise, let \( t^* > 0 \) be the unique solution to

\[
\frac{\phi}{\theta - \kappa} + \frac{\phi \left[ e^{-(\theta - r - \kappa)(T-t^*)} + e^{(\theta - \kappa)(T-t^*)} \frac{\theta - r - \kappa}{\theta - \kappa} \right]}{2\kappa - 2\theta + r} = -\frac{C e^{-(\theta - r - \kappa)(T-t^*)}}{\phi} a^{HM}
\]

The optimal effort is \( a_t = a^{HM} \) if \( t < t^* \) and \( a_t = \phi z_t \) if \( t \in [t^*, T] \), where

\[
z_t = e^{-(\theta - r - \kappa)(t-t^*)} \frac{a^{HM}}{\phi}, \quad t \in [t^*, T].
\]

The optimal level of effort implemented by the contract takes the following form. The manager’s tenure can be divided in two phases. In the early years of office, effort is relatively high, coinciding with the optimal effort implemented in absence of manipulation concerns, \( a^{HM} \). In the second phase, as the manager approaches \( T \) the manipulation constraint binds, forcing the principal to lower the short-term incentives, which in turn leads to a decreasing effort profile. Figure 6 shows the dynamics of long term incentives and effort.
B.1.1 Proof Proposition 7

The proof proceeds in several steps. First, we show that the optimal path of effort takes the form \( a_t = a^{HM} 1_{\{t < t^*\}} + \phi z_t 1_{\{t \geq t^*\}} \) and then we determine the value of \( t^* \) and the law of motion of \( z_t \).

**Lemma 3.** Suppose that \( \theta \geq r + \kappa \). If \( \phi z_t \leq a^{HM} \) then \( a_s = \phi z_s \) for all \( s \in [t, T] \).

**Proof.** Because \( \psi_t \leq 0 \) it follows that if \( \phi z_t \leq a^{HM} \) then it must be the case that \( a_t = \phi z_t \). To see why this is the case suppose first that \( \psi_t < 0 \). If \( \mu_t = 0 \) then we have that

\[
a_t = \frac{1 - r\gamma\psi_t}{1 + r\gamma\sigma^2} > a^{HM} \geq \phi z_t,
\]

which would violate the constraint. Then it must be the case that \( \mu_t > 0 \) which means that \( a_t = \phi z_t \). Suppose next that \( \psi_t = 0 \) so that

\[
a_t = \frac{1 - \mu_t}{1 + r\gamma\sigma^2}.
\]
If $\mu_t = 0$, we have $a_t = a^{HM}$ which can only satisfy the constraint if $\phi z_t = a^{HM}$, in which case the equality $a_t = \phi z_t$ is trivially satisfied. On the other hand, if $\phi z_t < a^{HM}$ then it must be the case that $\mu_t > 0$ which implies that $a_t = \phi z_t$. Hence, we can conclude that $a_t = \phi z_t$.

The next step is to show that $a_s = \phi z_s$ for $s \in (s,T]$. Replacing $a_t = \phi z_t$ in the law of motion for $z_t$ we get that $\dot{z}_t = (r + \kappa - r\gamma \phi) z_t = (r + \kappa - \theta) z_t \leq 0$. Hence, we have that $\phi z_s \leq a^{HM}$ for some $h$ and all $s \in (t,t+h)$. Repeating the same argument we get that $a_{t+h} = \phi z_{t+h}$ and $\dot{z}_{t+h} \leq 0$ so we can conclude that $\phi z_s \leq a_{nm}$ and so $a_s = \phi z_s$ for all $s \in [t,T]$.

**Lemma 4.** Let $t^* \equiv \inf\{t \in [0,T] : \phi z_t \leq a^{HM}\}$. The solution to the optimal control problem is $a_t = a^{HM}$ if $t \in [0,t^*)$ and $a_t = \phi z_t$ if $t \in [t^*,T]$.

**Proof.** We have already proven in Lemma 3 that $a_t = \phi z_t$ for $t \in [t^*,T]$. Hence, the only step left is to show that $a_t = a^{HM}$ for $t < t^*$. If $\phi z_0 \leq a^{HM}$ then by Lemma 3 we have $t^* = 0$ and there is nothing to prove. Suppose next that $\phi z_0 > a^{HM}$. Using the initial condition $\psi_0 = 0$ we get that the optimal effort at time zero satisfies the first order condition

$$a_0 = 1 - r\gamma \psi_0 - \mu_0 \leq a^{HM}.$$ 

If $\mu_t > 0$ then we have $\phi z_0 = a_0 < a^{HM}$ which contradicts the hypothesis that $\phi z_0 > a^{HM}$. Thus, we can conclude that $\mu_0 = 0$ and $a_0 = a^{HM}$. Replacing in the law of motion of $\psi_t$ we get that $\dot{\psi}_0 = 0$ and $\psi_0 = 0$. This means that we can extend the previous argument to the interval $[0,t^*)$ and get $\psi_t = 0$, $\mu_t = 0$, and $a_t = a^{HM}$ for all $t \in [0,t^*)$. 

**Proof of Proposition 7** Using the previous two lemmas, we can solve for the optimal path of effort, for $t < t_*$ we have that $\psi_t = 0$, $a_t = a^{HM}$, and

$$\dot{z}_t = (r + \kappa) z_t - r\gamma a^{HM}.$$ 

For $t \geq t_*$ we have that $a_t = \phi z_t$

$$\dot{z}_t = (r + \kappa - \theta) z_t,$$
and
\[ \dot{\psi}_t = -\kappa \psi_t - \phi \mu_t, \]
where \( \mu_t = 1 - r\gamma \psi_t - (1 + r\gamma \sigma^2) \phi z_t \). In addition, the system of equations satisfies the terminal condition \( \psi_T = -Cz_T \). For \( t \geq t^* \), the solution is
\[ z_t = e^{-(\theta - \kappa)(t-t^*)} z_{t^*}. \]

Solving for \( \psi_t \) and using the initial condition \( \psi_{t^*} = 0 \) we get
\[
\psi(t) = \frac{\phi}{\theta - \kappa} + \frac{\phi \left[ e^{-(\theta - \kappa+\theta)(t-t^*)} + e^{(\theta-\kappa)(t-t^*)} \frac{\theta - r - \kappa}{\theta - \kappa} \right]}{-2\theta + 2\kappa + r} \tag{38}
\]

Using the terminal condition we get the equation \( \psi_T = -Cz_T \) if and only if \( \phi z_0 \leq a^{HM} \) which means that \( t^* = 0 \) if and only if
\[
\phi^2 \frac{-2\theta + 2\kappa + r + (\theta - \kappa) e^{-(\theta - \kappa+\theta)^T} + e^{(\theta-\kappa)T} (\theta - r - \kappa)}{-C e^{-(\theta - \kappa)(T-t^*)} -2\theta + 2\kappa + r (\theta - \kappa)} \leq a^{HM}
\]
If this condition is not satisfied, then \( t^* > 0 \) and \( z_{t^*} = a^{HM}/\phi \). Using this relation, we get that \( t^* \) is the unique solution to
\[
\phi^2 \frac{\phi^2 \left[ e^{-(\theta - \kappa+\theta)(T-t^*)} + e^{(\theta-\kappa)(T-t^*)} \frac{\theta - r - \kappa}{\theta - \kappa} \right]}{-2\theta + 2\kappa + r} = -C e^{-(\theta - \kappa)(T-t^*)} a^{HM}
\]

Given \( t^* \), we can find \( z_0 \) solving the differential equation
\[ \dot{z}_t = (r + \kappa) z_t - r\gamma a^{HM}, \]
with the terminal condition \( z_{t^*} = a^{HM}/\phi \). This yields,
\[ z_t = \left[ \frac{r\gamma}{\kappa + r} - \frac{(\theta - r - \kappa) e^{-(r+\kappa)(t^*-t)}}{(r + \kappa) \phi} \right] a^{HM} \]

**B.2 Optimal Deterministic Contract**

In the remainder of this section we analyze the principal’s optimization problem. We solve the problem using Pontryagin’s maximum principle. Once again, as in Appendix B.1, because
the objective function is concave, and the dynamics and mixed constraint are linear, the necessary conditions in the maximum principle are also sufficient. The principal optimization problem reduces to

\[
\begin{align*}
\max_{z_0, a_t, m_t \geq 0} \int_0^T e^{-r t} \pi(a_t, m_t) \, dt - e^{-rT} \frac{1}{2} C z_T^2 \\
\text{subject to} \\
\dot{z}_t = (r + \kappa) z_t - r \gamma a_t \\
m_t \geq \frac{a_t - \phi z_t}{g},
\end{align*}
\]

where

\[\pi(a, m) \equiv a - \lambda m - \frac{gm^2}{2} - \frac{(1 + r \gamma \sigma^2) a^2}{2}.\]

This is an optimal control problem with mixed state constraints. The main challenge posed by this problem is the determination of the time intervals over which manipulation is positive and the manipulation constraint binds. We solve for the optimal path in closed form (up to the solution of a nonlinear equation). The current value Hamiltonian for this optimal control problem is

\[H(z_t, a_t, m_t, \psi_t) = \pi(a_t, m_t) + \psi_t((r + \kappa) z_t - r \gamma a_t).\]

We incorporate the different constraints by considering the augmented Hamiltonian

\[L(z_t, a_t, m_t, \psi_t, \eta_t, \nu_t) = H(z_t, a_t, m_t, \psi_t) + \eta_t \left( m_t - \frac{a_t - \phi z_t}{g} \right) + \nu_t m_t\]

where \(\eta_t\) and \(\nu_t\) are Lagrange multipliers. The co-state variable \(\psi_t\) solves the initial value problem

\[\dot{\psi}_t = -\kappa \psi_t - \frac{\phi}{g} \eta_t, \quad \psi_0 = 0.\]

together with the transversality condition \(\psi_T = -Cz_T\). By maximizing the augmented Hamiltonian, we find that effort and manipulation solves the first order conditions

\[
1 - \left(1 + r \gamma \sigma^2\right) a_t - r \gamma \psi_t - \frac{\eta_t}{g} = 0 \\
-\lambda - g m_t + \eta_t + \nu_t = 0
\]
and the complementary slackness conditions

\[ \eta_t = 0 \text{ if } m_t > \frac{a_t - \phi z_t}{g} \]
\[ \nu_t = 0 \text{ if } m_t > 0. \]

If \( m_t > 0 \) then \( \nu_t = 0 \) so effort is

\[ a_t = \frac{g - \lambda + \phi z_t - r \gamma g \psi_t}{1 + g (1 + r \gamma \sigma^2)}. \] (40)

This equation shows the determinants of effort. The relation between effort and \( \psi_t \) is intuitive: recall that \( \psi_t < 0 \) represents the shadow marginal cost of providing long-term incentives. The level of effort is higher when maintaining long term incentives is more costly because the firm has stronger incentives to accelerate vesting of long-term incentives. Similarly, providing more long term incentives (higher \( z_t \)) leads to higher effort as we can increase the amount of effort while keeping manipulation at moderate levels. Manipulation is interior only if \( m_t = (a_t - \phi z_t)/g > 0 \) which requires that \( a_t > \phi z_t \). Replace \( a_t \) we find the condition

\[ \frac{g - \lambda - r \gamma g \psi_t}{g (1 + r \gamma \sigma^2)} \geq \phi z_t \] (41)

If this condition is not satisfied, then \( m_t = 0 \) and we must consider the two cases that we already considered in Appendix B.1: i) \( a_t = \phi z_t \) and ii) \( a_t > \phi z_t \). If the solution is given by case i), then the multiplier \( \eta_t \) is determined by the first order condition for \( a_t \) and

\[ \eta_t = g (1 - (1 + r \gamma \sigma^2) \phi z_t - r \gamma \psi_t). \] (42)

The optimality condition is satisfied only if the multiplier \( \eta_t \) in (42) is positive. If neither inequality (41) holds nor \( \eta_t > 0 \), then the solution must be such \( a_t < \phi z_t \), which means that \( \eta_t = 0 \) so

\[ a_t = \frac{1 - r \gamma \psi_t}{1 + r \gamma \sigma^2}. \]

Figure 7 provides a summary of the previous analysis and illustrates the optimal effort and manipulation for different combinations of \((z_t, \psi_t)\) as we move through the phase diagram.
Figure 7: This figure summarizes the optimal effort and manipulation as we move through the phase diagram. There are three regions: in the first region we have positive manipulation, in the second region we have zero manipulation and effort is determined by the binding no-manipulation constraint, in the final region we have zero manipulation and the no-manipulation constraint is slack.

If we replace the solution for \((a_t, m_t)\) in the differential equation for \((z_t, \psi_t)\), we find a linear system of ordinary differential equations that can be solved in closed form for each time interval identified in the proposition. We identify the thresholds \(t^*\) and \(t^{**}\) using the transversality condition and pasting the solution for \((z_t, \psi_t)\) in the different intervals. The proof of Proposition 6 follows from an analysis of the behavior of this dynamic system as we move through the three regions identified in Figure 7. In particular, in Figure 7, we show that the system moves through the phase diagram from the northeast towards the southwest. The detailed formal analysis is provided next.

B.2.1 Proof Proposition 6

We start with 3 claims that we will use extensively in the proof.

Claim 1 If \(\phi z_t > a^{HM}\) and \(\psi_t = 0\) then \(a_t = a^{HM}\), \(m_t = 0\) and \(\dot{\psi}_t = 0\).
Proof. Suppose that $m_t > 0$. If this is the case, then we have that

\[
\begin{align*}
a_t - \phi z_t &= \frac{g - \lambda + \phi z_t}{1 + g(1 + r\gamma\sigma^2)} - \phi z_t \\
&= \frac{g - \lambda - \phi z_t g(1 + r\gamma\sigma^2)}{1 + g(1 + r\gamma\sigma^2)} \\
&< \frac{g - \lambda - a^{HM} g(1 + r\gamma\sigma^2)}{1 + g(1 + r\gamma\sigma^2)} \\
&= -\frac{\lambda}{1 + g(1 + r\gamma\sigma^2)} \leq 0,
\end{align*}
\]

which implies that $m_t < 0$, contradiction. Accordingly, we have

\[
a_t = \frac{1 - r\gamma\psi_t}{1 + r\gamma\sigma^2} = a^{HM}.
\]

Hence, we have that $m_t = 0 > a_t - \phi z_t$ which yields $\eta = 0$. Replacing in the law of motion for $\psi_t$ we find that $\dot{\psi}_t = 0$. □

Claim 2 If $\phi z_t \in \left[\frac{1 - \lambda/g - r\gamma\psi_t/g}{1 + r\gamma\sigma^2}, a^{HM}\right]$ then $a_t = \phi z_t$, $m_t = 0$, and $\dot{z}_t \leq 0$.

Proof. Suppose that $m_t > 0$, then $a_t - \phi z_t < 0$ which means that $m_t < 0$ a contradiction. Hence, it must be the case that $m_t = 0$. Using the first order condition $a$ we get

\[
a_t = \frac{1 + \eta/g}{1 + r\gamma\sigma^2}.
\]

If $\eta = 0$ we get $a_t = a^{HM}$ which violates the constraint $a_t \leq \phi z_t$ for $m_t = 0$. Hence, it must be the case that $\eta > 0$, which means that $a_t = \phi z_t$. Replacing in the ODE for $z_t$ we get $\dot{z}_t \leq 0$. □

Claim 3 $\phi z_t \in \left[\frac{1 - \lambda/g - r\gamma\psi_t/g}{1 + r\gamma\sigma^2}, a^{HM}\right] \Rightarrow m_t > 0$, and $\dot{z}_t < 0$.

Proof. Suppose that $m_t = 0$, then we have that $\eta = 0$ and

\[
a_t = \frac{1 - r\gamma\psi_t}{1 + r\gamma\sigma^2} \geq a^{HM} > \phi z_t
\]
which means that the constraint \( a_t \leq \phi z_t \) is violated. Thus, we have that \( m_t > 0, \eta_t > 0 \) and \( a_t > \phi z_t \). Using the law of motion of \( z_t \) we get that

\[
\dot{z}_t = (r + \kappa) z_t - r \gamma a_t < (r + \kappa) z_t - r \gamma \phi z_t = (r + \kappa - \theta) z_t \leq 0.
\]

\[\Box\]

**Proof Proposition 6** The first step is to show that \( \phi z_0 \leq \frac{\theta}{r+\kappa} a^{HM} \). Suppose this is not the case, then we have that \( \phi z_0 > \frac{\theta}{r+\kappa} a^{HM} > a^{HM} \). By Claim 1 we have that \( a_0 = a^{HM} \), \( \dot{\psi}_0 = 0 \) and \( \dot{z}_0 > 0 \). Repeating the same argument at any time \( t \) we find that \( \phi z_t > a^{HM} \), \( a_t = a^{HM} \) and \( \psi_t = 0 \), all \( t \in [0,T] \). However, this violates the transversality condition \( \psi_T = -\Psi'(z_T) \).

The previous upper bound implies that we only need to consider three cases:

**Case 1:** \( \phi z_0 \in \left[a^{HM}, \frac{\theta}{r+\kappa} a^{HM}\right] \)

**Case 2:** \( \phi z_0 \in \left[\frac{1-\lambda}{1+\gamma r^2}, a^{HM}\right] \)

**Case 3:** \( \phi z_0 < \frac{1-\lambda}{1+\gamma r^2} \)

First, we consider Case 1. From Claim 1 we have that \( a_0 = a^{HM} \), \( \dot{z}_0 \leq 0 \) and \( \dot{\psi}_0 = 0 \). We can also conclude from Claim 1 that \( a_t = a^{HM} \), \( \dot{z}_t \leq 0 \) and \( \dot{\psi}_t = 0 \) for all \( t \in [0,t^*] \) where \( t^* = \inf \{t > 0 : \phi z_t < a^{HM}\} \), where it must be the case that \( t^* < T \) because otherwise \( \psi_T = 0 \) which violates the transversality condition.

Next, Claim 2 implies that that \( a_{t^*} = \phi z_{t^*}, m_{t^*} = 0 \) and \( \dot{z}_{t^*} \leq 0 \). Let’s define \( t^{**} = \inf \{t > 0 : \phi z_t < \frac{1-\lambda}{1+\gamma r^2} \phi z_t\} \). If \( t^{**} > T \) then there is nothing else to prove. Hence, suppose that \( t^{**} < T \). Claim 2 implies that \( a_t = \phi z_t, m_t = 0 \) and \( \dot{z}_t \leq 0 \) for all \( t \in [t^*,t^{**}] \). Next, we show that it is also the case that \( \dot{\psi}_t < 0 \), all \( t \in [t^*,t^{**}] \). Using the law of motion of \( \psi_t \) we get that \( \dot{\psi}_t < 0 \). Differentiating \( \dot{\psi}_t \) we get that

\[
\ddot{\psi}_t = -\kappa \dot{\psi}_t - \frac{\phi}{g} \dot{\eta}_t,
\]

where

\[
\eta_t = g - g \left(1 + r \gamma \sigma^2\right) a_t - \psi_t r \gamma g = g - g \left(1 + r \gamma \sigma^2\right) \phi z_t - \psi_t r \gamma g
\]

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\[ \dot{\eta}_t = -g \left( 1 + r\gamma\sigma^2 \right) \phi z_t - \dot{\psi} r\gamma g. \]

Suppose there is \( \tilde{t} \in [t^*, t^{**}] \) such that \( \dot{\psi}_{\tilde{t}} > 0 \), then there is \( \tilde{t} < \tilde{t} \) such that \( \dot{\psi}_t = 0 \). Replacing in the law of motion of \( \eta \) and \( \dot{\psi}_t \) we find that \( \dot{\eta}_t > 0 \) and \( \ddot{\psi}_t < 0 \). Hence, \( \dot{\psi}_t \) can never cross zero which means that \( \dot{\psi}_t < 0 \) for all \( t \in [t^*, t^{**}] \).

The final step is to analyze the behavior of \((z_t, \dot{\psi}_t)\) for \( t \in [t^{**}, T] \). We know that \( \dot{z}_{t^{**}} < 0 \) and \( \dot{\psi}_{t^{**}} \leq 0 \), which means that \( \frac{d}{dt} \left( \phi z_t - \frac{1 - \lambda/g - r\gamma\psi_t/g}{1 + r\gamma\sigma^2} \right) \bigg|_{t=t^{**}} < 0 \). Accordingly, \( \phi z_t < \frac{1 - \lambda/g - r\gamma\psi_t/g}{1 + r\gamma\sigma^2} \) for \( t \in (t^{**}, t^{**} + \epsilon) \). Claim 3 implies that \( m_t > 0 \) and \( \dot{z}_t < 0 \) for all \( t \in (t^{**}, t^{**} + \epsilon) \). Differentiating \( \dot{\psi}_t \), we get that

\[ \ddot{\psi}_t = -\kappa \dot{\psi}_t - \frac{\phi}{g} \eta_t = -\kappa \dot{\psi}_t - \phi \dot{m}_t. \]

Differentiating the solution for \( m_t \) we get that \( \dot{\psi}_t = 0 \) implies \( \ddot{m}_t < 0 \). Thus, \( \dot{\psi}_t = 0 \) implies \( \dddot{\psi}_t < 0 \) which implies that \( \dot{\psi}_t < 0 \) for all \( t \in (t^{**}, t^{**} + \epsilon) \). We can repeat the argument beyond \( t^{**} + \epsilon \) to conclude that \( m_t > 0, \dot{z}_t < 0, \dot{\psi}_t < 0 \) all \( t \in (t^{**}, T] \). Manipulation is given by

\[ m_t = g - \lambda + \phi z_t - r\gamma g \psi_t 1 + g (1 + r\gamma\sigma^2) - \phi z_t \]
\[ = g - \lambda - \phi g \left( 1 + r\gamma\sigma^2 \right) z_t - r\gamma g \psi_t 1 + g (1 + r\gamma\sigma^2) \]

Hence, we get that

\[ \ddot{m}_t = -\phi g \left( 1 + r\gamma\sigma^2 \right) \dot{z}_t + r\gamma g \dot{\psi}_t 1 + g (1 + r\gamma\sigma^2) > 0. \]

Thus, whenever \( \phi z_0 \in [a^H_M, \frac{a^H_M}{r + \kappa^H_M}] \) the contract satisfies all the properties in Proposition 6.

We conclude the proof considering Cases 2 and 3. For Case 2, we repeat the argument in Case 1 using the initial condition \( \psi_0 = 0 \) and setting \( t^* = 0 \), and for Case 3 we repeat the argument Case 1 using the initial condition \( \psi_0 = 0 \) and setting \( t^{**} = 0 \).
B.3 Closed Form Solution Optimal Deterministic Contract

The first step is to solve the case with $m_t > 0$, all $t \in [0, T]$. That is, we start looking at the solution assuming that the inequality $\phi z_0 < \frac{1-\lambda/g}{1+r\gamma \sigma^2}$ holds.

Whenever effort and manipulation are interior they solve the first order condition

$$a_t = \frac{g - \lambda + \phi z_t - r \gamma g \psi_t}{1 + g(1 + r \gamma \sigma^2)}$$

$$m_t = \frac{a_t - \phi z_t}{g}.$$

Replacing $(a_t, m_t)$ in the differential equation for $z_t$ and $\psi_t$ we arrive to the forward-backward differential equations

$$\begin{pmatrix} \dot{z}_t \\ \dot{\psi}_t \end{pmatrix} = -\begin{pmatrix} \frac{r \gamma (g-\lambda)}{1+g(1+r \gamma \sigma^2)} + \frac{\lambda \phi g}{g(1+g(1+r \gamma \sigma^2))} \\ \frac{\phi (g-\lambda)}{1+g(1+r \gamma \sigma^2)} \end{pmatrix} + \begin{pmatrix} r + \kappa - \frac{\theta}{1+g(1+r \gamma \sigma^2)} \\ \frac{r^2 \gamma^2 g}{1+g(1+r \gamma \sigma^2)} - \frac{\theta}{1+g(1+r \gamma \sigma^2)} - \kappa \end{pmatrix} \begin{pmatrix} z_t \\ \psi_t \end{pmatrix}$$

with the initial condition $\psi_0 = 0$ and terminal condition $\psi_T = -C z_T$. The eigenvalues of the system are

$$\mu^- = \frac{1}{2} \left( r - \frac{\mathcal{D}}{1 + g(1 + r \gamma \sigma^2)} \right)$$

$$\mu^+ = \frac{1}{2} \left( r + \frac{\mathcal{D}}{1 + g(1 + r \gamma \sigma^2)} \right)$$

where

$$\mathcal{D} \equiv \sqrt{4\theta^2 g(1 + r \gamma \sigma^2) + \left((r + 2\kappa)(1 + g(1 + r \gamma \sigma^2)) - 2\theta\right)^2}.$$ 

with the associated eigenvectors are

$$v^- = \begin{pmatrix} \frac{(r + 2\kappa)(1 + g(1 + r \gamma \sigma^2)) - 2\theta - \mathcal{D}}{2\theta^2 (1 + r \gamma \sigma^2)} \\ 1 \end{pmatrix}$$

and

$$v^+ = \begin{pmatrix} \frac{(r + 2\kappa)(1 + g(1 + r \gamma \sigma^2)) - 2\theta + \mathcal{D}}{2\theta^2 (1 + r \gamma \sigma^2)} \\ 1 \end{pmatrix}.$$
Hence, by the theory of linear system of differential equations with constant coefficients we get that the solution is given by

\[
\begin{pmatrix} z_t \\ \psi_t \end{pmatrix} = \begin{pmatrix} \bar{z} \\ \bar{\psi} \end{pmatrix} + A^- v^- e^{\mu^- t} + A^+ v^+ e^{\mu^+ t},
\]

where \( v \equiv (v^- \ v^+) \),

\[
\begin{pmatrix} \bar{z} \\ \bar{\psi} \end{pmatrix} = \begin{pmatrix} r + \kappa - \frac{\theta}{1 + g(1 + r \gamma \sigma^2)} \\ \frac{\nu^2 - \phi^2 g}{g(1 + g(1 + r \gamma \sigma^2))} \end{pmatrix}^{-1} \begin{pmatrix} \frac{r \gamma (g - \lambda)}{1 + g(1 + r \gamma \sigma^2)} \\ \frac{\lambda \phi}{g} + \frac{\phi (g - \lambda)}{1 + g(1 + r \gamma \sigma^2)} \end{pmatrix}
\]

and \((A^-, A^+)\) constants to be determined. Using the initial conditions \( \psi_0 = 0 \) and given an arbitrary initial value \( z_0 \) we get that

\[
\begin{pmatrix} A^- \\ A^+ \end{pmatrix} = v^{-1} \begin{pmatrix} z_0 - \bar{z} \\ -\bar{\psi} \end{pmatrix}
\]

Similarly, using the terminal condition \( \psi_T = -C z_T \) we get

\[
\begin{pmatrix} A^- \\ A^+ \end{pmatrix} = (v e^{\mu^T})^{-1} \begin{pmatrix} z_T - \bar{z} \\ -C z_T - \bar{\psi} \end{pmatrix}
\]

where \( v e^{\mu^T} \equiv (e^{\mu^- T} v^-, e^{\mu^+ T} v^+) \). Combining these two expressions for \((A^-, A^+)\) we get and equation for \((z_0, z_T)\),

\[
v^{-1} \begin{pmatrix} z_0 - \bar{z} \\ -\bar{\psi} \end{pmatrix} = (v e^{\mu^T})^{-1} \begin{pmatrix} z_T - \bar{z} \\ -C z_T - \bar{\psi} \end{pmatrix}
\]

Finally, using the previous system of equations, we find that the solution \((z_0, z_T)\) is

\[
\begin{pmatrix} z_0 \\ z_T \end{pmatrix} = \left( v^{-1} e_1, -\left( (v e^{\mu^T})^{-1} e_1 - C (v e^{\mu^T})^{-1} e_2 \right) \right)^{-1} \left( v^{-1} - (v e^{\mu^T})^{-1} \right) \begin{pmatrix} \bar{z} \\ \bar{\psi} \end{pmatrix},
\]

where \( e_1 = (1, 0)^T \) and \( e_2 = (0, 1)^T \) are the unit vectors. Finally, using Proposition 6 we check if the condition \( \phi z_0 < \frac{1 - \lambda g}{1 + r \gamma \sigma^2} \) is satisfied. The next proposition provides a summary of
Proposition 8. If the optimal contract implements positive manipulation \((m_t > 0, \text{ all } t \in [0, T])\) then, the optimal effort and manipulation path are

\[
a_t = \frac{g - \lambda + \phi z_t - r \gamma g \psi_t}{1 + g(1 + r \gamma \sigma^2)}
\]

\[
m_t = \frac{a_t - \phi z_t}{g}.
\]

The long-term incentives, \(z_t\) and the co-state variable, \(\psi_t\) are

\[
\begin{pmatrix}
z_t \\
\psi_t
\end{pmatrix} = \begin{pmatrix}
\bar{z} \\
\bar{\psi}
\end{pmatrix} + A^- v^- e^{\mu^- t} + A^+ v^+ e^{\mu^+ t},
\]

where

\[
\mu^- = \frac{1}{2} \left( r - \frac{D}{1 + g(1 + r \gamma \sigma^2)} \right)
\]

\[
\mu^+ = \frac{1}{2} \left( r + \frac{D}{1 + g(1 + r \gamma \sigma^2)} \right)
\]

\[
D \equiv \sqrt{4 \theta^2 g (1 + r \gamma \sigma^2) + ((r + 2 \kappa)(1 + g(1 + r \gamma \sigma^2)) - 2 \theta_x)^2}
\]

are the eigenvalues of the system \((z_t, \psi_t)\) and

\[
v^- = \begin{pmatrix}
\frac{(r + 2 \kappa)(1 + g(1 + r \gamma \sigma^2)) - 2 \theta - D}{2 \theta_x^2(1 + r \gamma \sigma^2)} \\
1
\end{pmatrix}
\]

\[
v^+ = \begin{pmatrix}
\frac{(r + 2 \kappa)(1 + g(1 + r \gamma \sigma^2)) - 2 \theta + D}{2 \theta_x^2(1 + r \gamma \sigma^2)} \\
1
\end{pmatrix}
\]

are the associated eigenvectors. The long term steady state, \((\bar{z}, \bar{\psi})\) is

\[
\begin{pmatrix}
\bar{z} \\
\bar{\psi}
\end{pmatrix} = \left( \begin{array}{cc}
\frac{r + \kappa - \frac{\theta}{1 + g(1 + r \gamma \sigma^2)}}{\phi^2 g} & \frac{r^2 \gamma^2 g}{1 + g(1 + r \gamma \sigma^2)} - \kappa \\
\phi \frac{\theta}{g(1 + g(1 + r \gamma \sigma^2))} & \frac{1 + g(1 + r \gamma \sigma^2)}{\phi^2 g} - \kappa
\end{array} \right)^{-1} \left( \begin{array}{c}
\frac{r \gamma (g - \lambda)}{g(1 + g(1 + r \gamma \sigma^2))} \\
\frac{\phi (g - \lambda)}{g(1 + g(1 + r \gamma \sigma^2))}
\end{array} \right)
\]
The initial value \((z_0, \psi_0)\) and the constants \((A^-, A^+)\) are given by

\[
\begin{pmatrix}
A^- \\
A^+
\end{pmatrix}
= v^{-1}
\begin{pmatrix}
z_0 - \bar{z} \\
-\bar{\psi}
\end{pmatrix}
\]

\begin{align*}
z_0 &= e_1^\top \left( v^{-1} e_1, - \left( (ve^{\mu T})^{-1} e_1 - C (ve^{\mu T})^{-1} e_2 \right)^{-1} \left( v^{-1} - (ve^{\mu T})^{-1} \right) \right) \left( \bar{z} \quad \bar{\psi} \right) \\
\psi_0 &= 0
\end{align*}

where \(v \equiv (v^-, v^+)\), \(ve^{\mu T} \equiv (e^{\mu T} v^-, e^{\mu T} v^+)\), and \(e_1 = (1, 0)^\top\) and \(e_2 = (0, 1)^\top\) denote the unit vectors.

If the solution in the previous case does not satisfies the inequality \(\phi z_0 < \frac{1-\lambda/g}{1+r\gamma}\), then it must be the case that the optimal contract entails \(\phi z_0 \geq \frac{1-\lambda/g}{1+r\gamma}\). The next step is to solve the principal optimization problem assuming that \(\phi z_0 \in \left[ \frac{1-\lambda/g}{1+r\gamma}, a^{HM} \right]\). In this case, we have to intervals \([0, t^{**}]\) and \((t^{**}, T]\). Using the previous solution of the ode for \((z_t, \psi_t)\), we find that in the interval \((t^{**}, T]\) we have

\[
a_t = \frac{g - \lambda + \phi z_t - r\gamma g \psi_t}{1 + g \left( 1 + r\gamma \sigma^2 \right)} \\
m_t = \frac{a_t - \phi z_t}{g}
\]

and

\[
\begin{pmatrix}
z_t \\
\psi_t
\end{pmatrix}
= \begin{pmatrix}
\bar{z} \\
\bar{\psi}
\end{pmatrix}
+ A^- v^- e^{\mu^- (t-t^{**})} + A^+ v^+ e^{\mu^+ (t-t^{**})},
\]

where

\[
\begin{pmatrix}
A^- \\
A^+
\end{pmatrix}
= v^{-1}
\begin{pmatrix}
z_{t^{**}} - \bar{z} \\
\psi_{t^{**}} - \bar{\psi}
\end{pmatrix}
\]

(44)

Repeating computations similar to the ones we did in Proposition 8 we get the system of equations

\[
v^{-1}\begin{pmatrix}
z_{t^{**}} - \bar{z} \\
\psi_{t^{**}} - \bar{\psi}
\end{pmatrix}
= (ve^{\mu(T-t^{**})})^{-1}\begin{pmatrix}
z_T - \bar{z} \\
-C z_T - \bar{\psi}
\end{pmatrix}
\]

(45)
The solution to this system is:

\[
\begin{pmatrix}
  z_{t^{**}} \\
  z_T
\end{pmatrix} = \left( v^{-1}e_1, -\left( (ve^\mu(T-t^{**}))^{-1} e_1 - C (ve^\mu(T-t^{**}))^{-1} e_2 \right) \right)^{-1} \times \left[ \begin{pmatrix}
  \bar{z} \\
  \bar{\psi}
\end{pmatrix} - v^{-1}e_2 \psi_{t^{**}} \right],
\]

and in particular,

\[
z_{t^{**}} = e_1^\top \left( v^{-1}e_1, -\left( (ve^\mu(T-t^{**}))^{-1} e_1 - C (ve^\mu(T-t^{**}))^{-1} e_2 \right) \right)^{-1} \times \left[ \begin{pmatrix}
  \bar{z} \\
  \bar{\psi}
\end{pmatrix} - v^{-1}e_2 \psi_{t^{**}} \right]. \tag{47}
\]

After solving the system of equations in the interval \([t^{**}, T]\), we move backward and solve the solution in the interval \([0, t^{**}]\). In this interval, we have that \(a_t = \phi z_t\), which means that \(z_t = (r + \kappa - \theta) z_t\); hence, we have that \(z_t = z_0 e^{-(\theta - r - \kappa) t}\). In particular, we have that

\[
z_{t^{**}} = z_0 e^{-(\theta - r - \kappa) t^{**}}.
\]

The next step is to solve for \(\psi_t\). In the interval \([0, t^{**}]\), the constraint \(m_t \geq (a_t - \phi z_t)/g\) is binding; thus, we have that the multipliers \(\eta_t > 0\). In particular, we have that

\[
\eta_t = 1 - (1 + r\gamma\sigma^2)a_t - r\gamma \psi_t = 1 - (1 + r\gamma\sigma^2)\phi z_t - r\gamma \psi_t. \tag{48}
\]

Accordingly, \(\psi_t\) solves the following ODE

\[
\dot{\psi}_t = (\theta - \kappa)\psi_t - \phi + (1 + r\gamma\sigma^2)\phi^2 z_t, \quad \psi_0 = 0. \tag{49}
\]

This equations yields the solution

\[
\psi_{t^{**}} = \frac{\phi}{\theta - \kappa} \left(1 - e^{(\theta - \kappa) t^{**}}\right) \frac{(1 + r\gamma\sigma^2)\phi^2}{2\theta - 2\kappa - r} \left( e^{-(\theta - r - \kappa) t^{**}} - e^{(\theta - \kappa) t^{**}} \right) z_0 \tag{50}
\]

Step 2 in the proof of Proposition 6 implies that \(t^{**}\) is define as \(t^{**} = \inf\{ t > 0 : \phi z_t = \)
\[
\frac{1-\lambda/g-r\gamma\psi_t}{1+r\gamma\sigma^2} \}. \quad \text{Accordingly,}
\]
\[
\phi_0 z_0 = \frac{1 - \frac{\lambda}{g} + \frac{\theta}{\theta - \kappa} \left( e^{(\theta - \kappa)t^*} - 1 \right)}{(1 + r\gamma\sigma^2)e^{-(\theta - \kappa)t^*} - \frac{(1 + r\gamma\sigma^2)^2 \theta}{2\theta - 2\kappa - r} \left( e^{(\theta - \kappa)^t} - e^{-(\theta - \kappa)t^*} \right)} \text{ (51)}
\]

Finally, we can solve for \( t^* \) using \( \phi_0 z_0 \) in combination with \( (z_{t^*}, \psi_{t^*}) \) and Equation (47). The following Proposition summarizes the previous computations.

**Proposition 9.** Let \( t^* > 0 \) be a solution of the equation

\[
\begin{align*}
z_{t^*} & = e_1^T \left( v^{-1} e_1, - \left( \left( v e^\mu(T-t^*) \right)^{-1} e_1 - C \left( v e^\mu(T-t^*) \right)^{-1} e_2 \right) \right)^{-1} \\
& \times \left[ \left( v^{-1} - \left( v e^\mu(T-t^*) \right)^{-1} \right) \left( \bar{z} / \bar{\psi} \right) - v^{-1} e_2 \psi_{t^*} \right],
\end{align*}
\]

where

\[
\phi_0 z_0 = \frac{1 - \frac{\lambda}{g} + \frac{\theta}{\theta - \kappa} \left( e^{(\theta - \kappa)t^*} - 1 \right)}{(1 + r\gamma\sigma^2)e^{-(\theta - \kappa)t^*} - \frac{(1 + r\gamma\sigma^2)^2 \theta}{2\theta - 2\kappa - r} \left( e^{(\theta - \kappa)t^*} - e^{-(\theta - \kappa)t^*} \right)}
\]

\[
z_{t^*} = z_0 e^{-(\theta - \kappa)t^*}
\]

\[
\psi_{t^*} = -\frac{\phi}{\theta - \kappa} \left( e^{(\theta - \kappa)t^*} - 1 \right) - \frac{(1 + r\gamma\sigma^2)\phi^2}{2\theta - 2\kappa - r} \left( e^{-(\theta - \kappa)t^*} - e^{(\theta - \kappa)t^*} \right) z_0.
\]

If \( \phi_0 \in \left[ \frac{1-\lambda/g}{1+r\gamma\sigma^2}, a_{HM} \right] \) and \( t^* \leq T \), then the solution to the Principal’s problem is \( a_t = \phi z_t \), \( m_t = 0 \), \( z_t = z_0 e^{-(\theta - \kappa)t} \), all \( t \in [0,t^*] \), and

\[
a_t = \frac{g - \lambda + \phi z_t - r\gamma\psi_t}{1 + g \left( 1 + r\gamma\sigma^2 \right)},
\]

\[
m_t = \frac{a_t - \phi z_t}{g}
\]

for \( t \in (t^*, T] \), where

\[
\begin{pmatrix}
z_t \\
\psi_t
\end{pmatrix} = \begin{pmatrix}
\bar{z} \\
\bar{\psi}
\end{pmatrix} + A^- v^{-1}(t-t^*) + A^+ v e^\mu(t-t^*),
\]
and
\[
\begin{pmatrix}
A^- \\
A^+
\end{pmatrix} = v^{-1}
\begin{pmatrix}
\tilde{z} \\
\psi_{t^*} - \tilde{\psi}
\end{pmatrix}.
\]

If \( t^* > T \) and
\[
\phi z_0 = \frac{\frac{\sigma^2}{\theta - \kappa} (e^{(\theta - \kappa)T} - 1)}{\mathcal{C} e^{-(\theta - \kappa)T} \left( \frac{1 + r \gamma \sigma^2}{2 \theta - 2 \kappa - r} \right) (e^{-(\theta - \kappa)T} - e^{(\theta - \kappa)T})} \in \left[ 1 - \frac{\lambda}{g}, a^{HM} \right]
\]
then, \( a_t = \phi z_t \) and \( m_t = 0 \) for all \( t \in [0, T] \).

The final case is when \( \phi z_0 > a^{HM} \). In this case, we have that \( a_t = a^{HM} \) up to time \( t^* = \inf \{ t > 0 : \phi z_t = a^{HM} \} \). The value of \( z_t \) at time \( t \in [0, t^*] \) is given by
\[
\phi z_{t^*} = \phi z_0 e^{(r + \kappa)t^*} - \frac{\theta a^{HM}}{r + \kappa} \left( e^{(r + \kappa)t^*} - 1 \right) = a^{HM}.
\]

Hence, we have that
\[
\phi z_{t^*} = \phi z_0 e^{(r + \kappa)t^*} - \frac{\theta a^{HM}}{r + \kappa} \left( e^{(r + \kappa)t^*} - 1 \right) = a^{HM}.
\]

Replacing \( z_0 \) by \( z_{t^*} \) in Proposition 9 and using the condition \( a^{HM} = \phi z_{t^*} \) we get that the solution to the Principal problem is

**Proposition 10.** Let \( 0 < t^* < t^{**} \) be a solution to
\[
a^{HM} = \frac{1 - \frac{\lambda}{g} + \frac{\theta}{\theta - \kappa} \left( e^{(\theta - \kappa)(t^{**} - t^*)} - 1 \right)}{(1 + r \gamma \sigma^2)e^{-(\theta - \kappa)(t^{**} - t^*)} - \frac{1 + r \gamma \sigma^2}{2 \theta - 2 \kappa - r} \left( e^{-(\theta - \kappa)(t^{**} - t^*)} - e^{(\theta - \kappa)(t^{**} - t^*)} \right)}
\]
\[
z_{t^{**}} = e_1^T \left( v^{-1} e_1, -\left( ve^{\mu(T-t^{**})}\right)^{-1} e_1 - \mathcal{C} \left( ve^{\mu(T-t^{**})}\right)^{-1} e_2 \right)^{-1}
\]
\[
\times \left[ \left( v^{-1} - \left( ve^{\mu(T-t^{**})}\right)^{-1} \right) \left( \frac{\tilde{z}}{\tilde{\psi}} \right) - v^{-1} e_2 \psi_{t^{**}} \right],
\]

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where
\[
z_{t^*} = \frac{a_{HM}}{\phi} e^{-(\theta-r-\kappa)(t^*-t)}
\]
\[
\psi_{t^*} = -\frac{\phi}{\theta-\kappa} \left( e^{(\theta-\kappa)(t^*-t)} - 1 \right) - \frac{(1 + r\gamma^2\sigma^2)}{2\theta - 2\kappa - r} \left( e^{-(\theta-r-\kappa)(t^*-t)} - e^{(\theta-\kappa)(t^*-t)} \right) a_{HM}.
\]

The initial condition \(z_0\) is given by
\[
\phi z_0 = a_{HM} \left[ e^{-(r+\kappa)t^*} + \frac{\theta}{r + \kappa} \left( 1 - e^{-(r+\kappa)t^*} \right) \right].
\]

If \(\phi z_0 > a_{HM}\) and \(t^* \leq T\), then the solution to the Principal’s problem is
\[
a_t = a_{HM}, \quad m_t = 0, \quad \forall t \in [0, t^*]
\]
\[
a_t = \phi z_t, \quad m_t = 0, \quad \forall t \in (t^*, t^{**}]
\]
\[
a_t = g - \lambda + \phi z_t - r\gamma g\psi_t
\]
\[
m_t = \frac{a_t - \phi z_t}{g}
\]
for \(t \in (t^{**}, T]\), where
\[
\begin{pmatrix}
z_t \\
\psi_t
\end{pmatrix} = \begin{pmatrix}
\bar{z} \\
\bar{\psi}
\end{pmatrix} + A^- \mathbf{v}^- e^{\mu^-(t-t^*)} + A^+ \mathbf{v}^+ e^{\mu^+(t-t^{**})},
\]
and
\[
\begin{pmatrix}
A^- \\
A^+
\end{pmatrix} = \mathbf{v}^{-1} \begin{pmatrix}
z_{t^{**}} - \bar{z} \\
\psi_{t^{**}} - \bar{\psi}
\end{pmatrix}.
\]

If \(t^* > T\) and there is \(t^*\) such that
\[
a_{HM} = \frac{\phi^2}{\theta - \kappa} \left( e^{(\theta-\kappa)(T-t^*)} - 1 \right)
\]
\[
\frac{C e^{-(\theta-r-\kappa)(T-t^*)} - (1 + r\gamma^2\sigma^2)\phi^2 \left( e^{-(\theta-r-\kappa)(T-t^*)} - e^{(\theta-\kappa)(T-t^*)} \right)}{2\theta - 2\kappa - r}
\]
then \(m_t = 0, \forall t \in [0, T]\), \(a_t = a_{HM}, \forall t \in [0, t^*]\), \(a_t = \phi z_t, \forall t \in (t^*, T]\) and
\[
\phi z_0 = a_{HM} \left[ e^{-(r+\kappa)t} + \frac{\theta}{r + \kappa} \left( 1 - e^{-(r+\kappa)t} \right) \right]
\]

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C Numerical Computation HJB Equation

In this section we describe the numerical approach based on Barles and Souganidis (1991) (See Forsyth and Vetzal (2012) and Tourin (2013) for a more applied exposition with a focus on financial applications). We start writing the HJB equation in a way more suitable for computation. For computational purposes, it is convenient to measure the time backward, so we let \( s = T - t \), and define the value function \( f(z, s) = F(z, T - s) \). The infinitesimal generator is

\[
L^{a, \sigma} f = [(r + \kappa)z + ar\gamma(\sigma\sigma_z - 1)]f_z + \frac{1}{2}\sigma_z f_{zz} - rf,
\]

so the HJB equation can be written as

\[
f_s = \max_{a \in [0, \bar{a}]} \{ L^{a, \sigma} f + \pi(a, z) \}
\]

with a Dirichlet boundary condition

\[
f(z, 0) = -\frac{1}{2}Cz^2
\]

The upper bound \( \bar{\beta} \) implies that \( z_t \) is bounded so we can restrict the domain of the value function to a finite region. Unless \( \tau = \infty \), the upper bound on \( z_t \) varies over time. For the numerical implementation, it is convenient to consider a constant upper bound. On the other hand, the free disposal assumption implies that \( \beta_t \geq 0 \) so \( z_t \geq 0 \) (Formally, the free disposal implies that \( \beta_t \geq 0 \) for \( t \leq T \). In principle, it could be possible that \( \beta_t < 0 \) for \( t > T \) so \( z_t < 0 \) for some \( t \leq T \). However, it is easy to verify that it is never optimal to let \( z_t \) become negative on \([0, T]\)). Thus, we consider \( z \in Q \equiv [0, Z_{\text{max}}] \) for some large \( Z_{\text{max}} \).

Notice that, if we let \( T_Q \equiv \inf\{t > 0 : z_t \notin Q\} \), we have that \( T \wedge T_Q \to T \) when \( Z_{\text{max}} \to \infty \), which means that we can choose \( Z_{\text{max}} \) large enough so the constraint has an arbitrarily small effect on the solution of the contract. This type of boundary condition is suggested as an alternative by Kushner and Dupuis (2013). This means that we solve the problem

\[
F(z, 0) = \sup_{a, \sigma} E \left[ \int_0^{T \wedge T_Q} e^{-rt} \pi(a_t, z_t) dt - e^{-rT_Q} \frac{1}{2}Cz_t^2 1_{T \leq T_Q} + e^{-rT_Q} F(z_{T_Q}, T_Q) 1_{T > T_Q} \right]
\]
subject to

\[ dz_t = [(r + \kappa)z_t + r\gamma a_t(\sigma \sigma z_t - 1)]dt + \sigma z_t dB_t \]
\[ m_t \geq \frac{a_t - \phi z_t}{g} \]
\[ m_t \geq 0, \]

where \( \{0, Z_{\text{max}}\} \) are absorbing boundaries. Accordingly, at \( \{0, Z_{\text{max}}\} \) the condition \( dz_t = 0 \) must be satisfied so \( \sigma_z = 0 \) and \( a_{\text{max}} = (r + \kappa)Z_{\text{max}}/r\gamma \) when \( z = Z_{\text{max}} \) and \( a = 0 \) when \( z = 0 \). Thus, the value function at the boundaries is given by

\[ f(0, s) = 0 \]
\[ f(Z_{\text{max}}, s) = \pi \left( a_{\text{max}}, \max \left\{ 0, \frac{a_{\text{max}} - \phi Z_{\text{max}}}{g} \right\} \right) \left( 1 - e^{-rs} \right) - e^{-rs} \frac{1}{2} CZ_{\text{max}}^2. \]

We solve the partial differential equation using finite differences. We introduce the uniform grid \( \{z_0, \ldots, z_{N_z}\} \) such that \( z_0 = Z_{\text{min}} \) and \( z_{N_z} = Z_{\text{max}} \) and \( z_{i+1} - z_i = h \) for all \( 0 \leq i \leq N_z - 1 \). Similarly, we consider the grid for the time variable \( \{s_0, \ldots, s_{N_s}\} \) such that \( s_0 = 0, s_{N_s} = T \) and \( s_{n+1} - s_n = \Delta s \) for all \( 0 \leq n \leq N_s - 1 \). We denote the value function at the grid point \((z_i, s_n)\) by \( f_i^n \).

Next, we discretize the infinitesimal generator using an upwind scheme: In particular, we use the discretization

\[ (L_{h}^{a, \sigma} f^{n+1})_i = \alpha(a, \sigma_z, z_i) f^{n+1}_{i+1} + \rho(a, \sigma_z, z_i) \frac{\sigma_z}{2h^2} f^{n+1}_{i-1} - (\alpha(a, \sigma_z, z_i) + \rho(a, \sigma_z, z_i) + r)f^{n+1}_i. \]

where

\[ \alpha(a, \sigma_z, z_i) = \begin{cases} \frac{\sigma_z}{2h^2} + \frac{(r+\kappa)z_i + ar\gamma(\sigma z_i - 1)}{h} & \text{if } \frac{\sigma_z}{2h^2} + \frac{(r+\kappa)z_i + ar\gamma(\sigma z_i - 1)}{h} > 0 \\ \frac{\sigma_z}{2h^2} & \text{otherwise} \end{cases} \]
\[ \rho(a, \sigma_z, z_i) = \begin{cases} \frac{\sigma_z}{2h^2} - \frac{(r+\kappa)z_i + ar\gamma(\sigma z_i - 1)}{h} & \text{if } \frac{\sigma_z}{2h^2} + \frac{(r+\kappa)z_i + ar\gamma(\sigma z_i - 1)}{h} > 0 \\ \frac{\sigma_z}{2h^2} & \text{otherwise} \end{cases} \]

The previous discretization is such that whenever \( \frac{\sigma_z}{2h^2} + \frac{(r+\kappa)z_i + ar\gamma(\sigma z_i - 1)}{h} > 0 \) we have
\( (L_h^{a,\sigma z} f^{n+1})_i = [(r + \kappa)z_i + ar\gamma(\sigma \sigma_z - 1)] \frac{f^{n+1}_i - f^{n+1}_{i+1}}{h} + \frac{1}{2} \sigma_z f^{n+1}_i - 2f^{n+1}_{i-1} + f^{n+1}_{i-1} - rf^{n+1}_i. \)

and whenever \( \frac{\sigma_z^2}{2h^2} + \frac{(r + \kappa)z_i + ar\gamma(\sigma \sigma_z - 1)}{h} \leq 0 \) we have

\[ (L_h^{a,\sigma z} f^{n+1})_i = [(r + \kappa)z_i + ar\gamma(\sigma \sigma_z - 1)] \frac{f^{n+1}_i - f^{n+1}_{i+1}}{h} + \frac{1}{2} \sigma_z f^{n+1}_i - 2f^{n+1}_{i-1} + f^{n+1}_{i-1} - rf^{n+1}_i. \]

That is, the discretization \( L_h^{a,\sigma z} \) switches between forward and backward difference approximation of the derivatives to guarantee that the coefficients \( \alpha(a, \sigma_z, z_i) \) and \( \rho(a, \sigma_z, z_i) \) are always positive. The finite difference version of the HJB equation is

\[ \frac{f^{n+1}_i - f^n_i}{\Delta s} = \max_{a, \sigma z} \left\{ (L_h^{a,\sigma z} f^{n+1})_i + \pi (a, z_i) \right\}, \] (52)

The previous discretization is stable, consistent, and monotone, which means that the discretization (52) converges to the unique viscosity solution (Barles and Souganidis, 1991). The combination of forward and backward approximations in the discretized operator \( L_h^{a,\sigma z} \) makes difficult to solve the maximization problem in (52) analytically. We discretize the control set and consider controls in the grids \( a \in \{0, \ldots, a_K\} \) and \( \sigma_z \in \{\sigma_{z0}, \ldots, \sigma_{zK}\} \). Notice that once we have approximated the value function, we can use the closed form solution for the maximization in the original HJB equation to get a more precise approximation of the policy function. In the numerical implementation we use a uniform grid for \( a \) and a non-uniform grid for \( \sigma_z \); in particular, in the case of \( \sigma_z \) we use a grid that is finer near \( \sigma_z = 0 \).

The use of an implicit scheme in (52) requires to solve a highly nonlinear equation at each time step: We solve this nonlinear equation using policy iteration. Let \( f^n = (f^n_0, \ldots, f^n_{N_z}) \) be the value function at the time step \( n \). For \( n = 0 \) we set

\[ f^0_i = -\frac{1}{2} C z_i^2, \]

and for each time step \( n \geq 1 \), we use the following policy iteration algorithm

1. Set \((f^{n+1})^0 = f^n\)
2. Set \( \hat{f}^k = (f^{n+1})^k \)
3. For each $i \in \{1, \ldots, N_z - 1\}$, set

$$(a^k_i, \sigma^k_{zi}) = \arg \max_{a, \sigma} \left\{ (L^a_h \hat{f}^k)_i + \pi (a, z_i) \right\}$$

4. Find $(f^{n+1})^{k+1}$ solving the linear system of equations

$$(1 - \Delta s L^a_h \sigma^k_{zi}) (f^{n+1})^{k+1} = f^n_i + \Delta s \pi (a^k_i, z_i), \quad i = 1, \ldots, N_z - 1$$

$$(f^n_{0})^{k+1} = 0$$

$$(f^n_{N_z})^{k+1} = f(Z_{\text{max}}, s_{n+1}),$$

5. Stop if

$$\max_i \frac{|(f^n_{i})^{k+1} - \hat{f}^k_i|}{\max (1, |(f^n_{i})^{k+1}|)} < \text{tolerance},$$

otherwise go back to step 2.

The previous policy iteration algorithm converges monotonically to the unique solution of the nonlinear equation.