Construction of Schwartz space

Methods: Coinvariants

$\left.\begin{array}{l}\text{BK method} \\ \text{spectral method.}\end{array}\right\}$ focus on these.

F = local, drop ε from notation (or can just assume it is trivial)

§1 Coinvariants

Spose $\exists$ a space of f^ns S

- $G(F) \times S \to S$,
- $\mathcal{F}_S : S \to S$ commuting w/ $G(F)$
- $I : S_G \to C^\infty(X^\circ(F))$.

  w/ image containing $C_c^\infty(G(F))$.

Can define $S(X(F)) = I(S_G)$

& try to check $\mathcal{F}_S$ descends to $S(X(F))$:

$$\begin{tikzcd} S_G \arrow[r, "\mathcal{F}_S"] \arrow[d] & S_G \arrow[d] \\ S(X(F)) \arrow[r, "\mathcal{F}_X"] & S(X(F)) \end{tikzcd}$$

Example: $V \times G \to V$ (V a v. space)

G-equiv. isom $V \to V^\vee$

$S = S(V(F))$

$\mathcal{F}_S = \mathcal{F}_V$

$X = V /\!/ G$ (GIT quotient).

More complicated examples: see work of G, G, Liu, + Gu.

§2 BK method

First construct $\mathcal{F}_X : L^2(X^\circ(F))^\circlearrowleft$, assume of finite order

$$S(X^\circ(F)) < L^2(X^\circ(F))$$

Define $S_{BK}(X(F)) = \bigoplus_i \mathcal{F}_X^i(S(X^\circ(F)))$.

Obviously fixed under FT.

Not so natural. For $X = M_n$, doesn't yield full Schwartz space!

Slightly better: $\bigoplus_i \mathcal{F}_X^i(S(X^{sm}(F)))$

This at least recovers Schwartz space for $X = M_n$ (trivially).

Works well for $X = \overline{P^{an}\backslash G}^{aff}$,

see work of Chun-Hsien Hsu.

last method: The spectral method. First some prep.

<u>Prep for the spectral method</u>

§1. Harish-Chandra spaces.

(Bernstein: On the support of the Plancherel measure)

Choose
$$\begin{tikzcd} X \times G \arrow[r] \arrow[d, hook] & X \arrow[d, hook] \\ V \times GL_V \arrow[r] & V \end{tikzcd}$$

$x_0 \in X^\circ(F)$

$$r_G : G(F) \to \mathbb{R}_{>0}$$
$$g \longmapsto \log \max(|g_{ij}|, |(g^{-1})_{ij}|)$$

$$r_X : X^\circ(F) \to \mathbb{R}_{>0}$$
$$x \longmapsto \inf\{r_G(g) \mid x = x_0 g\}$$

The <u>Harish-Chandra space</u> $\mathcal{C}(X^\circ(F))$ is

$$\left\{ f \in C^\infty(X^\circ(F)) \;\middle|\; Df(1+r_X)^d \in L^2(X^\circ(F)) \ \forall d \geq 0,\ D \in U(\mathfrak{g}) \right\}$$

(when F non Arch omit D)

Bernstein claims this reduces to usual defⁿ when $X = G$,
I have not checked.

Expect $C_c^\infty(X^0(F)) \subsetneq S(X(F)) \subsetneq \mathcal{C}(X^0(F)) \subsetneq L^2(X^0(F))$.

How to construct?

Have $L^2(X^0(F)) = \int_{\hat{G}(F)} H_\pi \, d_X(\pi)$

for some Borel measure $d\mu_X(\pi)$. Here H_π are irrep's.
(see 0.2 Bernstein) (Bernstein, On the support of the Plancherel measure).

Eg: $L^2(\mathbb{R}) = \int_{\mathbb{R}} \mathbb{C}e^{it} \, dt$

$f \mapsto \underline{(t \to \int_{\mathbb{R}} f(x) e^{-itx} dx)}$

only defined a.e.

FT well-defined $\forall t$ if $f \in C_c^\infty(\mathbb{R})$.

Want a large s.space of $L^2(X^0(F))$ on which decomp. is well-defined.

<u>Propⁿ</u> (§2.3 of Bernstein) Let $\alpha : C_c^\infty(X^0(F)) \to L^2(X^0(F))$ be the incl.

$\exists$ a family of maps $\alpha_\pi : C_c^\infty(X^0(F)) \to H_\pi \ \forall \pi$

s.t. $\alpha = \int_{\hat{G}(F)} \alpha_\pi \cdot d\mu(\pi)$.

This is a bit abstract... for $f \in S(X^0(F))$ $\alpha_\pi(f)$ is no longer a f^n on $X^0(F)$

Given a Hilbert rep^n π w/ space π,

$$\mathrm{Hom}_G(C_c^\infty(X^0(F)), V) \leftrightarrow \mathrm{Hom}_G(V^\infty, C^\infty(X^0(F)))$$

$$\alpha \longmapsto \alpha^T$$

$$(v, \alpha(\phi)) = (\alpha^T(v), \phi)_{L^2(X^0(F))}$$

Can write Plancherel decomp as

$$f = \int_{\hat{C}(F)} \underbrace{\alpha_\pi^T \circ \alpha_\pi(f)}\, d\mu(\pi)$$

this is a f^n on $X^0(F)$.

Want to extend the prop^n a bit.

Def^n: $X^0(F)$ has polynomial growth if

$$dx(\{x \in X^0(F) \mid r(x) \le t\}) \ll (1+t^d) \text{ for some } d \ge 0.$$

If $X^0(F)$ has polynomial growth,

Bernstein proves

$$f = \int_{\hat{C}(F)} \iota_\pi^T \circ \partial_\pi(f) \, d\mu(\pi)$$

holds for $f \in \mathcal{C}(X^0(F))$.

Lets explicate $\hat{G}(F)$, or more precisely $T_p(G)$.

§2 The tempered spectrum

Let's describe $T_t(G)$, following Beuzart-Plessis.

(Define parabolic subgp, standard parabolic, conj thm)

$A_0 < G$ maximal split torus.

A parabolic subgp P is semistandard if

$A_0 < P$. Can then write

$P = MN$ w/ M reductive & N unipotent

(Levi decomposition).

M is the Levi subgp.

We say M is semistandard if

$M > A_0$. A semistandard parabolic contains a unique semistandard Levi subgp.

Example: GL_3. $A_0(R) = \left\{ \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \,\middle|\, a, b, c \in R^\times \right\}$

$$\left\{ \begin{pmatrix} a & b & c \\ & d & e \\ & & f \end{pmatrix} \in GL_3(R) \right\} = \left\{ \underset{N}{\begin{pmatrix} 1 & b & c \\ & 1 & e \\ & & 1 \end{pmatrix}} \right\} \cdot \left\{ \underset{M}{\begin{pmatrix} a & & \\ & d & \\ & & f \end{pmatrix}} \right\}$$ is semistandard

so is $\left\{ \begin{pmatrix} a & & c \\ b & d & e \\ & & f \end{pmatrix} \right\}$,

also $$\left\{ \begin{pmatrix} a & b & e \\ c & d & f \\ & & g \end{pmatrix} \right\} = \left\{ \overset{M}{\begin{pmatrix} I_2 & & e \\ & & f \\ & & 1 \end{pmatrix}} \right\} \cdot \left\{ \overset{N}{\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ & & g \end{pmatrix}} \right\}$$

Let $\mathfrak{a}_M^* = \mathrm{Hom}(X^*(M), \mathbb{R})$ ⟵ real v. space.

$$H_M : M(F) \to \mathfrak{a}_M^*$$

$$\langle H_M(m), \chi \rangle = |\chi(m)|.$$

By Iwasawa decomposition,

$$G(F) = N(F)M(F)K,$$

where $K \leq G(F)$ is a suitable max'l compact subgp.

$$H_P(nmk) := H_M(m).$$

If σ an adm. rep^n of $M(F)$, $\lambda \in \mathfrak{a}_{M\mathbb{C}}^*$, define the <u>induced rep^n</u>

$$I_P^G(\sigma_\lambda) = \{ \varphi : G(F) \to \sigma \mid \varphi(nmg) = e^{\langle H_P(m), \lambda + \rho_P \rangle} \varphi(g) \}.$$

If σ is $\square$-integrable & $\lambda \in i\mathfrak{a}_M^*$ then $I_P^G(\sigma_\lambda)$ is tempered.

Not always irreducible, but irreducible outside a measure 0 set of λ.

$T_P(G) = \{ I_P^G(\sigma) \mid P = MN$ a parabolic subgp, $\sigma \in \Pi_2(M) \}$

Isom class of $I_P^G(\sigma)$

depends only on M.

$I_P^G(\sigma) \leftrightarrow (M, \sigma)$.

(M, σ) & (M', σ') correspond to the same

isom class $\Leftrightarrow$ they are $G(F)$-conj.

On the factors of the form (M, σ) have an

action of $i\mathfrak{a}_M^*$.

Gives $T_P(G)$ an orbifold structure

w/ ∞'ly many components.

Thus $T_P(G)$ is an orbifold w/ ∞'ly many components.

We say a fn on $T_P(G)$ is smooth, Schwartz, etc,

if the pullback $i\mathfrak{a}_M^* \to \mathbb{C}$ is smooth,

Schwartz, etc.

Notation for later: $\Pi_2(M)_{\mathbb{C}} = \{ \sigma_\lambda : \lambda \in \mathfrak{a}_{M\mathbb{C}}^* \}$

8.3 A slightly different perspective.

There is a v. bundle

$$Z \longrightarrow T_P \quad w/$$

$$Z(\pi) = \mathcal{L}^T(\pi) \subset \mathcal{C}(X^\circ(F))$$

Should always have a Plancherel isom

$$HP: \quad \mathcal{C}(X^\circ(F)) \xrightarrow{\sim} \mathcal{C}(T_P, Z)$$

where $\mathcal{C}(T_P, Z)$ denotes smooth sections T s.t.

(1) When F is nonArch, T vanishes outside a finite set of components

(2) When F is Arch,

$\forall$ semistandard $P = MN_P$, diff op D on $i\mathfrak{a}^*_M$ w/ constant coef & every $u \in \mathcal{U}(\mathfrak{g})$

$$P_{D,u,N}(T) := \sup_{\sigma \in \Pi_2(M)} \|(DT)_\sigma * u\| (1 + \|\sigma\|)^N$$

is finite.

Here $\|\sigma\|$ is essentially the norm of the inf. char.

Let's explain the isom.

Let $K < G(F)$ be a maxl compact subgp &

$\mathfrak{J}$ a finite set of K types.

$$\mathcal{C}(X(F))_{\mathfrak{J}} = \bigoplus_{\tau \in \mathfrak{J}} \mathcal{C}(X^\circ(F))(\tau).$$

For a semistandard M, $\sigma \in \Pi_2(M)$,

let B_σ be an orthonormal basis of the

space of $I(\sigma)$ $\forall\ \lambda \in \mathfrak{a}_M^*$, obtain

ONB $\quad B_{\sigma_\lambda} = B_\sigma e^{\langle H_P(\ ), \lambda \rangle}$ of $I(\sigma_\lambda)$.

Assume B_σ consists of K-finite vectors.

Let $\quad Z(f, \varphi) = \displaystyle\int_{X^\circ(F)} f(x)\, \jmath^T(\varphi)(x)\, dx$

Should have

$$HP(f)(I(\sigma_\lambda)) = \sum_{\varphi \in B} Z(f, \varphi)\, \jmath^T(\varphi).$$

Many cases are in the literature (Bernstein-Plessis,

Delorme - Harinck - Sakellaridis)

Should also have

$$HP^{-1}(T) = \int_{T_P(G)} T(I(\sigma)) \sum_{\varphi \in B_\sigma} \jmath^T(\varphi)\ d\mu(I(\sigma))$$

§4 Algebraic structures.

Let X, Y be reduced schemes of finite type $/\mathbb{C}$.
We say a map $X(\mathbb{C}) \to Y(\mathbb{C})$ is algebraic or polynomial if it is induced by a morphism $X \to Y$. A partially defined map $X(\mathbb{C}) \dashrightarrow Y(\mathbb{C})$ is rational if it is induced by a rational map. Use this in the following.

Thm: Assume F is non Arch.

Then $\Pi_2(M)_{\mathbb{C}}$ can be given the structure of the complex pts of a scheme of finite type over $\mathbb{C}$ in a canonical manner. Real pts corr. to $\Pi_2(M)$

Reference: Renard, Rep^n des groupes réductifs p-adiques, §V.2.4

First, an unramified quasi-char of $M(F)$ is a continuous hom

$$\chi : M(F) \to \mathbb{C}^\times \text{ s.t.}$$

χ is trivial on

$$M(F)' := \{\ker |\chi| \mid \chi \in X^*(M)\}$$

Here $X^*(M) = \mathrm{Hom}(M, \mathbb{G}_m)$

Let $\mathrm{Hom}_{unr}(M, \mathbb{C}^\times)$ be the gp of unram. char

Have an exact sequence

$$\begin{tikzcd}
1 \arrow[r] & M(F)' \arrow[r] & M(F) \arrow[r, "H_M"] & \Lambda(M) \arrow[r] & 1 \\
& & & \mathfrak{a}_M^* \arrow[u, phantom, sloped, "\subset"] &
\end{tikzcd}$$

Thus $$\mathrm{Hom}_{unr}(M, \mathbb{C}^\times) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(\Lambda(M), \mathbb{C}^\times)$$
$$u \circ H_M \longleftarrow\!\shortmid\ u$$

$\Lambda(M)$ is a free abelian gp of finite rank,

$$\mathrm{Hom}_{\mathbb{Z}}(\Lambda(M), \mathbb{C}^\times) \simeq \mathrm{Hom}_{Alg}(\mathbb{C}[\Lambda(M)], \mathbb{C})$$

this is the complex pts of a torus, say T_M.

If $M = \mathbb{G}_m$, then $\Lambda(M) = \mathbb{Z}$.

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{C}^\times) \cong \mathrm{Hom}(\mathbb{C}[\mathbb{Z}], \mathbb{C})$$

$$\& \quad \mathbb{C}[\mathbb{Z}] \cong \mathbb{C}[x, x^{-1}]$$

$$\text{so } T_M \cong \mathbb{G}_m.$$

One checks that

$$ev_g : \mathrm{Hom}_{un}(G, \mathbb{C}^\times) \longrightarrow \mathbb{C}^\times$$
$$\chi \longmapsto \chi(g)$$

is algebraic.

Have a homom.

$$\mathfrak{a}^*_{M\mathbb{C}} = \mathrm{Hom}_{\mathbb{Z}}(X^*(M), \mathbb{C}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\Lambda(M), \mathbb{C}^\times)$$
$$\| $$
$$\mathrm{Hom}_{\mathbb{Z}}(\Lambda(M), \mathbb{C})$$
$$\lambda \longmapsto q^{-\lambda}$$

full rank

that is holom & maps ia_M^* to $\mathrm{Hom}_\pi(\Lambda(M), S^1)$

So, what about $\Pi_2(M)_\mathbb{C}$?

$$\Pi_2(M) = \bigsqcup_{\mathcal{O}} \mathcal{O} \quad ,$$ union over ia_M^*-orbits.

. .

Let $W(M, G) = N_G(M)(F) / C_G(M)(F)$

Fact: for σ tempered

$$I(\sigma) \cong I(\sigma')$$

$$\Leftrightarrow \quad w.\sigma(g) := \sigma(w^{-1} g w) \cong \sigma'$$

for some $w \in W(M, G)$.

$$\Rightarrow \mathcal{O} \cong \left(ia_M^* / \ker \right) / W(M, G)_\sigma$$

where $\ker = \ker\left(i\mathfrak{a}_M \to \mathrm{Hom}_{unr}(C(F), \mathbb{C}^*) \right)$