“The government of the United States under the present administration has withdrawn oil lands from entry in order to conserve this asset, and has also taken steps toward prosecuting a group of California oil companies for conspiring to maintain unduly high prices, thus restricting production. Though these moves may at first sight appear contradictory in intent, they are really aimed at two distinct evils, a Scylla and Charybdis between which public policy must be steered.”

*Harold Hotelling, The Economics of Exhaustible Resources (1931)*
I. Introduction

Many studies of nonrenewable resource production have assumed there is a fixed reserve base of the resource, which can be exploited over time. In “The Economics of Exhaustible Resources,” Harold Hotelling was the first to show that with constant marginal extraction cost, price minus marginal cost should rise at the discount rate in a competitive market, and rent should rise at the discount rate in a monopolistic market. In addition, he showed that the monopoly price will initially be higher (and later will be lower) than the competitive price. The degree to which the two prices differ depends on the level of production cost and the way in which demand elasticities alter as resources are depleted.

Since then, the literature has expanded the Hotelling model to investigate other topics. For instance, some literature has considered what effect uncertainty would have on the rate of resource extraction. These studies have shown that when the reserve base is uncertain, a resource should be extracted more slowly. Reserve uncertainty has been examined by Gilbert (1976a), Heal (1978), and Loury (1976). In addition, Dasgupta and Stiglitz (1976), Heal (1978), and Hoel (1976) studied optimal extraction paths when a substitute for the resource may be introduced at some uncertain point in time. These studies have followed Hotelling in assuming a fixed reserve base.

In this paper, Pindyck allows the reserve base to change and treats it as the basis for production. Exploratory activity acts as the means of increasing or maintaining a level of reserves. To include the concept of depletion, Pindyck assumes that reserve additions resulting from exploration decrease as cumulative discoveries increase. This paper examines exploration and production and their dynamics. In turn, this examination gives a description of the entire price and reserve profile for a nonrenewable resource.

II. Exploration and Production under Competition and Monopoly

Competitive Producers

Competitive producers take price, \( p \), as given, and choose rate of production, \( q \), from a proved reserve based, \( R \). The average cost of production, \( C_1(R) \) increases as \( R \) falls.

Reserves may be replenished in response to exploratory effort, \( w \). The rate of flow of additions to \( R \) depends on \( w \) and cumulative reserve additions, \( x \), meaning, \( \dot{x} = f(w, x) \) with \( f_w > 0 \) and \( f_x < 0 \). The latter first order condition implies that as exploration continues over time, it becomes increasingly difficult to make new discoveries.
The cost of exploratory effort $C_2(w)$ increases with $w$, and it is assumed that $C''_2(w) \geq 0$. In addition, it is assumed that $C'(w)/f_w$, the marginal discovery cost, increases as $w$ increases. Finally, Pindyck assumes that $C_1(R) \to \infty$ as $R \to 0$. With these assumptions and definitions, we can set up the producer’s problem.

Producers solve:

$$\max_{q, w} W = \int_0^\infty [qp - C_1(R)q - C_2(w)]e^{-\delta t} \, dt \quad (1)$$

subject to:

$$\dot{R} = \dot{x} - q \quad (2)$$
$$\dot{x} = f(w, x) \quad (3)$$

and:

$$R \geq 0, q \geq 0, w \geq 0, x \geq 0. \quad (4)$$

Given this producer problem, we get the following present value Hamiltonian.

$$H = qpe^{-\delta t} - C_1(R)qe^{-\delta t} - C_2(w)e^{-\delta t} + \lambda_1[f(w, x) - q] + \lambda_2 f(w, x). \quad (5)$$

Because $H$ is a linear function of $q$, the producers’ choice of $q$ will be a bang-bang solution rather than an interior solution. This means that producers either produce either nothing or at some maximum capacity level. This will depend on whether the marginal discounted benefit of production, $p e^{-\delta t}$, exceeds the marginal discounted cost, $C_1(R)e^{-\delta t} + \lambda_1$. This relationship will depend on price, and market clearing will ensure that:

$$pe^{-\delta t} - C_1(R)e^{-\delta t} - \lambda_1 = 0. \quad (6)$$

Differentiating $H$ with respect to our state variables, $R$ and $x$, gives us the following dynamic equations for $\lambda_1$ and $\lambda_2$:

$$\dot{\lambda}_1 = C'_1(R)q e^{-\delta t} \quad (6)$$

and

$$\dot{\lambda}_2 = -(\lambda_1 + \lambda_2)f_x. \quad (7)$$

$\lambda_i$ is the change in the present value of future profits resulting from an additional unit of reserves. $\lambda_i$ is always positive, but $\dot{\lambda}_i$ is always negative because $C'(R) < 0$ by assumption. This means that at some point production will stop even though further exploration could yield more reserves.
Using the above equations, we can derive the equation describing the dynamics of the price path:

\[ \dot{p} = \delta p - \delta C'_1(R) + C'_1(R)f(w, x). \]  

(9)

Pindyck proved in an earlier paper that if extraction costs rise as reserves fall, but there is no \( w \), price follows the following equation:

\[ \dot{p} = \delta p - \delta C'_1(R) \]  

(9’)

So, we can see that prices with exploration rise more slowly than without exploration. Also, we can see that if \( C'(R) = 0 \), ie production costs do not depend on reserves, the rate of change of the price path is unaffected by exploration and is identical with that in the constant-cost Hotelling problem. The level of the price path, however, will be affected by the by exploration.

Because in this situation of exploration “planned” reserves will be greater than initial reserves, our producer can set the initial price at a lower level. This gives us the relationship in figure 1 which shows price trajectories with and without exploration in a situation of constant extraction costs.

![Price paths for constant extraction costs](image)

**Fig. 1.—** Price paths for constant extraction costs

Next we can determine path for the optimal rate of exploration. Because \( H \) is nonlinear in \( w \), we will not have a bang-bang solution. We instead start by
setting \( \frac{\partial H}{\partial w} = 0 \). Substituting equation (8) for \( \lambda \), we get the following relationship:

\[
\lambda_2 = \frac{C_2'(w)}{f_w} e^{-\delta t} - pe^{-\delta t} + C_1(R)e^{-\delta t}.
\]  
(10)

Using equations (8) and (10) we can rewrite equation (7) as:

\[
\lambda_2 = -\frac{f_x}{f_w} C_2'(w)e^{-\delta t}.
\]  
(11)

Further manipulation of our equations allows us to get the following equation:

\[
\dot{\lambda}_2 = -C_2'(w) \left( \frac{f_{wx}}{f_w^2} \right) e^{-\delta t} + \frac{f_w c_w C_2''(w) - C_2'(w) f_{ww}}{(f_w)^2} e^{-\delta t} - \delta \frac{C_2'(w)}{f_w} e^{-\delta t} - C_1'(R) q e^{-\delta t}.
\]  
(12)

Finally, equating (12) with (11) and rearranging gives us our \( \dot{w} \) equation:

\[
\dot{w} = C_2'(w) \left[ (f_{wx}/f_w) \cdot f - f_x + \delta \right] + C_1'(R) q f_w.
\]  
(13)

The characteristics of the boundary conditions for equations (9) and (13) will depend on whether or not \( C_2'(0)/f_w(0) \), the marginal discovery cost with 0 exploratory effort, is zero. To see why this is, we will consider two cases: \( C_2'(0)/f_w(0) = 0 \) and \( C_2'(0)/f_w(0) = \phi > 0 \).

Consider the situation where \( C_2'(0)/f_w(0) = 0 \). Production will stop at some terminal time T. At time T, w must be zero because further exploratory effort will have no value. In addition, from the transversality condition, we know that because there is no terminal cost associated with x, cumulative discoveries, \( \lambda_2(T) = 0 \). Plugging this into equation (10), we get:

\[
\lambda_2(T) = \frac{C_2'(0)}{f_w(0)} e^{-\delta T} - p(T)e^{-\delta T} + C_1(R(T))e^{-\delta T}.
\]
0 = 0 − p(T)e^{−δt} + C_1(R(T))e^{−δt}

p(T) = C_1(R(T))

This implies that price rises and reserves fall until the profit on the last bit of the resource is zero. This implies with equation (8) that \( \lambda_1(T) \), discounted rent at time \( T \), is zero.

Consider instead the situation where \( C'_2(0) / f_w'(0) = φ > 0 \). In this case, exploratory effort will become zero before production does. Denote \( T_1 < T \) as the time at which \( w \) becomes 0. The transversality condition will still imply that \( \lambda_2(T) = 0 \). In addition, as long as \( w = 0 \), \( \dot{\lambda}_2 = 0 \) implying that \( \lambda_2(T_1) = 0 \). Then, using equation (10), we see that for \( t \geq T_1 \):

\[
\lambda_2(t) = \frac{C'_2(0)}{f_w'(0)} e^{−δt} − p(t)e^{−δt} + C_1(R(t))e^{−δt}
\]

\[0 = φe^{−δt} − p(t)e^{−δt} + C_1(R(t))e^{−δt}\]

\[φ = p(t) − C_1(R(t))\]

Combining this result with equation (8) we find that \( p(t) − C_1(R(t)) = \lambda e^{δt} = φ \), and \( \dot{\lambda}_1 = −δλ_1 \). Then, using equation (6), we see that \( C'_2(R(t))q = −δφ \). These equations describe the behavior of \( w \), \( q \), and \( p \). Exploratory effort, \( w \), becomes zero at \( T_1 \) just as \( p − C_1(R) \rightarrow φ \) and \( −C'_1(R)q / δ \rightarrow φ \). This implies that for \( t \geq T_1 \), both \( p − C_1(R) \) and \( C'_1(R)q \) remain constant. Note that conditions (1) and (2) can be interpreted through recognizing that new reserves can have value either through being extracted and sold or through being stored, which reduces extraction costs. So, the last additional unit of reserves should be discovered when its marginal discovery cost \( (φ) \) equals (1) the net revenue that would be obtained by extracting and selling the unit and (2) the storage value of the unit.

Given particular functional forms for \( f \), \( C_1 \), and \( C_2 \), as well as a demand function relating \( p \) and \( q \), then equations (9) and (13) can be solved together with boundary conditions described earlier to yield optimal paths for price and exploratory effort, \( w \). The particular pattern of exploratory effort, price, and production that occurs will depend on the initial value of \( R \). This will be examined in detail in Section III.
Monopolistic Producer
The monopolistic producer will choose \( q \) and \( w \) to maximize the sum of discounted profits in (1), but he faces a demand function \( p(q) \) with \( p'(q) < 0 \).
The Hamiltonian will not be linear in \( q \) as a result, so we no longer have a bang-bang solution. Equations (6) and (7) still apply to this case, but maximizing \( H \) with respect to \( q \) gives us:

\[
\lambda_1 = MR e^{-\delta t} - C_1(R) e^{-\delta t},
\]

(14)

where \( MR = p + q(dp/dq) \). Differentiating (14) with respect to time and equating with (6) yields an equation describing the dynamics of marginal revenue:

\[
\dot{MR} = \delta MR - \delta C_1(R) + C_1'(R) f(w, x).
\]

(15)

As before, if extraction costs do not depend on \( R \), \( MR \) follows the same path as in the standard Hotelling problem: marginal revenue net of extraction cost rises at the discount rate. As in the competitive case, given any initial reserve level, exploration permits the initial price (and therefore MR) to be lower.

Maximizing \( H \) with respect to \( w \) and substituting (14) for \( \lambda_1 \) gives us an expression for \( \lambda_2 \):

\[
\lambda_2 = \frac{C_2'(w)}{f_w} e^{-\delta t} - MR e^{-\delta t} + C_1(R) e^{-\delta t}.
\]

(16)

Differentiating this equation with respect to time, and equating with (7) yields the following differential equation:

\[
\dot{w} = \frac{C_2'(w) [\langle f_{wx} f_w \rangle f_x - f_x + \delta] + C_1'(R) q f_w}{C_2''(w) - \frac{C_2'(w) f_{ww}}{f_w}}.
\]

(17)

This is identical to equation (13), but this does not mean the pattern of \( w \) is the same under monopoly as in the competitive case. As long as \( q \) is initially lower for the monopolist, \( \dot{w} \) will be larger since \( C_1'(R) < 0 \). This implies that whether initial \( R \) is small or large, the monopolist will initially exert less exploratory effort than the competitive firms, but later exert more effort.

III. The Behavior of Optimal Exploration and Production
In the solution of the typical exhaustible resource problem for competitive producers, price will rise slowly over time as reserves fall so that demand is choked off just as the last unit of reserves is extracted (assuming constant extraction costs), or just as the profit on the last unit extracted becomes zero (assuming extraction costs rise) and \( R \) declines. In this model, the price path will
depend on initial $R$ and on the magnitude and behavior of extraction costs, $C_1(R)$.

If initial $R$ is large enough so that extraction costs are low, price will start low and slowly rise over time as in the Hotelling model. Otherwise, if reserves are initially small and extraction costs depend on reserves, price will begin high, fall as reserves increase, and then rise slowly as reserves decline. These different cases will now be investigated more closely.

Case 1: Price Steadily Increasing
In figure 1, we saw that if extraction costs are constant, the form of the solution is the same as in the Hotelling problem. Even if extraction costs depend on reserves, however, price can increase steadily if initial $R$ is high enough.

If initial $R$ is large, this implies that $C_1(R)$ and $C'_1(R)$ are small, meaning that $\dot{p}$ will be positive. Also, $\psi$ will be positive initially. This is evident from the fact that the denominator of the right-hand side of (13) is always positive, the first term in the numerator is positive, and the second term is very small (since $R$ is large). Consequently, $w$ will begin growing at a very small level. When reserves are large, new discoveries are not needed initially, meaning the firm can defer the cost of exploration and discount it. Because initially $w$ is small, reserves will fall. They will fall more and more slowly as exploration increases. At some point after reserves have become small enough, $\psi$ will become negative ($C'_1(R)$ is large so that the second term in the numerator of (13) will outweigh the first term). Exploration will decline towards zero as most of the reserves are used up. Price will rise until demand is choked off just as profit on the last unit of resource is zero and just as $w$ becomes zero. At this point, it no longer pays to search for more reserves even though the resource is not “exhausted.” This scenario is outlined by the solid lines in figure 2.

Now presume that extraction costs are small relative to price and to the cost of exploration. This diminishes the value of holding a large stock of reserves, and further encourages firms to postpone exploratory activity. This scenario is outlined by the dotted lines in figure 2.
Case 2: U–Shaped Price Path

If initial reserves are very small, price will begin falling from a high level, since $C_1(R)$ and $C'_1(R)$ are large. Exploration will begin declining from a high level, because $C'_1(R)$ is large. Reserves will initially increase in response to exploration. However, in the later stages of resource use, reserves will decreases as exploration falls and the average product of exploration falls. As $R$ falls, price will increase until demand, exploration, and the profit on the last extracted unit of resources all simultaneously become zero. This is represented by the solid lines in figure 3.

Suppose that extraction costs are small. This means exploration can decline more rapidly since there is less need to build up a large reserve. Later, as production increases, $\dot{w}$ can become positive. Exploration increases so that the stock of reserves does not fall to zero too quickly.

As the returns from exploration lessen, $C'_1(R)$ will dominate the numerator of (13), causing $\dot{w}$ to become negative and exploration will fall to zero. Price will follow a U-shaped path. This is illustrated by the dotted lines in figure 3.
Our case 1 and case 2 are summarized by the phase diagram in figure 4. Our equations indicate that the $\dot{R} = 0$ isocline is nearly vertical for large values of $R$, but as $R$ becomes small, $q$ becomes small, so that the isocline bends toward the origin. Equation (13) indicates that the $\dot{w} = 0$ isocline will be downward sloping, a consequence of the fact that increased $R$ and $w$ make $\dot{w}$ larger.

The $\dot{w}$ isocline will shift to the left if $q$ decreases or if $x$ increases. The isocline will be closer to the origin if extraction costs are relatively low. In figure 4, $(\dot{w} = 0)_1$ corresponds to large extraction costs, $(\dot{w} = 0)_2$ corresponds to low extraction costs with $q$ small and/or $x$ large, and $(\dot{w} = 0)_3$ corresponds to small extraction costs with $q$ large and/or $x$ small.

Curve A denotes the optimal trajectory for when reserves are initially large. Reserves decrease continuously, but exploration increases and then decreases over the trajectory. If reserves are initially small, the optimal trajectory will depend on extraction costs. If the costs are large, exploration will be at a high level and will continually decrease, as in curve B. If, however, the costs are small, exploration can decrease, increase, and decrease again as shown in curve C.
IV. The Case of No Depletion

Recall that in this model we look at resources not as exhaustible but nonrenewable. Thus our notion of depletion refers to cases where discoveries from exploration no longer maintain our level of reserves and reserves decline. This happens because new discoveries are harder to come by if our cumulative discoveries are already high.

In this section we review cases where it does not get harder to make new discoveries as our cumulative discoveries increase and thus production can in essence go on indefinitely. An example is bauxite where at least in the intermediate term depletion is not an issue and we can say \( f_x = 0 \).

As producers of this type of resource our primary goal is to figure out our steady state level of reserves, \( \bar{R} \), and our steady state level of production \( \bar{q} \). Once we get to \( \bar{R} \) the goal of exploration will be to replenish extraction so as to maintain \( \bar{R} \). That is, \( \bar{q} = \bar{w} \) and \( \bar{w} = 0 \).

Since it does not get harder to make new discoveries \( (f_x = 0) \), our \( \bar{w} = 0 \) isocline will not shift as we saw in the previous section.

In figure 5 we see the trajectories to the steady state in the cases of large initial reserves (A) and small initial reserves (B). Any other trajectory will not lead to the long run equilibrium of constant reserves and production. Alternatively, any other trajectory leads either to reserves and exploration that grow limitlessly or decline toward 0.
Setting $f_x = 0$ and $\dot{w} = 0$ in equation 13 (this described how our effort changed over time) we find that

$$\frac{C'_2(w)}{f_w} = -\frac{C'_1(\bar{R})\bar{q}}{\delta}$$

The RHS of equation (18) above is the discounted present value of the annual flow of extraction cost savings from one extra unit of reserves i.e. the marginal benefit from one extra unit of reserves in the steady state.

The LHS of equation (18) above is the marginal discovery cost in maintaining the extra unit of reserves. We could start off in one of two situations:

- $MC > MB$ This is a case with initial reserves larger than necessary ($R > \bar{R}$).
  In this case we should have $w < \bar{w}$ to let the reserves decline to $\bar{R}$ and so $\dot{w} > 0$. (Trajectory A)

- $MC < MB$ This is a case with initial reserves smaller than necessary ($R < \bar{R}$).
  so we should have $w > \bar{w}$ and $\dot{w} < 0$. (Trajectory B)

Note that our steady state levels of $\bar{R}, \bar{w}$ and $\bar{q}$ are independent of initial reserves. We simply adjust from our initial reserves to get to these levels.

To figure out our optimal $\bar{w}$ we maximize $\pi$ at our steady state.
and take the derivative with respect to $\bar{w}$

$$\bar{w} = g(R, \bar{p}).$$

From setting $\bar{w} = 0$ equation (13) becomes

$$\delta C_2'(\bar{w}) + C_1'(\bar{R}) f(\bar{w}) f'(\bar{w}) = 0. \quad (21)$$

Equations (20) and (21) combined with $f(\bar{w}) = \bar{q}$ and $\bar{p} = p(\bar{q})$ describe a unique solution for our steady state variables $\bar{w}, \bar{R}, \bar{q},$ and $\bar{p}$ that are independent of initial conditions. This independence from initial reserves can be thought of as a “Golden Rule” of accumulation.

V. Measuring Resource Scarcity

“An appropriate scarcity measure ‘should summarize the sacrifices required to obtain a unit of the resource.’” Fisher (1977)

Economic incentives have been stressed throughout the paper and are key to understanding the notion of scarcity. Resource scarcity in this context is not just about potential reserves but how hard/costly it is to get to the potential reserves.

To fully capture the sacrifice Fisher refers to in the quote above we look at “rent” $(P - C_1)$ as a signal of scarcity. Remember producers produce until $\pi = 0$ and then stop. Rearranging (8) we see

$$\rho = C_1(R) + \lambda_1 e^{lt}. \quad (24)$$

and from $\frac{\partial h}{\partial w} = 0$ we see that

$$\lambda_1 e^{lt} = \frac{C_2'(\bar{w})}{f(\bar{w})} - \lambda_2 e^{lt}. \quad (25)$$

The RHS of (25) is made up of the marginal discovery cost and the shadow price of an additional unit of cumulative discoveries i.e. the change to my future stream of profits from discovering one more unit of the resource. Also you can think of this as how an additional unit of resource discovered impacts future marginal discovery costs (simply put-opportunity cost). Usually $\lambda_2$ is negative since it gets harder to make new discoveries if there have been a lot of discoveries already.

Leaving out the first term on the RHS of (25) would disregard the rising difficulty in making new discoveries. From (11) we see (assuming $\lambda_2 < 0$), that the magnitude of the opportunity cost of additional cumulative discoveries is
shrinking over time while the actual value of the marginal discovery cost increases. As we near the last phase of the production cycle, reserves are very low, the marginal cost of discovery is very high and resource use will be low (prices will be high).

This rationale applies in the current case however if we have the case where our resource is a factor of production (or consumption good), then \( P \) is a better measure of scarcity. In this case the extraction costs are part of the sacrifices required to obtain the resource.

VI. Conclusion
The paper distinguishes between exhaustible and nonrenewable resources because producers are not endowed with a fixed reserve base; they can actually increase or decrease the reserves through exploration activity provided they have the economic incentives to do so. Viewing the resource in this way enables to describe the entire history of its use.

The optimal rates of exploration and production depend on each other and must be jointly determined. According to equations (13) and (17), firms do a cost benefit analysis and build up reserves up to a level where extraction costs are reduced.

The decline in real oil prices over time confirms the U-shaped price profile. We find that prices fall as reserves increase, and then rise as reserves fall (unlike the Hotelling problem where prices gradually increase and production decreases). However, if the initial resource endowment is large, then price does actually increase over time.

In the case where a commodity is not prone to depletion (salt or sodium chloride, for example), the optimal level of reserves is independent of the initial reserve endowment.

The model does not take into consideration market structures other than the two extremes of a monopoly and perfect competition. In the real world, resource exploration and production is actually dominated by a few firms, and not just a single one. Also, uncertainty caused by political instability, natural disasters, wars etc. which have drastic effects on commodity prices are not taken covered.
Appendix

An empirical example is given from the oil-rich Permian region in Western Texas to give support to the model’s predictions. The dataset contains figures for 10 years: 1965-1974

- \( C_1(R) = \frac{m}{R} \)
- Average production cost = $1.25 per barrel \implies m = 7170 \times 1.25 = 8960 \)
- Reserves = 7170 million barrels

Three equations are estimated:

One representing the exploratory cost per well drilled because \( \omega \) = number of wells drilled

\[
\frac{C_2(\omega)}{\omega} = 0.0670 + \frac{103.2}{\omega} \\
R^2 = .458 \quad SE = .0039 \quad F(1, 7) = 5.90
\] (A1)

The second represents reserve additions through the variable DISC

\[
\log \text{DISC} = 2.389 + 0.599 \log \omega - 0.0002258x \\
R^2 = .837 \quad SE = 0.172 \quad F(2, 7) = 17.93
\] (A2)

The third one is the estimated demand curve

\[
q = 660 - 20p. \\
(A3)
\]

To obtain the numerical solutions, they write difference equation approximations to the differential equations as given by equations (9), 1(13), (15), and (17). Also, using equations (2) and (3), the simulations are run and the following results are obtained:

- Competitive price is initially lower but later higher than the monopoly price
- A monopolists’ initial exploratory effort is lower as production is lower and fewer discoveries are needed to maintain the reserve base
- Competitive exploration and production ceases much quicker than the monopolists’. In a competitive regime, production ceases in 55 years but for a monopolist, it is 92 years.
- Cumulative discoveries of reserves are the same in both cases
If oil was nondepletable, as highlighted in section IV, production, price, and reserves approach their steady state levels, according to equations (20) to (23). The following table shows what happens to these variables in both cases.

<table>
<thead>
<tr>
<th></th>
<th>Price (p)</th>
<th>Exploratory effort (ω)</th>
<th>Reserves (R)</th>
<th>Production (q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Competitive market</td>
<td>16.7</td>
<td>288</td>
<td>43</td>
<td>326</td>
</tr>
<tr>
<td>Monopoly</td>
<td>0.43</td>
<td>913</td>
<td>54.1</td>
<td>651.4</td>
</tr>
</tbody>
</table>

When we compare the optimal and actual prices and levels of exploration, we see that the optimal ω is much larger than actual ω but later on, they are very similar. This clearly suggests that it takes much longer to exploit the available resources than the model suggests.

**TABLE A1**

<table>
<thead>
<tr>
<th>Year</th>
<th>Production (10^6 Barrels/Year)</th>
<th>Price ($/Barrel)</th>
<th>Rent* ($/Barrel)</th>
<th>Wells Drilled</th>
<th>Reserves (10^6 Barrels)</th>
<th>Cumulative Discoveries (10^6 Barrels)</th>
<th>Profits (10^6 $/Year)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1965</td>
<td>552.0</td>
<td>5.400</td>
<td>4.150</td>
<td>9.253</td>
<td>7.170</td>
<td>0.0</td>
<td>1,556</td>
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<td>1966</td>
<td>557.0</td>
<td>5.146</td>
<td>4.177</td>
<td>4.779</td>
<td>9.243</td>
<td>2690</td>
<td>1,901</td>
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<td>551.9</td>
<td>5.402</td>
<td>4.488</td>
<td>3.794</td>
<td>3.981</td>
<td>4295</td>
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<td>548.5</td>
<td>5.373</td>
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<td>9.022</td>
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<td>544.7</td>
<td>5.340</td>
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<td>9.763</td>
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<td>5.883</td>
<td>3.554</td>
<td>8.892</td>
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<td>1990</td>
<td>417.0</td>
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<td>10.391</td>
<td>5.411</td>
<td>5.117</td>
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* Marginal discovery cost and opportunity cost of additional cumulative discoveries.

**TABLE A2**

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<tr>
<th>Year</th>
<th>Production (10^6 Barrels/Year)</th>
<th>Price ($/Barrel)</th>
<th>Rent* ($/Barrel)</th>
<th>Wells Drilled</th>
<th>Reserves (10^6 Barrels)</th>
<th>Cumulative Discoveries (10^6 Barrels)</th>
<th>Profits (10^6 $/Year)</th>
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</table>

* Marginal discovery cost and opportunity cost of additional cumulative discoveries.
Group Hicks: The equations: filling in the steps

Producer’s problem under competition

\[ Max_{q,w} W = \int_0^\infty [q p - C_1(R)q - C_2(w)] e^{-\delta t} dt \]

st:
\[ R = \dot{x} - q \]
\[ \dot{x} = f(w,x) \]
and
\[ R \geq 0, q \geq 0, w \geq 0, x \geq 0 \]

You should recognize that q and w are our control variables. R and x are our state variables. This problem should give us the following Hamiltonian.

\[ H = qpe^{-\delta t} - C_1(R)qe^{-\delta t} - C_2(w)e^{-\delta t} + \lambda_1 f(w,x) - \lambda_2 f(w,x) \]

To solve this using dynamic programming, we should take the following first order conditions:

\[ \frac{\partial H}{\partial w} = 0, \quad \frac{\partial H}{\partial q} = 0, \quad \frac{\partial H}{\partial R} = -\lambda_1, \quad \frac{\partial H}{\partial x} = -\lambda_2 \]

As stated earlier in the handout, note that because H is linear in q, the solution for q will be a bang-bang solution. By assuming market clearing, we get the following equation:

\[ pe^{-\delta t} - C_1(r)e^{-\delta t} - \lambda_1 = 0 \]

And we have our state variable’s FOCs:

\[ \dot{\lambda}_1 = C_1'(R)q e^{-\delta t} \]
\[ \dot{\lambda}_2 = -(\lambda_1 + \lambda_2) f_x \]

We will now derive the equation for \( \dot{p} \):

\[ \frac{d}{dt} \left[ pe^{-\delta t} - C_1(r)e^{-\delta t} - \lambda_1 = 0 \right] \]

\[ 0 = \dot{p} e^{-\delta t} - \delta pe^{-\delta t} - C_1(R)(-\delta)e^{-\delta t} - \dot{R}C_1'(R)e^{-\delta t} - \dot{\lambda}_1 \]
\begin{align*}
0 &= \dot{p} e^{-\delta t} - \delta p e^{-\delta t} - C_1(R)(-\delta) e^{-\delta t} - (\dot{x} - q) C_1'(R) e^{-\delta t} - \dot{\lambda}_1 \\
\Rightarrow \dot{\lambda}_1 &= \dot{p} e^{-\delta t} - \delta p e^{-\delta t} - C_1(R)(-\delta) e^{-\delta t} - (\dot{x} - q) C_1'(R) e^{-\delta t} \\

\text{Equate this with equation (6)} \\
C_1'(R) q e^{-\delta t} &= \dot{p} e^{-\delta t} - d p e^{-\delta t} + C_1(R)(\delta) e^{-\delta t} - \dot{x} C_1'(R) e^{-\delta t} + q C_1'(R) e^{-\delta t} \\
\dot{p} &= \delta p - \delta C_1(R) + \dot{x} C_1'(R) \\

\text{Then, using equation (3)} \\
\dot{p} &= \delta p - \delta C_1(R) + C_1'(R) f(w,x) \\

\text{Which is our desired equation for } \dot{p}.
\end{align*}
Deriving exploration path:

\[ \frac{\partial H}{\partial w} = -C_2'(w)e^{-\sigma t} + \lambda_1 \dot{w} + \gamma w = 0 \]

- \[ \lambda_1 \dot{w} = \gamma w - C_2'(w)e^{-\sigma t} \]
- \[ \lambda_1 = \frac{C_2'(w)e^{-\sigma t}}{\dot{w}} - \lambda_2 \]

Sub equation (8)

\[ pe^{-\sigma t} - C_1(R)e^{-\sigma t} = C_2'(w)e^{-\sigma t} - \lambda_2 \]

\[ \Rightarrow \lambda_2 = \frac{C_2'(w)e^{-\sigma t}}{\dot{w}} - pe^{-\sigma t} + C_1(R)e^{-\sigma t} \] (10)

Then use (8) \& (10) to write (7) as:

\[ \dot{\lambda}_2 = - (\lambda_1 + \lambda_2) \dot{w} \] (7)

\[ \Rightarrow \dot{\lambda}_2 = - \left( pe^{-\sigma t} - C_1(R)e^{-\sigma t} + \frac{C_2'(w)e^{-\sigma t}}{\dot{w}} - pe^{-\sigma t} + C_1(R)e^{-\sigma t} \right) \dot{w} \]

\[ \lambda_2 = -C_2'(w) \left( \frac{\dot{w}}{\dot{w}} \right) e^{-\sigma t} \] (11)

diff. (10) wrt time and Sub (2), (3), (9) for \( R, \dot{x}, \dot{y}, \dot{r} \)

\[ \lambda_2 = \frac{C_2'(w)e^{-\sigma t}}{\dot{w}} - pe^{-\sigma t} + C_1(R)e^{-\sigma t} \]

\[ \lambda_2 = \frac{C_2''(w)}{\dot{w}} e^{-\sigma t} - \frac{\dot{w} C_2'(w) \dot{w}}{(\dot{w})^2} e^{-\sigma t} - \frac{\ddot{w} C_2'(w) \dot{w} x e^{-\sigma t}}{(\dot{w})^3} \]

\[ - \dot{\sigma} C_2'(w) e^{-\sigma t} - pe^{-\sigma t} + \dot{w} e^{-\sigma t} + R C_1(R)e^{-\sigma t} - \dot{\sigma} C_1(R)e^{-\sigma t} \]
\[ \dot{\varphi} = \left[ \frac{c''(\varphi)}{fw} - \frac{c'(\varphi)fw}{(fw)^2} \right] \dot{\varphi} e^{-\sigma t} - \frac{\delta c'(\varphi)}{fw} e^{-\sigma t} \]

\[-(\dot{c}'(R) + c'(R) f(\varphi,x)) e^{-\sigma t} \quad \text{dp} = \dot{\varphi} e^{-\sigma t} + RC_1'(R) e^{-\sigma t} - \delta c'(\varphi) e^{-\sigma t} \]

\[ \dot{\varphi} = \left[ \frac{fw c''(\varphi) - c'(\varphi)fw}{(fw)^2} \right] \dot{\varphi} e^{-\sigma t} - \frac{c'(\varphi)fwx \cdot f}{(fw)^2} e^{-\sigma t} \]

\[-\frac{\delta c'(\varphi)}{fw} - q c'(R) e^{-\sigma t} \quad (12) \]

equate (12) with C11) and rearrange

\[-\frac{fx}{fw} c'(\varphi) e^{-\sigma t} = \left[ \frac{fw c''(\varphi) - c'(\varphi)fw}{(fw)^2} \right] \dot{\varphi} e^{-\sigma t} - \frac{c'(\varphi)fwx \cdot f}{(fw)^2} e^{-\sigma t} \]

\[-\frac{\delta c'(\varphi)}{fw} - C_1'(R) q e^{-\sigma t} \]

\[ \left[ \frac{fw c''(\varphi) - c'(\varphi)fw}{(fw)^2} \right] \dot{\varphi} = C_1'(R) q + \frac{c'(\varphi)fwx \cdot f}{(fw)^2} - \frac{fx}{fw} c'(\varphi) \]

\[ \left[ \frac{fw c''(\varphi) - c'(\varphi)fw}{(fw)^2} \right] \dot{\varphi} = (fw)^2 C_1'(R) q + C_1'(\varphi)fwx \cdot f - \frac{fx}{fw} c'(\varphi) \]

\[ \dot{\varphi} = \frac{fw C_1'(R) q + c'(\varphi)}{fw C''(\varphi) - c'(\varphi)fw \cdot \left[ \frac{fx}{fw} - \frac{\delta}{fw} \right] \cdot \frac{fw}{fw} \}

\[ \dot{\varphi} = \frac{C_1'(\varphi)fwx \cdot f - \frac{fx}{fw} - \delta}{fw C''(\varphi) - c'(\varphi)fw \cdot \left[ \frac{fw C'_1(\varphi)q + c'(\varphi)}{fw C''(\varphi) - c'(\varphi)fw \cdot \left[ \frac{fx}{fw} - \frac{\delta}{fw} \right] \cdot \frac{fw}{fw} \} \]

\[ \dot{\varphi} = C_2'(\varphi) \left[ \frac{fw x}{fw} \cdot f - \frac{fx}{fw} - \delta \right] + C_1'(R) q \frac{fw}{fw} \]

\[ C_2''(\varphi) - C_2'(\varphi) \frac{fw}{fw} \]

\[ \dot{\varphi} = \frac{C_2'(\varphi) \left[ \frac{fw x}{fw} \cdot f - \frac{fx}{fw} - \delta \right] + C_1'(R) q \frac{fw}{fw} \]

\[ \frac{C_2''(\varphi) - C_2'(\varphi) \frac{fw}{fw} \]