• I. Objective

• II. Model Development
  1) Solow model: saving rate
  2) Solow model: population growth
  3) Solow model: technological progress

• III. Main Conclusions and Policy Advice
Sustainable Growth Model

- Equation System

production function: \( Y = AK^a L^{1-a} \)

transformed production: \( y_t = A_t k_t^a \) due to CRS \( k = \frac{K}{L} \) \( y = \frac{Y}{L} \)

production allocation: \( y_t = i_t + c_t \)

investment equals savings: \( i_t = s \times y_t \)

capital accumulation: \( \Delta k_t = i_t - \delta k_t - nk_t \)

\( k_{t+1} = k_t + \Delta k_t \)

population growth: \( L = L_0 \)
### Symbol System

#### Exogenous Variables (set parameters)

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>2</td>
<td>technology parameter</td>
</tr>
<tr>
<td>alpha</td>
<td>0.6</td>
<td>capital exponent</td>
</tr>
<tr>
<td>$s$</td>
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</tr>
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<td>$\delta$</td>
<td>0.6</td>
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</tr>
<tr>
<td>initial $k$</td>
<td>6</td>
<td>initial capital-labor ratio</td>
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#### Endogenous Variables

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>capital per worker</td>
</tr>
<tr>
<td>$y$</td>
<td>output per worker</td>
</tr>
<tr>
<td>$c$</td>
<td>consumption per worker</td>
</tr>
<tr>
<td>$i$</td>
<td>investment per worker</td>
</tr>
<tr>
<td>$\delta k$</td>
<td>depreciation per worker</td>
</tr>
</tbody>
</table>
We start from a point with more capital per person than equilibrium. Capital depreciation exceeds capital accumulation, so that capital per person gradually decreased to equilibrium point. Since the 32 year, the economy enters a steady growth state with an equilibrium $k^*$ of 2.05309.

<table>
<thead>
<tr>
<th>Year</th>
<th>$k$</th>
<th>$y$</th>
<th>$%\Delta y$</th>
<th>$c=(1-s)y$</th>
<th>$i=sy=sf(k)$</th>
<th>$\delta k=\delta(K/L)$</th>
<th>$\Delta k=i-\delta k$</th>
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<td>5.860312</td>
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<td>3.5161873</td>
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<td>-1.25588</td>
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<td>4.744125</td>
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<td>-13.14%</td>
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<td>4.54893</td>
<td>-10.63%</td>
<td>2.729358</td>
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<td>2.4976689</td>
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<td>-0.37071</td>
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<tr>
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<td>3.883606</td>
<td>-6.71%</td>
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<td>-0.49%</td>
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<td>1.263939</td>
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<td>1.8548165</td>
<td>1.2365443</td>
<td>1.239798</td>
<td>-0.00325</td>
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<tr>
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<td>3.082282</td>
<td>-0.03%</td>
<td>1.8493689</td>
<td>1.2329126</td>
<td>1.233735</td>
<td>-0.00082</td>
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<td>3.079982</td>
<td>-0.01%</td>
<td>1.8479893</td>
<td>1.2319929</td>
<td>1.232201</td>
<td>-0.00021</td>
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<tr>
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<td>2.05346</td>
<td>3.079795</td>
<td>-0.01%</td>
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<td>1.2319179</td>
<td>1.232076</td>
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</tr>
<tr>
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<td>2.053302</td>
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<td>1.8476771</td>
<td>1.2317848</td>
<td>1.231854</td>
<td>-6.95E-05</td>
</tr>
</tbody>
</table>
Graphical Description

Figure 4: Income distribution over time

Figure 5: Capital and sustainable condition
Approach 2: Algebra

The total change in capital–labor ratio is
\[ \Delta k = sf(k) - \delta k \]

Our CRS Cobb–Douglas production function is
\[ f(k) = Ak^\alpha \]

So we have: \( \Delta k = sAk^\alpha - \delta k \)

Equilibrium Condition: find the \( k^* \)
\[ s.t. \quad \Delta k^* = 0, \quad i.e., \quad sAk^{*\alpha} - \delta k^* = 0 \]

\[ \therefore sAk^{*\alpha} = \delta k^*, \text{ rearrange: } k^{*\alpha-1} = \frac{\delta}{sA} \]

\[ \therefore k^* = \left( \frac{\delta}{sA} \right)^{\frac{1}{\alpha-1}} \text{ reduced form of the Solow Model} \]
The equilibrium solution is not affected by initial value of capital per person since $k^* = (\delta/sA)^{(1/\alpha-1)}$. But it could influence the time to converge. Country with a higher initial capital per person may take longer to time to reach the equilibrium status.
We can see that the country with a higher saving rate converges to a stable economy more quickly and reaches equilibrium with higher output and higher capital per person.
Sustainable Growth Model-population & technology progress added

- Equation System

  production function: \( Y = AK^a L^{1-a} \)

  transformed production: \( y_t = A_t k_t^a \) due to CRS \( k = \frac{K}{L} \) \( y = \frac{Y}{L} \)

  Technology Progress: \( A_{t+1} = t \times A_t \)

  production allocation: \( y_t = i_t + c_t \)

  investment equals savings: \( i_t = s \times y_t \)

  capital accumulation: \( \Delta k_t = i_t - \delta k_t - nk_t \)

  \( k_{t+1} = k_t + \Delta k_t \)

  population growth: \( L_{t+1} = L_t (1 + n) \)
## Symbol System

### Exogenous Variables (set parameters)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>alpha</td>
<td>0.6</td>
<td>capital exponent</td>
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<tr>
<td>s</td>
<td>0.4</td>
<td>savings rate per worker</td>
</tr>
<tr>
<td>( \delta )</td>
<td>0.6</td>
<td>depreciation rate per worker</td>
</tr>
<tr>
<td>initial ( A )</td>
<td>2</td>
<td>initial technology parameter</td>
</tr>
<tr>
<td>t</td>
<td>0</td>
<td>technology progress rate</td>
</tr>
<tr>
<td>n</td>
<td>0.01</td>
<td>population growth rate</td>
</tr>
<tr>
<td>initial ( k )</td>
<td>6</td>
<td>initial capital-labor ratio</td>
</tr>
</tbody>
</table>

### Endogenous Variables

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
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<tr>
<td>( k )</td>
<td>capital per worker</td>
</tr>
<tr>
<td>( y )</td>
<td>output per worker</td>
</tr>
<tr>
<td>( c )</td>
<td>consumption per worker</td>
</tr>
<tr>
<td>( i )</td>
<td>investment per worker</td>
</tr>
<tr>
<td>( A )</td>
<td>technology parameter</td>
</tr>
<tr>
<td>( \delta k )</td>
<td>depreciation per worker</td>
</tr>
</tbody>
</table>
Table 2: Equilibrium Condition: $\Delta k=0$ (population growth rate =0.01)

<table>
<thead>
<tr>
<th>Year</th>
<th>k</th>
<th>A</th>
<th>y</th>
<th>$%\Delta y$</th>
<th>c</th>
<th>i</th>
<th>$\delta k=\delta (K/L)$</th>
<th>$\Delta k=i-\delta k-nk$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>2</td>
<td>5.8603121</td>
<td>—</td>
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<td>0.06</td>
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<tr>
<td>5</td>
<td>2.9217931</td>
<td>2</td>
<td>3.8055696</td>
<td>-6.98%</td>
<td>2.2833418</td>
<td>1.5222278</td>
<td>1.7530758</td>
<td>0.0292179</td>
</tr>
<tr>
<td>10</td>
<td>2.1778488</td>
<td>2</td>
<td>3.190411</td>
<td>-1.90%</td>
<td>1.9142466</td>
<td>1.2761644</td>
<td>1.3067093</td>
<td>0.0217785</td>
</tr>
<tr>
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<td>2.0195701</td>
<td>2</td>
<td>3.0491961</td>
<td>-0.48%</td>
<td>1.8295176</td>
<td>1.2196784</td>
<td>1.211742</td>
<td>0.0201957</td>
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<td>-0.12%</td>
<td>1.808979</td>
<td>1.205986</td>
<td>1.1891547</td>
<td>0.0198192</td>
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<td>1.9727143</td>
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<td>1.8039303</td>
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<td>1.1836286</td>
<td>0.0197271</td>
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<td>-0.01%</td>
<td>1.802684988</td>
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<td>1.182267</td>
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<td>3.0043089</td>
<td>-0.01%</td>
<td>1.802585366</td>
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</tr>
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<td>1.9701264</td>
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<td>0.00%</td>
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<td>1.1820758</td>
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<td>1.80237754</td>
<td>1.2015851</td>
<td>1.181931</td>
<td>0.0196988</td>
</tr>
</tbody>
</table>

- Assuming an annual population growth rate of 1%, the economy would enter a steady state since the 32 year.
We start from point B (Figure 6), where demand for capital, including capital for the newborn and refill for depreciated capital, exceeds savings.

Figure. 7 Solow Growth Model
• The capital available for everyone gradually declined and reached the equilibrium point around the 32nd year, where capital accumulation equals the demand for capital injection and the growth of output per capita is nearly zero.
Approach 2: Algebra

The total change in capital–labor ratio is

\[ \Delta k = s f(k) - \delta k - nk \]

Our CRS Cobb–Douglas production function is

\[ f(k) = A k^\alpha \]

So we have: \[ \Delta k = s A k^\alpha - \delta k - nk \]

Equilibrium Condition: find the \( k^* \)

s.t. \[ \Delta k^* = 0, \text{ i.e., } s A k^{\alpha*} - \delta k^* - nk^* = 0 \]

\[ \therefore s A k^{\alpha*} = (\delta + n) k^*, \text{ rearrange: } k^{\alpha* - 1} = \frac{\delta + n}{s A} \]

\[ \therefore k^* = \left( \frac{\delta + n}{s A} \right)^{\frac{1}{\alpha - 1}} \text{ reduced form of the Solow Model} \]
The country with slower population growth (B) reaches equilibrium at a similar speed as the country without population growth (A). This also applies to the situation where the economy converges to an equilibrium output higher than initial output.

Economy with higher population growth rate will reach an equilibrium with lower capital and output per person. It seems that people's standard of living are pushed downwards in a country with fast population growth.
Country with higher technology level converges to an equilibrium with higher capital and output per person. It also converges more quickly than the country with lower technology level.
## Solow Model: Technological Progress

<table>
<thead>
<tr>
<th>Year</th>
<th>k</th>
<th>A</th>
<th>y</th>
<th>%Δy</th>
<th>Δk</th>
<th>k</th>
<th>A</th>
<th>y</th>
<th>%Δy</th>
<th>Δk</th>
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<td>—</td>
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<td>5.86</td>
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<td>-1.31587516</td>
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<td>-0.03%</td>
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<td>2</td>
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<td>0.00%</td>
<td>-5.93E-05</td>
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</table>
A country could not reach a stable growth status if there is also technological progress because the assumption of diminishing returns to capital may not hold. However, the country will enter a situation with constant growth rate of output per capita around the 33rd year.
III. Main Conclusions and Policy Advice

1. Country with a **higher saving rate** converges to a stable economy more quickly and reaches equilibrium with higher output per person. Government should encourage savings.

2. The country with **population growth** reaches equilibrium at a similar speed, but with lower output per person with population growth. Therefore, it is sensible for policy-makers to control population growth as to increase the welfare for everyone.

3. Country with **higher technology level** converges to equilibrium with higher capital and output per person. Advice: a developing country allocates some resources for technology development, even though at the cost of reduction of initial capital per person.

4. A country could **not reach a stable** growth status if there is also technological progress.

5. **Golden rule growth.** The idea that with depreciation and growth in the labor force, it is possible to get such a big capital stock that steady state consumption falls.
• Thanks!