Supplement to "Inference for Subvectors and Other Functions of Partially Identified Parameters in Moment Inequality Models"*

Federico A. Bugni
Department of Economics
Duke University
federico.bugni@duke.edu

Ivan A. Canay
Department of Economics
Northwestern University
iacanay@northwestern.edu

Xiaoxia Shi
Department of Economics
University of Wisconsin, Madison
xshi@ssc.wisc.edu

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Abstract

This document provides auxiliary lemmas and their proofs for the authors' paper "Inference for Subvectors and Other Functions of Partially Identified Parameters in Moment Inequality Models".

KEYWORDS: Partial Identification, Moment Inequalities, Subvector Inference, Hypothesis Testing.

JEL CLASSIFICATION: C01, C12, C15.

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S.1 Notation

Throughout the Appendix we employ the notation defined in Bugni et al. (2016, Section A). For the reader's convenience, we restate the main elements in the table below.

$\mathcal{P}_0$	$\{F \in \mathcal{P} : \Theta_I(F) \neq \emptyset\}$
$\Sigma_F(\theta)$	$\mathrm{Var}_F(m(W,\theta))$
$D_F(\theta)$	$\mathrm{diag}(\Sigma_F(\theta))$
$Q_F(\theta)$	$S(E_F[m(W,\theta)], \Sigma_F(\theta))$
$\Theta_I^{\ln \kappa_n}(F)$	$\{\theta \in \Theta : S(\sqrt{n} E_F[m(W,\theta)], \Sigma_F(\theta)) \leq \ln \kappa_n\}$
$\Theta_I(F,\lambda)$	$\Theta(\lambda) \cap \Theta_I(F)$
$\Gamma_{n,F}(\lambda)$	$\{(\theta,\ell) \in \Theta(\lambda) \times \mathbb{R}^k : \ell = \sqrt{n} D_F^{-1/2}(\theta) E_F[m(W_i,\theta)]\}$
$\Gamma_{b_n,F}^{\mathrm{SS}}(\lambda)$	$\{(\theta,\ell) \in \Theta(\lambda) \times \mathbb{R}^k : \ell = \sqrt{b_n} D_F^{-1/2}(\theta) E_F[m(W,\theta)]\}$
$\Gamma_{n,F}^{\mathrm{PR}}(\lambda)$	$\{(\theta,\ell) \in \Theta(\lambda) \times \mathbb{R}^k : \ell = \kappa_n^{-1}\sqrt{n} D_F^{-1/2}(\theta) E_F[m(W_i,\theta)]\}$
$\Gamma_{n,F}^{\mathrm{DR}}(\lambda)$	$\{(\theta,\ell) \in \Theta(\lambda) \cap \Theta_I^{\ln \kappa_n}(F) \times \mathbb{R}^k : \ell = \kappa_n^{-1}\sqrt{n} D_F^{-1/2}(\theta) E_F[m(W,\theta)]\}$
$v_{n,j}(\theta)$	$n^{-1/2}\sigma_{F,j}^{-1}(\theta)\sum_{i=1}^n (m_j(W_i,\theta) - E_F[m_j(W_i,\theta)]), \quad j = 1,\ldots,k$
$\Omega_F(\theta,\theta')_{[j_1,j_2]}$	$E_F\left[\left(\frac{m_{j_1}(W,\theta)-E_F[m_{j_1}(W,\theta)]}{\sigma_{F,j_1}(\theta)}\right)\left(\frac{m_{j_2}(W,\theta')-E_F[m_{j_2}(W,\theta')]}{\sigma_{F,j_2}(\theta')}\right)\right]$

Table 1: Important Notation

S.2 Auxiliary Theorems

Theorem S.2.1. *Let $\Gamma_{n,F}^{\mathrm{PR}}(\lambda)$ be as in Table 1 and $T_n^{PR}(\lambda)$ be as in (4.6). Let $\{(\lambda_n, F_n) \in \mathcal{L}_0\}_{n\geq 1}$ be a (sub)sequence of parameters such that for some $(\Gamma^{\mathrm{PR}}, \Omega) \in \mathcal{S}(\Theta \times \mathbb{R}^k_{[\pm\infty]}) \times \mathcal{C}(\Theta^2)$: (i) $\Omega_{F_n} \xrightarrow{u} \Omega$ and (ii) $\Gamma_{n,F_n}^{\mathrm{PR}}(\lambda_n) \xrightarrow{H} \Gamma^{\mathrm{PR}}$. Then, there exists a further subsequence $\{u_n\}_{n\geq 1}$ of $\{n\}_{n\geq 1}$ such that, along $\{F_{u_n}\}_{n\geq 1}$,*

$$\{T_{u_n}^{PR}(\lambda_{u_n}) | \{W_i\}_{i=1}^n\} \xrightarrow{d} J(\Gamma^{\mathrm{PR}}, \Omega) \equiv \inf_{(\theta,\ell)\in\Gamma^{\mathrm{PR}}} S(v_\Omega(\theta) + \ell, \Omega(\theta)), \ \ a.s. \ ,$$

where $v_\Omega : \Theta \to \mathbb{R}^k$ is a tight Gaussian process with covariance (correlation) kernel Ω.

Theorem S.2.2. *Let $\Gamma_{n,F}^{\mathrm{PR}}(\lambda)$ and $\Gamma_{n,F}^{\mathrm{DR}}(\lambda)$ be as in Table 1. Let $T_n^{PR}(\lambda)$ be as in (4.6) and define*

$$\tilde{T}_n^{DR}(\lambda) \equiv \inf_{\theta \in \Theta(\lambda)\cap\Theta_I^{\ln\kappa_n}(F)} S(v_n^*(\theta) + \varphi^*(\kappa_n^{-1}\sqrt{n}\hat{D}_n^{-1/2}(\theta)\bar{m}_n(\theta)), \hat{\Omega}_n(\theta)) \ , \tag{S.1}$$

where $v_n^*(\theta)$ is as in (2.8), $\varphi^*(\cdot)$ as in Assumption A.1, and $\Theta_I^{\ln\kappa_n}(F)$ as in Table 1. Let $\{(\lambda_n, F_n) \in \mathcal{L}_0\}_{n\geq 1}$ be a (sub)sequence of parameters such that for some $(\Gamma^{\mathrm{DR}}, \Gamma^{\mathrm{PR}}, \Omega) \in \mathcal{S}(\Theta \times \mathbb{R}^k_{[\pm\infty]})^2 \times \mathcal{C}(\Theta^2)$: (i) $\Omega_{F_n} \xrightarrow{u} \Omega$, (ii) $\Gamma_{n,F_n}^{\mathrm{DR}}(\lambda_n) \xrightarrow{H} \Gamma^{\mathrm{DR}}$, and (iii) $\Gamma_{n,F_n}^{\mathrm{PR}}(\lambda_n) \xrightarrow{H} \Gamma^{\mathrm{PR}}$. Then, there exists a further subsequence $\{u_n\}_{n\geq 1}$ of $\{n\}_{n\geq 1}$ such that, along $\{F_{u_n}\}_{n\geq 1}$,

$$\{\min\{\tilde{T}_{u_n}^{DR}(\lambda_{u_n}), T_{u_n}^{PR}(\lambda_{u_n})\} | \{W_i\}_{i=1}^n\} \xrightarrow{d} J(\Gamma^{\mathrm{MR}}, \Omega) \equiv \inf_{(\theta,\ell)\in\Gamma^{\mathrm{MR}}} S(v_\Omega(\theta) + \ell, \Omega(\theta)), \ \ a.s. \ ,$$

where $v_\Omega : \Theta \to \mathbb{R}^k$ *is a tight Gaussian process with covariance (correlation) kernel* Ω*,*

$$\Gamma^{\mathrm{MR}} \equiv \Gamma^{\mathrm{DR}}_{*} \cup \Gamma^{\mathrm{PR}} \quad \text{and} \quad \Gamma^{\mathrm{DR}}_{*} \equiv \{(\theta,\ell) \in \Theta \times \mathbb{R}^k_{[\pm\infty]} : \ell = \varphi^*(\ell') \text{ for some } (\theta,\ell') \in \Gamma^{\mathrm{DR}}\} \ . \tag{S.2}$$

Theorem S.2.3. *Let* $\Gamma^{\mathrm{SS}}_{b_n,F}(\lambda)$ *be as in Table 1 and* $T^{SS}_{b_n}(\lambda)$ *be the subsampling test statistic. Let* $\{(\lambda_n, F_n) \in \mathcal{L}_0\}_{n\geq 1}$ *be a (sub)sequence of parameters such that for some* $(\Gamma^{\mathrm{SS}}, \Omega) \in \mathcal{S}(\Theta \times \mathbb{R}^k_{[\pm\infty]}) \times \mathcal{C}(\Theta^2)$*: (i)* $\Omega_{F_n} \overset{u}{\to} \Omega$ *and (ii)* $\Gamma^{\mathrm{SS}}_{b_n,F_n}(\lambda_n) \overset{H}{\to} \Gamma^{\mathrm{SS}}$*. Then, there exists a further subsequence* $\{u_n\}_{n\geq 1}$ *of* $\{n\}_{n\geq 1}$ *such that, along* $\{F_{u_n}\}_{n\geq 1}$*,*

$$\{T^{SS}_{u_n}(\lambda_{u_n}) | \{W_i\}_{i=1}^n\} \overset{d}{\to} J(\Gamma^{\mathrm{SS}}, \Omega) \equiv \inf_{(\theta,\ell)\in\Gamma^{\mathrm{SS}}} S(v_\Omega(\theta) + \ell, \Omega(\theta,\theta)), \ \text{a.s.} \ ,$$

where $v_\Omega : \Theta \to \mathbb{R}^k$ *is a tight Gaussian process with covariance (correlation) kernel* Ω*.*

Theorem S.2.4. *Let* $\Gamma_{n,F}(\lambda)$ *be as in Table 1 and* $T_n(\lambda)$ *be as in (4.1). Let* $\{(\lambda_n, F_n) \in \mathcal{L}_0\}_{n\geq 1}$ *be a (sub)sequence of parameters such that for some* $(\Gamma, \Omega) \in \mathcal{S}(\Theta \times \mathbb{R}^k_{[\pm\infty]}) \times \mathcal{C}(\Theta^2)$*: (i)* $\Omega_{F_n} \overset{u}{\to} \Omega$ *and (ii)* $\Gamma_{n,F_n}(\lambda_n) \overset{H}{\to} \Gamma$*. Then, there exists a further subsequence* $\{u_n\}_{n\geq 1}$ *of* $\{n\}_{n\geq 1}$ *such that, along* $\{F_{u_n}\}_{n\geq 1}$*,*

$$T_{u_n}(\lambda_{u_n}) \overset{d}{\to} J(\Gamma, \Omega) \equiv \inf_{(\theta,\ell)\in\Gamma} S(v_\Omega(\theta) + \ell, \Omega(\theta)), \ \text{as } n \to \infty \ ,$$

where $v_\Omega : \Theta \to \mathbb{R}^k$ *is a tight Gaussian process with zero-mean and covariance (correlation) kernel* Ω*.*

S.3 Auxiliary Lemmas

Lemma S.3.1. *Let* $\{F_n \in \mathcal{P}\}_{n\geq 1}$ *be a (sub)sequence of distributions s.t.* $\Omega_{F_n} \overset{u}{\to} \Omega$ *for some* $\Omega \in \mathcal{C}(\Theta^2)$*. Then,*

1. $v_n \overset{d}{\to} v_\Omega$ *in* $l^\infty(\Theta)$*, where* $v_\Omega : \Theta \to \mathbb{R}^k$ *is a tight zero-mean Gaussian process with covariance (correlation) kernel* Ω*. In addition,* v_Ω *is a uniformly continuous function, a.s.*
2. $\hat{\Omega}_n \overset{p}{\to} \Omega$ *in* $l^\infty(\Theta)$*.*
3. $D^{-1/2}_{F_n}(\cdot)\hat{D}^{1/2}_n(\cdot) - I_k \overset{p}{\to} \boldsymbol{0}_{k\times k}$ *in* $l^\infty(\Theta)$*.*
4. $\hat{D}^{-1/2}_n(\cdot)D^{1/2}_{F_n}(\cdot) - I_k \overset{p}{\to} \boldsymbol{0}_{k\times k}$ *in* $l^\infty(\Theta)$*.*
5. *For any arbitrary sequence* $\{a_n \in \mathbb{R}_{++}\}_{n\geq 1}$ *s.t.* $a_n \to \infty$*,* $a_n^{-1} v_n \overset{p}{\to} \boldsymbol{0}_k$ *in* $l^\infty(\Theta)$*.*
6. *For any arbitrary sequence* $\{a_n \in \mathbb{R}_{++}\}_{n\geq 1}$ *s.t.* $a_n \to \infty$*,* $a_n^{-1} \tilde{v}_n \overset{p}{\to} \boldsymbol{0}_k$ *in* $l^\infty(\Theta)$*.*
7. $\{v^*_n | \{W_i\}_{i=1}^n\} \overset{d}{\to} v_\Omega$ *in* $l^\infty(\Theta)$ *a.s., where* v_Ω *is the tight Gaussian process described in part 1.*
8. $\{\tilde{v}^{SS}_{b_n} | \{W_i\}_{i=1}^n\} \overset{d}{\to} v_\Omega$ *in* $l^\infty(\Theta)$ *a.s., where*

$$\tilde{v}^{SS}_{b_n}(\theta) \equiv \frac{1}{\sqrt{b_n}} \sum_{i=1}^{b_n} D^{-1/2}_{F_n}(\theta)(m(W^{SS}_i, \theta) - \bar{m}_n(\theta)) \ , \tag{S.1}$$

$\{W^{SS}_i\}_{i=1}^{b_n}$ *is a subsample of size* b_n *from* $\{W_i\}_{i=1}^n$*, and* v_Ω *is the tight Gaussian process described in part 1.*

9. For $\tilde{\Omega}^{SS}_{b_n}(\theta) \equiv D^{-1/2}_{F_n}(\theta)\hat{\Sigma}^{SS}_{b_n}(\theta)D^{-1/2}_{F_n}(\theta)$, $\{\tilde{\Omega}^{SS}_{b_n}|\{W_i\}_{i=1}^n\} \overset{p}{\to} \Omega$ in $l^\infty(\Theta)$ a.s.

Lemma S.3.2. *For any sequence $\{(\lambda_n, F_n) \in \mathcal{L}\}_{n\geq 1}$ there exists a subsequence $\{u_n\}_{n\geq 1}$ of $\{n\}_{n\geq 1}$ s.t. $\Omega_{F_{u_n}} \overset{u}{\to} \Omega$, $\Gamma_{u_n,F_{u_n}}(\lambda_{u_n}) \overset{H}{\to} \Gamma$, $\Gamma^{\mathrm{PR}}_{u_n,F_{u_n}}(\lambda_{u_n}) \overset{H}{\to} \Gamma^{\mathrm{PR}}$, and $\Gamma^{\mathrm{DR}}_{u_n,F_{u_n}}(\lambda_{u_n}) \overset{H}{\to} \Gamma^{\mathrm{DR}}$, for some $(\Gamma, \Gamma^{\mathrm{DR}}, \Gamma^{\mathrm{PR}}, \Omega) \in \mathcal{S}(\Theta \times \mathbb{R}^k_{[\pm\infty]})^3 \times \mathcal{C}(\Theta^2)$, where $\Gamma_{n,F_n}(\lambda)$, $\Gamma^{\mathrm{DR}}_{n,F_n}(\lambda)$, and $\Gamma^{\mathrm{PR}}_{n,F_n}(\lambda)$ are defined in Table 1.*

Lemma S.3.3. *Let $\{F_n \in \mathcal{P}\}_{n\geq 1}$ be an arbitrary (sub)sequence of distributions and let $X_n(\theta) : \Omega \to l^\infty(\Theta)$ be any stochastic process such that $X_n \overset{p}{\to} 0$ in $l^\infty(\Theta)$. Then, there exists a subsequence $\{u_n\}_{n\geq 1}$ of $\{n\}_{n\geq 1}$ such that $X_{u_n} \overset{a.s.}{\to} 0$ in $l^\infty(\Theta)$.*

Lemma S.3.4. *Let the set A be defined as follows:*

$$A \equiv \left\{ x \in \mathbb{R}^p_{[+\infty]} \times \mathbb{R}^{k-p} : \max\left\{ \max_{j=1,\ldots,p}\{[x_j]_-\}, \max_{s=p+1,\ldots,k}\{|x_s|\}\right\} = 1 \right\} . \tag{S.2}$$

Then, $\inf_{(x,\Omega)\in A\times\Psi} S(x, \Omega) > 0$.

Lemma S.3.5. *If $S(x, \Omega) \leq 1$ then there exist a constant $\varpi > 0$ such that $x_j \geq -\varpi$ for all $j \leq p$ and $|x_j| \leq \varpi$ for all $j > p$.*

Lemma S.3.6. *The function S satisfies the following properties: (i) $x \in (-\infty, \infty]^p \times \mathbb{R}^{k-p}$ implies $\sup_{\Omega\in\Psi} S(x, \Omega) < \infty$, (ii) $x \notin (-\infty, \infty]^p \times \mathbb{R}^{k-p}$ implies $\inf_{\Omega\in\Psi} S(x, \Omega) = \infty$.*

Lemma S.3.7. *Let $(\Gamma, \Gamma^{\mathrm{DR}}, \Omega) \in \mathcal{S}(\Theta \times \mathbb{R}^k_{[\pm\infty]})^2 \times \mathcal{C}(\Theta^2)$ be such that $\Omega_{F_n} \overset{u}{\to} \Omega$, $\Gamma_{n,F_n}(\lambda_n) \overset{H}{\to} \Gamma$, and $\Gamma^{\mathrm{DR}}_{n,F_n}(\lambda_n) \overset{H}{\to} \Gamma^{\mathrm{DR}}$, for some $\{(\lambda_n, F_n) \in \mathcal{L}_0\}_{n\geq 1}$. Then, Assumptions A.1 and A.3 imply that for all $(\theta, \ell) \in \Gamma^{\mathrm{DR}}$ there exists $(\theta, \tilde{\ell}) \in \Gamma$ with $\tilde{\ell}_j \geq \varphi^*_j(\ell_j)$ for $j \leq p$ and $\tilde{\ell}_j = \ell_j \equiv 0$ for $j > p$, where $\varphi^*(\cdot)$ is defined in Assumption A.1.*

Lemma S.3.8. *Let $(\Gamma, \Gamma^{\mathrm{PR}}, \Omega) \in \mathcal{S}(\Theta \times \mathbb{R}^k_{[\pm\infty]})^2 \times \mathcal{C}(\Theta^2)$ be such that $\Omega_{F_n} \overset{u}{\to} \Omega$, $\Gamma_{n,F_n}(\lambda_n) \overset{H}{\to} \Gamma$, and $\Gamma^{\mathrm{PR}}_{n,F_n}(\lambda_n) \overset{H}{\to} \Gamma^{\mathrm{PR}}$, for some $\{(\lambda_n, F_n) \in \mathcal{L}_0\}_{n\geq 1}$. Then, Assumption A.3 implies that for all $(\theta, \ell) \in \Gamma^{\mathrm{PR}}$ with $\ell \in \mathbb{R}^p_{[+\infty]} \times \mathbb{R}^{k-p}$, there exists $(\theta, \tilde{\ell}) \in \Gamma$ with $\tilde{\ell}_j \geq \ell_j$ for $j \leq p$ and $\tilde{\ell}_j = \ell_j$ for $j > p$.*

Lemma S.3.9. *Let Assumptions A.3-A.5 hold. For $\lambda_0 \in \Gamma$ and $\{\lambda_n \in \Gamma\}_{n\geq 1}$ as in Assumption A.5, assume that $\Omega_{F_n} \overset{u}{\to} \Omega$, $\Gamma_{n,F_n}(\lambda_0) \overset{H}{\to} \Gamma$, $\Gamma^{\mathrm{PR}}_{n,F_n}(\lambda_0) \overset{H}{\to} \Gamma^{\mathrm{PR}}$, $\Gamma^{\mathrm{SS}}_{b_n,F_n}(\lambda_0) \overset{H}{\to} \Gamma^{\mathrm{SS}}$, $\Gamma^{\mathrm{PR}}_{n,F_n}(\lambda_n) \overset{H}{\to} \Gamma^{\mathrm{PR}}_A$, and $\Gamma^{\mathrm{SS}}_{b_n,F_n}(\lambda_n) \overset{H}{\to} \Gamma^{\mathrm{SS}}_A$ for some $(\Gamma, \Gamma^{\mathrm{SS}}, \Gamma^{\mathrm{PR}}, \Gamma^{\mathrm{SS}}_A, \Gamma^{\mathrm{PR}}_A, \Omega) \in \mathcal{S}(\Theta \times \mathbb{R}^k_{[\pm\infty]})^5 \times \mathcal{C}(\Theta^2)$. Then,*

$$c_{(1-\alpha)}(\Gamma^{\mathrm{PR}}, \Omega) \leq c_{(1-\alpha)}(\Gamma^{\mathrm{SS}}, \Omega) \ .$$

Lemma S.3.10. *Let Assumptions A.3-A.7 hold. Then,*

$$\liminf_{n\to\infty} \left(E_{F_n}[\phi^{PR}_n(\lambda_0)] - E_{F_n}[\phi^{SS}_n(\lambda_0)]\right) \ > \ 0 \ .$$

S.4 Proofs of Theorems in Section S.2

Proof of Theorem S.2.1. Step 1. To simplify expressions, let $\Gamma^{\mathrm{PR}}_n \equiv \Gamma^{\mathrm{PR}}_{n,F_n}(\lambda_n)$. Consider the following derivation,

$$T^{PR}_n(\lambda_n) = \inf_{\theta\in\Theta(\lambda_n)} S\left(v^*_n(\theta) + \mu_{n,1}(\theta) + \mu_{n,2}(\theta)'\kappa_n^{-1}\sqrt{n}D^{-1/2}_{F_n}(\theta)E_{F_n}[m(W,\theta)], \hat{\Omega}_n(\theta)\right)$$

$$= \inf_{(\theta,\ell)\in\Gamma_n^{\rm PR}} S\left(v_n^*(\theta) + \mu_{n,1}(\theta) + \mu_{n,2}(\theta)'\ell, \hat{\Omega}_n(\theta)\right) ,$$

where $\mu_n(\theta) = (\mu_{n,1}(\theta), \mu_{n,2}(\theta))$, $\mu_{n,1}(\theta) \equiv \kappa_n^{-1}\tilde{v}_n(\theta)$, $\mu_{n,2}(\theta) \equiv \{\hat{\sigma}_{n,j}^{-1}(\theta)\sigma_{F_n,j}(\theta)\}_{j=1}^k$, and $\tilde{v}_n(\theta) \equiv \sqrt{n}\hat{D}_n^{-1}(\theta)(\bar{m}_n(\theta) - E_F[m(W,\theta)])$. Note that $\hat{D}_n^{-1/2}(\theta)$ and $D_{F_n}^{1/2}(\theta)$ are both diagonal matrices.

Step 2. We now show that there is a subsequence $\{a_n\}_{n\geq 1}$ of $\{n\}_{n\geq 1}$ s.t. $\{(v_{a_n}^*, \mu_{a_n}, \hat{\Omega}_{a_n})|\{W_i\}_{i=1}^{a_n}\} \overset{d}{\to} (v_\Omega, (\mathbf{0}_k, \mathbf{1}_k), \Omega)$ in $l^\infty(\theta)$ a.s. By part 8 in Lemma S.3.1, $\{v_n^*|\{W_i\}_{i=1}^n\} \overset{d}{\to} v_\Omega$ in $l^\infty(\theta)$. Then the result would follow from finding a subsequence $\{a_n\}_{n\geq 1}$ of $\{n\}_{n\geq 1}$ s.t. $\{(\mu_{a_n}, \hat{\Omega}_{a_n})|\{W_i\}_{i=1}^{a_n}\} \to ((\mathbf{0}_k, \mathbf{1}_k), \Omega)$ in $l^\infty(\theta)$ a.s. Since $(\mu_n, \hat{\Omega}_n)$ is conditionally non-random, this is equivalent to finding a subsequence $\{a_n\}_{n\geq 1}$ of $\{n\}_{n\geq 1}$ s.t. $(\mu_{a_n}, \hat{\Omega}_{a_n}) \overset{a.s.}{\to} ((\mathbf{0}_k, \mathbf{1}_k), \Omega)$ in $l^\infty(\theta)$. In turn, this follows from step 1, part 5 of Lemma S.3.1, and Lemma S.3.3.

Step 3. Since $\Theta_I(F_n, \lambda_n) \neq \emptyset$, there is a sequence $\{\theta_n \in \Theta(\lambda_n)\}_{n\geq 1}$ s.t. for $\ell_{n,j} \equiv \kappa_n^{-1}\sqrt{n}\sigma_{F_n,j}^{-1}(\theta_n)E_{F_n}[m_j(W,\theta_n)]$,

$$\limsup_{n\to\infty} \ell_{n,j} \equiv \bar{\ell}_j \geq 0, \quad \text{for } j \leq p, \qquad \text{and} \quad \lim_{n\to\infty} |\ell_{n,j}| \equiv \bar{\ell}_j = 0, \quad \text{for } j > p\ . \tag{S.1}$$

By compactness of $(\Theta \times \mathbb{R}_{[\pm\infty]}^k, d)$, there is a subsequence $\{k_n\}_{n\geq 1}$ of $\{a_n\}_{n\geq 1}$ s.t. $d((\theta_{k_n}, \ell_{k_n}), (\bar{\theta}, \bar{\ell})) \to 0$ for some $(\bar{\theta}, \bar{\ell}) \in \Theta \times \mathbb{R}_{+,\infty}^p \times \mathbf{0}_{k-p}$. By step 2, $\lim(v_{k_n}(\theta_{k_n}), \mu_{k_n}(\theta_{k_n}), \Omega_{k_n}(\theta_{k_n})) = (v_\Omega(\bar{\theta}), (\mathbf{0}_k, \mathbf{1}_k), \Omega(\bar{\theta}))$, and so

$$T_{k_n}^{PR}(\lambda_{k_n}) \leq S(v_{k_n}(\theta_{k_n}) + \mu_{k_n,1}(\theta_{k_n}) + \mu_{k_n,2}(\theta_{k_n})'\ell_{k_n}, \Omega_{k_n}(\theta_{k_n})) \to S(v_\Omega(\bar{\theta}) + \bar{\ell}, \Omega(\bar{\theta}))\ , \tag{S.2}$$

where the convergence occurs because by the continuity of $S(\cdot)$ and the convergence of its argument. Since $(v_\Omega(\bar{\theta}) + \bar{\ell}, \Omega(\bar{\theta})) \in \mathbb{R}_{[+\infty]}^p \times \mathbb{R}^{k-p} \times \Psi$, we conclude that $S(v_\Omega(\bar{\theta}) + \bar{\ell}, \Omega(\bar{\theta}))$ is bounded.

Step 4. Let $\mathcal{D}$ denote the space of functions that map Θ onto $\mathbb{R}^k \times \Psi$ and let $\mathcal{D}_0$ be the space of uniformly continuous functions that map Θ onto $\mathbb{R}^k \times \Psi$. Let the sequence of functionals $\{g_n\}_{n\geq 1}$ with $g_n : \mathcal{D} \to \mathbb{R}$ given by

$$g_n(v(\cdot), \mu(\cdot), \Omega(\cdot)) \equiv \inf_{(\theta,\ell)\in\Gamma_n^{\rm PR}} S(v(\theta) + \mu_1(\theta) + \mu_2(\theta)'\ell, \Omega(\theta))\ . \tag{S.3}$$

Let the functional $g : \mathcal{D}_0 \to \mathbb{R}$ be defined by

$$g(v(\cdot), \mu(\cdot), \Omega(\cdot)) \equiv \inf_{(\theta,\ell)\in\Gamma^{\rm PR}} S(v(\theta) + \mu_1(\theta) + \mu_2(\theta)'\ell, \Omega(\theta))\ .$$

We now show that if the sequence of (deterministic) functions $\{(v_n(\cdot), \mu_n(\cdot), \Omega_n(\cdot)) \in \mathcal{D}\}_{n\geq 1}$ satisfies

$$\lim_{n\to\infty} \sup_{\theta\in\Theta} ||(v_n(\theta), \mu_n(\theta), \Omega_n(\theta)) - (v(\theta), (\mathbf{0}_k, \mathbf{1}_k), \Omega(\theta))|| = 0\ , \tag{S.4}$$

for some $(v(\cdot), \Omega(\cdot)) \in \mathcal{D}_0$, then $\lim_{n\to\infty} g_n(v_n(\cdot), \mu_n(\cdot), \Omega_n(\cdot)) = g(v(\cdot), (\mathbf{0}_k, \mathbf{1}_k), \Omega(\cdot))$. To prove this we show that $\liminf_{n\to\infty} g_n(v_n(\cdot), \mu_n(\cdot), \Omega_n(\cdot)) \geq g(v(\cdot), (\mathbf{0}_k, \mathbf{1}_k), \Omega(\cdot))$. Showing the reverse inequality for the lim sup is similar and therefore omitted. Suppose not, i.e., suppose that $\exists \delta > 0$ and a subsequence $\{a_n\}_{n\geq 1}$ of $\{n\}_{n\geq 1}$ s.t. $\forall n \in \mathbb{N}$,

$$g_{a_n}(v_{a_n}(\cdot), \mu_{a_n}(\cdot), \Omega_{a_n}(\cdot)) < g(v(\cdot), (\mathbf{0}_k, \mathbf{1}_k), \Omega(\cdot)) - \delta\ . \tag{S.5}$$

By definition, $\exists\ \{(\theta_{a_n}, \ell_{a_n})\}_{n\geq 1}$ that approximates the infimum in (S.3), i.e., $\forall n \in \mathbb{N}$, $(\theta_{a_n}, \ell_{a_n}) \in \Gamma^{\text{PR}}_{a_n}$ and

$$|g_{a_n}(v_{a_n}(\cdot), \mu_{a_n}(\cdot), \Omega_{a_n}(\cdot)) - S(v_{a_n}(\theta_{a_n}) + \mu_1(\theta_{a_n}) + \mu_2(\theta_{a_n})'\ell_{a_n}, \Omega_{a_n}(\theta_{a_n}))| \leq \delta/2\ . \tag{S.6}$$

Since $\Gamma^{\text{PR}}_{a_n} \subseteq \Theta \times \mathbb{R}^k_{[\pm\infty]}$ and $(\Theta \times \mathbb{R}^k_{[\pm\infty]}, d)$ is a compact metric space, there exists a subsequence $\{u_n\}_{n\geq 1}$ of $\{a_n\}_{n\geq 1}$ and $(\theta^*, \ell^*) \in \Theta \times \mathbb{R}^k_{[\pm\infty]}$ s.t. $d((\theta_{u_n}, \ell_{u_n}), (\theta^*, \ell^*)) \to 0$. We first show that $(\theta^*, \ell^*) \in \Gamma^{\text{PR}}$. Suppose not, i.e. $(\theta^*, \ell^*) \notin \Gamma^{\text{PR}}$, and consider the following argument

$$\begin{aligned} d((\theta_{u_n}, \ell_{u_n}), (\theta^*, \ell^*)) + d_H(\Gamma^{\text{PR}}_{u_n}, \Gamma^{\text{PR}}) &\geq d((\theta_{u_n}, \ell_{u_n}), (\theta^*, \ell^*)) + \inf_{(\theta,\ell)\in\Gamma^{\text{PR}}} d((\theta, \ell), (\theta_{u_n}, \ell_{u_n})) \\ &\geq \inf_{(\theta,\ell)\in\Gamma^{\text{PR}}} d((\theta, \ell), (\theta^*, \ell^*)) > 0\ , \end{aligned}$$

where the first inequality follows from the definition of Hausdorff distance and the fact that $(\theta_{u_n}, \ell_{u_n}) \in \Gamma^{\text{PR}}_{u_n}$, and the second inequality follows by the triangular inequality. The final strict inequality follows from the fact that $\Gamma^{\text{PR}} \in \mathcal{S}(\Theta \times \mathbb{R}^k_{[\pm\infty]})$, i.e., it is a compact subset of $(\Theta \times \mathbb{R}^k_{[\pm\infty]}, d)$, $d((\theta, \ell), (\theta^*, \ell^*))$ is a continuous real-valued function, and Royden (1988, Theorem 7.18). Taking limits as $n \to \infty$ and using that $d((\theta_{u_n}, \ell_{u_n}), (\theta^*, \ell^*)) \to 0$ and $\Gamma^{PR}_{u_n} \overset{H}{\to} \Gamma^{\text{PR}}$, we reach a contradiction.

We now show that $\ell^* \in \mathbb{R}^p_{[+\infty]} \times \mathbb{R}^{k-p}$. Suppose not, i.e., suppose that $\exists j = 1, \ldots, k$ s.t. $l^*_j = -\infty$ or $\exists j > p$ s.t. $\ell^*_j = \infty$. Let $\mathbf{J}$ denote the set of indices $j = 1, \ldots, k$ s.t. this occurs. For any $\ell \in \mathbb{R}^k_{[\pm\infty]}$ define $\Xi(\ell) \equiv \max_{j\in\mathbf{J}} ||\ell_j||$. By definition of $\Gamma^{\text{PR}}_{u_n,F_{u_n}}$, $\ell_{u_n} \in \mathbb{R}^k$ and thus, $\Xi(\ell_{u_n}) < \infty$. By the case under consideration, $\lim \Xi(\ell_{u_n}) = \Xi(\ell^*) = \infty$. Since $(\Theta, ||\cdot||)$ is a compact metric space, $d((\theta_{u_n}, \ell_{u_n}), (\theta^*, \ell^*)) \to 0$ implies that $\theta_{u_n} \to \theta^*$. Then,

$$\begin{aligned} &||(v_{u_n}(\theta_{u_n}), \mu_{u_n}(\theta_{u_n}), \Omega_{u_n}(\theta_{u_n})) - (v(\theta^*), (\mathbf{0}_k, \mathbf{1}_k), \Omega(\theta^*))|| \\ &\leq ||(v_{u_n}(\theta_{u_n}), \mu_{u_n}(\theta_{u_n}), \Omega_{u_n}(\theta_{u_n})) - (v(\theta_{u_n}), (\mathbf{0}_k, \mathbf{1}_k), \Omega(\theta_{u_n}))|| + ||(v(\theta_{u_n}), \Omega(\theta_{u_n})) - (v(\theta^*), \Omega(\theta^*))|| \\ &\leq \sup_{\theta\in\Theta} ||(v_{u_n}(\theta), \mu_{u_n}(\theta), \Omega_{u_n}(\theta)) - (v(\theta), (\mathbf{0}_k, \mathbf{1}_k), \Omega(\theta))|| + ||(v(\theta_{u_n}), \Omega(\theta_{u_n})) - (v(\theta^*), \Omega(\theta^*))|| \to 0\ , \end{aligned}$$

where the last convergence holds by (S.4), $\theta_{u_n} \to \theta^*$, and $(v(\cdot), \Omega(\cdot)) \in \mathcal{D}_0$.

Since $(v(\cdot), \Omega(\cdot)) \in \mathcal{D}_0$, the compactness of Θ implies that $(v(\theta^*), \Omega(\theta^*)$ is bounded. Since $\lim \Xi(\ell_{u_n}) = \Xi(\ell^*) = \infty$ and $\lim v_{u_n}(\theta_{u_n}) = v(\theta^*) \in \mathbb{R}^k$, it then follows that $\lim \Xi(\ell_{u_n})^{-1}||v_{u_n}(\theta_{u_n})|| = 0$. By construction, $\{\Xi(\ell_{u_n})^{-1}\ell_{u_n}\}_{n\geq 1}$ is s.t. $\lim \Xi(\ell_{u_n})^{-1}\,[\ell_{u_n,j}]_- = 1$ for some $j \leq p$ or $\lim \Xi(\ell_{u_n})^{-1}\,|\ell_{u_n,j}| = 1$ for some $j > p$. By this, it follows that $\{\Xi(\ell_{u_n})^{-1}(v_{u_n}(\theta_{u_n}) + \ell_{u_n}), \Omega_{u_n}(\theta_{u_n})\}_{n\geq 1}$ with $\lim \Omega_{u_n}(\theta_{u_n}) = \Omega(\theta^*) \in \Psi$ and $\lim \Xi(\ell_{u_n})^{-1}[v_{u_n,j}(\theta_{u_n}) + \ell_{u_n,j}]_- = 1$ for some $j \leq p$ or $\lim \Xi(h_{u_n})^{-1}\,|v_{u_n,j}(\theta_{u_n}) + \ell_{u_n,j}| = 1$ for some $j > p$. This implies that,

$$S(v_{u_n}(\theta_{u_n}) + \ell_{u_n}, \Omega_{u_n}(\theta_{u_n})) = \Xi(\ell_{u_n})^\chi S(\Xi(\ell_{u_n})^{-1}(v_{u_n}(\theta_{u_n}) + \ell_{u_n}), \Omega_{u_n}(\theta_{u_n})) \to \infty\ .$$

Since $\{(\theta_{u_n}, \ell_{u_n})\}_{n\geq 1}$ is a subsequence of $\{(\theta_{a_n}, \ell_{a_n})\}_{n\geq 1}$ that approximately achieves the infimum in (S.3),

$$g_n(v_n(\cdot), \mu_n(\cdot), \Sigma_n(\cdot)) \to \infty\ . \tag{S.7}$$

However, (S.7) violates step 3 and is therefore a contradiction.

We then know that $d((\theta_{a_n}, \ell_{a_n}), (\theta^*, \ell^*)) \to 0$ with $\ell^* \in \mathbb{R}^p_{[+\infty]} \times \mathbb{R}^{k-p}$. By repeating previous arguments, we conclude that $\lim(v_{u_n}(\theta_{u_n}), \mu_{u_n}(\theta_{u_n}), \Omega_{u_n}(\theta_{u_n})) = (v(\theta^*), (\mathbf{0}_k, \mathbf{1}_k), \Omega(\theta^*)) \in \mathbb{R}^k \times \Psi$. This implies that $\lim(v_{u_n}(\theta_{u_n}) + \mu_{u_n,1}(\theta_{u_n}) + \mu_{u_n,2}(\theta_{u_n})'\ell_{u_n}, \Omega_{u_n}(\theta_{u_n})) = (v(\theta^*) + \ell^*, \Omega(\theta^*)) \in (\mathbb{R}^k_{[\pm\infty]} \times \Psi)$, i.e., $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$||S(v_{u_n}(\theta_{u_n}) + \mu_{u_n,1}(\theta_{u_n}) + \mu_{u_n,2}(\theta_{u_n})'\ell_{u_n}, \Omega_{u_n}(\theta_{u_n})) - S(v(\theta^*) + \ell^*, \Omega(\theta^*))|| \leq \delta/2 \ . \tag{S.8}$$

By combining (S.6), (S.8), and the fact that $(\theta^*, \ell^*) \in \Gamma^{\text{PR}}$, it follows that $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$g_{u_n}(v_{u_n}(\cdot), \mu_{u_n}(\cdot), \Omega_{u_n}(\cdot)) \geq S(v_\Omega(\theta^*) + \ell^*, \Omega(\theta^*)) - \delta \geq g(v(\cdot), (\mathbf{0}_k, \mathbf{1}_k), \Omega(\cdot)) - \delta \ ,$$

which is a contradiction to (S.5).

Step 5. The proof is completed by combining the representation in step 1, the convergence result in step 2, the continuity result in step 4, and the extended continuous mapping theorem (see, e.g., van der Vaart and Wellner, 1996, Theorem 1.11.1). In order to apply this result, it is important to notice that parts 1 and 5 in Lemma S.3.1 and standard convergence results imply that $(v(\cdot), \Omega(\cdot)) \in \mathcal{D}_0$ a.s. □

Proof of Theorem S.2.2. Step 1. To simplify expressions let $\Gamma_n^{\text{PR}} \equiv \Gamma^{\text{PR}}_{n,F_n}(\lambda_n)$, $\Gamma_n^{\text{DR}} \equiv \Gamma^{\text{DR}}_{n,F_n}(\lambda_n)$, and consider the following derivation,

$$\begin{aligned}
&\min\{\tilde{T}_n^{DR}(\lambda_n), T_n^{PR}(\lambda_n)\} \\
&= \min\left\{ \begin{array}{c} \inf_{\theta \in \Theta(\lambda_n) \cap \Theta_I^{\ln \kappa_n}(F_n)} S(v_n^*(\theta) + \varphi^*(\kappa_n^{-1}\sqrt{n}\hat{D}_n^{-1/2}(\theta)\bar{m}_n(\theta)), \hat{\Omega}_n(\theta)), \\ \inf_{\theta \in \Theta(\lambda_n)} S(v_n^*(\theta) + \kappa_n^{-1}\sqrt{n}\hat{D}_n^{-1/2}(\theta)\bar{m}_n(\theta), \hat{\Omega}_n(\theta)) \end{array} \right\} \\
&= \min\left\{ \begin{array}{c} \inf_{\theta \in \Theta(\lambda_n) \cap \Theta_I^{\ln \kappa_n}(F_n)} S(v_n^*(\theta) + \varphi^*(\mu_{n,1}(\theta) + \mu_{n,2}(\theta)'\kappa_n^{-1}D_{F_n}^{-1/2}(\theta)\sqrt{n}(E_{F_n}m(W,\theta))), \hat{\Omega}_n(\theta)), \\ \inf_{\theta \in \Theta(\lambda_n)} S(v_n^*(\theta) + \mu_{n,1}(\theta) + \mu_{n,2}(\theta)'\kappa_n^{-1}D_{F_n}^{-1/2}(\theta)\sqrt{n}(E_{F_n}m(W,\theta)), \hat{\Omega}_n(\theta)) \end{array} \right\} \\
&= \min\left\{ \inf_{(\theta,\ell) \in \Gamma_n^{\text{DR}}} S(v_n^*(\theta) + \varphi^*(\mu_{n,1}(\theta) + \mu_{n,2}(\theta)'\ell), \hat{\Omega}_n(\theta)), \inf_{(\theta,\ell) \in \Gamma_n^{\text{PR}}} S(v_n^*(\theta) + \mu_{n,1}(\theta) + \mu_{n,2}(\theta)'\ell, \hat{\Omega}_n(\theta)) \right\}
\end{aligned}$$

where $\mu_n(\theta) \equiv (\mu_{n,1}(\theta), \mu_{n,2}(\theta))$, $\mu_{n,1}(\theta) \equiv \kappa_n^{-1}\hat{D}_n^{-1/2}(\theta)\sqrt{n}(\bar{m}_n(\theta) - E_{F_n}m(W,\theta)) \equiv \kappa_n^{-1}\tilde{v}_n(\theta)$, and $\mu_{n,2}(\theta) \equiv \{\sigma_{n,j}^{-1}(\theta)\sigma_{F_n,j}(\theta)\}_{j=1}^k$. Note that we used that $D_{F_n}^{-1/2}(\theta)$ and $\hat{D}_n^{-1/2}(\theta)$ are both diagonal matrices.

Step 2. There is a subsequence $\{a_n\}_{n\geq1}$ of $\{n\}_{n\geq1}$ s.t. $\{(\hat{v}^*_{a_n}, \mu_{a_n}, \hat{\Omega}_{a_n})|\{W_i\}_{i=1}^{a_n}\} \to^d (v_\Omega, (\mathbf{0}_k, \mathbf{1}_k), \Omega)$ in $l^\infty(\Theta)$ a.s. This step is identical to Step 2 in the proof of Theorem S.2.1.

Step 3. Let $\mathcal{D}$ denote the space of bounded functions that map Θ onto $\mathbb{R}^{2k} \times \Psi$ and let $\mathcal{D}_0$ be the space of bounded uniformly continuous functions that map Θ onto $\mathbb{R}^{2k} \times \Psi$. Let the sequence of functionals $\{g_n\}_{n\geq1}$, $\{g_n^1\}_{n\geq1}$, $\{g_n^2\}_{n\geq1}$ with $g_n : \mathcal{D} \to \mathbb{R}$, $g_n^1 : \mathcal{D} \to \mathbb{R}$, and $g_n^2 : \mathcal{D} \to \mathbb{R}$ be defined by

$$\begin{aligned}
g_n(v(\cdot), \mu(\cdot), \Omega(\cdot)) &\equiv \min\left\{g_n^1(v(\cdot), \mu(\cdot), \Omega(\cdot)), g_n^2(v(\cdot), \mu(\cdot), \Omega(\cdot))\right\} \\
g_n^1(v(\cdot), \mu(\cdot), \Omega(\cdot)) &\equiv \inf_{(\theta,\ell) \in \Gamma_n^{\text{DR}}} S(v_n^*(\theta) + \varphi^*(\mu_{n,1}(\theta) + \mu_{n,2}(\theta)'\ell), \Omega(\theta)) \\
g_n^2(v(\cdot), \mu(\cdot), \Omega(\cdot)) &\equiv \inf_{(\theta,\ell) \in \Gamma_n^{\text{PR}}} S(v_n^*(\theta) + \mu_{n,1}(\theta) + \mu_{n,2}(\theta)'\ell, \Omega(\theta)) \ .
\end{aligned}$$

Let the functional $g : \mathcal{D}_0 \to \mathbb{R}$, $g^1 : \mathcal{D}_0 \to \mathbb{R}$, and $g^2 : \mathcal{D}_0 \to \mathbb{R}$ be defined by:

$$\begin{aligned} g(v(\cdot),\mu(\cdot),\Omega(\cdot)) &\equiv \min\left\{g^1(v(\cdot),\mu(\cdot),\Omega(\cdot)), g^2(v(\cdot),\mu(\cdot),\Omega(\cdot))\right\} \\ g^1(v(\cdot),\mu(\cdot),\Omega(\cdot)) &\equiv \inf_{(\theta,\ell)\in\Gamma^{\rm DR}} S(v_\Omega(\theta)+\varphi^*(\mu_1(\theta)+\mu_2(\theta)'\ell),\Omega(\theta)) \\ g^2(v(\cdot),\mu(\cdot),\Omega(\cdot)) &\equiv \inf_{(\theta,l)\in\Gamma^{\rm PR}} S(v_\Omega(\theta)+\mu_1(\theta)+\mu_2(\theta)'\ell,\Omega(\theta)) \ . \end{aligned}$$

If the sequence of deterministic functions $\{(v_n(\cdot),\mu_n(\cdot),\Omega_n(\cdot))\}_{n\geq 1}$ with $(v_n(\cdot),\mu_n(\cdot),\Omega_n(\cdot)) \in \mathcal{D}$ for all $n \in \mathbb{N}$ satisfies

$$\lim_{n\to\infty}\sup_{\theta\in\Theta}||(v_n(\theta),\mu_n(\theta),\Omega_n(\theta)) - (v_\Omega(\theta),(\mathbf{0}_k,\mathbf{1}_k),\Omega(\theta))|| = 0 \ ,$$

for some $(v(\cdot),(\mathbf{0}_k,\mathbf{1}_k),\Omega(\cdot)) \in \mathcal{D}_0$ then $\lim_{n\to\infty}||g^s_n(v_n(\cdot),\mu_n(\cdot),\Omega_n(\cdot)) - g^s(v(\cdot),(\mathbf{0}_k,\mathbf{1}_k),\Omega(\cdot))|| = 0$ for $s = 1,2$, respectively. This follows from similar steps to those in the proof of Theorem S.2.1, step 4. By continuity of the minimum function,

$$\lim_{n\to\infty}||g_n(v_n(\cdot),\mu_n(\cdot),\Omega_n(\cdot)) - g(v(\cdot),(\mathbf{0}_k,\mathbf{1}_k),\Omega(\cdot))|| = 0 \ .$$

Step 4. By combining the representation of $\min\{\tilde{T}^{DR}_n(\lambda_n),T^{PR}_n(\lambda_n)\}$ in step 1, the convergence results in steps 2 and 3, Theorem S.2.1, and the extended continuous mapping theorem (see, e.g., Theorem 1.11.1 of van der Vaart and Wellner (1996)) we conclude that

$$\{\min\{\tilde{T}^{DR}_n(\lambda_n),T^{PR}_n(\lambda_n)\}|\{W_i\}_{i=1}^n\} \overset{d}{\to} \min\left\{J(\Gamma^{\rm DR}_*,\Omega),J(\Gamma^{\rm PR},\Omega)\right\} \ a.s.,$$

where

$$J(\Gamma^{\rm DR}_*,\Omega) \equiv \inf_{(\theta,\ell)\in\Gamma^{\rm DR}_*} S(v_\Omega(\theta)+\ell,\Omega(\theta)) = \inf_{(\theta,\ell')\in\Gamma^{\rm DR}} S(v_\Omega(\theta)+\varphi^*(\ell'),\Omega(\theta)) \ . \tag{S.9}$$

The result then follows by noticing that,

$$\begin{aligned} \min\left\{J(\Gamma^{\rm DR}_*,\Omega),J(\Gamma^{\rm PR},\Omega)\right\} &= \min\left\{\inf_{(\theta,\ell)\in\Gamma^{\rm DR}_*} S(v_\Omega(\theta)+\ell,\Omega(\theta)), \inf_{(\theta,\ell)\in\Gamma^{\rm PR}} S(v_\Omega(\theta)+\ell,\Omega(\theta))\right\} \\ &= \inf_{(\theta,\ell)\in\Gamma^{\rm DR}_*\cup\Gamma^{\rm PR}} S(v_\Omega(\theta)+\ell,\Omega(\theta)) = J(\Gamma^{\rm MR},\Omega) \ . \end{aligned}$$

This completes the proof. □

Proof of Theorem S.2.3. This proof is similar to that of Theorem S.2.1. For the sake of brevity, we only provide a sketch that focuses on the main differences. From the definition of $T^{SS}_{b_n}(\lambda_n)$, we can consider the following derivation,

$$\begin{aligned} T^{SS}_{b_n}(\lambda_n) &\equiv \inf_{\theta\in\Theta(\lambda_n)} Q^{SS}_{b_n}(\theta) = \inf_{\theta\in\Theta(\lambda_n)} S(\sqrt{b_n}\bar{m}^{SS}_{b_n}(\theta),\hat{\Sigma}^{SS}_{b_n}(\theta)) \\ &= \inf_{(\theta,\ell)\in\Gamma^{\rm SS}_{b_n}} S(\tilde{v}^{SS}_{b_n}(\theta)+\mu_n(\theta)+\ell,\tilde{\Omega}^{SS}_{b_n}(\theta)) \ , \end{aligned}$$

where $\mu_n(\theta) \equiv \sqrt{b_n}D^{-1/2}_{F_n}(\theta)(\bar{m}_n(\theta) - E_{F_n}[m(W,\theta)])$, $\tilde{v}^{SS}_{b_n}(\theta)$ is as in (S.1), and $\tilde{\Omega}^{SS}_n(\theta) \equiv D^{-1/2}_{F_n}(\theta)\hat{\Sigma}^{SS}_{b_n}(\theta)D^{-1/2}_{F_n}(\theta)$. From here, we can repeat the arguments used in the proof of Theorem S.2.1.

The main difference in the argument is that the reference to parts 2 and 7 in Lemma S.3.1 need to be replaced by parts 9 and 8, respectively. □

Proof of Theorem S.2.4. The proof of this theorem follows by combining arguments from the proof of Theorem S.2.1 with those from Bugni et al. (2014, Theorem 3.1). It is therefore omitted. □

S.5 Proofs of Lemmas in Section S.3

We note that Lemmas S.3.2-S.3.5 correspond to Lemmas D3-D7 in Bugni et al. (2014) and so we do not include the proofs of those lemmas in this paper.

Proof of Lemma S.3.1. The proof of parts 1-7 follow from similar arguments to those used in the proof of Bugni et al. (2014, Theorem D.2). Therefore, we now focus on the proof of parts 8-9.

Part 9. By the argument used to prove Bugni et al. (2014, Theorem D.2 (part 1)), $\mathcal{M}(F) \equiv \{D_F^{-1/2}(\theta)m(\cdot,\theta) : \mathcal{W} \to \mathbb{R}^k\}$ is Donsker and pre-Gaussian, both uniformly in $F \in \mathcal{P}$. Thus, we can extend the arguments in the proof of van der Vaart and Wellner (1996, Theorem 3.6.13 and Example 3.6.14) to hold under a drifting sequence of distributions $\{F_n\}_{n\geq 1}$ along the lines of van der Vaart and Wellner (1996, Section 2.8.3). From this, it follows that:

$$\left\{\sqrt{\frac{n}{(n-b_n)}}\tilde{v}_{b_n}^{SS}(\theta)\bigg|\{W_i\}_{i=1}^n\right\} \overset{d}{\to} v_\Omega(\theta) \quad \text{in } l^\infty(\Theta) \quad a.s. \tag{S.1}$$

To conclude the proof, note that,

$$\sup_{\theta\in\Theta}\left\|\sqrt{\frac{n}{(n-b_n)}}\tilde{v}_{b_n}^{SS}(\theta)-\tilde{v}_{b_n}^{SS}(\theta)\right\| = \sup_{\theta\in\Theta}\|\tilde{v}_{b_n}^{SS}(\theta)\|\sqrt{\frac{b_n/n}{(1-b_n/n)}}\,.$$

In order to complete the proof, it suffices to show that the RHS of the previous equation is $o_p(1)$ a.s. In turn, this follows from $b_n/n = o(1)$ and (S.1) as they imply that $\{\sup_{\theta\in\Theta}\|\tilde{v}_{b_n}^{SS}(\theta)\||\{W_i\}_{i=1}^n\} = O_p(1)$ a.s.

Part 10. This result follows from considering the subsampling analogue of the arguments used to prove Bugni et al. (2014, Theorem D.2 (part 2)). □

Proof of Lemma S.3.6. Part 1. Suppose not, that is, suppose that $\sup_{\Omega\in\Psi} S(x,\Omega)=\infty$ for $x\in(-\infty,\infty]^p\times\mathbb{R}^{k-p}$. By definition, there exists a sequence $\{\Omega_n\in\Psi\}_{n\geq1}$ s.t. $S(x,\Omega_n)\to\infty$. By the compactness of Ψ, there exists a subsequence $\{k_n\}_{n\geq1}$ of $\{n\}_{n\geq1}$ s.t. $\Omega_{k_n}\to\Omega^*\in\Psi$. By continuity of S on $(-\infty,\infty]^p\times\mathbb{R}^{k-p}\times\Psi$ it then follows that $\lim S(x,\Omega_{k_n})=S(x,\Omega^*)=\infty$ for $(x,\Omega^*)\in(-\infty,\infty]^p\times\mathbb{R}^{k-p}\times\Psi$, which is a contradiction to $S:(-\infty,\infty]^p\times\mathbb{R}^{k-p}\to\mathbb{R}_+$.

Part 2. Suppose not, that is, suppose that $\sup_{\Omega\in\Psi} S(x,\Omega)=B<\infty$ for $x\notin(-\infty,\infty]^p\times\mathbb{R}^{k-p}$. By definition, there exists a sequence $\{\Omega_n\in\Psi\}_{n\geq1}$ s.t. $S(x,\Omega_n)\to\infty$. By the compactness of Ψ, there exists a subsequence $\{k_n\}_{n\geq1}$ of $\{n\}_{n\geq1}$ s.t. $\Omega_{k_n}\to\Omega^*\in\Psi$. By continuity of S on $\mathbb{R}^k_{[\pm\infty]}\times\Psi$ it then follows that $\lim S(x,\Omega_{k_n})=S(x,\Omega^*)=B<\infty$ for $(x,\Omega^*)\in\mathbb{R}^k_{[\pm\infty]}\times\Psi$. Let $\mathbf{J}\in\{1,\ldots k\}$ be set of coordinates s.t. $x_j=-\infty$ for $j\leq p$ or $|x_j|=\infty$ for $j>p$. By the case under consideration, there is at least one such coordinate. Define $M\equiv\max\{\max_{j\notin\mathbf{J},j\leq p}[x_j]_-,\max_{j\notin\mathbf{J},j>p}|x_j|\}<\infty$. For any $C>M$, let $x'(C)$ be defined as follows. For $j \notin \mathbf{J}$, set $x'_j(C) = x_j$ and for $j \in \mathbf{J}$, set $x'_j(C)$ as follows $x'_j(C) = -C$ for $j \leq p$ and $|x'_j(C)| = C$ for $j > p$. By definition, $\lim_{C\to\infty} x'(C) = x$ and by continuity properties of the function S, $\lim_{C\to\infty} S(x'(C), \Omega^*) = S(x, \Omega^*) = B < \infty$. By homogeneity properties of the function S and by Lemma S.3.4, we have that

$$S(x'(C), \Omega^*) = C^\chi S(C^{-1}x'(C), \Omega^*) \geq C^\chi \inf_{(x,\Omega)\in A\times\Psi} S(x, \Omega) > 0,$$

where A is the set in Lemma S.3.4. Taking $C \to \infty$ the RHS diverges to infinity, producing a contradiction. □

Proof of Lemma S.3.7. The result follows from similar steps to those in Bugni et al. (2014, Lemma D.10) and is therefore omitted. □

Proof of Lemma S.3.8. Let $(\theta, \ell) \in \Gamma^{\text{PR}}$ with $\ell \in \mathbb{R}^p_{[+\infty]} \times \mathbb{R}^{k-p}$. Then, there is a subsequence $\{a_n\}_{n\geq1}$ of $\{n\}_{n\geq1}$ and a sequence $\{(\theta_n, \ell_n)\}_{n\geq1}$ such that $\theta_n \in \Theta(\lambda_n)$, $\ell_n \equiv \kappa_n^{-1}\sqrt{n}D_{F_n}^{-1/2}(\theta_n)E_{F_n}[m(W, \theta_n)]$, $\lim_{n\to\infty} \ell_{a_n} = \ell$, and $\lim_{n\to\infty} \theta_{a_n} = \theta$. Also, by $\Omega_{F_n} \overset{u}{\to} \Omega$ we get $\Omega_{F_n}(\theta_n) \to \Omega(\theta)$. By continuity of $S(\cdot)$ at $(\ell, \Omega(\theta))$ with $\ell \in \mathbb{R}^p_{[+\infty]} \times \mathbb{R}^{k-p}$,

$$\kappa_{a_n}^{-\chi} a_n{}^{\chi/2} Q_{F_{a_n}}(\theta_{a_n}) = S(\kappa_{a_n}^{-1}\sqrt{a_n}\sigma^{-1}_{F_{a_n},j}(\theta_{a_n})E_{F_{a_n}}[m_j(W, \theta_{a_n})], \Omega_{F_{a_n}}(\theta_{a_n})) \to S(\ell, \Omega(\theta)) < \infty\ . \quad \text{(S.2)}$$

Hence $Q_{F_{a_n}}(\theta_{a_n}) = O(\kappa^\chi_{a_n} a_n{}^{-\chi/2})$. By this and Assumption A.3(a), it follows that

$$O(\kappa^\chi_{a_n} a_n{}^{-\chi/2}) = c^{-1}Q_{F_{a_n}}(\theta_{a_n}) \geq \min\{\delta, \inf_{\tilde{\theta}\in\Theta_I(F_{a_n},\lambda_{a_n})} ||\theta_{a_n} - \tilde{\theta}||\}^\chi \quad \Rightarrow \quad ||\theta_{a_n} - \tilde{\theta}_{a_n}|| \leq O(\kappa_{a_n}/\sqrt{a_n})\ , \quad \text{(S.3)}$$

for some sequence $\{\tilde{\theta}_{a_n} \in \Theta_I(F_{a_n}, \lambda_{a_n})\}_{n\geq1}$. By Assumptions A.3(b)-(c), the intermediate value theorem implies that there is a sequence $\{\theta^*_n \in \Theta(\lambda_n)\}_{n\geq1}$ with θ^*_n in the line between θ_n and $\tilde{\theta}_n$ such that

$$\kappa_n^{-1}\sqrt{n}D_{F_n}^{-1/2}(\theta_n)E_{F_n}[m(W, \theta_n)] = G_{F_n}(\theta^*_n)\kappa_n^{-1}\sqrt{n}(\theta_n - \tilde{\theta}_n) + \kappa_n^{-1}\sqrt{n}D_{F_n}^{-1/2}(\tilde{\theta}_n)E_{F_n}[m(W, \tilde{\theta}_n)]\ .$$

Define $\hat{\theta}_n \equiv (1 - \kappa_n^{-1})\tilde{\theta}_n + \kappa_n^{-1}\theta_n$ or, equivalently, $\hat{\theta}_n - \tilde{\theta}_n \equiv \kappa_n^{-1}(\theta_n - \tilde{\theta}_n)$. We can write the above equation as

$$G_{F_n}(\theta^*_n)\sqrt{n}(\hat{\theta}_n - \tilde{\theta}_n) = \kappa_n^{-1}\sqrt{n}D_{F_n}^{-1/2}(\theta_n)E_{F_n}[m(W, \theta_n)] - \kappa_n^{-1}\sqrt{n}D_{F_n}^{-1/2}(\tilde{\theta}_n)E_{F_n}[m(W, \tilde{\theta}_n)]\ . \quad \text{(S.4)}$$

By convexity of $\Theta(\lambda_n)$ and $\kappa_n^{-1} \to 0$, $\{\hat{\theta}_n \in \Theta(\lambda_n)\}_{n\geq1}$ and by (S.3), $\sqrt{a_n}||\hat{\theta}_{a_n} - \tilde{\theta}_{a_n}|| = O(1)$. By the intermediate value theorem again, there is a sequence $\{\theta^{**}_n \in \Theta(\lambda_n)\}_{n\geq1}$ with θ^{**}_n in the line between $\hat{\theta}_n$ and $\tilde{\theta}_n$ such that

$$\begin{aligned}\sqrt{n}D_{F_n}^{-1/2}(\hat{\theta}_n)E_{F_n}[m(W, \hat{\theta}_n)] &= G_{F_n}(\theta^{**}_n)\sqrt{n}(\hat{\theta}_n - \tilde{\theta}_n) + \sqrt{n}D_{F_n}^{-1/2}(\tilde{\theta}_n)E_{F_n}[m(W, \tilde{\theta}_n)] \\ &= G_{F_n}(\theta^*_n)\sqrt{n}(\hat{\theta}_n - \tilde{\theta}_n) + \sqrt{n}D_{F_n}^{-1/2}(\tilde{\theta}_n)E_{F_n}[m(W, \tilde{\theta}_n)] + \epsilon_{1,n}\ ,\end{aligned} \quad \text{(S.5)}$$

where the second equality holds by $\epsilon_{1,n} \equiv (G_{F_n}(\theta^{**}_n) - G_{F_n}(\theta^*_n))\sqrt{n}(\hat{\theta}_n - \tilde{\theta}_n)$. Combining (S.4) with (S.5) we get

$$\sqrt{n}D_{F_n}^{-1/2}(\hat{\theta}_n)E_{F_n}[m(W,\hat{\theta}_n)] = \kappa_n^{-1}\sqrt{n}D_{F_n}^{-1/2}(\theta_n)E_{F_n}[m(W,\theta_n)] + \epsilon_{1,n} + \epsilon_{2,n} \ , \tag{S.6}$$

where $\epsilon_{2,n} \equiv (1-\kappa_n^{-1})\sqrt{n}D_{F_n}^{-1/2}(\tilde{\theta}_n)E_{F_n}[m(W,\tilde{\theta}_n)]$. From $\{\tilde{\theta}_{a_n} \in \Theta_I(F_{a_n},\lambda_{a_n})\}_{n\geq 1}$ and $\kappa_n^{-1} \to 0$, it follows that $\epsilon_{2,a_n,j} \geq 0$ for $j \leq p$ and $\epsilon_{2,a_n,j} = 0$ for $j > p$. Moreover, Assumption A.3(c) implies that $||G_{F_{a_n}}(\theta_{a_n}^{**}) - G_{F_{a_n}}(\theta_{a_n}^{*})|| = o(1)$ for any sequence $\{F_{a_n} \in \mathcal{P}_0\}_{n\geq 1}$ whenever $||\theta_{a_n}^{*} - \theta_{a_n}^{**}|| = o(1)$. Using $\sqrt{a_n}||\hat{\theta}_{a_n} - \tilde{\theta}_{a_n}|| = O(1)$, we have

$$||\epsilon_{1,a_n}|| \leq ||G_{F_{a_n}}(\theta_{a_n}^{**}) - G_{F_{a_n}}(\theta_{a_n}^{*})||\sqrt{a_n}||\hat{\theta}_{a_n} - \tilde{\theta}_{a_n}|| = o(1) \ . \tag{S.7}$$

Finally, since $(\mathbb{R}^k_{[\pm\infty]}, d)$ is compact, there is a further subsequence $\{u_n\}_{n\geq 1}$ of $\{a_n\}_{n\geq 1}$ s.t. $\sqrt{u_n}D_{F_{u_n}}^{-1/2}(\hat{\theta}_{u_n})E_{F_{u_n}}[m(W,\hat{\theta}_{u_n})]$ and $\kappa_{u_n}^{-1}\sqrt{u_n}D_{F_{u_n}}^{-1/2}(\theta_{u_n})E_{F_{u_n}}[m(W,\theta_{u_n})]$ converge. Then, from (S.6), (S.7), and the properties of ϵ_{2,a_n} we conclude that

$$\lim_{n\to\infty} \tilde{\ell}_{u_n,j} \equiv \lim_{n\to\infty} \sqrt{u_n}\sigma_{F_{u_n},j}^{-1}(\hat{\theta}_{u_n})E_{F_{u_n}}[m_j(W,\hat{\theta}_{u_n})] \geq \lim_{n\to\infty} \kappa_{u_n}^{-1}\sqrt{u_n}\sigma_{F_{u_n},j}^{-1}(\theta_{u_n})E_{F_{u_n}}[m_j(W,\theta_{u_n})], \quad \text{for } j \leq p \ ,$$
$$\lim_{n\to\infty} \tilde{\ell}_{u_n,j} \equiv \lim_{n\to\infty} \sqrt{u_n}\sigma_{F_{u_n},j}^{-1}(\hat{\theta}_{u_n})E_{F_{u_n}}[m_j(W,\hat{\theta}_{u_n})] = \lim_{n\to\infty} \kappa_{u_n}^{-1}\sqrt{u_n}\sigma_{F_{u_n},j}^{-1}(\theta_{u_n})E_{F_{u_n}}[m_j(W,\theta_{u_n})], \quad \text{for } j > p \ ,$$

which completes the proof, as $\{(\hat{\theta}_{u_n},\tilde{\ell}_{u_n}) \in \Gamma_{u_n,F_{u_n}}(\lambda_{u_n})\}_{n\geq 1}$ and $\hat{\theta}_{u_n} \to \theta$. □

Proof of Lemma S.3.9. We divide the proof into four steps.

Step 1. We show that $\inf_{(\theta,\ell)\in\Gamma^{ss}} S(v_\Omega(\theta)+\ell,\Omega(\theta)) < \infty$ a.s. By Assumption A.5, there exists a sequence $\{\tilde{\theta}_n \in \Theta_I(F_n,\lambda_n)\}_{n\geq 1}$, where $d_H(\Theta(\lambda_n),\Theta(\lambda_0)) = O(n^{-1/2})$. Then, there exists another sequence $\{\theta_n \in \Theta(\lambda_0)\}_{n\geq 1}$ s.t. $\sqrt{n}||\theta_n - \tilde{\theta}_n|| = O(1)$ for all $n \in \mathbb{N}$. Since Θ is compact, there is a subsequence $\{a_n\}_{n\geq 1}$ s.t. $\sqrt{a_n}(\theta_{a_n} - \tilde{\theta}_{a_n}) \to \lambda \in \mathbb{R}^{d_\theta}$, and $\theta_{a_n} \to \theta^*$ and $\tilde{\theta}_{a_n} \to \theta^*$ for some $\theta^* \in \Theta$. For any $n \in \mathbb{N}$, let $\ell_{a_n,j} \equiv \sqrt{b_{a_n}}\sigma_{F_{a_n},j}^{-1}(\theta_{a_n})E_{F_{a_n}}[m_j(W,\theta_{a_n})]$ for $j = 1,\ldots,k$, and note that

$$\ell_{a_n,j} = \sqrt{b_{a_n}}\sigma_{F_{a_n},j}^{-1}(\tilde{\theta}_{a_n})E_{F_{a_n}}[m_j(W,\tilde{\theta}_{a_n})] + \Delta_{a_n,j} \tag{S.8}$$

by the intermediate value theorem, where $\hat{\theta}_{a_n}$ lies between θ_{a_n} and $\tilde{\theta}_{a_n}$ for all $n \in \mathbb{N}$, and

$$\Delta_{a_n,j} \equiv \frac{\sqrt{b_{a_n}}}{\sqrt{a_n}}(G_{F_{a_n},j}(\hat{\theta}_{a_n}) - G_{F_{a_n},j}(\theta^*))\sqrt{a_n}(\theta_{a_n} - \tilde{\theta}_{a_n}) + \frac{\sqrt{b_{a_n}}}{\sqrt{a_n}}G_{F_{a_n},j}(\theta^*)\sqrt{a_n}(\theta_{a_n} - \tilde{\theta}_{a_n}) \ .$$

Letting $\Delta_{a_n} = \{\Delta_{a_n,j}\}_{j=1}^k$, it follows that

$$||\Delta_{a_n}|| \leq \frac{\sqrt{b_{a_n}}}{\sqrt{a_n}}||G_{F_{a_n}}(\hat{\theta}_{a_n}) - G_{F_{a_n}}(\theta^*)|| \times ||\sqrt{a_n}(\theta_{a_n} - \tilde{\theta}_{a_n})|| + ||\frac{\sqrt{b_{a_n}}}{\sqrt{a_n}}G_{F_{a_n}}(\theta^*)|| \times ||\sqrt{a_n}(\theta_{a_n} - \tilde{\theta}_{a_n})|| = o(1) \ , \tag{S.9}$$

where $b_n/n \to 0$, $\sqrt{a_n}(\theta_{a_n} - \tilde{\theta}_{a_n}) \to \lambda$, $\sqrt{b_{a_n}}G_{F_{a_n}}(\theta^*)/\sqrt{a_n} = o(1)$, $\hat{\theta}_{a_n} \to \theta^*$, and $||G_{F_{a_n}}(\hat{\theta}_{a_n}) - G_{F_{a_n}}(\theta^*)|| = o(1)$ for any sequence $\{F_{a_n} \in \mathcal{P}_0\}_{n\geq 1}$ by Assumption A.3(c). Thus, for all $j \leq k$,

$$\lim_{n\to\infty} \ell_{a_n,j} \equiv \lim_{n\to\infty} \sqrt{b_{a_n}}\sigma_{F_{a_n},j}^{-1}(\theta_{a_n})E_{F_{a_n}}[m_j(W,\theta_{a_n})] = \ell_j^* \equiv \lim_{n\to\infty} \sqrt{b_{a_n}}\sigma_{F_{a_n},j}^{-1}(\tilde{\theta}_{a_n})E_{F_{a_n}}[m_j(W,\tilde{\theta}_{a_n})] \ .$$

Since $\{\tilde{\theta}_n \in \Theta_I(F_n,\lambda_n)\}_{n\geq 1}$, $\ell_j^* \geq 0$ for $j \leq p$ and $\ell_j^* = 0$ for $j > p$. Let $\ell^* \equiv \{\ell_j^*\}_{j=1}^k$. By definition,

$\{(\theta_{a_n}, \ell_{a_n}) \in \Gamma^{SS}_{b_{a_n},F_{a_n}}(\lambda_0)\}_{n\geq 1}$ and $d((\theta_{a_n}, \ell_{a_n}), (\theta^*, \ell^*)) \to 0$, which implies that $(\theta^*, \ell^*) \in \Gamma^{SS}$. From here, we conclude that

$$\inf_{(\theta,\ell)\in\Gamma^{SS}} S(v_\Omega(\theta) + \ell, \Omega(\theta)) \leq S(v_\Omega(\theta^*) + \ell^*, \Omega(\theta^*)) \leq S(v_\Omega(\theta^*), \Omega(\theta^*)) ,$$

where the first inequality follows from $(\theta^*, \ell^*) \in \Gamma^{SS}$, the second inequality follows from the fact that $\ell^*_j \geq 0$ for $j \leq p$ and $\ell^*_j = 0$ for $j > p$ and the properties of $S(\cdot)$. Finally, the RHS is bounded as $v_\Omega(\theta^*)$ is bounded a.s.

Step 2. We show that if $(\bar{\theta}, \bar{\ell}) \in \Gamma^{SS}$ with $\bar{\ell} \in \mathbb{R}^p_{[+\infty]} \times \mathbb{R}^{k-p}$, $\exists(\bar{\theta}, \ell^*) \in \Gamma^{PR}$ where $\ell^*_j \geq \bar{\ell}_j$ for $j \leq p$ and $\ell^*_j = \bar{\ell}_j$ for $j > p$. As an intermediate step, we use the limit sets under the sequence $\{(\lambda_n, F_n)\}_{n\geq 1}$, denoted by Γ^{SS}_A and Γ^{PR}_A in the statement of the lemma.

We first show that $(\bar{\theta}, \bar{\ell}) \in \Gamma^{SS}_A$. Since $\Gamma^{SS}_{b_n,F_n}(\lambda_0) \overset{H}{\to} \Gamma^{SS}$, there exist a subsequence $\{(\theta_{a_n}, \ell_{a_n}) \in \Gamma^{SS}_{b_{a_n},F_{a_n}}(\lambda_0)\}_{n\geq 1}$, $\theta_{a_n} \to \bar{\theta}$, and $\ell_{a_n} \equiv \sqrt{b_{a_n}} D^{-1/2}_{F_{a_n}}(\theta_{a_n}) E_{F_{a_n}}[m(W, \theta_{a_n})] \to \bar{\ell}$. To show that $(\bar{\theta}, \bar{\ell}) \in \Gamma^{SS}_A$, we now find a subsequence $\{(\theta'_{a_n}, \ell'_{a_n}) \in \Gamma^{SS}_{b_{a_n},F_{a_n}}(\lambda_n)\}_{n\geq 1}$, $\theta'_{a_n} \to \bar{\theta}$, and $\ell'_{a_n} \equiv \sqrt{b_{a_n}} D^{-1/2}_{F_{a_n}}(\theta'_{a_n}) E_{F_{a_n}}[m(W, \theta'_{a_n})] \to \bar{\ell}$. Notice that $\{(\theta_{a_n}, \ell_{a_n}) \in \Gamma^{SS}_{b_{a_n},F_{a_n}}(\lambda_0)\}_{n\geq 1}$ implies that $\{\theta_{a_n} \in \Theta(\lambda_0)\}_{n\geq 1}$. This and $d_H(\Theta(\lambda_n), \Theta(\lambda_0)) = O(n^{-1/2})$ implies that there is $\{\theta'_{a_n} \in \Theta(\lambda_{a_n})\}_{n\geq 1}$ s.t. $\sqrt{a_n}||\theta'_{a_n} - \theta_{a_n}|| = O(1)$ which implies that $\theta'_{a_n} \to \bar{\theta}$. By the intermediate value theorem there exists a sequence $\{\theta^*_n \in \Theta\}_{n\geq 1}$ with θ^*_n in the line between θ_n and θ'_n such that

$$\begin{aligned} \ell'_{a_n} &\equiv \sqrt{b_{a_n}} D^{-1/2}_{F_{a_n}}(\theta'_{a_n}) E_{F_{a_n}}[m(W, \theta'_{a_n})] = \sqrt{b_{a_n}} D^{-1/2}_{F_{a_n}}(\theta_{a_n}) E_{F_{a_n}}[m(W, \theta_{a_n})] + \sqrt{b_{a_n}} G_{F_{a_n}}(\theta^*_{a_n})(\theta'_{a_n} - \theta_{a_n}) \\ &= \ell_{a_n} + \Delta_{a_n} \to \bar{\ell} , \end{aligned}$$

where we have defined $\Delta_{a_n} \equiv \sqrt{b_{a_n}} G_{F_{a_n}}(\theta^*_{a_n})(\theta'_{a_n} - \theta_{a_n})$ and $\Delta_{a_n} = o(1)$ holds by similar arguments to those in (S.9). This proves $(\bar{\theta}, \bar{\ell}) \in \Gamma^{SS}_A$.

We now show that $\exists(\bar{\theta}, \ell^*) \in \Gamma^{PR}_A$ where $\ell^*_j \geq \bar{\ell}_j$ for $j \leq p$ and $\ell^*_j = \bar{\ell}_j$ for $j > p$. Using similar arguments to those in (S.2) and (S.3) in the proof of Lemma S.3.8, we have that $Q_{F_{a_n}}(\theta'_{a_n}) = O(b^{-\chi/2}_{a_n})$ and that there is a sequence $\{\tilde{\theta}_n \in \Theta_I(F_n, \lambda_n)\}_{n\geq 1}$ s.t. $\sqrt{b_{a_n}}||\theta'_{a_n} - \tilde{\theta}_{a_n}|| = O(1)$.

Following similar steps to those leading to (S.4) in the proof of Lemma S.3.8, it follows that

$$\kappa^{-1}_n \sqrt{n} G_{F_n}(\theta^*_n)(\hat{\theta}_n - \tilde{\theta}_n) = \sqrt{b_n} D^{-1/2}_{F_n}(\theta'_n) E_{F_n}[m(W, \theta'_n)] - \sqrt{b_n} D^{-1/2}_{F_n}(\tilde{\theta}_n) E_{F_n}[m(W, \tilde{\theta}_n)] , \tag{S.10}$$

where $\{\theta^*_n \in \Theta(\lambda_n)\}_{n\geq 1}$ lies in the line between θ'_n and $\tilde{\theta}_n$, and $\hat{\theta}_n \equiv (1 - \kappa_n\sqrt{b_n/n})\tilde{\theta}_n + \kappa_n\sqrt{b_n/n}\theta'_n$. By Assumption A.4, $\hat{\theta}_n$ is a convex combination of $\tilde{\theta}_n$ and θ'_n for n sufficiently large. Note also that $\sqrt{b_{a_n}}||\hat{\theta}_{a_n} - \tilde{\theta}_{a_n}|| = o(1)$. By doing yet another intermediate value theorem expansion, there is a sequence $\{\theta^{**}_n \in \Theta(\lambda_n)\}_{n\geq 1}$ with θ^{**}_n in the line between $\hat{\theta}_n$ and $\tilde{\theta}_n$ such that

$$\kappa^{-1}_n \sqrt{n} D^{-1/2}_{F_n}(\hat{\theta}_n) E_{F_n}[m(W, \hat{\theta}_n)] = \kappa^{-1}_n \sqrt{n} G_{F_n}(\theta^{**}_n)(\hat{\theta}_n - \tilde{\theta}_n) + \kappa^{-1}_n \sqrt{n} D^{-1/2}_{F_n}(\tilde{\theta}_n) E_{F_n}[m(W, \tilde{\theta}_n)] . \tag{S.11}$$

Since $\sqrt{b_{a_n}}||\theta^*_{a_n} - \tilde{\theta}_{a_n}|| = O(1)$ and $\sqrt{b_{a_n}}||\tilde{\theta}_{a_n} - \theta^{**}_{a_n}|| = o(1)$, it follows that $\sqrt{b_{a_n}}||\theta^*_{a_n} - \theta^{**}_{a_n}|| = O(1)$. Next,

$$\begin{aligned} \kappa^{-1}_n \sqrt{n} D^{-1/2}_{F_n}(\hat{\theta}_n) E_{F_n}[m(W, \hat{\theta}_n)] &= \kappa^{-1}_n \sqrt{n} G_{F_n}(\theta^*_n)(\hat{\theta}_n - \tilde{\theta}_n) + \kappa^{-1}_n \sqrt{n} D^{-1/2}_{F_n}(\tilde{\theta}_n) E_{F_n}[m(W, \tilde{\theta}_n)] + \Delta_{n,1} \\ &= \sqrt{b_n} D^{-1/2}_{F_n}(\theta'_n) E_{F_n}[m(W, \theta'_n)] + \Delta_{n,1} + \Delta_{n,2} , \end{aligned} \tag{S.12}$$

where the first equality follows from (S.11) and $\Delta_{n,1} \equiv \kappa_n^{-1}\sqrt{n}(G_{F_n}(\theta_n^{**}) - G_{F_n}(\theta_n^*))(\hat{\theta}_n - \tilde{\theta}_n)$, and the second holds by (S.10) and $\Delta_{n,2} \equiv \kappa_n^{-1}\sqrt{n}(1-\kappa_n\sqrt{b_n/n})D_{F_n}^{-1/2}(\tilde{\theta}_n)E_{F_n}[m(W,\tilde{\theta}_n)]$. By similar arguments to those in the proof of Lemma S.3.8, $||\Delta_{a_n,1}|| = o(1)$. In addition, Assumption A.4 and $\{\tilde{\theta}_n \in \Theta_I(F_n,\lambda_n)\}_{n\geq 1}$ imply that $\Delta_{n,2,j} \geq 0$ for $j \leq p$ and n sufficiently large, and that $\Delta_{n,2,j} = 0$ for $j > p$ and all $n \geq 1$.

Now define $\ell''_{a_n} \equiv \kappa_{a_n}^{-1}\sqrt{a_n}D_{F_{a_n}}^{-1/2}(\hat{\theta}_{a_n})E_{F_n}[m(W,\hat{\theta}_{a_n})]$ so that by compactness of $(\mathbb{R}^k_{[\pm\infty]}, d)$ there is a further subsequence $\{u_n\}_{n\geq 1}$ of $\{a_n\}_{n\geq 1}$ s.t. $\ell''_{u_n} = \kappa_{u_n}^{-1}\sqrt{u_n}D_{F_{u_n}}^{-1/2}(\hat{\theta}_{u_n})E_{F_{u_n}}[m(W,\hat{\theta}_{u_n})]$ and $\Delta_{u_n,1}$ converges. We define $\ell^* \equiv \lim_{n\to\infty}\ell''_{u_n}$. By (S.12) and properties of $\Delta_{n,1}$ and $\Delta_{n,2}$, we conclude that

$$\lim_{n\to\infty}\ell''_{u_n,j} = \lim_{n\to\infty}\kappa_{u_n}^{-1}\sqrt{u_n}\sigma_{F_{u_n},j}^{-1}(\hat{\theta}_{u_n})E_{F_{u_n}}[m_j(W,\hat{\theta}_{u_n})] \geq \lim_{n\to\infty}\sqrt{b_{u_n}}\sigma_{F_{u_n},j}^{-1}(\theta'_{u_n})E_{F_{u_n}}[m_j(W,\theta'_{u_n})] = \bar{\ell}_j,\ \text{for } j \leq p\ ,$$
$$\lim_{n\to\infty}\ell''_{u_n,j} = \lim_{n\to\infty}\kappa_{u_n}^{-1}\sqrt{u_n}\sigma_{F_{u_n},j}^{-1}(\hat{\theta}_{u_n})E_{F_{u_n}}[m_j(W,\hat{\theta}_{u_n})] = \lim_{n\to\infty}\sqrt{b_{u_n}}\sigma_{F_{u_n},j}^{-1}(\theta'_{u_n})E_{F_{u_n}}[m_j(W,\theta'_{u_n})] = \bar{\ell}_j,\ \text{for } j > p\ ,$$

Thus, $\{(\hat{\theta}_{u_n},\ell''_{u_n}) \in \Gamma^{\text{PR}}_{u_n,F_{u_n}}(\lambda_n)\}_{n\geq 1}$, $\hat{\theta}_{u_n} \to \bar{\theta}$, and $\ell''_{u_n} \to \ell^*$ where $\ell^*_j \geq \bar{\ell}_j$ for $j \leq p$ and $\ell^*_j = \bar{\ell}_j$ for $j > p$, and $(\bar{\theta},\ell^*) \in \Gamma^{\text{PR}}_A$.

We conclude the step by showing that $(\bar{\theta},\ell^*) \in \Gamma^{\text{PR}}$. To this end, find a subsequence $\{(\theta^\dagger_{u_n},\ell^\dagger_{u_n}) \in \Gamma^{\text{PR}}_{b_{u_n},F_{u_n}}(\lambda_0)\}_{n\geq 1}$, $\theta^\dagger_{u_n} \to \bar{\theta}$, and $\ell^\dagger_{u_n} \equiv \kappa_{u_n}^{-1}\sqrt{u_n}D_{F_{u_n}}^{-1/2}(\theta^\dagger_{u_n})E_{F_{u_n}}[m(W,\theta^\dagger_{u_n})] \to \ell^*$. Notice that $\{(\hat{\theta}_{u_n},\ell''_{u_n}) \in \Gamma^{\text{PR}}_{u_n,F_{u_n}}(\lambda_n)\}_{n\geq 1}$ implies that $\{\hat{\theta}_{u_n} \in \Theta(\lambda_{u_n})\}_{n\geq 1}$. This and $d_H(\Theta(\lambda_n),\Theta(\lambda_0)) = O(n^{-1/2})$ implies that there is $\{\theta^\dagger_{u_n} \in \Theta(\lambda_0)\}_{n\geq 1}$ s.t. $\sqrt{u_n}||\hat{\theta}_{u_n} - \theta^\dagger_{u_n}|| = O(1)$ which implies that $\theta^\dagger_{u_n} \to \bar{\theta}$. By the intermediate value theorem there exists a sequence $\{\theta_n^{***} \in \Theta\}_{n\geq 1}$ with θ_n^{***} in the line between $\hat{\theta}_n$ and $\theta_n^\dagger$ such that

$$\begin{aligned}\ell^\dagger_{u_n} &\equiv \kappa_{u_n}^{-1}\sqrt{u_n}D_{F_{u_n}}^{-1/2}(\theta^\dagger_{u_n})E_{F_{u_n}}[m(W,\theta^\dagger_{u_n})] = \kappa_{u_n}^{-1}\sqrt{u_n}D_{F_{u_n}}^{-1/2}(\theta^\dagger_{u_n})E_{F_{u_n}}[m(W,\theta^\dagger_{u_n})] + \kappa_{u_n}^{-1}\sqrt{u_n}G_{F_{u_n}}(\theta_{u_n}^{***})(\theta^\dagger_{u_n} - \hat{\theta}_{u_n}) \\ &= \ell''_{u_n} + \Delta_{u_n} \to \ell^*\ ,\end{aligned}$$

where we have define $\Delta_{u_n} \equiv \kappa_{u_n}^{-1}\sqrt{u_n}G_{F_{u_n}}(\theta_{u_n}^{***})(\theta^\dagger_{u_n} - \hat{\theta}_{u_n})$ and $\Delta_{u_n} = o(1)$ holds by similar arguments to those used before. By definition, this proves that $(\bar{\theta},\ell^*) \in \Gamma^{\text{PR}}$.

Step 3. We show that $\inf_{(\theta,\ell)\in\Gamma^{\text{SS}}} S(v_\Omega(\theta)+\ell,\Omega(\theta)) \geq \inf_{(\theta,\ell)\in\Gamma^{\text{PR}}} S(v_\Omega(\theta)+\ell,\Omega(\theta))$ a.s. Since v_Ω is a tight stochastic process, there is a subset of the sample space $\mathcal{W}$, denoted $\mathcal{A}_1$, s.t. $P(\mathcal{A}_1) = 1$ and $\forall\omega \in \mathcal{A}_1$, $\sup_{\theta\in\Theta}||v_\Omega(\omega,\theta)|| < \infty$. By step 1, there is a subset of $\mathcal{W}$, denoted $\mathcal{A}_2$, s.t. $P(\mathcal{A}_2) = 1$ and $\forall\omega \in \mathcal{A}_2$,

$$\inf_{(\theta,\ell)\in\Gamma^{\text{SS}}} S(v_\Omega(\omega,\theta)+\ell,\Omega(\theta)) < \infty\ .$$

Define $\mathcal{A} \equiv \mathcal{A}_1 \cap \mathcal{A}_2$ and note that $P(\mathcal{A}) = 1$. In order to complete the proof, it then suffices to show that $\forall\omega \in \mathcal{A}$,

$$\inf_{(\theta,\ell)\in\Gamma^{\text{SS}}} S(v_\Omega(\omega,\theta)+\ell,\Omega(\theta)) \geq \inf_{(\theta,\ell)\in\Gamma^{\text{PR}}} S(v_\Omega(\omega,\theta)+\ell,\Omega(\theta))\ . \tag{S.13}$$

Fix $\omega \in \mathcal{A}$ arbitrarily and suppose that (S.13) does not occur, i.e.,

$$\Delta \equiv \inf_{(\theta,\ell)\in\Gamma^{\text{PR}}} S(v_\Omega(\omega,\theta)+\ell,\Omega(\theta)) - \inf_{(\theta,\ell)\in\Gamma^{\text{SS}}} S(v_\Omega(\omega,\theta)+\ell,\Omega(\theta)) > 0\ . \tag{S.14}$$

By definition of infimum, $\exists(\bar{\theta},\bar{\ell}) \in \Gamma^{\text{SS}}$ s.t. $\inf_{(\theta,\ell)\in\Gamma^{\text{SS}}} S(v_\Omega(\omega,\theta)+\ell,\Omega(\theta)) + \Delta/2 \geq S(v_\Omega(\omega,\bar{\theta})+\bar{\ell},\Omega(\bar{\theta}))$, and so, from this and (S.14) it follows that

$$S(v_\Omega(\omega,\bar{\theta})+\bar{\ell},\Omega(\bar{\theta})) \leq \inf_{(\theta,\ell)\in\Gamma^{\mathrm{PR}}} S(v_\Omega(\omega,\theta)+\ell,\Omega(\theta)) - \Delta/2 \ . \tag{S.15}$$

We now show that $\bar{\ell}\in\mathbb{R}^p_{[+\infty]}\times\mathbb{R}^{k-p}$. Suppose not, i.e., suppose that $\bar{\ell}_j=-\infty$ for some $j<p$ and $|\bar{\ell}_j|=\infty$ for some $j>p$. Since $\omega\in\mathcal{A}\subseteq\mathcal{A}_1$, $||v_\Omega(\omega,\bar{\theta})||<\infty$. By part 2 of Lemma S.3.6 it then follows that $S(v_\Omega(\omega,\bar{\theta})+\bar{\ell},\Omega(\bar{\theta}))=\infty$. By (S.15), $\inf_{(\theta,\ell)\in\Gamma^{\mathrm{SS}}} S(v_\Omega(\omega,\theta)+\ell,\Omega(\theta))=\infty$, which is a contradiction to $\omega\in\mathcal{A}_2$.

Since $\bar{\ell}\in\mathbb{R}^p_{[+\infty]}\times\mathbb{R}^{k-p}$, step 2 implies that $\exists(\bar{\theta},\ell^*)\in\Gamma^{\mathrm{PR}}$ where $\ell^*_j\geq\bar{\ell}_j$ for $j\leq p$ and $\ell^*_j=\bar{\ell}_j$ for $j>p$. By properties of $S(\cdot)$,

$$S(v_\Omega(\omega,\bar{\theta})+\ell^*,\Omega(\bar{\theta})) \leq S(v_\Omega(\omega,\bar{\theta})+\bar{\ell},\Omega(\bar{\theta})) \ . \tag{S.16}$$

Combining (S.14), (S.15), (S.16), and $(\bar{\theta},\ell^*)\in\Gamma^{\mathrm{PR}}$, we reach the following contradiction,

$$\begin{aligned} 0<\Delta/2 &\leq \inf_{(\theta,\ell)\in\Gamma^{\mathrm{PR}}} S(v_\Omega(\omega,\theta)+\ell,\Omega(\theta)) - S(v_\Omega(\omega,\bar{\theta})+\bar{\ell},\Omega(\bar{\theta})) \\ &\leq \inf_{(\theta,\ell)\in\Gamma^{\mathrm{PR}}} S(v_\Omega(\omega,\theta)+\ell,\Omega(\theta)) - S(v_\Omega(\omega,\bar{\theta})+\ell^*,\Omega(\bar{\theta})) \leq 0 \ . \end{aligned}$$

Step 4. Suppose the conclusion of the lemma is not true. This is, suppose that $c_{(1-\alpha)}(\Gamma^{\mathrm{PR}},\Omega) > c_{(1-\alpha)}(\Gamma^{\mathrm{SS}},\Omega)$. Consider the following derivation

$$\begin{aligned} &\alpha < P(J(\Gamma^{\mathrm{PR}},\Omega) > c_{(1-\alpha)}(\Gamma^{\mathrm{SS}},\Omega)) \\ &\quad\leq P(J(\Gamma^{\mathrm{SS}},\Omega) > c_{(1-\alpha)}(\Gamma^{\mathrm{SS}},\Omega)) + P(J(\Gamma^{\mathrm{PR}},\Omega) > J(\Gamma^{\mathrm{SS}},\Omega)) = 1 - P(J(\Gamma^{\mathrm{SS}},\Omega)\leq c_{(1-\alpha)}(\Gamma^{\mathrm{SS}},\Omega)) \leq \alpha \ , \end{aligned}$$

where the first strict inequality holds by definition of quantile and $c_{(1-\alpha)}(\Gamma^{\mathrm{PR}},\Omega) > c_{(1-\alpha)}(\Gamma^{\mathrm{SS}},\Omega)$, the last equality holds by step 3, and all other relationships are elementary. Since the result is contradictory, the proof is complete. □

Proof of Lemma S.3.10. By Theorem 4.3, $\liminf(E_{F_n}[\phi^{PR}_n(\lambda_0)] - E_{F_n}[\phi^{SS}_n(\lambda_0)]) \geq 0$. Suppose that the desired result is not true. Then, there is a further subsequence $\{u_n\}_{n\geq1}$ of $\{n\}_{n\geq1}$ s.t.

$$\lim E_{F_{u_n}}[\phi^{PR}_{u_n}(\lambda_0)] = \lim E_{F_{u_n}}[\phi^{SS}_{u_n}(\lambda_0)] \ . \tag{S.17}$$

This sequence $\{u_n\}_{n\geq1}$ will be referenced from here on. We divide the remainder of the proof into steps.

Step 1. We first show that there is a subsequence $\{a_n\}_{n\geq1}$ of $\{u_n\}_{n\geq1}$ s.t.

$$\{T^{SS}_{a_n}(\lambda_0)|\{W_i\}^{a_n}_{i=1}\} \xrightarrow{d} S(v_\Omega(\theta^*)+(g,\mathbf{0}_{k-p}),\Omega(\theta^*)),\ \text{a.s.} \tag{S.18}$$

Conditionally on $\{W_i\}^n_{i=1}$, Assumption A.7(c) implies that

$$T^{SS}_n(\lambda_0) = S(\sqrt{b_n}D^{-1}_{F_n}(\hat{\theta}^{SS}_n)\bar{m}^{SS}_{b_n}(\hat{\theta}^{SS}_n),\tilde{\Omega}^{SS}_{b_n}(\hat{\theta}^{SS}_n)) + o_p(1),\ \text{a.s.} \tag{S.19}$$

By continuity of the function S, (S.18) follows from (S.19) if we find a subsequence $\{a_n\}_{n\geq 1}$ of $\{u_n\}_{n\geq 1}$ s.t.

$$\{\tilde{\Omega}^{SS}_{a_n}(\hat{\theta}^{SS}_{a_n})|\{W_i\}_{i=1}^{a_n}\} \xrightarrow{p} \Omega(\theta^*),\ \text{a.s.} \tag{S.20}$$

$$\{\sqrt{b_{a_n}}D^{-1/2}_{F_{a_n}}(\hat{\theta}^{SS}_{a_n})\bar{m}^{SS}_{a_n}(\hat{\theta}^{SS}_{a_n})|\{W_i\}_{i=1}^{a_n}\} \xrightarrow{d} v_\Omega(\theta^*) + (g, \mathbf{0}_{k-p}),\ \text{a.s.} \tag{S.21}$$

To show (S.20), note that

$$||\tilde{\Omega}^{SS}_n(\hat{\theta}^{SS}_n) - \Omega(\theta^*)|| \leq \sup_{\theta\in\Theta}||\tilde{\Omega}^{SS}_n(\theta,\theta) - \Omega(\theta,\theta)|| + ||\Omega(\hat{\theta}^{SS}_n) - \Omega(\theta^*)||\ .$$

the first term on the RHS is conditionally $o_p(1)$ a.s. by Lemma S.3.1 (part 5) and the second term is conditionally $o_p(1)$ a.s. by $\Omega \in \mathcal{C}(\Theta^2)$ and $\{\hat{\theta}^{SS}_n|\{W_i\}_{i=1}^n\} \xrightarrow{p} \theta^*$ a.s. Then, (S.20) holds for the original sequence $\{n\}_{n\geq 1}$.

To show (S.21), note that

$$\sqrt{b_n}D^{-1/2}_{F_n}(\hat{\theta}^{SS}_n)\bar{m}^{SS}_n(\hat{\theta}^{SS}_n) = \tilde{v}^{SS}_n(\theta^*) + (g, \mathbf{0}_{k-p}) + \mu_{n,1} + \mu_{n,2}\ ,$$

where

$$\begin{aligned}
\mu_{n,1} &\equiv \tilde{v}_n(\hat{\theta}^{SS}_n)\sqrt{b_n/n}\\
\mu_{n,2} &\equiv (\tilde{v}^{SS}_{b_n}(\hat{\theta}^{SS}_n) - \tilde{v}^{SS}_{b_n}(\theta^*)) + \sqrt{b_n}(D^{-1/2}_{F_n}(\hat{\theta}^{SS}_n)E_{F_n}[m(W,\hat{\theta}^{SS}_n)] - D^{-1/2}_{F_n}(\tilde{\theta}^{SS}_n)E_{F_n}[m(W,\tilde{\theta}^{SS}_n)])\\
&\quad +(\sqrt{b_n}D^{-1/2}_{F_n}(\tilde{\theta}^{SS}_n)E_{F_n}[m(W,\tilde{\theta}^{SS}_n)] - (g,\mathbf{0}_{k-p}))\ .
\end{aligned}$$

Lemma S.3.1 (part 9) implies that $\{\tilde{v}^{SS}_{b_n}(\theta^*)|\{W_i\}_{i=1}^n\} \xrightarrow{d} v_\Omega(\theta^*)$ a.s. and so, (S.21) follows from

$$\{\mu_{a_n,1}|\{W_i\}_{i=1}^{a_n}\} = o_p(1),\ \text{a.s.} \tag{S.22}$$

$$\{\mu_{a_n,2}|\{W_i\}_{i=1}^{a_n}\} = o_p(1),\ \text{a.s.} \tag{S.23}$$

By Lemma S.3.1 (part 7), $\sup_{\theta\in\Theta}||\tilde{v}_n(\theta)||\sqrt{b_n/n} = o_p(1)$, and by taking a further subsequence $\{a_n\}_{n\geq 1}$ of $\{n\}_{n\geq 1}$, $\sup_{\theta\in\Theta}||\tilde{v}_{a_n}(\theta)||\sqrt{b_{a_n}/a_n} = o_{a.s.}(1)$. Since $\tilde{v}_n(\cdot)$ is conditionally non-stochastic, this result implies (S.23).

By Assumption A.7(c), (S.23) follows from showing that $\{\tilde{v}^{SS}_n(\theta^*) - \tilde{v}^{SS}_n(\hat{\theta}^{SS}_n)|\{W_i\}_{i=1}^n\} = o_p(1)$ a.s., which we now show. Fix $\mu > 0$ arbitrarily, it suffices to show that

$$\limsup P_{F_n}(||\tilde{v}^{SS}_n(\theta^*) - \tilde{v}^{SS}_n(\hat{\theta}^{SS}_n)|| > \varepsilon|\{W_i\}_{i=1}^n) < \mu \text{ a.s.} \tag{S.24}$$

Fix $\delta > 0$ arbitrarily. As a preliminary step, we first show that

$$\lim P_{F_n}(\rho_{F_n}(\theta^*, \hat{\theta}^{SS}_n) \geq \delta|\{W_i\}_{i=1}^n) = 0\ \text{ a.s.}\ , \tag{S.25}$$

where ρ_{F_n} is the intrinsic variance semimetric in (A-1). Then, for any $j = 1, \ldots, k$,

$$V_{F_n}(\sigma^{-1}_{F_n,j}(\hat{\theta}^{SS}_n)m_j(W,\hat{\theta}^{SS}_n) - \sigma^{-1}_{F_n,j}(\theta^*)m_j(W,\theta^*)) = 2(1 - \Omega_{F_n}(\theta^*,\hat{\theta}^{SS}_n)_{[j,j]})\ .$$

By (A-1), this implies that

$$P_{F_n}(\rho_{F_n}(\theta^*,\hat{\theta}_n^{SS})\geq\delta|\{W_i\}_{i=1}^n)\leq\sum_{j=1}^{k}P_{F_n}(1-\Omega_{F_n}(\theta^*,\hat{\theta}_n^{SS})_{[j,j]}\geq\delta^2 2^{-1}k^{-1}|\{W_i\}_{i=1}^n)\ . \tag{S.26}$$

For any $j=1,\ldots,k$, note that

$$\begin{aligned} P_{F_n}(1-\Omega_{F_n}(\theta^*,\hat{\theta}_n^{SS})_{[j,j]}\geq\delta^2 2^{-1}k^{-1}|\{W_i\}_{i=1}^n) &\leq P_{F_n}(1-\Omega(\theta^*,\hat{\theta}_n^{SS})_{[j,j]}\geq\delta^2 2^{-2}k^{-1}|\{W_i\}_{i=1}^n)+o(1)\\ &\leq P_{F_n}(||\theta^*-\hat{\theta}_n^{SS}||>\tilde{\delta}|\{W_i\}_{i=1}^n)+o(1)=o_{a.s.}(1)\ , \end{aligned}$$

where we have used that $\Omega_{F_n}\overset{u}{\to}\Omega$ and so $\sup_{\theta,\theta'\in\Theta}||\Omega(\theta,\theta')_{[j,j]}-\Omega_{F_n}(\theta,\theta')_{[j,j]}||<\delta^2 2^{-2}k^{-1}$ for all sufficiently large n, that $\Omega\in\mathcal{C}(\Theta^2)$ and so $\exists\tilde{\delta}>0$ s.t. $||\theta^*-\hat{\theta}_n^{SS}||\leq\tilde{\delta}$ implies that $1-\Omega(\theta^*,\hat{\theta}_n^{SS})_{[j,j]}\leq\delta^2 2^{-2}k^{-1}$, and that $\{\hat{\theta}_n^{SS}|\{W_i\}_{i=1}^n\}\overset{p}{\to}\theta^*$ a.s. Combining this with (S.26), (S.25) follows.

Lemma S.3.1 (part 1) implies that $\{\tilde{v}_n^{SS}(\cdot)|\{W_i\}_{i=1}^n\}$ is asymptotically ρ_F-equicontinuous uniformly in $F\in\mathcal{P}$ (a.s.) in the sense of van der Vaart and Wellner (1996, page 169). Then, $\exists\delta>0$ s.t.

$$\limsup_{n\to\infty}P^*_{F_n}(\sup_{\rho_{F_n}(\theta,\theta')<\delta}||\tilde{v}_n^{SS}(\theta)-\tilde{v}_n^{SS}(\theta')||>\varepsilon|\{W_i\}_{i=1}^n)<\mu\ \text{ a.s.} \tag{S.27}$$

Based on this choice, consider the following argument:

$$\begin{aligned} P^*_{F_n}(||\tilde{v}_n^{SS}(\theta^*)-\tilde{v}_n^{SS}(\hat{\theta}_n^{SS})||>\varepsilon|\{W_i\}_{i=1}^n)\leq{}& P^*_{F_n}(\sup_{\rho_{F_n}(\theta,\theta')<\delta}||\tilde{v}_n^{SS}(\theta^*)-\tilde{v}_n^{SS}(\hat{\theta}_n)||>\varepsilon|\{W_i\}_{i=1}^n)\\ &+P^*_{F_n}(\rho_{F_n}(\theta^*,\hat{\theta}_n)\geq\delta|\{W_i\}_{i=1}^n)\ . \end{aligned}$$

From this, (S.25), and (S.27), (S.24) follows.

Step 2. For arbitrary $\varepsilon>0$ and for the subsequence $\{a_n\}_{n\geq1}$ of $\{u_n\}_{n\geq1}$ in step 1 we want show that

$$\lim P_{F_{a_n}}(|c_{a_n}^{SS}(\lambda_0,1-\alpha)-c_{(1-\alpha)}(g,\Omega(\theta^*))|\leq\varepsilon)=1\ , \tag{S.28}$$

where $c_{(1-\alpha)}(g,\Omega(\theta^*))$ denotes the $(1-\alpha)$-quantile of $S(v_\Omega(\theta^*)+(g,\mathbf{0}_{k-p}),\Omega(\theta^*))$. By our maintained assumptions and Assumption A.7(b.iii), it follows that $c_{(1-\alpha)}(g,\Omega(\theta^*))>0$.

Fix $\bar{\varepsilon}\in(0,\min\{\varepsilon,c_{(1-\alpha)}(g,\Omega(\theta^*))\})$. By our maintained assumptions, $c_{(1-\alpha)}(g,\Omega(\theta^*))-\bar{\varepsilon}$ and $c_{(1-\alpha)}(g,\Omega(\theta^*))+\bar{\varepsilon}$ are continuity points of the CDF of $S(v_\Omega(\theta^*)+(g,\mathbf{0}_{k-p}),\Omega(\theta^*))$. Then,

$$\lim P_{F_{a_n}}(T_{a_n}^{SS}(\lambda_0)\leq c_{(1-\alpha)}(g,\Omega(\theta^*))+\bar{\varepsilon}|\{W_i\}_{i=1}^{a_n})=P(S(v_\Omega(\theta^*)+(g,\mathbf{0}_{k-p}),\Omega(\theta^*))\leq c_{(1-\alpha)}(g,\Omega(\theta^*))+\bar{\varepsilon})>1-\alpha\ , \tag{S.29}$$

where the equality holds a.s. by step 1, and the strict inequality holds by $\bar{\varepsilon}>0$. By a similar argument,

$$\lim P_{F_{a_n}}(T_{a_n}^{SS}(\lambda_0)\leq c_{(1-\alpha)}(g,\Omega(\theta^*))-\bar{\varepsilon}|\{W_i\}_{i=1}^{n})=P(S(v_\Omega(\theta^*)+(g,\mathbf{0}_{k-p}),\Omega(\theta^*))\leq c_{(1-\alpha)}(g,\Omega(\theta^*))-\bar{\varepsilon})<1-\alpha\ , \tag{S.30}$$

where, as before, the equality holds a.s. by step 1. Next, notice that

$$\{\lim P_{F_{a_n}}(T_{a_n}^{SS}(\lambda_0)\leq c_{(1-\alpha)}(g,\Omega(\theta^*))+\bar{\varepsilon}|\{W_i\}_{i=1}^{a_n})>1-\alpha\}\subseteq\{\liminf\{c_{a_n}^{SS}(\lambda_0,1-\alpha)<c_{(1-\alpha)}(g,\Omega(\theta^*))+\bar{\varepsilon}\}\}\ ,$$

with the same result holding with $-\bar{\varepsilon}$ replacing $+\bar{\varepsilon}$. By combining this result with (S.29) and (S.30), we get

$$\{\liminf\{|c_{a_n}^{SS}(\lambda_0,1-\alpha)-c_{(1-\alpha)}(g,\Omega(\theta^*))|\leq\bar{\varepsilon}\}\}\ \text{ a.s.}$$

From this result, $\bar{\varepsilon} < \varepsilon$, and Fatou's Lemma, (S.28) follows.

Step 3. For an arbitrary $\varepsilon > 0$ and for a subsequence $\{w_n\}_{n\geq 1}$ of $\{a_n\}_{n\geq 1}$ in step 2 we want to show that

$$\lim P_{F_{w_n}}(c_{(1-\alpha)}(\pi,\Omega(\theta^*)) + \varepsilon \geq c^{PR}_{w_n}(\lambda_0, 1-\alpha)) = 1 \ , \tag{S.31}$$

where $c_{(1-\alpha)}(\pi,\Omega(\theta^*))$ denotes the $(1-\alpha)$-quantile of $S(v_\Omega(\theta^*) + (\pi, \mathbf{0}_{k-p}), \Omega(\theta^*))$ and $\pi \in \mathbb{R}^p_{[+,+\infty]}$ is a parameter to be determined that satisfies $\pi \geq g$ and $\pi_j > g_j$ for some $j = 1,\ldots,p$.

The arguments required to show this are similar to those used in steps 1-2. For any $\theta \in \Theta(\lambda_0)$, define $\tilde{T}^{PR}_n(\theta) \equiv S(v^*_n(\theta) + \kappa_n^{-1}\sqrt{n}\hat{D}_n^{-1/2}(\theta)\bar{m}_n(\theta), \hat{\Omega}_n(\theta))$. We first show that there is a subsequence $\{w_n\}_{n\geq1}$ of $\{a_n\}_{n\geq1}$ s.t.

$$\{\tilde{T}^{PR}_{w_n}(\hat{\theta}^{SS}_{w_n})|\{W_i\}_{i=1}^{w_n}\} \xrightarrow{d} S(v_\Omega(\theta^*) + (\pi, \mathbf{0}_{k-p}), \Omega(\theta^*)) \text{ a.s.} \tag{S.32}$$

Consider the following derivation:

$$\begin{aligned}
\tilde{T}^{PR}_n(\hat{\theta}^{SS}_n) &= S(v^*_n(\hat{\theta}^{SS}_n) + \kappa_n^{-1}\sqrt{n}\hat{D}_n^{-1/2}(\hat{\theta}^{SS}_n)\bar{m}_n(\hat{\theta}^{SS}_n), \hat{\Omega}_n(\hat{\theta}^{SS}_n)) \\
&= S(D_{F_n}^{-1/2}(\hat{\theta}^{SS}_n)\hat{D}_n^{1/2}(\hat{\theta}^{SS}_n)v^*_n(\hat{\theta}^{SS}_n) + \kappa_n^{-1}\sqrt{n}D_{F_n}^{-1/2}(\hat{\theta}^{SS}_n)\bar{m}_n(\hat{\theta}^{SS}_n), \tilde{\Omega}_n(\hat{\theta}^{SS}_n)) \ .
\end{aligned}$$

By continuity of the function S, (S.31) would follow from if we find a subsequence $\{w_n\}_{n\geq1}$ of $\{a_n\}_{n\geq1}$ s.t.

$$\begin{aligned}
&\{\tilde{\Omega}_{w_n}(\hat{\theta}^{SS}_{w_n})|\{W_i\}_{i=1}^{w_n}\} \xrightarrow{p} \Omega(\theta^*), \text{ a.s.} \\
&\{D_{F_{w_n}}^{-1/2}(\hat{\theta}^{SS}_{w_n})\hat{D}_{w_n}^{1/2}(\hat{\theta}^{SS}_{w_n})v^*_{w_n}(\hat{\theta}^{SS}_{w_n}) + \kappa_{w_n}^{-1}\sqrt{w_n}D_{F_{w_n}}^{-1/2}(\hat{\theta}^{SS}_{w_n})\bar{m}_{w_n}(\hat{\theta}^{SS}_{w_n})|\{W_i\}_{i=1}^{w_n}\} \xrightarrow{d} v_\Omega(\theta^*) + (\pi, \mathbf{0}_{k-p}), \text{ a.s.}
\end{aligned}$$

The first statement is shown as in (S.20) in step 1. To show the second statement, note that

$$D_{F_{w_n}}^{-1/2}(\hat{\theta}^{SS}_{w_n})\hat{D}_{w_n}^{1/2}(\hat{\theta}^{SS}_{w_n})v^*_{w_n}(\hat{\theta}^{SS}_{w_n}) + \kappa_{w_n}^{-1}\sqrt{w_n}D_{F_{w_n}}^{-1/2}(\hat{\theta}^{SS}_{w_n})\bar{m}_{w_n}(\hat{\theta}^{SS}_{w_n}) = v^*_{w_n}(\theta^*) + (\pi, \mathbf{0}_{k-p}) + \mu_{w_n,3} + \mu_{w_n,4},$$

where

$$\begin{aligned}
\mu_{w_n,3} &= D_{F_{w_n}}^{-1/2}(\hat{\theta}^{SS}_{w_n})\hat{D}_{w_n}^{1/2}(\hat{\theta}^{SS}_{w_n})v^*_{w_n}(\hat{\theta}^{SS}_{w_n}) - v^*_{w_n}(\theta^*) + \kappa_{w_n}^{-1}\tilde{v}_{w_n}(\hat{\theta}^{SS}_{w_n}) \\
&\quad + \kappa_{w_n}^{-1}\sqrt{w_n}(D_{F_{w_n}}^{-1/2}(\hat{\theta}^{SS}_{w_n})E_{F_{w_n}}[m(W,\hat{\theta}^{SS}_{w_n})] - D_{F_{w_n}}^{-1/2}(\tilde{\theta}^{SS}_{w_n})E_{F_{w_n}}[m(W,\tilde{\theta}^{SS}_{w_n})]), \\
\mu_{w_n,4} &= (\kappa_{w_n}^{-1}\sqrt{w_n/b_{w_n}})\sqrt{b_{w_n}}D_{F_{w_n}}^{-1/2}(\tilde{\theta}^{SS}_{w_n})E_{F_{w_n}}[m(W,\tilde{\theta}^{SS}_{w_n})] - (\pi, \mathbf{0}_{k-p}).
\end{aligned}$$

By Lemma S.3.1 (part 9), we have $\{v^*_{w_n}(\theta^*)|\{W_i\}_{i=1}^n\} \xrightarrow{d} v_\Omega(\theta^*)$ a.s. By the same arguments as in step 1, we have that $\{\mu_{w_n,3}|\{W_i\}_{i=1}^{w_n}\} = o_p(1)$, a.s. By possibly considering a subsequence, $\kappa_{w_n}^{-1}\sqrt{w_n/b_{w_n}} \to K^{-1} \in (1,\infty]$ by Assumption A.7(d) and $\{\sqrt{b_n}D_{F_{w_n}}^{-1/2}(\tilde{\theta}^{SS}_{w_n})E_{F_{w_n}}[m(W,\tilde{\theta}^{SS}_{w_n})]|\{W_i\}_{i=1}^{w_n}\} = (g, \mathbf{0}_{k-p}) + o_p(1)$ a.s. by Assumption A.7(c,iii), with $g \in \mathbb{R}^p_{[+,+\infty]}$ by step 1. By combining these two and by possibly considering a further subsequence, we conclude that $\{(\kappa_{w_n}^{-1}\sqrt{w_n/b_{w_n}})\sqrt{b_n}D_{F_{w_n}}^{-1/2}(\tilde{\theta}^{SS}_{w_n})E_{F_{w_n}}[m(W,\tilde{\theta}^{SS}_{w_n})]|\{W_i\}_{i=1}^{w_n}\} = (\pi, \mathbf{0}_{k-p}) + o_p(1)$ a.s. where $\pi \in \mathbb{R}^p_{[+,+\infty]}$. Since $K^{-1} > 1$, $\pi_j \geq g_j \geq 0$ for all $j = 1,\ldots,p$. By Assumption A.7(c,iii), there is $j = 1,\ldots,p$, s.t. $g_j \in (0,\infty)$ and so $\pi_j = K^{-1}g_j > g_j$. From this, we have that $\{\mu_{w_n,4}|\{W_i\}_{i=1}^{w_n}\} = o_p(1)$, a.s.

Let $\tilde{c}^{PR}_n(\theta, 1-\alpha)$ denote the conditional $(1-\alpha)$-quantile of $\tilde{T}^{PR}_n(\theta)$. On the one hand, (S.32) and the arguments in step 2 imply that $\lim P_{F_{w_n}}(|\tilde{c}^{PR}_{w_n}(\hat{\theta}^{SS}_{w_n}, 1-\alpha) - c_{(1-\alpha)}(\pi,\Omega(\theta^*))| \leq \varepsilon) = 1$ for any $\varepsilon > 0$.

On the other hand, $T_n^{PR}(\lambda_0) = \inf_{\theta\in\Theta(\lambda_0)} \tilde{T}_n^{PR}(\theta)$ and $\{\hat{\theta}_n^{SS} \in \Theta(\lambda_0)\}_{n\geq 1}$ imply that $\tilde{c}_n^{PR}(\hat{\theta}_n^{SS}, 1-\alpha) \geq c_n^{PR}(\lambda_0, 1-\alpha)$. By combining these, (S.31) follows.

We conclude by noticing that by $c_{(1-\alpha)}(g,\Omega(\theta^*)) > 0$ (by step 2) and $\pi \geq g$ with $\pi_j > g_j$ for some $j = 1,\ldots,p$, our maintained assumptions imply that $c_{(1-\alpha)}(g,\Omega(\theta^*)) > c_{(1-\alpha)}(\pi,\Omega(\theta^*))$.

Step 4. We now conclude the proof. By Assumption A.7(a) and arguments similar to step 1 we deduce that

$$T_{w_n}(\lambda_0) \xrightarrow{d} S(v_\Omega(\theta^*) + (\lambda, \mathbf{0}_{k-p}), \Omega(\theta^*)) \ . \tag{S.33}$$

Fix $\varepsilon \in (0, \min\{c_{(1-\alpha)}(g,\Omega(\theta^*)), (c_{(1-\alpha)}(g,\Omega(\theta^*)) - c_{(1-\alpha)}(\pi,\Omega(\theta^*)))/2\}$ (possible by steps 2-3), and note that

$$P_{F_{w_n}}(T_{w_n}(\lambda_0) \leq c_{w_n}^{SS}(\lambda_0, 1-\alpha)) \leq P_{F_{w_n}}(T_{w_n}(\lambda_0) \leq c_{(1-\alpha)}(g,\Omega(\theta^*)) + \varepsilon) + P_{F_{w_n}}(|c_{w_n}^{SS}(\lambda_0,1-\alpha) - c_{(1-\alpha)}(g,\Omega(\theta^*))| > \varepsilon) \ ,$$

By (S.28), (S.33), and our maintained assumptions, it follows that

$$\limsup P_{F_{w_n}}(T_{w_n}(\lambda_0) \leq c_{w_n}^{SS}(\lambda_0, 1-\alpha)) \leq P(S(v_\Omega(\theta^*) + (\lambda, \mathbf{0}_{k-p}), \Omega(\theta^*)) \leq c_{(1-\alpha)}(g,\Omega(\theta^*)) + \varepsilon) \ , \tag{S.34}$$

$$\liminf P_{F_{w_n}}(T_{w_n}(\lambda_0) \leq c_{w_n}^{SS}(\lambda_0, 1-\alpha)) \geq P(S(v_\Omega(\theta^*) + (\lambda, \mathbf{0}_{k-p}), \Omega(\theta^*)) \leq c_{(1-\alpha)}(g,\Omega(\theta^*)) - \varepsilon) \ . \tag{S.35}$$

Since (S.34), (S.35), $c_{(1-\alpha)}(g,\Omega(\theta^*)) > 0$, and our maintained assumptions,

$$\lim E_{F_{w_n}}[\phi_{w_n}^{SS}(\lambda_0)] = P(S(v_\Omega(\theta^*) + (\lambda, \mathbf{0}_{k-p}), \Omega(\theta^*)) > c_{(1-\alpha)}(g,\Omega(\theta^*))) \ . \tag{S.36}$$

We can repeat the same arguments to deduce an analogous result for the Penalize Resampling Test. The main difference is that for Test PR we do not have a characterization of the minimizer, which is not problematic as we can simply bound the asymptotic rejection rate using the results from step 3. This is,

$$\lim E_{F_{w_n}}[\phi_{w_n}^{PR}(\lambda_0)] \geq P(S(v_\Omega(\theta^*) + (\lambda, \mathbf{0}_{k-p}), \Omega(\theta^*)) > c_{(1-\alpha)}(\pi,\Omega(\theta^*))) \ . \tag{S.37}$$

By our maintained assumptions, $c_{(1-\alpha)}(g,\Omega(\theta^*)) > c_{(1-\alpha)}(\pi,\Omega(\theta^*))$, (S.36), and (S.37), we conclude that

$$\begin{aligned} \lim E_{F_{w_n}}[\phi_{w_n}^{PR}(\lambda_0)] &\geq P(S(v_\Omega(\theta^*) + (\lambda, \mathbf{0}_{k-p}), \Omega(\theta^*)) > c_{(1-\alpha)}(\pi,\Omega(\theta^*))) \\ &> P(S(v_\Omega(\theta^*) + (\lambda, \mathbf{0}_{k-p}), \Omega(\theta^*)) > c_{(1-\alpha)}(g,\Omega(\theta^*))) = \lim E_{F_{w_n}}[\phi_{w_n}^{SS}(\lambda_0)] \ . \end{aligned}$$

Since $\{w_n\}_{n\geq 1}$ is a subsequence of $\{u_n\}_{n\geq 1}$, this is a contradiction to (S.17) and concludes the proof. □

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