Notes for “School Choice: A Mechanism Design Approach,” by Atila Abdulkadiroğlu and Tayfun Sönmez

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Introduction

- Traditionally, a student whose family cannot pay for private school is assigned to a public school according to where she lives
- Recent rise in popularity of two types of school choice:
  - *Intra-district choice*: select schools throughout the district where the student lives
  - *Inter-district choice*: select schools outside of the student’s own district
- A student assignment mechanism is necessary in states or districts with school choice
  - Many currently used mechanisms are legally ill-defined or establish procedures which induce parents to play complicated admissions games; often, misrepresenting their true preferences is in their best interest
- In an attempt to resolve these problems, the authors propose two competing student assignment mechanisms: the Gale-Shapley student optimal stable mechanism and the top trading cycles mechanism
- Related problem: allocation of dormitory rooms
  - Mechanism: random serial dictatorship (RSD)
    - Order students with a lottery and assign the first student her top choice, the next student her top choice among the remaining slots, and so on.
    - RSD is *Pareto efficient*: no student can be made better off without making another student worse off
    - RSD is *strategy-proof*: truthful preference revelation is a dominant strategy for the students
- Difficulty with school choice: the priority ordering of a student can be different at different schools
  - Priority may be determined by the proximity of the student’s home to the school, the presence of a sibling at the school, the bilingual needs of the student, etc.
- Related problem: college admissions
  - Stability: there should be no unmatched student-school pair \((i, s)\) where student \(i\) prefers school \(s\) to her assignment and school \(s\) prefers student \(i\) to one or more of its admitted students
- Application of stability to school choice: there should be no unmatched student-school pair \((i, s)\) where student \(i\) prefers school \(s\) to her assignment and she has higher priority than some other student who is assigned a seat at school \(s\)
• **Justified envy**: the situation in which student \(i\) prefers school \(s\) to her assignment and has higher priority than student \(j\), but student \(j\) is nonetheless assigned a seat at school \(s\)
  
  o A stable matching in school choice eliminates justified envy

I. School Choice

• Assumptions:
  
  o A number of students will be assigned to have a seat in a number of schools
  
  o Each school has a maximum capacity, but the total number of seats across schools is enough to accommodate all the students
  
  o Each student has a strict preference over all the schools. Also, each school has a strict priority over all the students. But schools’ priorities do not represent the schools’ preferences since their priorities are determined by local laws. The priority ordering of two students with the same level of priority under the law will be determined by a lottery

• Outcome (matching):
  
  o Each student is assigned to one school, and no school is assigned to students over its capacity. This is a matching. A matching is *Pareto efficient* if no other matching exists such that each student is assigned to a weakly better school and one student a strictly better school

• Two School Choice Mechanisms:
  
  o Boston Student Assignment Mechanism (Direct Mechanism)
    
    ▪ Each student reports a school ranking that reveals the preference
    
    ▪ Schools determine their priorities as follows:
      
      • First priority: sibling and walk zone
      • Second priority: sibling
      • Third priority: walk zone
      • Fourth priority: other students
      • Students with identical priorities are ordered by a lottery
    
    ▪ Matching rules
      
      • Round 1: Each school considers the students that take it as the first choice and assign seats to these students following their priority order until either there are no seats left or no student is left who has listed it as his or her first choice
      
      • Round 2: Consider the remaining students. Each school considers the students that take it as the second choice and assign remaining seats to these students following their priority order until either there are no seats left or no student left who has listed it as his or her second choice
      
      • Round \(k\): Consider the remaining students. Each school considers the students that take it as the \(k^{th}\) choice and assign remaining seats
to these students following their priority order until either there are no seats left or no student left who has listed it as his or her $k^{th}$ choice

- **Problems:**
  - It is not strategy-proof. Even if a student has very high priority at school $s$, unless she lists it as her top choice she loses her priority to students who have listed $s$ as their top choice. Hence, students are likely to misrepresent their preferences by increasing their ranking of those schools for which they have high priority.
  - Since students (or parents) are induced to misrepresent their preferences, the outcome is unlikely to be Pareto efficient.

- **Columbus Student Assignment Mechanism (Indirect Mechanism)**
  - Each student may apply to up to three different schools.
  - For some schools, seats are guaranteed for students who live in the school’s regular assignment area, and the priority among remaining applicants is determined by a random lottery. For the remaining schools, the priority among all applicants is determined by a random lottery.
  - For each school, available seats are offered to students with the highest priority by a lottery, and the remaining applications are put on a waiting list.
  - After receiving an offer, a student has three days to accept or decline. If she accepts an offer, she is assigned a seat; she is removed from the waiting list of other schools to which she has applied. As soon as seats become available at schools because of declined offers, the lottery office makes offers to students on the waiting lists.
  - **Problems**
    - The optimal application strategy of a student is unclear. It is unclear whether accepting or declining the offer from her second or third choice is an optimal choice. Also, it is unclear what type of school list is optimal.
    - Inefficiency problem. When a student gets an offer from another student’s first-choice school, she may accept it even if it is not her first choice, because she doesn’t know whether she will receive better offers. This results in inefficient matching.

**II. Two Competing Mechanisms**

- **Gale-Shapley Student Optimal Stable Mechanism**
  - Version of the Gale-Shapley deferred acceptance algorithm.
  - Colleges are treated as agents with preferences while schools with priorities are regarded as “objects” to be consumed by the students.
• Mathematically, college preferences analogous to school priorities
• Exhibits stability in the college admission context
• Eliminates justified envy in the school choice context

○ Step 1
  • Each student proposes to her first choice. Each school tentatively assigns its seats to its proposers one at a time following their priority order. Any remaining proposers are rejected

○ Step k
  • Each student who was rejected in the previous step proposes to her next choice. Each school considers the students it has been holding together with its new proposers and tentatively assigns its seats to these students one at a time following their priority order. Any remaining proposers are rejected

○ Last Step:
  • No student proposal is rejected
  • Each student is given final assignment

○ Proposition 1: Gale-Shapley student optimal stable mechanism Pareto-dominates any other mechanism that eliminates justified envy.

  • Proof (from the college admissions problem): Let us call a college “possible” for a particular applicant if there is a stable assignment that sends him there. Assume that up to a given point in the procedure no applicant has yet been turned away from a college that is possible for him. At this point suppose that college $A$, having received applications from a full quota of better-qualified applicants $b_1, b_2, \ldots, b_q$, rejects applicant $a$. We must show that $A$ is impossible for $a$. We know that each $b_i$ prefers college $A$ to all the others, except for those that have previously rejected him, and hence (by assumption) are impossible for him. Consider a hypothetical assignment that sends $a$ to $A$ and everyone else to colleges that are possible for them. At least one of the $b_i$s will have to go to a less desirable place than $A$. But this arrangement is unstable, since $b_i$ and $A$ could upset it to the benefit of both. Hence the hypothetical assignment is unstable and $A$ is impossible for $a$. The conclusion is that our procedure only rejects applicants from colleges which they could not possibly be admitted to in any stable assignment. Therefore, the resulting assignment is optimal (Gale and Shapley, 1962)

○ Proposition 2: Gale-Shapley student optimal stable mechanism is strategy-proof.

  • Students do not risk losing their priority if they do not rank that school high in their preferences. This allows students to consider admissions as independent of preference ranking, enabling them to truthfully reveal their preferences. No room for strategic play.
Stability – Pareto efficiency trade-off:

Let students $i_k$ order their school $s_k$ preferences as

$i_1: s_2, s_1, s_3; \ i_2: s_1, s_2, s_3; \ i_3: s_1, s_2, s_3$

while schools $s_k$ order their student $i_k$ priorities as

$s_1: i_1 - i_3 - i_2; s_2: i_2 - i_1 - i_3; s_3: i_2 - i_1 - i_3$

yielding a stable matching $\left( \begin{array}{ccc} i_1 & i_2 & i_3 \\ s_1 & s_2 & s_3 \end{array} \right)$.

However, the Pareto-efficient matching is $\left( \begin{array}{ccc} i_1 & i_2 & i_3 \\ s_2 & s_1 & s_3 \end{array} \right)$.

This illustrates how complete elimination of justified envy may result in a Pareto-inefficient matching. Student 1 prefers school $s_2$ to her assignment, $s_1$, and student 2 prefers $s_1$ to her assignment, $s_2$. So, if given the chance, students 1 and 2 would trade school assignments, making both better off while leaving student 3 with the same level of utility. Thus prioritizing stability can lead to Pareto inefficiency. If eliminating justified envy is the policymaker’s first goal, she should use the Gale-Shapley mechanism, recognizing that there may be welfare losses.

The above observation led the authors to investigate which formulation would be Pareto-efficient. Not surprisingly, a relaxation in the definition of student priority (higher priorities representing better opportunity to receive assignment) would suffice.

- **Top Trading Cycles Mechanism**
  - Direct mechanism
  - Pareto efficient
  - Allows for justified envy
  - Students have strict preferences over schools
  - Generalized version of the random serial dictatorship mechanism, allowing for different priorities among schools.
  - Variant of house allocation model with existing tenant

Step 1

- Counter keeps track of how many seats are still available at the school. Set counters equal to capacity. Each student points to her favorite school under her announced preferences. Each school points to the student who has the highest priority for the school. Since the number of students and schools is finite, there is at least one cycle. (A cycle is an ordered list of distinct schools and distinct students $\{s_1, i_1, s_2, i_2, \ldots, s_k, i_k\}$ where $s_1$
points to $i_1$, $i_1$ points to $s_2$, ... $s_k$ points to $i_k$, $i_k$ points to $s_1$.) Each school can be part of at most one cycle. Similarly, each student can be part of at most one cycle. Every student in a cycle is assigned a seat at the school she points to and is removed. The counter of each school in a cycle is reduced by one and if it reduces to zero, the school is also removed. Counters of all other schools stay put.

- **Step k**
  - Each remaining student points to her favorite school among the remaining schools and each remaining school points to the student with highest priority among the remaining students. There is at least one cycle. Every student in a cycle is assigned a seat at the school that she points to and is removed. The counter of each school in a cycle is reduced by one and if it reduces to zero the school is also removed. Counters of all other schools stay put.

- **Last Step:**
  - All students are assigned a seat.

- **Proposition 3:** Top trading cycles mechanism is Pareto efficient.
  - **Proof (appendix):** Any student removed at Step 1 is assigned her top choice by definition and, thus, cannot be made better off. Any student removed at Step 2 is assigned her top choice among the remaining ones. As preferences are strict, this student cannot be better off without making anyone else worse off. By the time the algorithm ends, no student can be made better off without hurting someone from the earlier Steps.

- **Proposition 4:** Top trading cycles mechanism is strategy-proof.
  - This follows naturally from the set-up of the cycles. A student removed at Step $k$ has qualifications that would not allow her to be removed any earlier. Misrepresenting her preferences would not alter assignments as other more preferable assignments would be already given to students with higher priority; truthful revelation of preferences is a dominant strategy. As a matter of fact, manipulation of preferences may only harm the student.
  - **Proof (appendix):** Consider a student $i$ with true preferences $P_i$. Fix an announced preference profile $Q_{-i} = (Q_j)_{j \in P \setminus \{i\}}$ for every student except $i$. Let $T$ be the step at which student $i$ leaves under $Q_i$, $(s, i_1, s_1, ..., s_k, i)$ be the cycle she joins, and $s$ be her school assignment. Let $T^*$ be the step at which she leaves under her true preferences $P_i$. We want to show the assignment under $P_i$ is as least as good as school $s$.

  **Case 1:** $T^* \geq T$
Suppose student $i$ announces her true preferences $P_i$. Consider Step $T$. The remaining students and schools at the beginning of Step $T$ are the same whether student $i$ announces $Q_i$ or $P_i$. Thus, at $T$ school $s$ points to student $i_1$, student $i_1$ points to school $s_1$, ..., school $s_k$ points to student $i$. As long as student $i$ remains unassigned, the procedure iterates. As long as she reveals her true preferences in each step, the student will eventually be assigned to a school as least as good as school $s$. Otherwise, joining the cycle $(s, i_1, s_1, ..., s_k, i)$ would assign her school $s$ as initially.

Case 2: $T^* < T$

Consider Step $T^*$. The remaining students and schools at the beginning of Step $T^*$ are the same whether student $i$ announces $Q_i$ or $P_i$ (as in Case 1, only now the set of schools is potentially bigger). At Step $T^*$, the student is assigned to her favorite school by revealing her true preferences $P_i$. Again, this assignment (under $P_i$) is at least as good as $s$.

- Which Mechanism Shall Be Chosen?
  - As both proposed mechanisms are strategy-proof, the criterion for choosing between them is the policymaker’s interpretation of the priorities.
  - When priority is given to complete elimination of justified envy rather than Pareto efficiency, then Gale-Shapley student optimal stable mechanism is more appropriate (public universities).
  - When moving from a system of separate intra-district school programs to an inter-district program, the Gale-Shapley student optimal stable mechanism would ensure a smoother transition. As long as inter-district priorities are lower than the intra-district ones, the outcome of the intra-district Gale-Shapley student optimal stable mechanism remains stable on the inter-district level. However, a similar transition may hurt some students under the top trading cycles mechanism.
  - When two patterns of assignments are followed and some students are guaranteed a seat within their district, the top trading cycles mechanism may be more useful. Under true preferences, some students enjoy seats at least as good as the ones guaranteed within their district. Welfare gains are possible when these students are allowed to trade their school preferences with others. Furthermore, any instability is necessarily due to the randomly obtained priorities.

### III. Controlled Choice

- Controlled choice attempts to provide choice to parents while maintaining racial, ethnic and/or ability balance.
In many school districts, controlled choice constraints are implemented via rigid or flexible racial quotas. Both Gale-Shapley and the top trading cycles mechanism can be easily modified to accommodate this kind of constraints.

- **Gale-Shapley Student Optimal Stable Mechanism with Type-Specific Quotas**
  - If quotas are rigid, one can implement the mechanism separately for each type.
  - The modified mechanism (described on Page 739) eliminates all justified envy between students of the same type. Now the fairness requirement has changed. If there is an unmatched student-school pair \((i, s)\) where student \(i\) prefers school \(s\) to her assignment and she has higher priority than some other student \(j\) who is assigned a seat at school \(s\), then students \(i\) and \(j\) are of different types and the quota for the type of student \(i\) is full at school \(s\).
  - **Proof of elimination of justified envy:**
    - Suppose there is an unmatched student-school pair \((i, s)\) where student \(j\) is assigned school \(s\) and student \(i\) is assigned school \(t\). Student \(i\) must have proposed to school \(s\) before \(t\) because he prefers \(s\) to \(t\), and he must have been rejected by \(s\) because he ends up at school \(t\) instead. If student \(j\) proposes to \(s\) before \(i\), then when \(i\) proposes to school \(s\), \(j\) must be rejected if \(i\) is rejected because \(i\) has a higher priority than \(j\) at school \(s\). If student \(j\) proposes to \(s\) after \(i\) is rejected by \(s\), \(j\) cannot get in because at this time school \(s\) has already been filled with students with priorities higher than \(i\), and hence higher than \(j\). Therefore we have a contradiction; \(j\) cannot have a seat at school \(s\).
  - **Proposition 5:** Gale-Shapley student optimal mechanism with type-specific quotas is strategy-proof.
    - The equivalent of the modified mechanism is described on Page 740. Here we clarify a bit the extension of student preference. If \(s\) is preferred to \(t\) then each of \(s^1, s^2, s^3\) is preferred to each of \(t^1, t^2, t^3\). This means \(s^1 > s^2 > s^3 > t^1 > t^2 > t^3\), in which order a student who prefers \(s\) to \(t\) would propose. An intuitive explanation for the equivalence is: in this mechanism, it does not affect the results even if a student is absent in some rounds as long as he proposes to schools according to his true preference. If he gets in \(s\), it means he has higher priority than anyone who does not get in while preferring \(s\) to his assignment. For example, student \(j\) is a type 1 student, so even he would get rejected at \(s^2\) for sure; it does not hurt his outcome to apply in the order specified above.

- **Top Trading Cycles Mechanism with Type-Specific Quotas**
  - No need to modify in the case of rigid quotas.
  - The modified mechanism (described on Page 740) may induce efficiency loss due to the constraints.
Here we provide a simple proof that there is at least one cycle at each step of the top trading cycles mechanism. Suppose we have \( n \) schools and \( m \) students at step \( k \), and \( n \) is less than or equal to \( m \). \( s^1 \) must point to some student, without loss of generality (since we can always relabel schools and students with a one-to-one mapping), let \( s^1 \) point to \( i^1 \). If \( i^1 \) points to \( s^1 \), then we have one cycle. If not, \( i^1 \) must be pointing to some other school, which can be labeled as \( s^2 \). If \( s^2 \) points to \( i^1 \), then we have one cycle. If not, then \( s^2 \) must be pointing to some other students, which can be labeled as \( i^2 \). Now we have an open cycle \((s^1, i^1, s^2, i^2, \ldots)\), if a school (student) points to a previous student (school), we have one cycle. If not, we would finally arrive at \((s^1, i^1, \ldots, s^n, i^n)\). Since we have fewer schools than students, student \( i^n \) must be pointing to some school, and we have at least one cycle. Interestingly, there is still at least one cycle even if we put different types of students together in one step. That is, we do not need to separately apply this mechanism for different types of students.

**Proposition 6:** The outcome of the modified top trading cycles mechanism is constrained efficient.

- Constrained efficient means there is no other matching that satisfies the controlled choice constraints which assigns all students a weakly better school and at least one student a strictly better school. Given the constraints, the proof is similar to Proposition 3.

**Proposition 7:** The outcome of the modified top trading cycles mechanism is also strategy-proof.

- Given the constraints, the proof is similar to Proposition 4.
Controlled Choice Game!

There are seven students ($i_1, \ldots, i_7$) and three schools ($s_1, s_2, s_3$). There are two types of students, type 1 and type 2. Students $i_1$ through $i_4$ are type 1, and $i_5$ through $i_7$ are type 2. Schools $s_1$ and $s_3$ have two seats each and $s_2$ has three seats. School $s_1$ will enroll one type 1 and one type 2. School $s_2$ will enroll no more than two type 1 students and no more than two type 2 students. School $s_3$ does not have a type quota.

School priorities are:

$S_1 : i_2 - i_1 - i_3 - i_5 - i_4 - i_7 - i_6$

$S_2 : i_3 - i_7 - i_6 - i_5 - i_2 - i_1 - i_4$

$S_3 : i_6 - i_4 - i_7 - i_1 - i_3 - i_2 - i_5$

Student preferences are:

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Determine the final matchings given by the Gale-Shapley mechanism and the top trading cycles mechanism.

Appendix

Alvin Roth (1982) proves that there are matching procedures which always yield stable outcomes, efficient matching procedures in which truthful revelation is a dominant strategy for every agent, but no matching procedure exists which meets both these requirements. However, it is possible to find a stable matching mechanism when confining a given set of agents’ strategic actions. Note that high schools (unlike universities) are “objects to be consumed” by students due to their public nature. In our matching problem, this translates to keeping the set of high-school priorities fixed which holds as these are exogenously predetermined by policymakers. We need to define a procedure where students’ stated preferences are revealed in a manner such that, in the resulting non-cooperative game, it is a dominant strategy for each student $i_k$ in $I$ to state his true preference profile $P$. 
The following lemma states that if a misrepresentation by $i_k$ leaves $i_k$ at least as well off as at $s_1$, then no student will suffer, i.e., every student likes school $s_2 = g(P')$ resulting from the misrepresentation at least as well as school $s_1 = g(P)$.

**Lemma 2:** If $P'(i_k)$ is a misrepresentation such that $s_2 = g(P')$ and either $s_2P(i_k)s_1$ or $s_{2,k} = s_{1,k}$ then, for each $i_l$ in I, either $s_2P(i_l)s_1$ or $s_{2,l} = s_{1,l}$.

Proof: Suppose, to the contrary, that $s_1P(i_l)s_2$ for some $i_l$ in I, i.e., $i_l$ is worse off in school $s_2$ than in school $s_1$. Since every student other than $i_l$ states the same preferences in the profiles $P$ and $P'$, it must be that $i_l$ is rejected by $s_1$ at some step of the procedure $G(P')$. Let $j$ be the first step of $G(P')$ at which some $i_l$ is rejected by $s_1$. Then, $s_1$ must have received an application in step $j$ of $G(P')$ from student $i_m$ who had not applied in $G(P)$ such that $i_mP(s_1)i_l$, i.e. from a student $i_m$ who $s_1$ prefers to $i_l$. The fact that $i_m$ did not propose to $s_1$ in $G(P)$ means $s_2P(i_m)s_1$ and so $i_m$ must have been rejected by $s_2$ prior to step $j$ which contradicts the choice of $j$ as the first such period. Consequently, no $i_l$ is rejected by $s_1$ in $G(P')$ which completes lemma’s proof.

**Proposition 2:** Gale-Shapley student optimal stable mechanism is strategy-proof.

Proof: Let $s_1 = g(P)$ assignment result from the algorithm in $t$ steps, $G(P)$ terminates at step $t$. It suffices to show that if $P' \neq P$ only in that $P'(i_k)$, say, replaces student $i_k$’s true preferences $P(i_k)$, then student $i_k$ does not prefer the assignment $s_2 = g(P')$ to $s_1 = g(P)$. That is, we need to show that no successful misrepresentation of preferences is possible, where a misrepresentation $P'(i_k)$ is defined to be a successful misrepresentation by student $i_k$ if $s_2P'(i_k)s_1$. That is, a misrepresentation $P'(i_k)$ is successful if $i_k$ truthfully prefers school $s_2$ she’s matched to when she misrepresents her preferences to the school $s_1$.

Assuming that students only consider manipulations that don’t harm themselves, $s_2P'(i_k)s_1$, we need to show that these don’t help them either, $s_{2,k} = s_{1,k}$ for any $i_k$ in I who gets assigned at step $t$ in $G(P)$. Since $t$ was the final step of $G(P)$, $i_k$ was the only student not asked for assignment $s_1$ in $G(P)$. Otherwise, one more step would have occurred. However, by the Lemma, no student does worse in $G(P')$ than at $G(P)$. Hence, only $i_k$ proposes to $s_1$ in $G(P')$ – since $s_1$ receives at least one application – and, so $s_2,k = s_{1,k}$. The same holds for any other $i_j$ who is the only one to apply at $s_1$ in $G(P)$, regardless of the period at which she is assigned to a school. So, if the manipulator $i_t$ got his preferable assignment at step $t$ of $G(P)$ or if he got assigned at a school that had received no other applications in $G(P)$ then his manipulation would not be any beneficiary.

Alternatively, suppose that $i_t$ gets an assignment at some other step $k$ of $G(P)$ prior to $t$. Let $r$ be a step of $G(P)$ such that $k \leq r < t$. The induction hypothesis states that if
for every $i_k$ who get assigned at step $r + 1$ through $t$ of $G(P)$ then $s_{2,k} = s_{1,k}$ for every $i_k$ who get assigned at step $r$ of $G(P)$. Let $i_q$ be a student assigned at step $r$, $l'$ the subset of students who do not receive assignments at $s_1$, i.e., $l' = \{i_k \in I | s_1 P(i) s_\alpha \}$ or, equivalently, the ones who prefer $s_1$ to their final assignment $s_\alpha$. If $l'$ is empty then $s_2 = s_1$ by the previous paragraph. If not, let $i_\psi$ be the student in $l'$ such that $i_\psi P(s_1) i_\psi \forall i_\psi \in l'$, i.e., $i_\psi$ is preferred by $s_1$ to all other students earlier not admitted in $G(P)$. Then, $i_\psi$ may only get his assignment after step $r$ of $G(P)$ since he is not rejected by $s_1$ prior to $r$ and $s_{2,\psi} = s_{1,\psi}$ by the induction hypothesis.

As $i_\psi$ does not misrepresent his preferences ($i_\psi \neq i_t$) it means that $i_\psi$ requests assignment $s_1$ in $G(P')$ and gets rejected in favor who $s_1$. But since no student applies at $s_y$ in $G(P')$ who did not also applies in $G(P)$ this means that $s_1$ rejects $i_\psi$ in favor of $i_q$, so $s_{2,q} = s_{1,q}$. Thus, by induction $s_{2,k} = s_{1,k} \forall i_k$ who gets assigned to a school at step $k$ or later. Therefore, manipulation cannot be successful as no better outcome can be achieved.

**Glossary**

*Unstable:* An assignment of applicants to colleges will be called unstable if there are two students, $\alpha$ and $\beta$, who are assigned to schools $A$ and $B$, respectively, although $\beta$ prefers $A$ to $B$ and $A$ gives higher priority to $\beta$ than to $\alpha$. An unstable assignment can be upset by a school and a student acting together in a manner which benefits both.

*Optimal:* A stable assignment is called optimal if every applicant is at least as well off under it as under any other stable assignment.

*Strategy-proof:* A mechanism is called strategy-proof if no student can ever benefit by unilaterally misrepresenting her preferences.

*Direct:* A mechanism is called direct if it requires students to reveal their preferences over schools and selects a matching based on these submitted preferences and student priorities.

*Cycle:* A cycle is when an ordered list of distinct schools, each pointing to its highest priority student, and distinct students, each pointing to her most preferred school, results in one school or student appearing more than once, completing the “cycle.”