LOGO

Deflection of a dilute stream of particles

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Outline

1. Introduction
2. Formulation
3. Numerical results
4. Theoretical approach
5. Validity and comparison
6. Mechanism and conclusion

Introduction

- **Granular material**

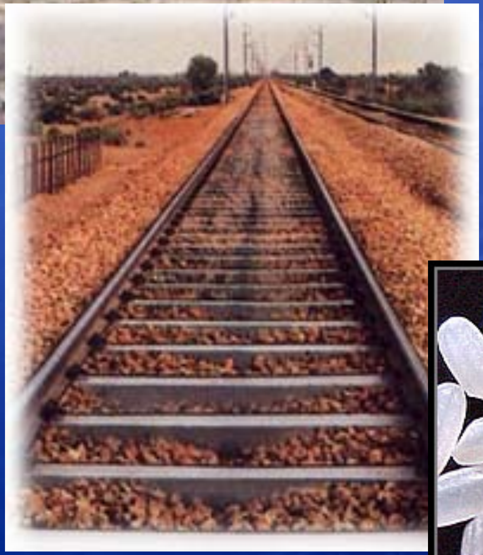
  A granular material is a conglomeration of discrete solid, macroscopic particles characterized by a loss of energy whenever the particles interact.

- **Granular flows**

  Flows formed by Granular materials

Granular materials

- Granular materials are everywhere



Rapid Granular flows

1

2

3

Motivation and goal

- Oblique deflectors such as deflecting dams are usually employed to divert particle avalanches.

- The design of passive protection methods often does not have a solid theoretical underpinning.

- Average force experienced by the oblique wall.

- Investigate the system at the particle scale to understand the dynamics of the deflection process.

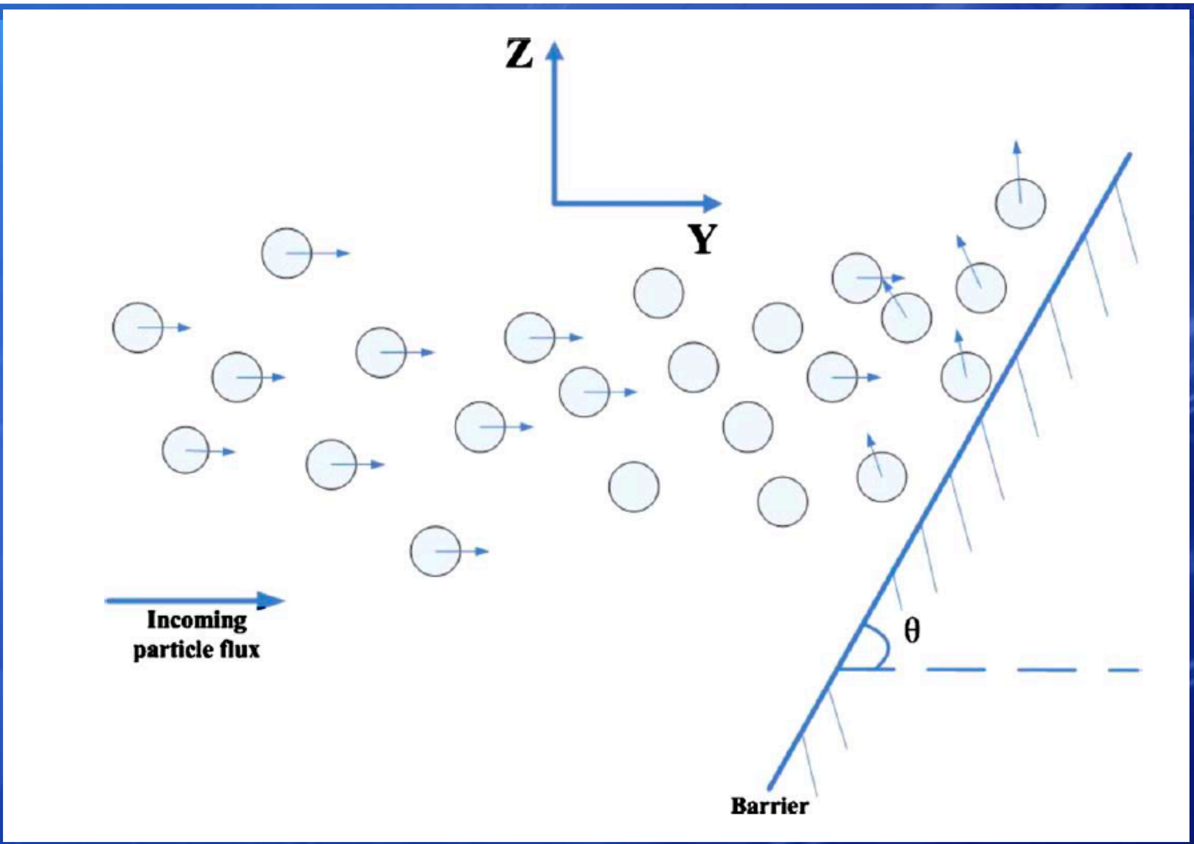
Enormous amount of work of very important work on dense granular materials.

But dilute granular materials have been studied much less.

Generally believed that dilute flows are very simple

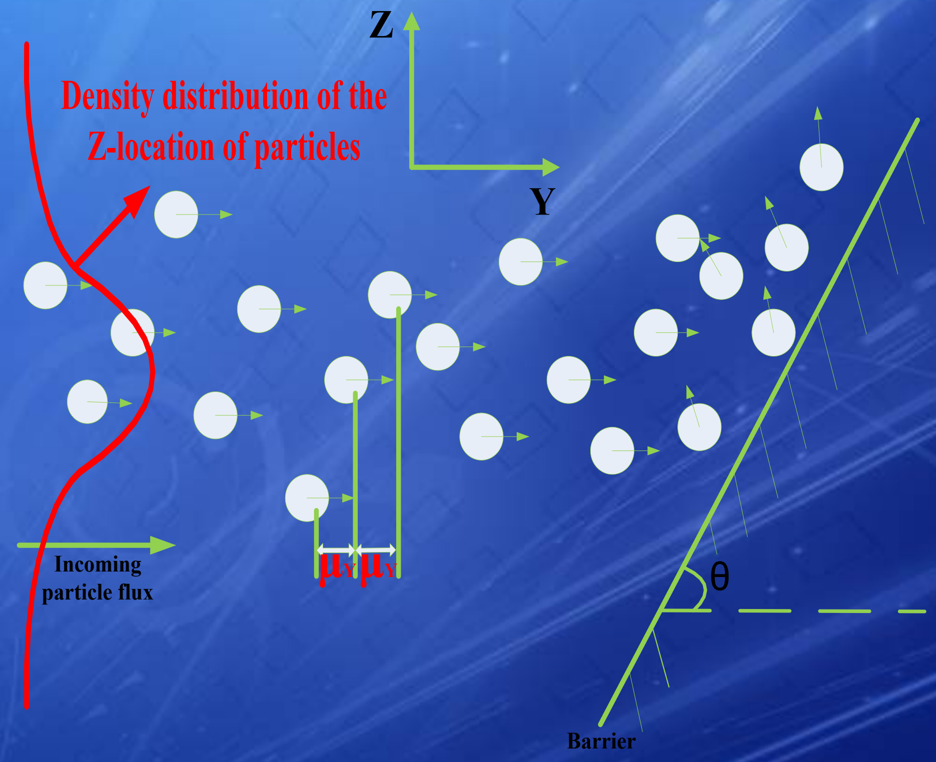
I hope to convince you that this is completely untrue

Simple Model System



- The particle stream consists of identical smooth spheres of radius and mass.
- All incoming particles move in the same direction and have the same initial velocity.

Geometric Parameters



- Y-direction: distance between adjacent particles has a given distribution (eg. Non-random)   V=(Mean Separation in Y)/radius
- Z-direction: input location has a given distribution (eg. Gaussian) with standard deviation   S=(standard deviation Separation in Z)/radius

Particle Parameters

- Collisions between particles are inelastic with a restitution coefficient

$$0 \leq e \leq 1$$

- Collisions between particles and the wall are also inelastic with a restitution coefficient

$$0 \leq e_w \leq 1$$

Numerical methods

- We consider a finite domain that is much larger than μ_Y and σ_Z .
- Two methods for propagating particles

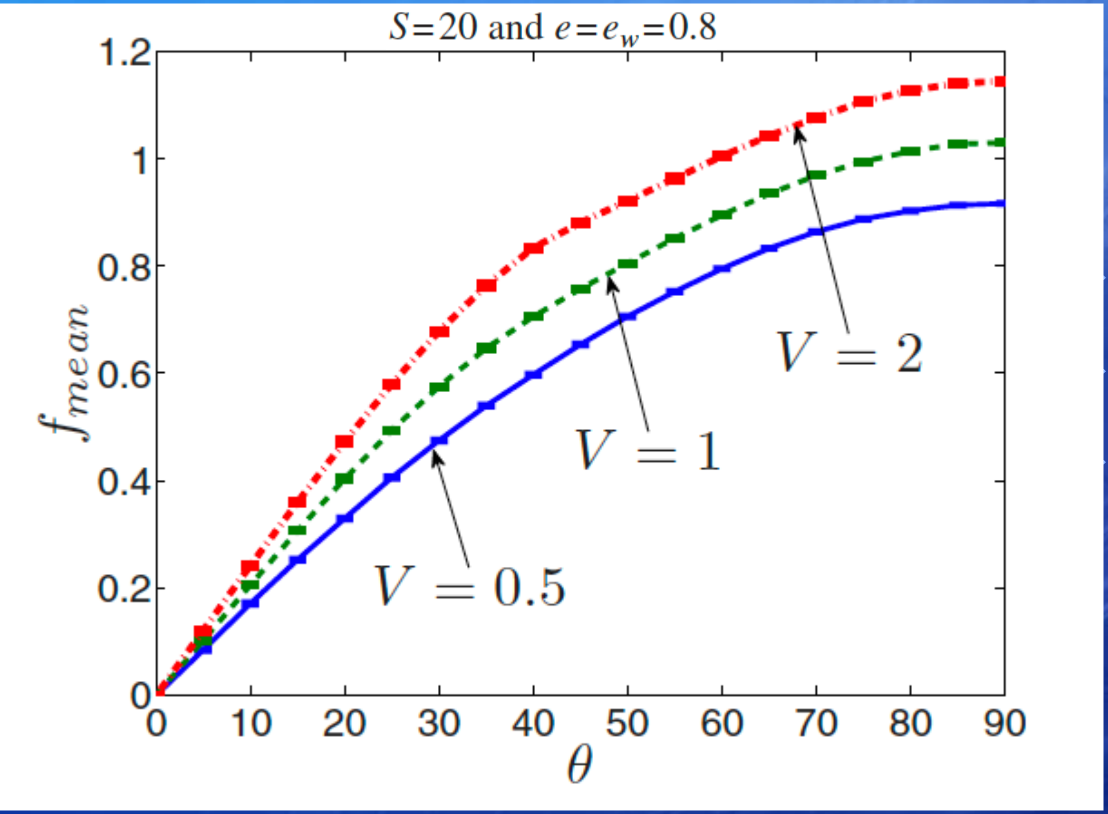
Exact Method

- Hard spheres
- Binary events
- Rapid
- Relative dilute system

Potential Method

- Soft spheres
- Potential force
- Slow
- Relative dense system

Numerical results for Force on Barrier



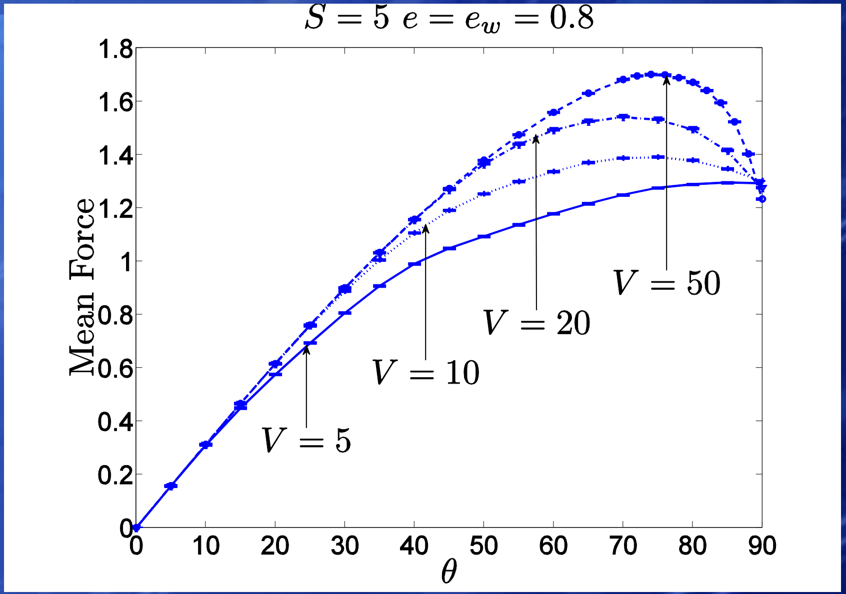
- S=transverse separation
- V=longitudinal separation
- Theta=barrier angle

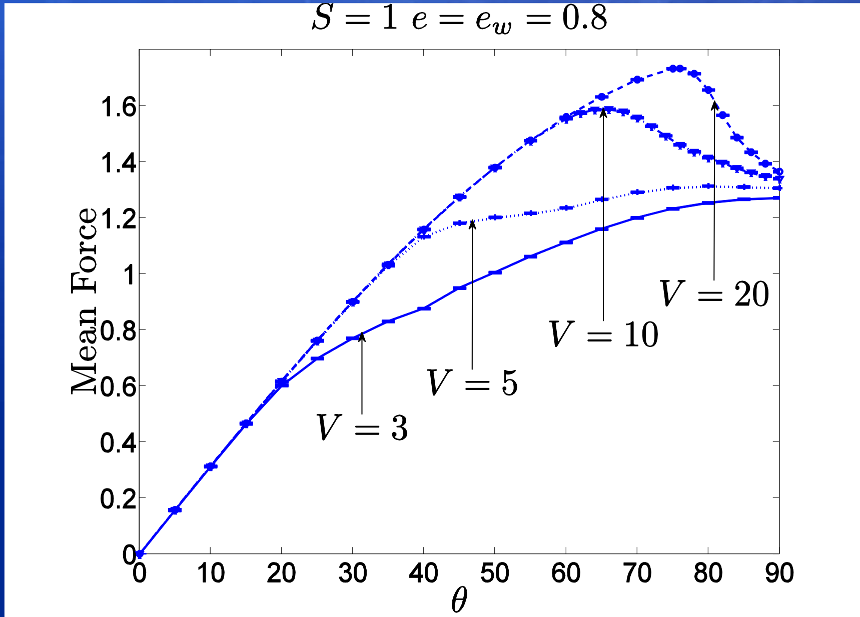
increase barrier angle => increase the force on wall

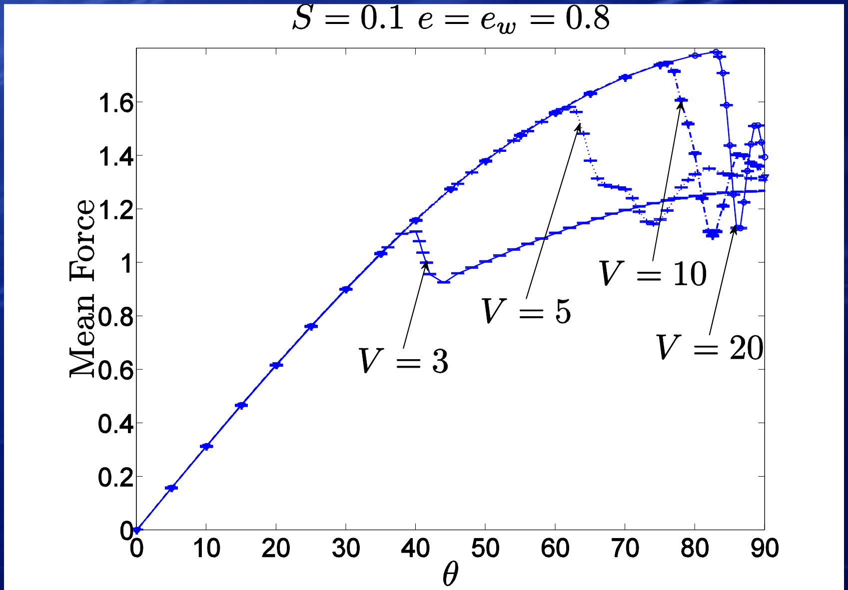
No surprise

Numerical results for Force on Barrier

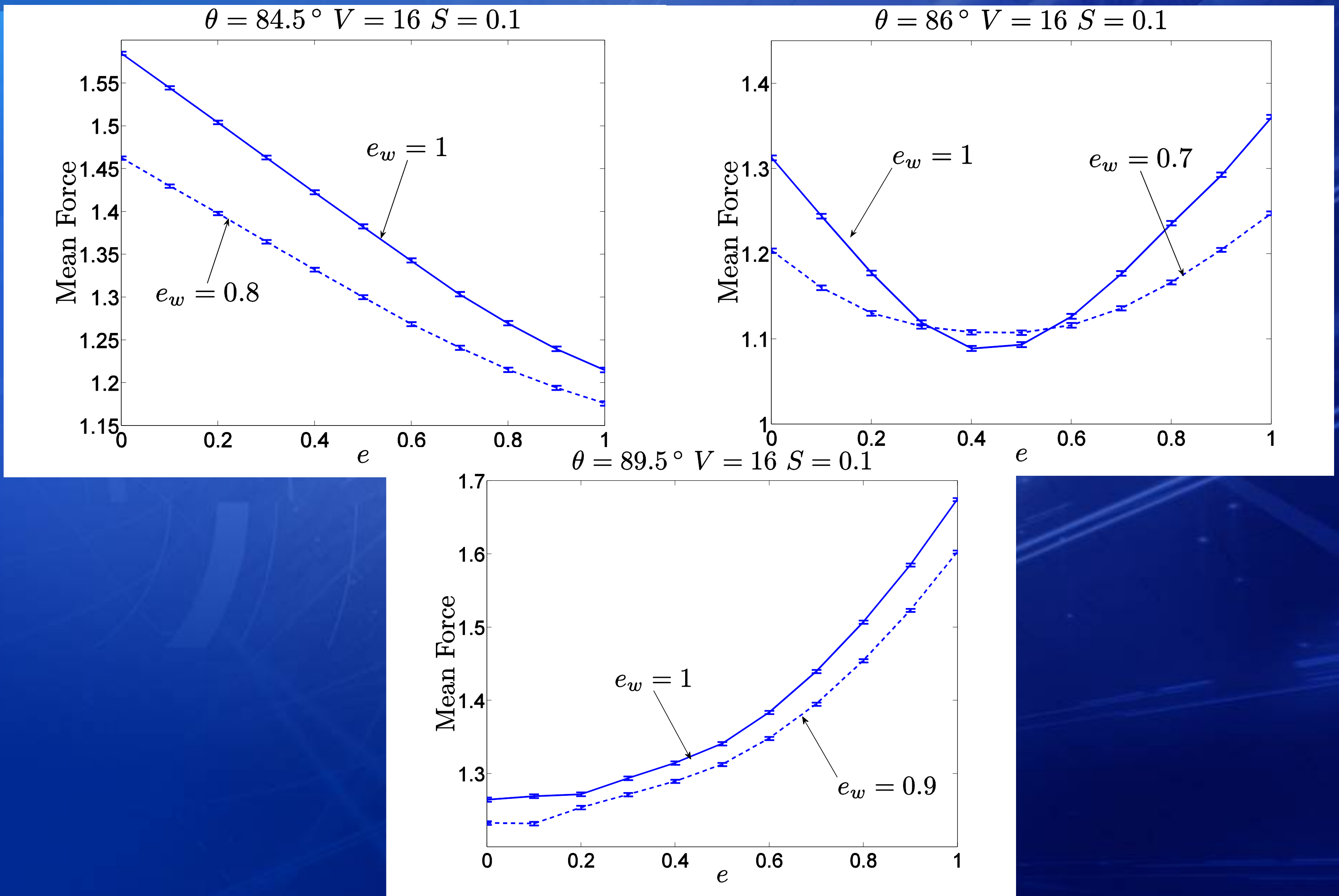
But, for more focused jets we can get very surprising phenomomena







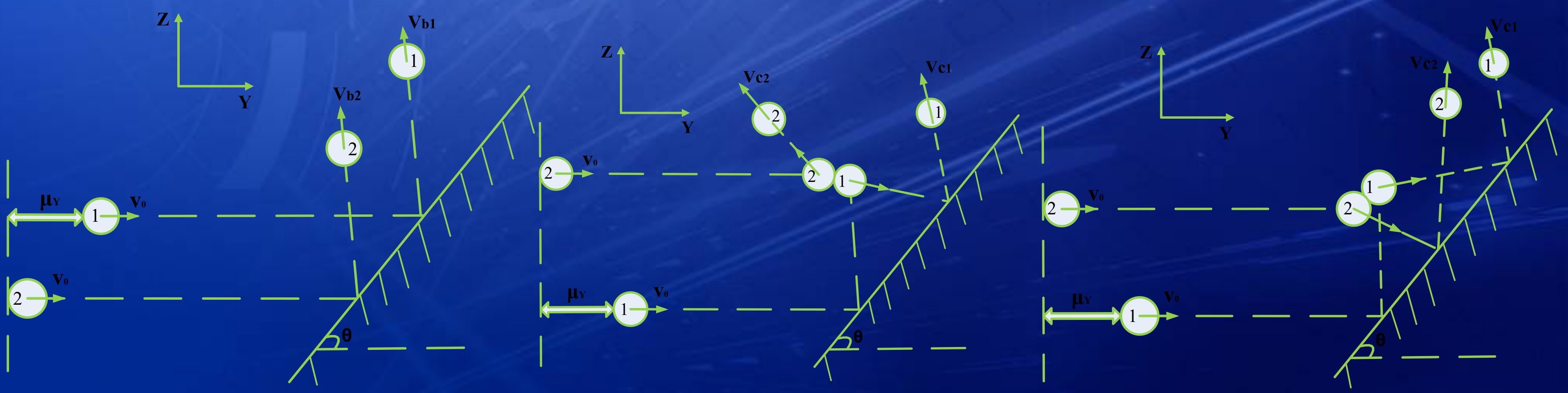
Numerical results for Force on Barrier



Theoretical approach

- We Assume that any given particle may experience at most one particle-particle collision. After this collision, one or both of the two particles can hit the wall again, but we will neglect further particle-particle collisions.

- Three possible outcomes for the two particles passing through the system



Theoretical approach

- We will derive the formula for the probability of a given particle experiencing a particle-particle collision under our assumption, and hence derive an exact expression for the mean force on the wall.

- Let B_n denote the n-th particle in the particle stream. Let $C_{n,j}$ and D_n $(\forall n,\ j \in \mathbf{Z})$ denote the following events:

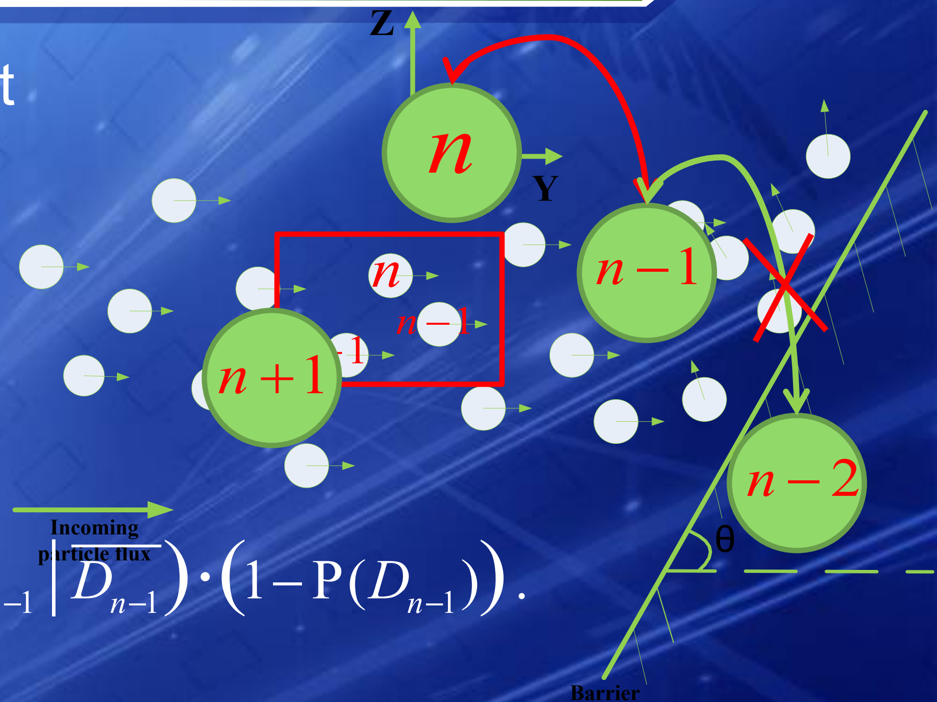
$$C_{n,j} = \left\{ B_n \text{ collides with } B_j \right\}$$

$$D_n = \left\{ B_n \text{ collides with any of the previous particles} \right\} .$$

Then we obtain

$$D_n = \bigcup_{j=0}^{n-1} C_{n,j} \quad \text{and} \quad C_{n,j} \cap C_{n,k} = \varnothing \quad (\forall\, j \neq k,\ \ n, j, k \in \mathbf{Z}) .$$

Interactions with nearest neighbors



- B_n may only collide with its nearest neighbors B_{n-1} or B_{n+1}, that is

  $C_{n,n-j} = \varnothing \quad (2 \le j \le n, \forall n, j \in \mathbb{Z})$.

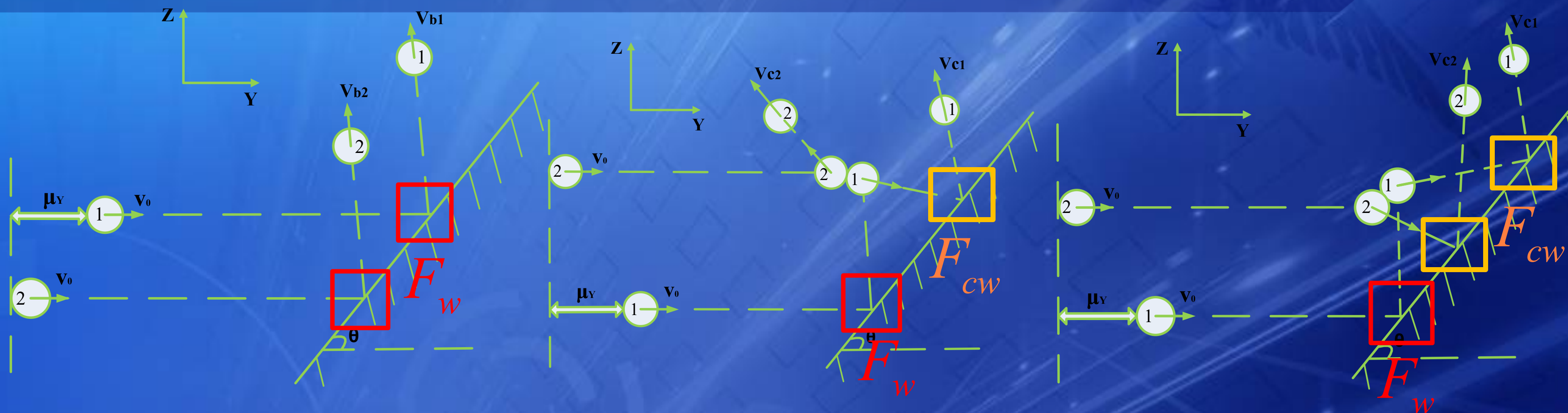
- In this case, note that $C_{n,n-1} \Leftrightarrow \overline{D_{n-1}}$, then we obtain

$$\mathrm{P}(D_n) = \mathrm{P}\Big(\bigcup_{j=0}^{n-1} C_{n,j}\Big) = \mathrm{P}(C_{n,n-1}) = \mathrm{P}\Big(C_{n,n-1} \Big| \overline{D_{n-1}}\Big) \cdot \Big(1 - \mathrm{P}(D_{n-1})\Big).$$

- Define p_1 to be the probability of a single particle colliding with its previous nearest neighboring particle in a steady state. Then

$$p_1 = \lim_{n \to +\infty} \mathrm{P}(D_n) = \frac{p_\tau}{1 + p_\tau} , \qquad p_\tau = \mathrm{P}\Big(C_{n,n-1} \Big| \overline{D_{n-1}}\Big)$$

Interactions with nearest neighbors



- Let $F_{mean}^{(1)}$ denote the mean force on the surface when we only include the interactions with nearest neighbors.

$$\begin{aligned} F_{mean}^{(1)} &= F_w \lim_{n\to+\infty} \mathrm{P}(\overline{D_n}) + F_{cw} \lim_{n\to+\infty} \mathrm{P}(\overline{D_{n-1}}) \\ &= \left(1 - \lim_{n\to+\infty} \mathrm{P}(D_n)\right)\cdot\left(F_w + F_{cw}\right) \\ &= (1-p_1)\left(F_w + F_{cw}\right) = \frac{1}{1+p_\tau}\left(F_w + F_{cw}\right). \end{aligned}$$

Interactions with next nearest neighbors

- B_n may collide with its two previous particles B_{n-1} and B_{n-2} or two later particles B_{n+1} and B_{n+2}. That is $C_{n,n-j} = \varnothing \ (3 \le j \le n, \forall n, j \in \mathbb{Z})$. Then

$$\mathrm{P}(D_n) = \mathrm{P}\Big(\bigcup_{j=0}^{n-1} C_{n,j}\Big) = \mathrm{P}(C_{n,n-1} \cup C_{n,n-2}) = \mathrm{P}(C_{n,n-1}) + \mathrm{P}(C_{n,n-2}) .$$

- Let p_2 be the probability of a single particle colliding with its previous nearest and next nearest neighboring particles in a steady state.

$$p_2 = \lim_{n\to+\infty} \mathrm{P}(D_n) = \lim_{n\to+\infty} \mathrm{P}(C_{n,n-1}) + \lim_{n\to+\infty} \mathrm{P}(C_{n,n-2}) = \frac{p_\tau + B}{1 + p_\tau + B},$$

where $B = \dfrac{p_{2\tau}(1-p_\tau)^2(1-p_\tau+p_{2\tau})}{1-p_\tau p_{2\tau} + (1-p_\tau)^2 p_{2\tau}}$, $\quad p_{2\tau} = \mathrm{P}\Big(C_{n,n-2} \,\Big|\, \overline{C_{n,n-1}} \cap \overline{C_{n-1,n-2}} \cap \overline{D_{n-2}}\Big)$.

- Let $F_{mean}^{(2)}$ denote the mean force on the wall when the interactions with the nearest and the next nearest neighbors are both included. Then

$$F_{mean}^{(2)} = \frac{1}{1+p_\tau+B}\left[F_w + \frac{1-p_\tau p_{2\tau}}{1-p_\tau p_{2\tau}+(1-p_\tau)^2 p_{2\tau}} F_{cw1} + \frac{(1-p_\tau)^2(1+p_{2\tau})}{1-p_\tau p_{2\tau}+(1-p_\tau)^2 p_{2\tau}} F_{cw2}\right].$$

Theoretical results

- In our theoretical derivation, we only consider the interaction with the nearest neighboring particles. Then by analyzing pair-wise particle-particle collisions, we can obtain the following formula for calculating the mean force F_{mean},

$$F_{mean} = \frac{1}{1+p_{\tau}}\left(F_{w} + F_{cw}\right)$$

$$= F_{b}\left(\mathbf{v_b}(z_1, z_2)\right) + \iint_{A} F_{p}\left(\mathbf{v_p}(z_1, z_2)\right)\rho(z_1, z_2)\mathrm{d}z_1\mathrm{d}z_2$$

where $A = \left\{(z_1, z_2) \in \sim^{2} \middle| C_{2,1}\right\}$. F_b Denotes the directly impact on the wall, F_p denotes the impact on the wall following a particle-particle collision and $\rho(z_1, z_2)$ is the joint distribution density of the initial heights of the two particles.

Theoretical results

- Performing some integral and algebra , the analytical solution of dimensionless mean force f_{mean} is given by

$$f_{mean}=\frac{(1+e_w)\sin\theta}{4+2\mathrm{erf}(C_-)-4\mathrm{erf}(C_+)}\cdot\Big\{4+\big((1+e)-e_w(1-e)\big)\big(\mathrm{erf}(D_-)-\mathrm{erf}(D_+)\big)$$

$$+\big((1-e)-e_w(1+e)\big)\Big[\mathrm{erf}(C_-)-\mathrm{erf}(C_+)+\mathrm{H}\Big(e_w-\frac{1-e}{1+e}\Big)\big(\mathrm{erf}(K_+)-\mathrm{erf}(K_-)\big)\Big]$$

$$+(1+e)(1+e_w)\Bigg[\Big(\frac{S^2\sin^2\theta}{2}+\frac{V^2\cos^2\theta}{4}\Big)\Big(\mathrm{erf}(C_-)-\mathrm{erf}(C_+)+\mathrm{erf}(D_+)-\mathrm{erf}(D_-)+\mathrm{H}\Big(e_w-\frac{1-e}{1+e}\Big)$$

$$\big(\mathrm{erf}(K_+)-\mathrm{erf}(K_-)\big)\Big)+\frac{VS\sin\theta\cos\theta}{\sqrt{\pi}}\Big(e^{-C_+^2}-e^{-C_-^2}+e^{-D_-^2}-e^{-D_+^2}+\mathrm{H}\Big(e_w-\frac{1-e}{1+e}\Big)\big(e^{-K_-^2}-e^{-K_+^2}\big)\Big)$$

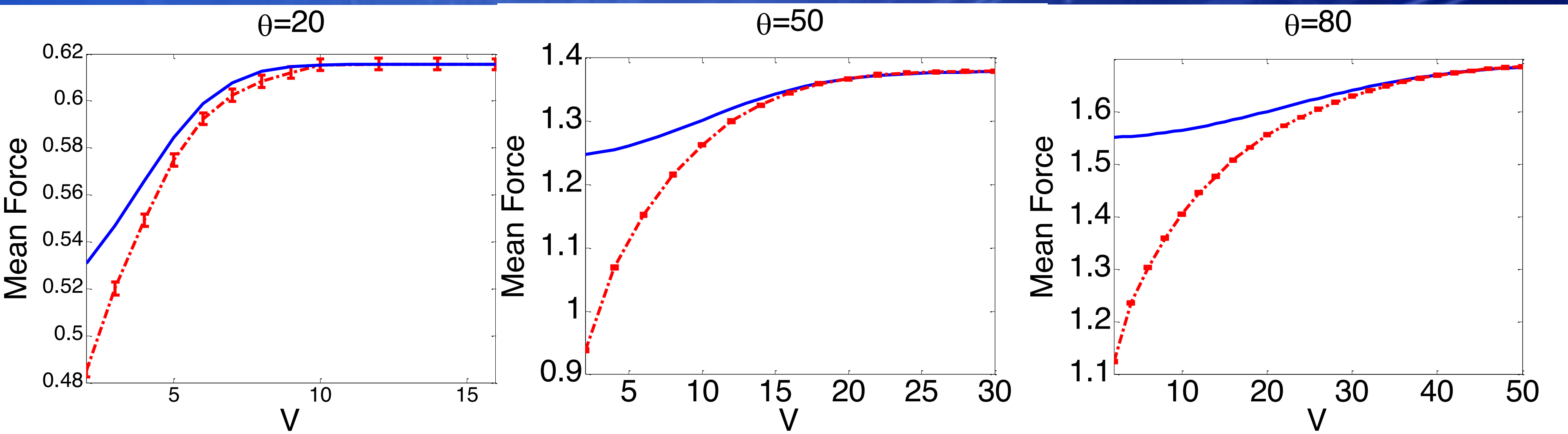
$$+\frac{S^2\sin^2\theta}{\sqrt{\pi}}\Big(C_+e^{-C_+^2}-C_-e^{-C_-^2}+D_-e^{-D_-^2}-D_+e^{-D_+^2}+\mathrm{H}\Big(e_w-\frac{1-e}{1+e}\Big)\big(K_-e^{-K_-^2}-K_+e^{-K_+^2}\big)\Big)\Bigg]\Bigg\}$$

where $\mathrm{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,\mathrm{d}t$, $\mathrm{H(x)}=\begin{cases}1 & \text{if } x\geq 0\\ 0 & \text{if } x<0\end{cases}$ and

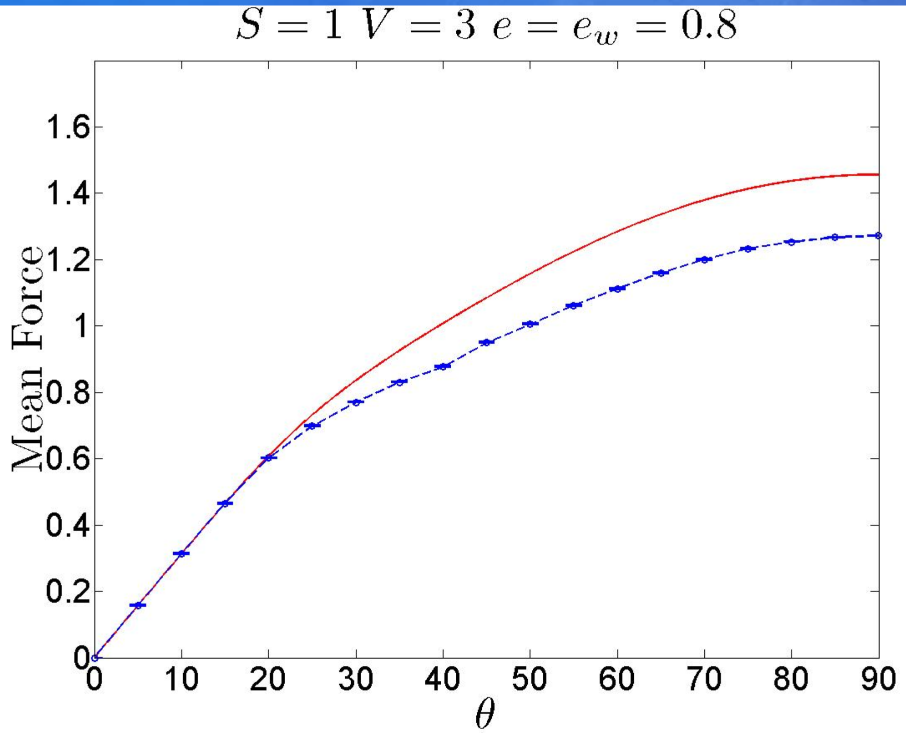
$$C_\pm=\frac{\mp 1-\frac{V\cos\theta}{2}}{S\sin\theta},\quad D_\pm=\frac{\mp\sqrt{\frac{(1+e)-e_w(1-e)}{(1+e)(1+e_w)}}-\frac{V\cos\theta}{2}}{S\sin\theta},\quad K_\pm=\frac{\mp\sqrt{\frac{e_w(1+e)-(1-e)}{(1+e)(1+e_w)}}-\frac{V\cos\theta}{2}}{S\sin\theta}.$$

Validity

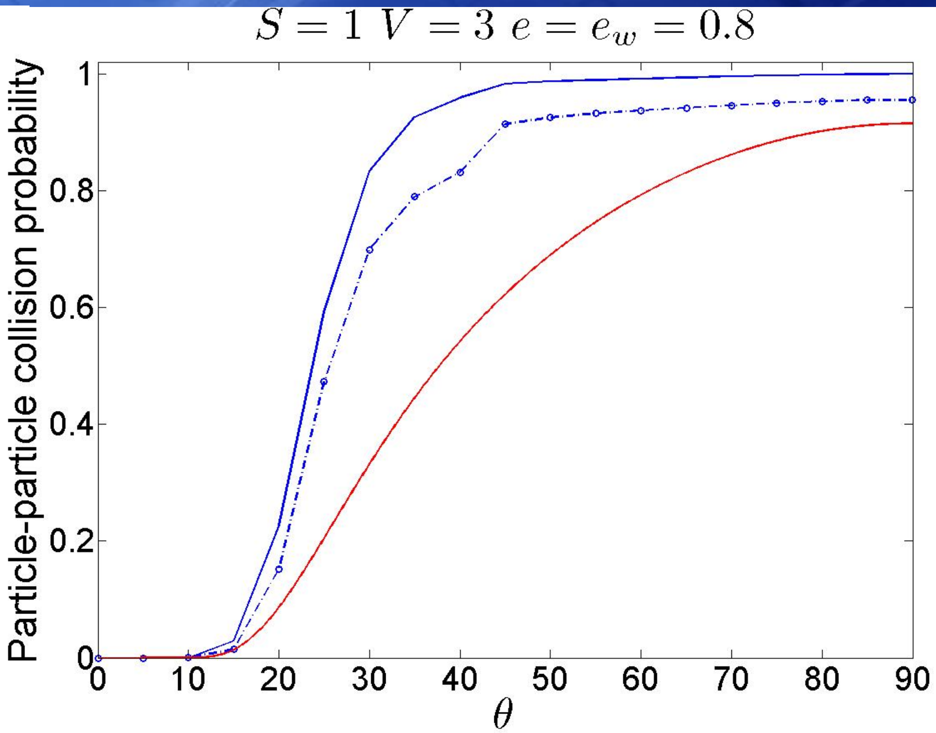
- Our theory gives a good approximation when the fraction of particles that experience at least two particle-particle collisions is low.
- if $V\cos\theta \gg 1$, then the probability of a particle experiencing a particle-particle collision is low. Thus the probability of particles experiencing at least two particle-particle collisions must also be low, and so our theory will be valid.
- Figures of f_{mean}vs. V for different $\theta\,(S=5,\ e=e_w=0.8)$.



Comparison

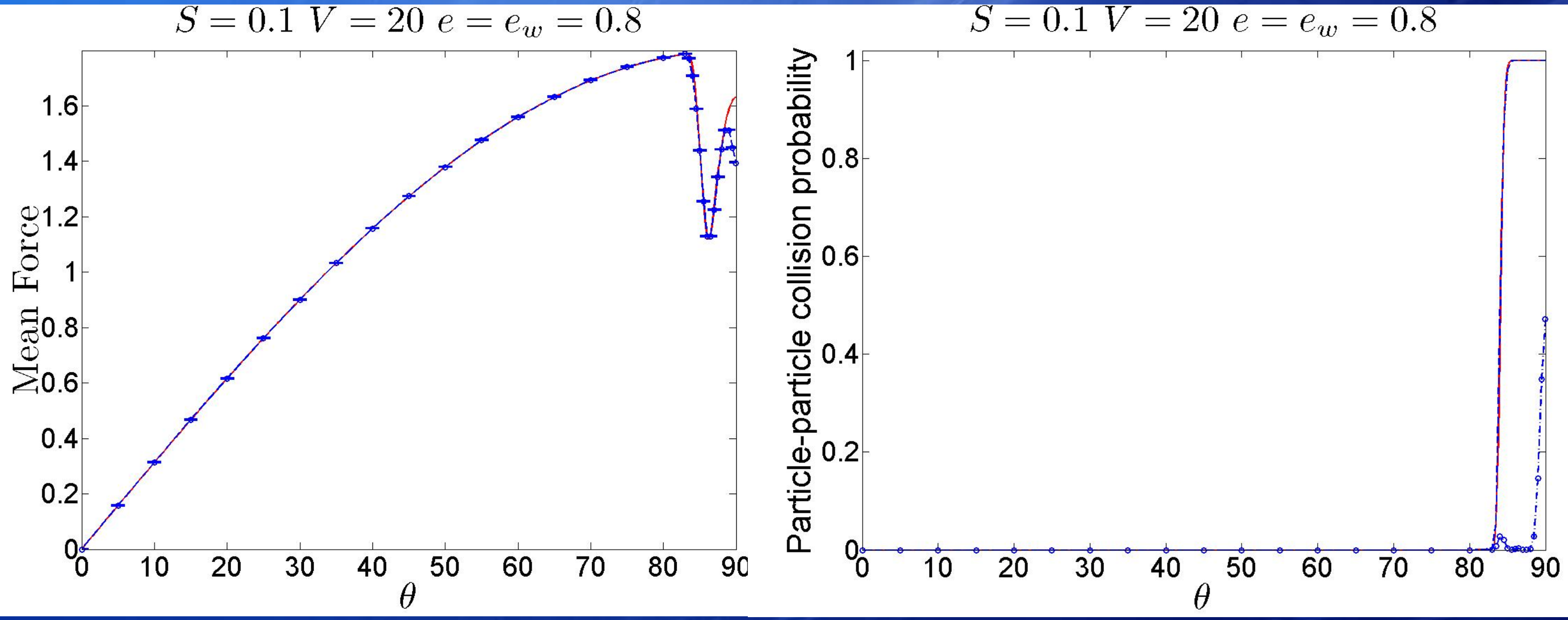


$S = 1\ V = 3\ e = e_w = 0.8$
Mean Force
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
0
10
20
30
40
50
60
70
80
90
θ

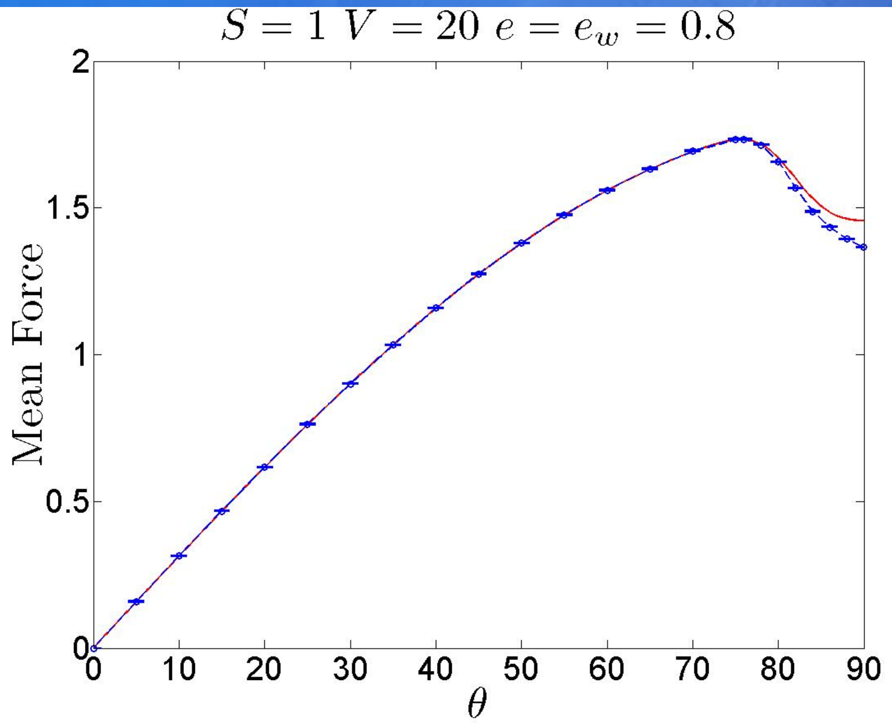


$S = 1\ V = 3\ e = e_w = 0.8$
Particle-particle collision probability
0
0.2
0.4
0.6
0.8
1
0
10
20
30
40
50
60
70
80
90
θ

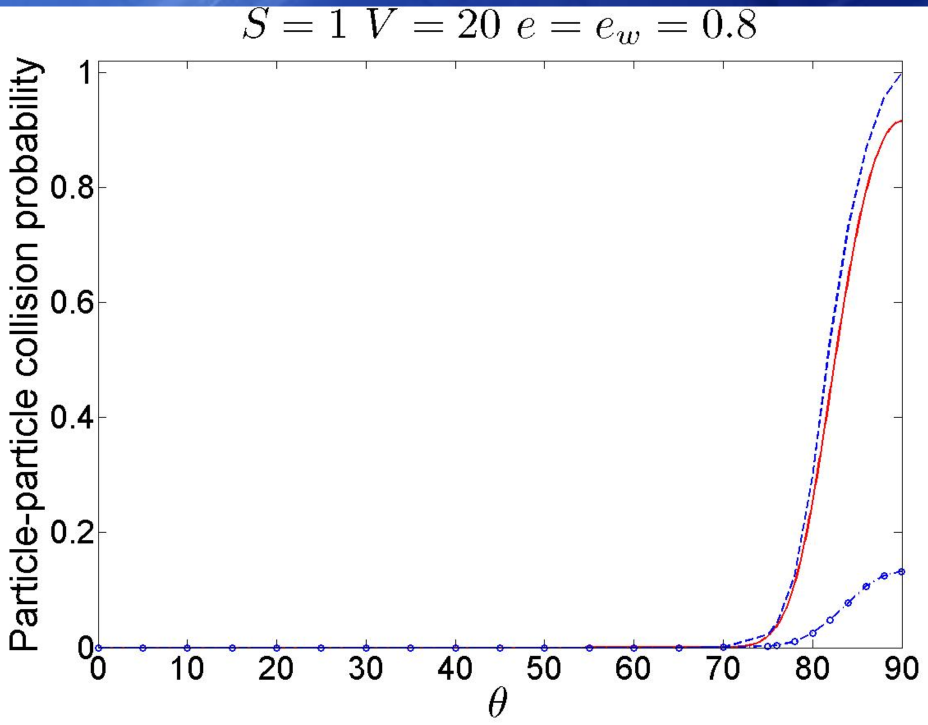
Comparison


$S = 0.1\ V = 20\ e = e_w = 0.8$
Mean Force
θ
$S = 0.1\ V = 20\ e = e_w = 0.8$
Particle-particle collision probability
θ

Comparison



$S = 1\ V = 20\ e = e_w = 0.8$
Mean Force
θ
0
10
20
30
40
50
60
70
80
90
0
0.5
1
1.5
2



$S = 1\ V = 20\ e = e_w = 0.8$
Particle-particle collision probability
θ
0
10
20
30
40
50
60
70
80
90
0
0.2
0.4
0.6
0.8
1

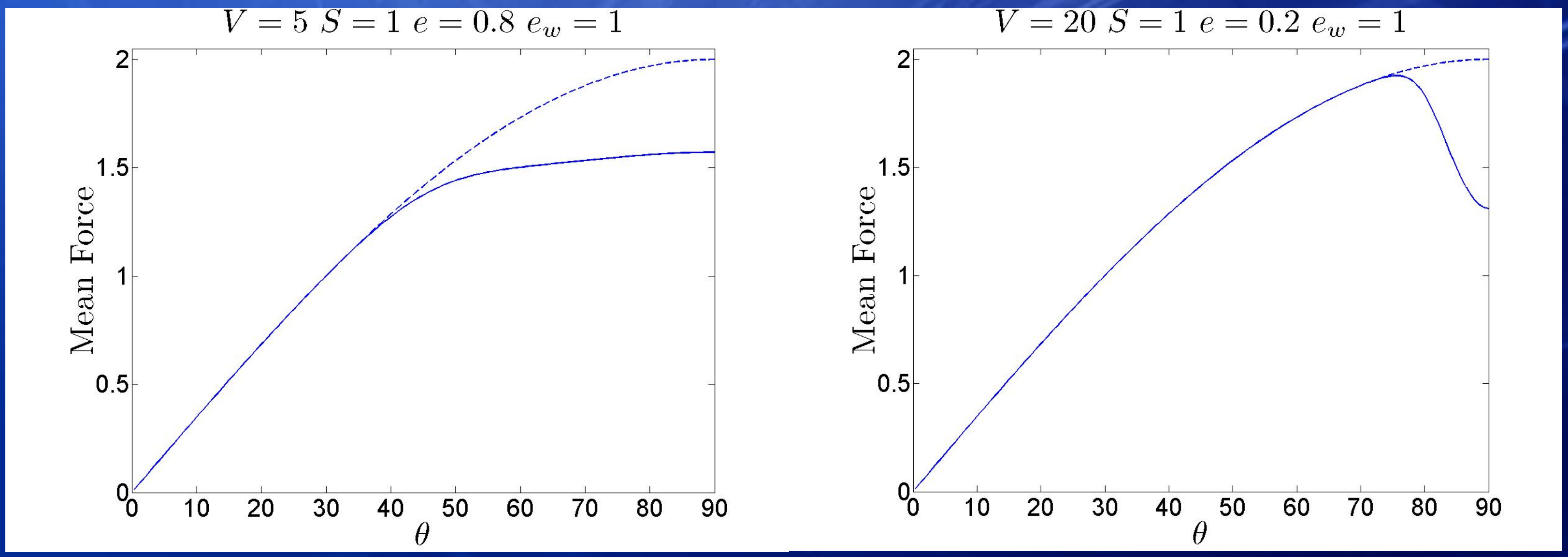
Geometric effects

- When the particle-wall restitution coefficient is unity $(e_w = 1)$, then we obtain

$$f_{mean} = f_g \cdot f_s \qquad \text{where} \quad f_g = 2\sin\theta, \quad f_s = \frac{1 + \int_\Omega \left| e - (1+e)q^2 \right| \rho(\xi)\mathrm{d}\xi}{1 + \int_\Omega \rho(\xi)\mathrm{d}\xi}$$

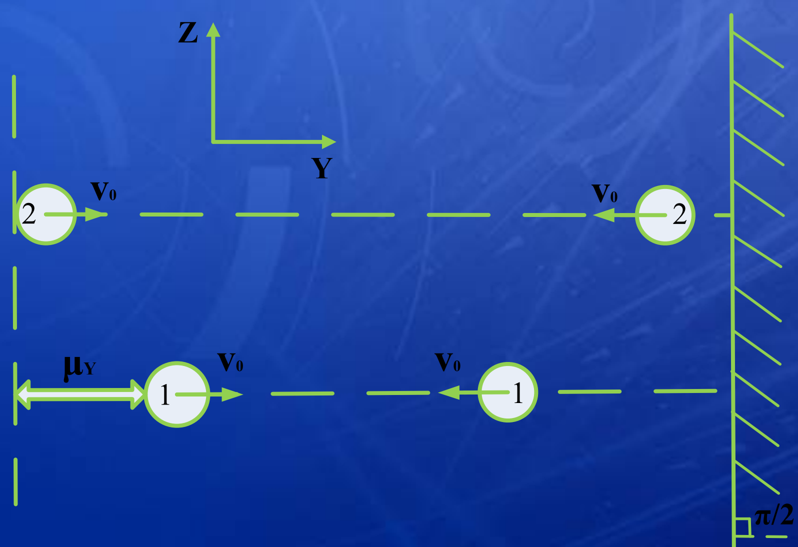
$$\Omega = \left[\frac{V\cos\theta - 2}{\sqrt{2}S\sin\theta}, \frac{V\cos\theta + 2}{\sqrt{2}S\sin\theta}\right], \quad q = \frac{V\cos\theta - \sqrt{2}\xi S\sin\theta}{2}, \quad \rho(\xi) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\xi^2}{2}\right).$$

- Larger deflector angle implies a larger velocity component perpendicular to the wall, and hence larger impulse. We refer to this as geometric effects.

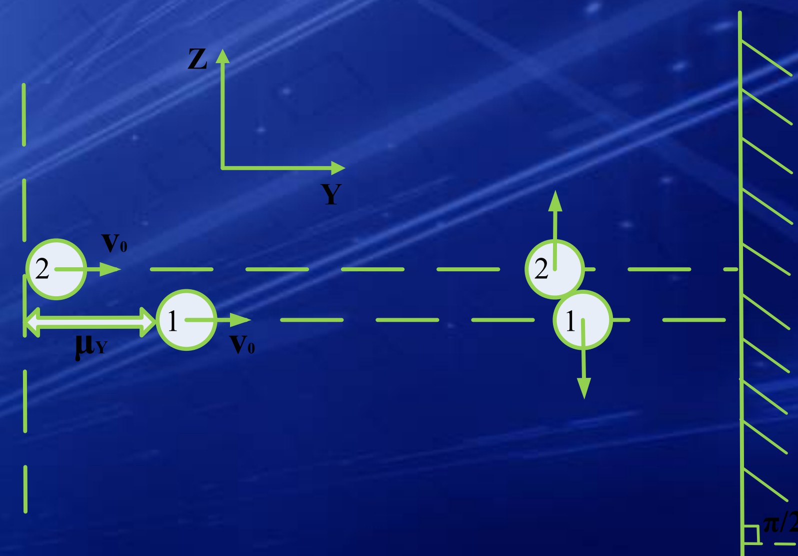


Shielding effects

- The impulse experienced by the wall is the net change of momentum experienced by the particles. When particles rebound from the wall they can collide with incoming ones and be scattered. This scattering may reduce the net impulse on the wall. We refer to this effect as shielding.
- The simplest example of shielding effect

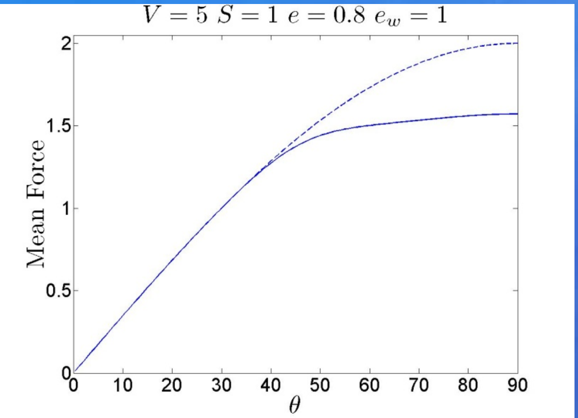


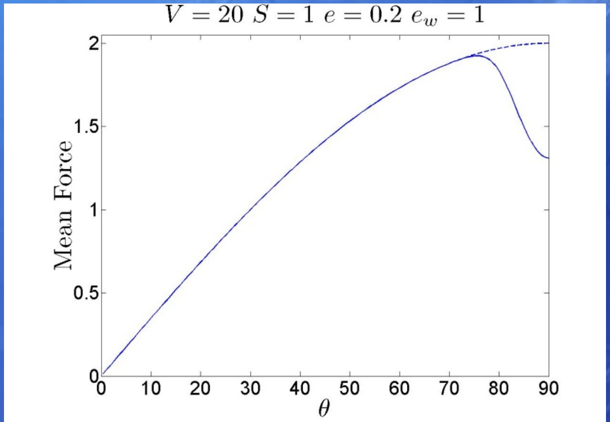
$$\text{Net impulse} = 4v_0$$

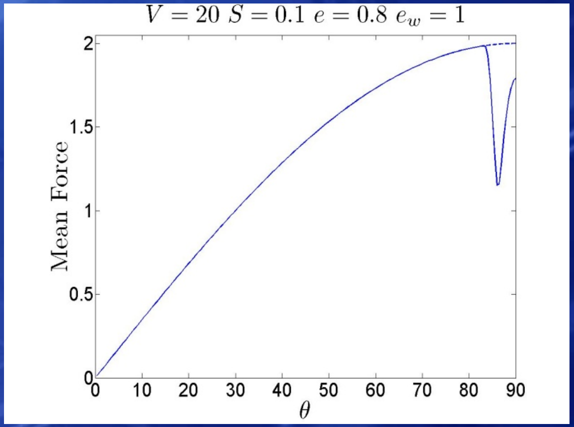


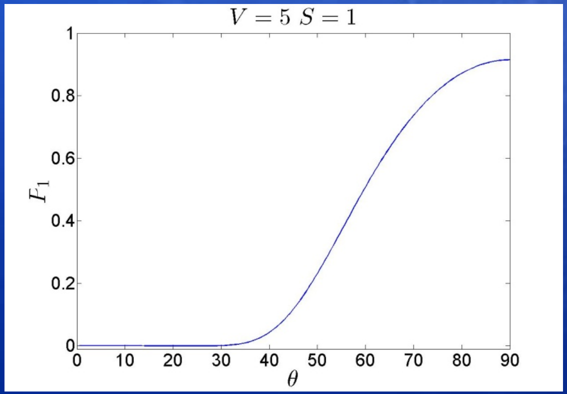
$$\text{Net impulse} = 2v_0$$

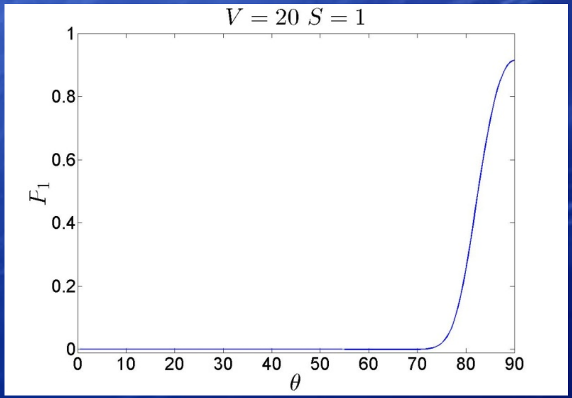
Mechanism

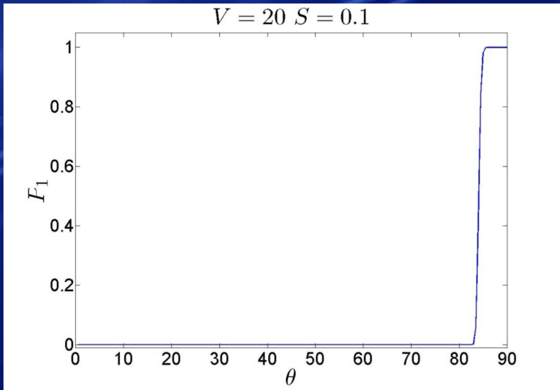

$V = 5$ $S = 1$ $e = 0.8$ $e_w = 1$
Mean Force
θ


$V = 20$ $S = 1$ $e = 0.2$ $e_w = 1$
Mean Force
θ

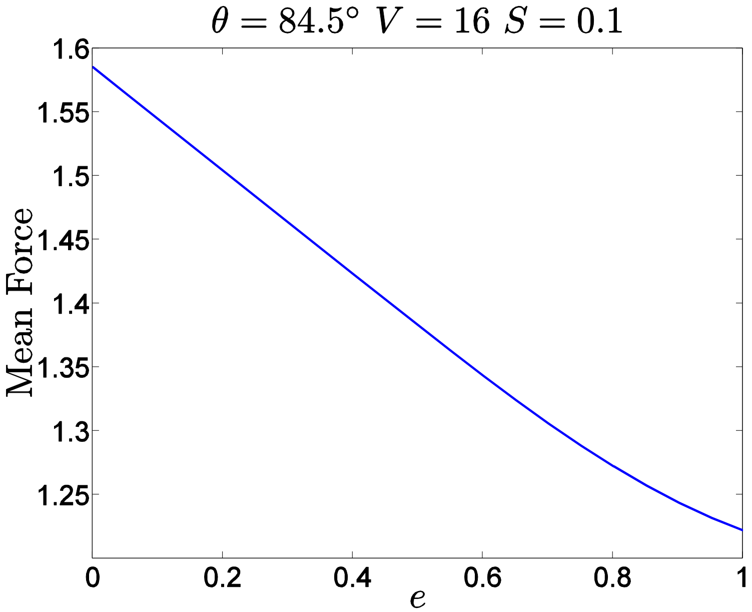

$V = 20$ $S = 0.1$ $e = 0.8$ $e_w = 1$
Mean Force
θ


$V = 5$ $S = 1$
F_1
θ

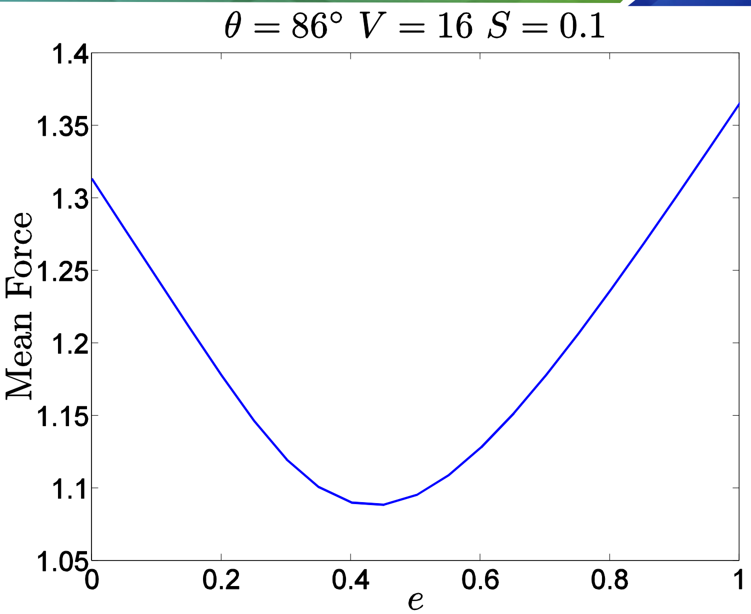

$V = 20$ $S = 1$
F_1
θ


$V = 20$ $S = 0.1$
F_1
θ

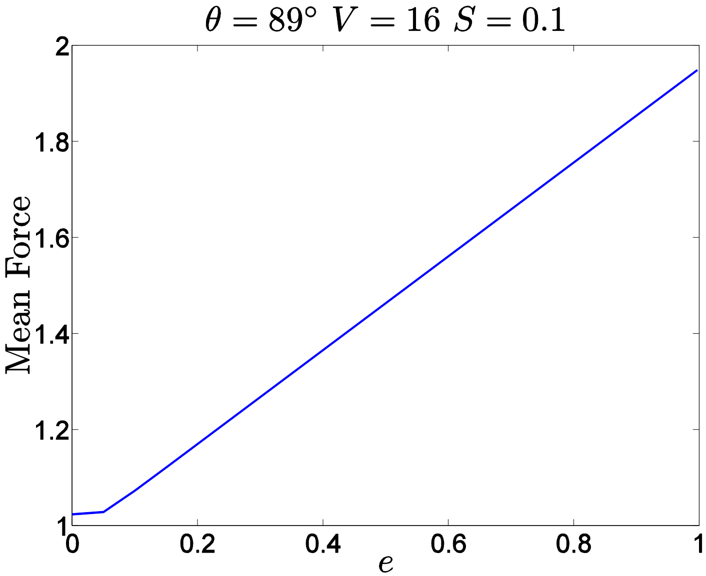
Mechanism



$\theta = 84.5° \; V = 16 \; S = 0.1$
Mean Force
1.6
1.55
1.5
1.45
1.4
1.35
1.3
1.25
0
0.2
0.4
0.6
0.8
1
e

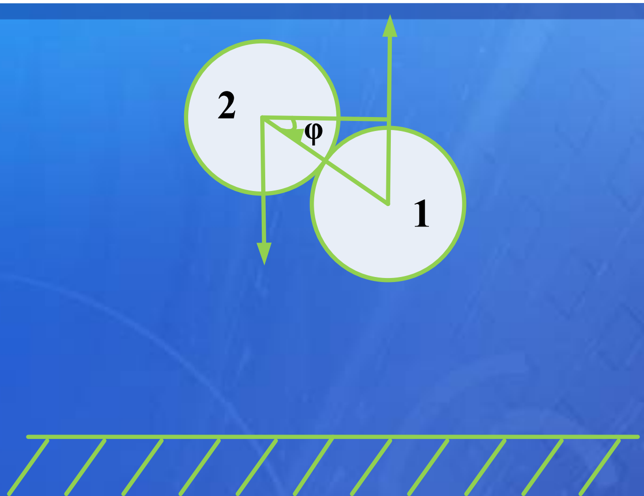


$\theta = 86° \; V = 16 \; S = 0.1$
Mean Force
1.4
1.35
1.3
1.25
1.2
1.15
1.1
1.05
0
0.2
0.4
0.6
0.8
1
e

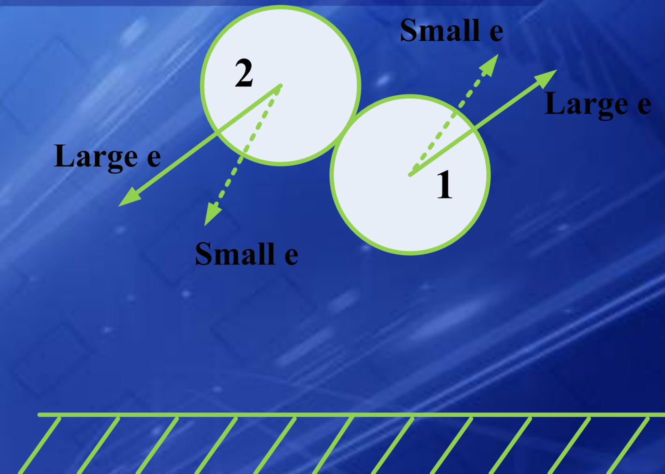


$\theta = 89° \; V = 16 \; S = 0.1$
Mean Force
2
1.8
1.6
1.4
1.2
1
0
0.2
0.4
0.6
0.8
1
e

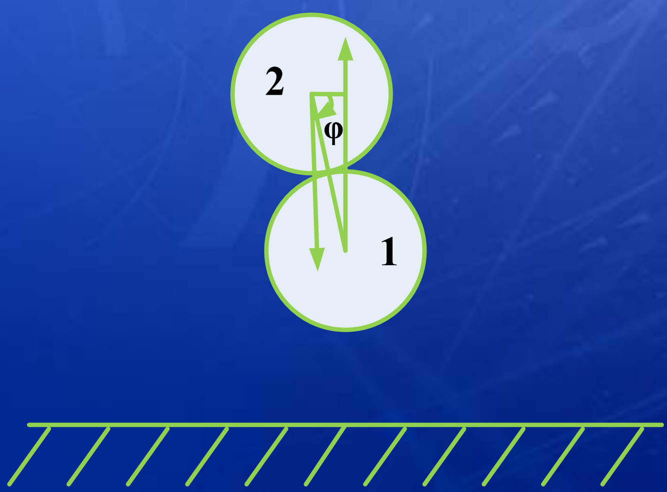
Glancing and head-on collisions



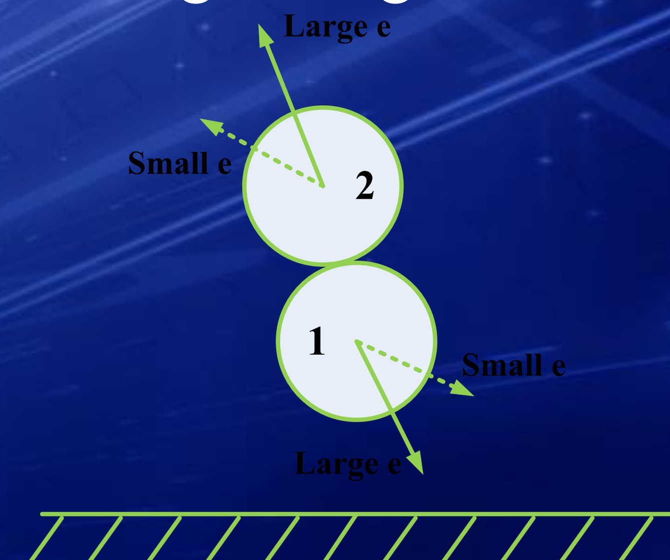
Before glancing collision



After glancing collision



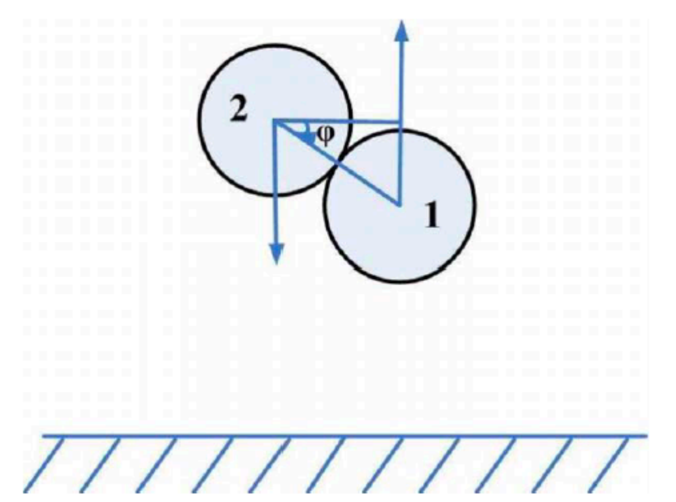
Before head - on collision



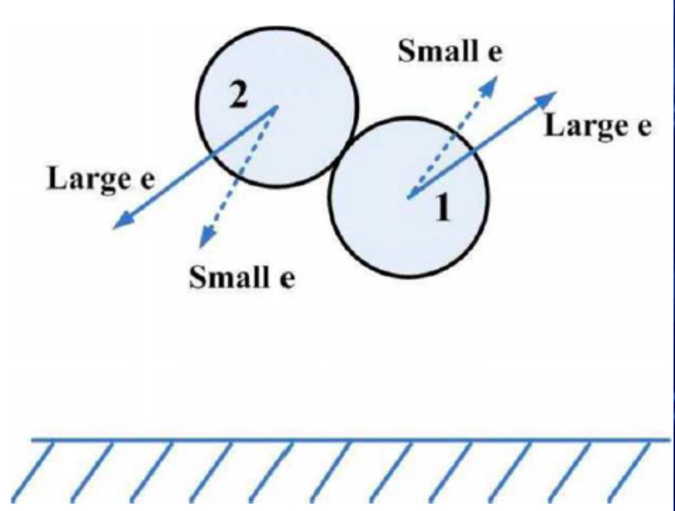
After head - on collision

Glancing and head-on collisions

Glancing collision (rotated frame)



Before glancing collision



After glancing collision

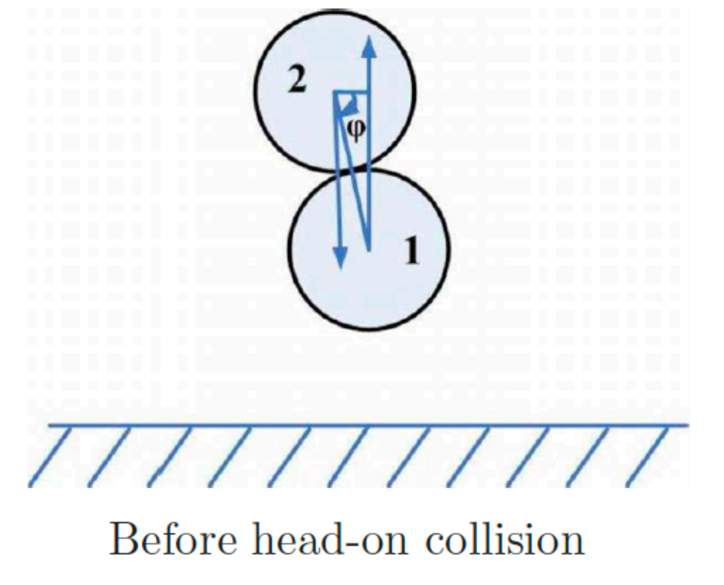
Glancing collisions:

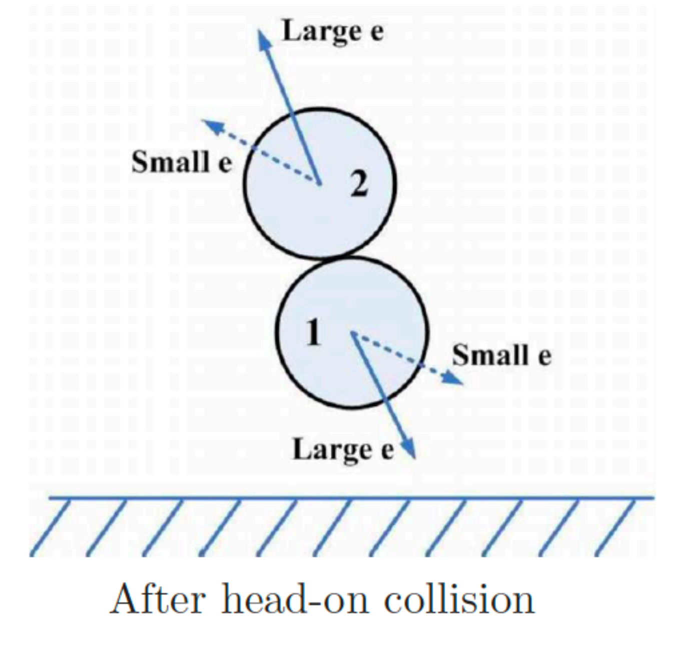
larger e => the 2nd particle is deflected at a larger

=> less impact on the barrier

Glancing and head-on collisions

Head-on collision (rotated frame)





Head-on collisions:

larger e => the 1st particle bounces back faster

=> more impact on the barrier

Conclusion

- We have considered a system where a dilute stream of particles collides with an oblique planar wall (application: diverting particle avalanches).
- For dilute particle streams, large deflector angle may actually decrease the mean force (opposite of intuition).
- We derived an exact solution for the mean force on the wall. Good agreement with our numerical results if the system is dilute.
- Explicitly quantify both of the geometric and shielding effects. Explain the interesting features we have found.
- Understand how geometric effect and the shielding effect compete with each other.

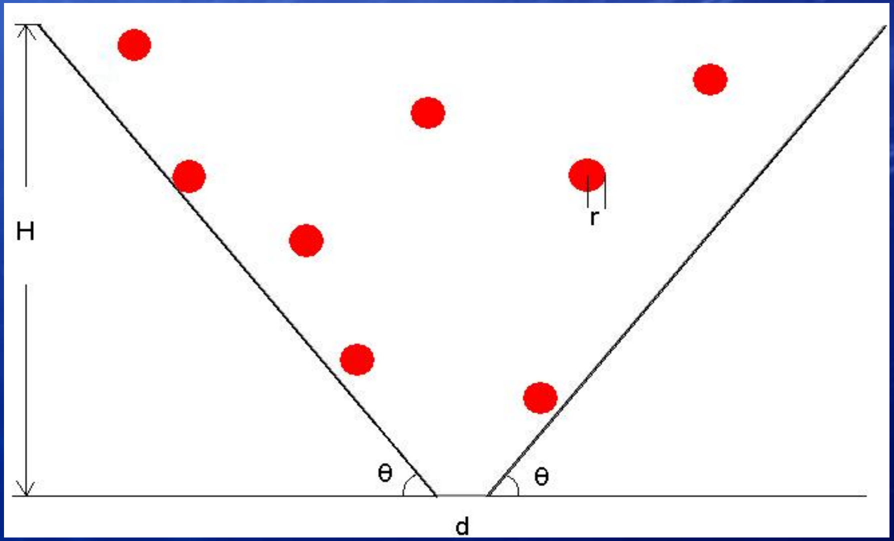
Inelastic particles falling through a hopper

Joint work with Qiang ZHANG, Yuan FANG and Ming GAO

Physical system

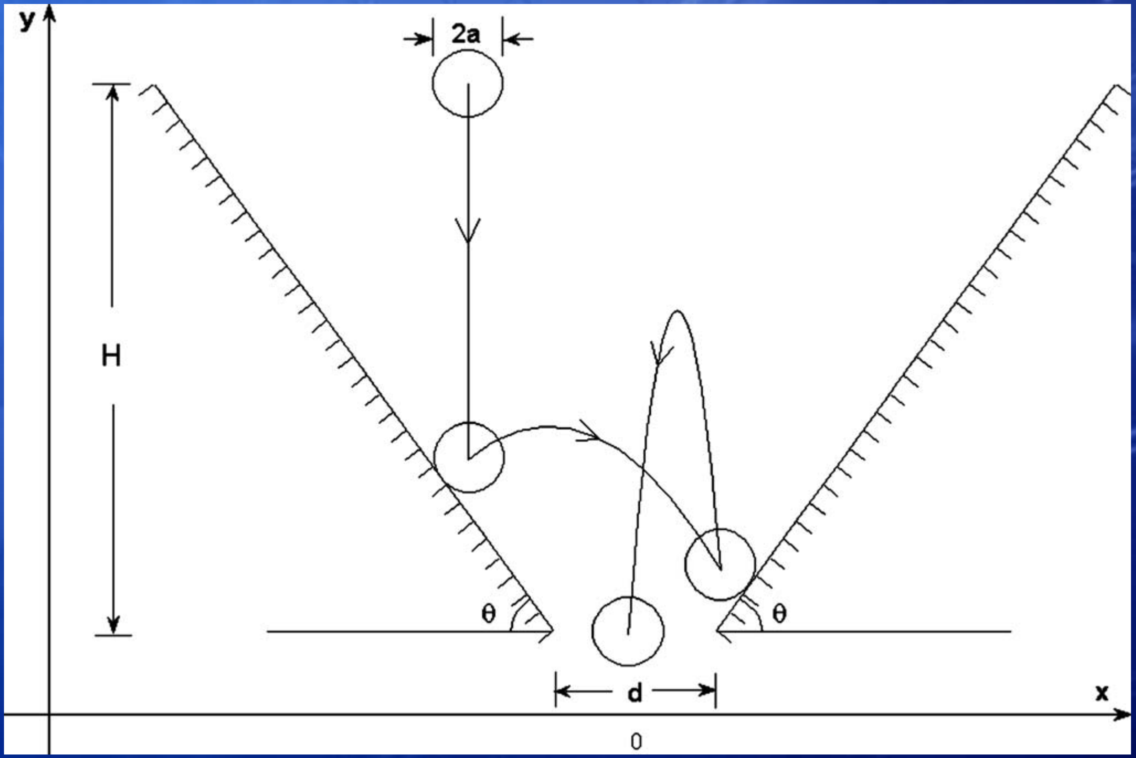
Particles are randomly dropped into a hopper from the top.
The particles fall under gravity.
When the particle falls through the hole, it leaves the hopper.

How does the residence duration depend on various physical parameters?



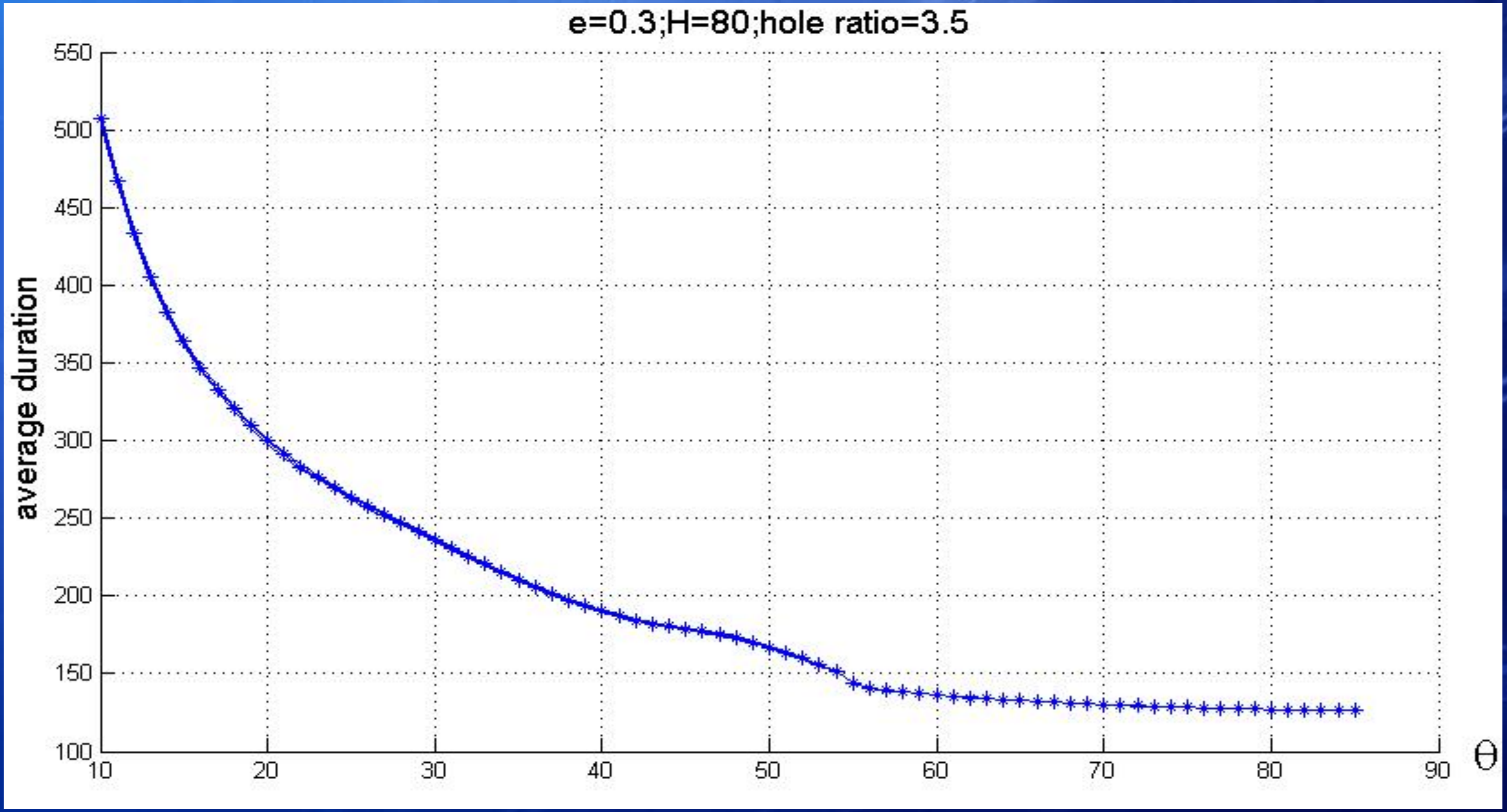
One-particle System

We consider a simple one-particle system.
What could be simpler that this?



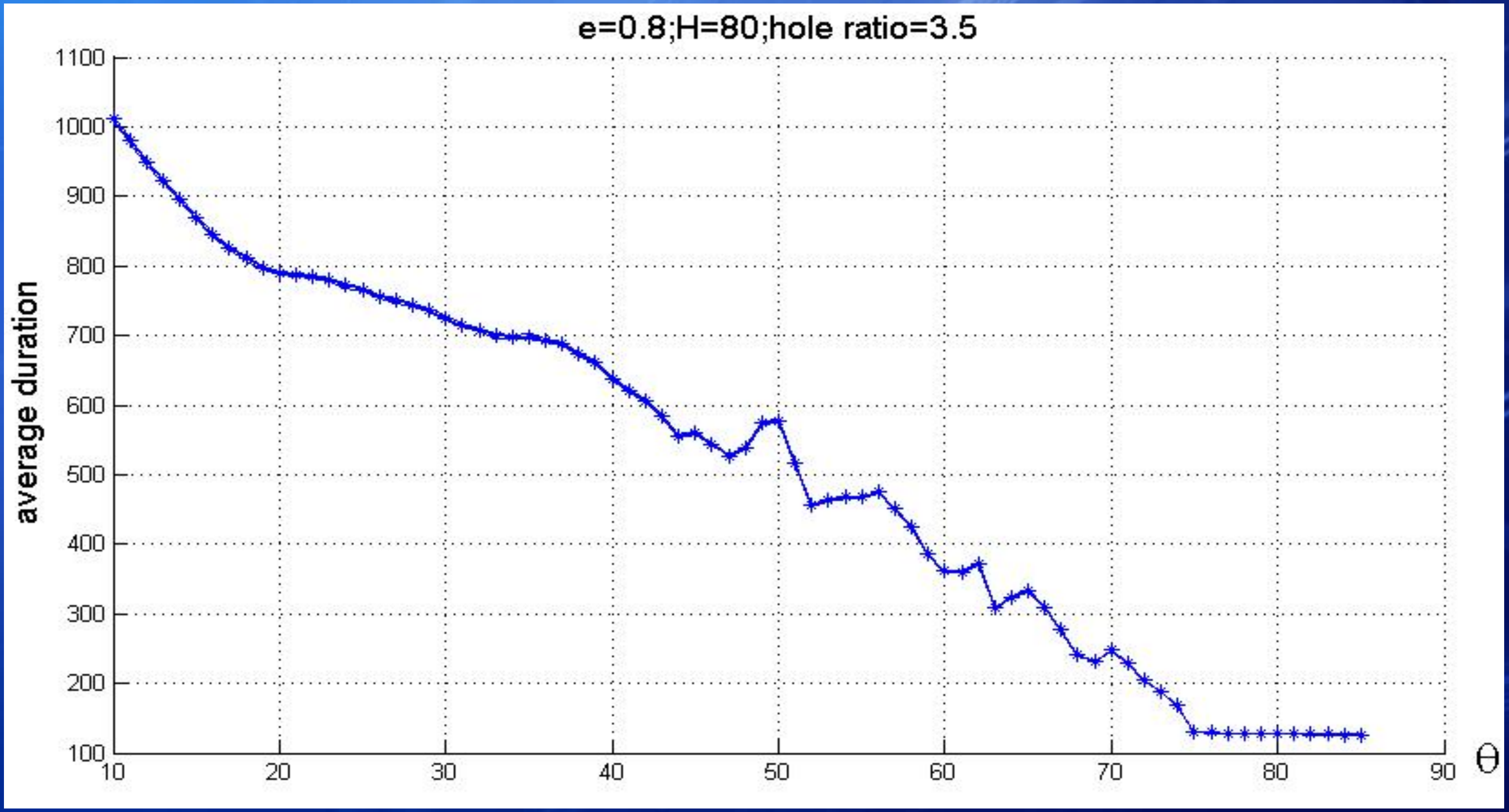
Intuitive phenomenon

The steeper the slope is, the shorter the particle will stay in the system

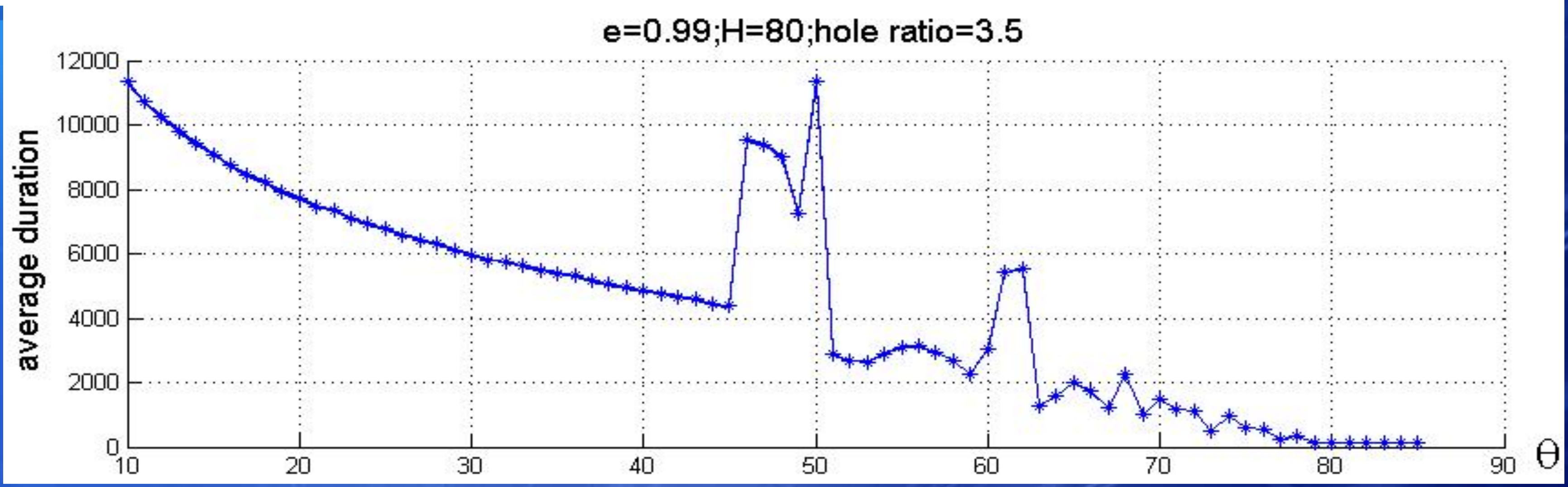


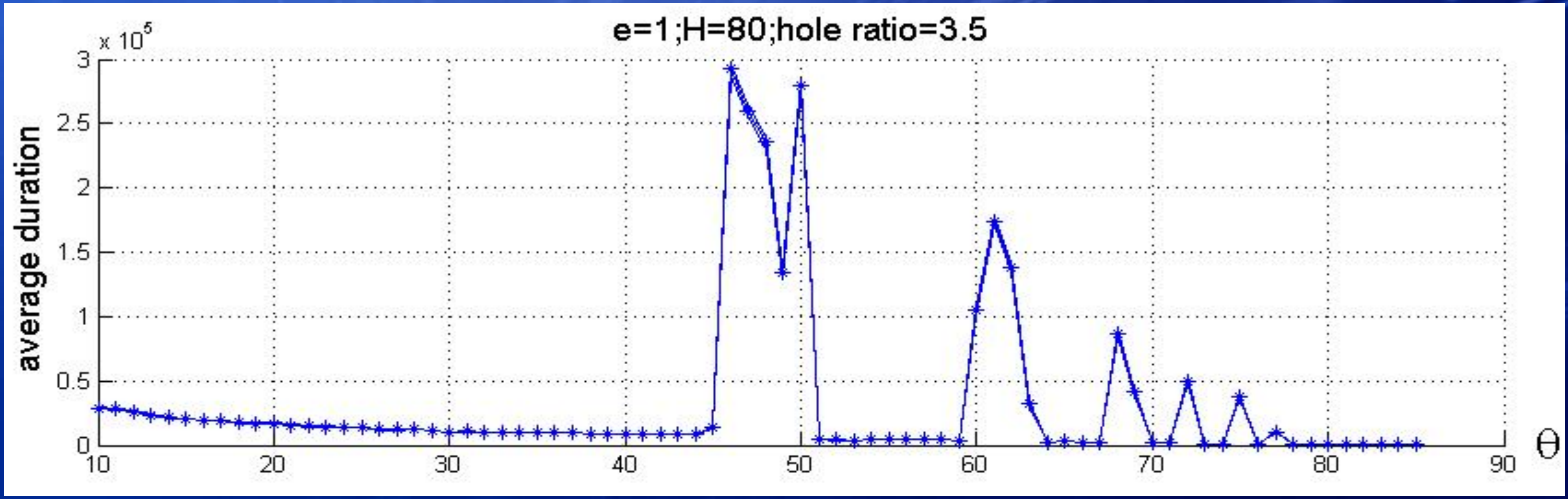
Seems obvious! What else could possibly happen?

Surprising phenomenon

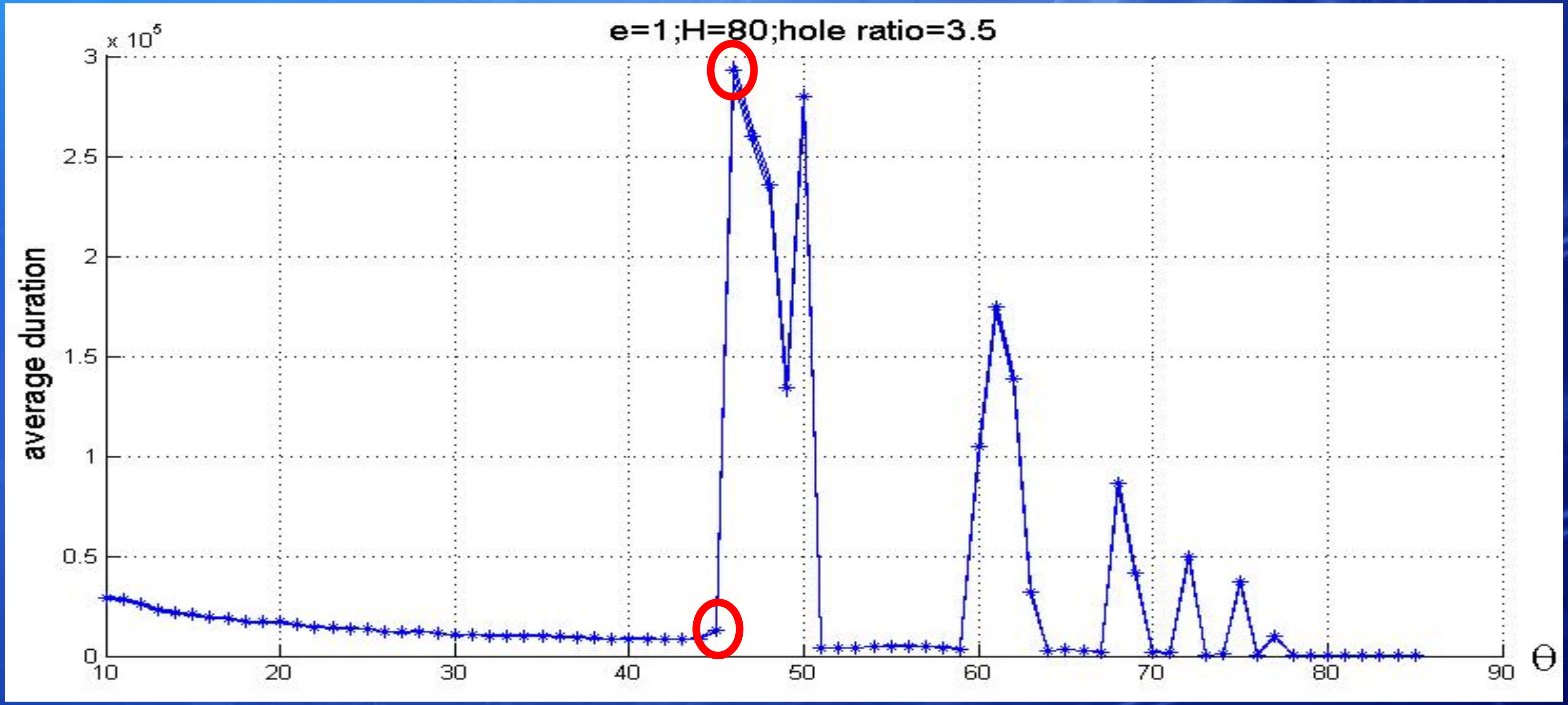


Surprising phenomenon





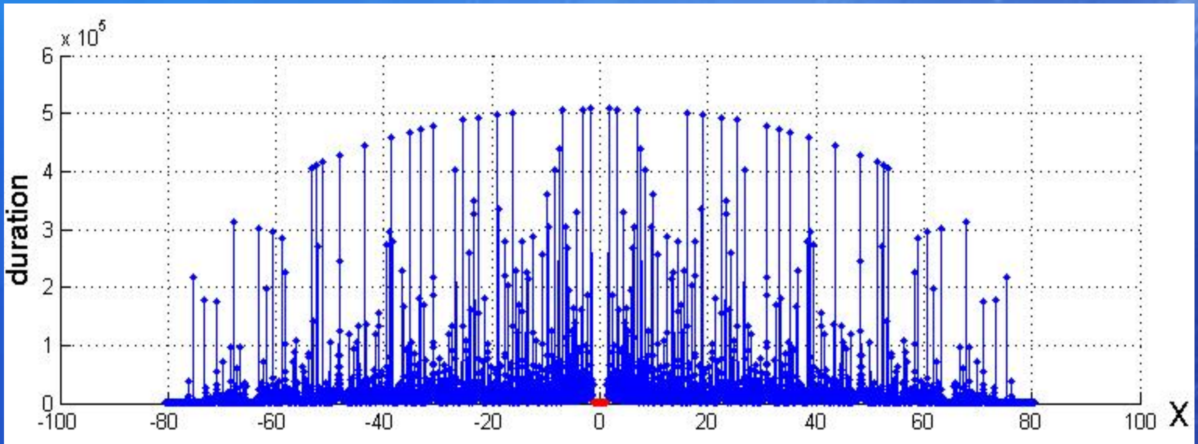
e=1



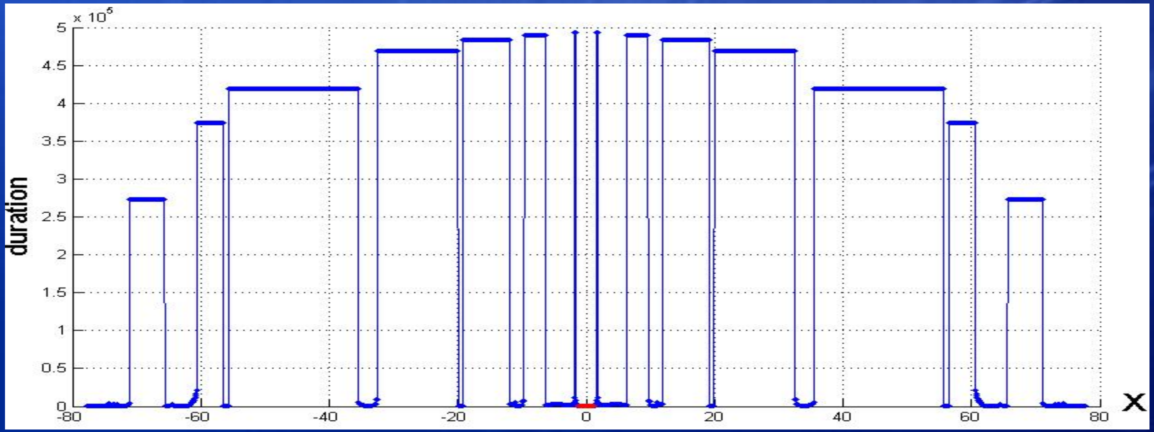
Because we input the particles uniformly on the top boundary of the hopper, we draw the following figures to help us analyze.

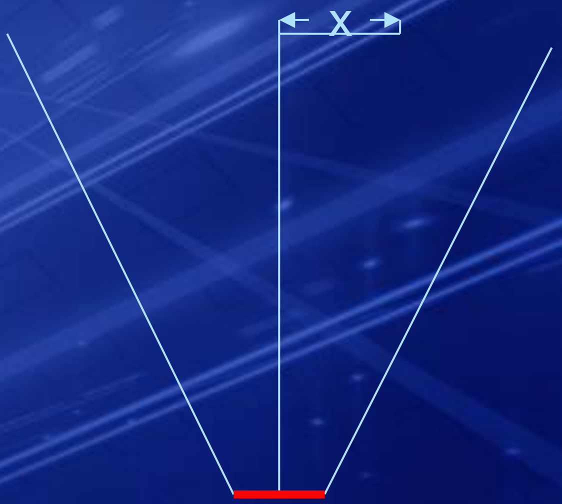
Duration versus injection location

e=1, θ=45

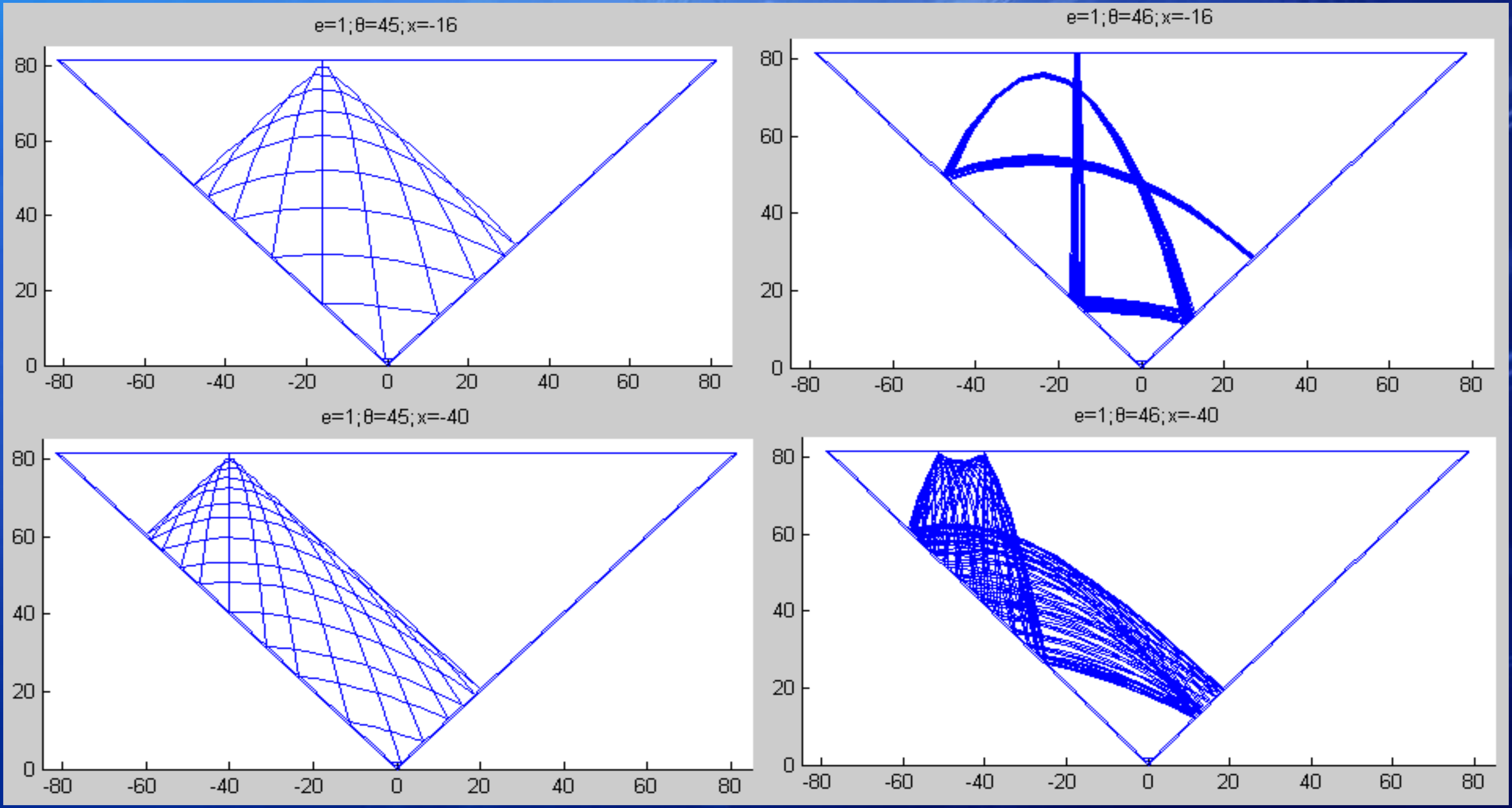


e=1, θ=46

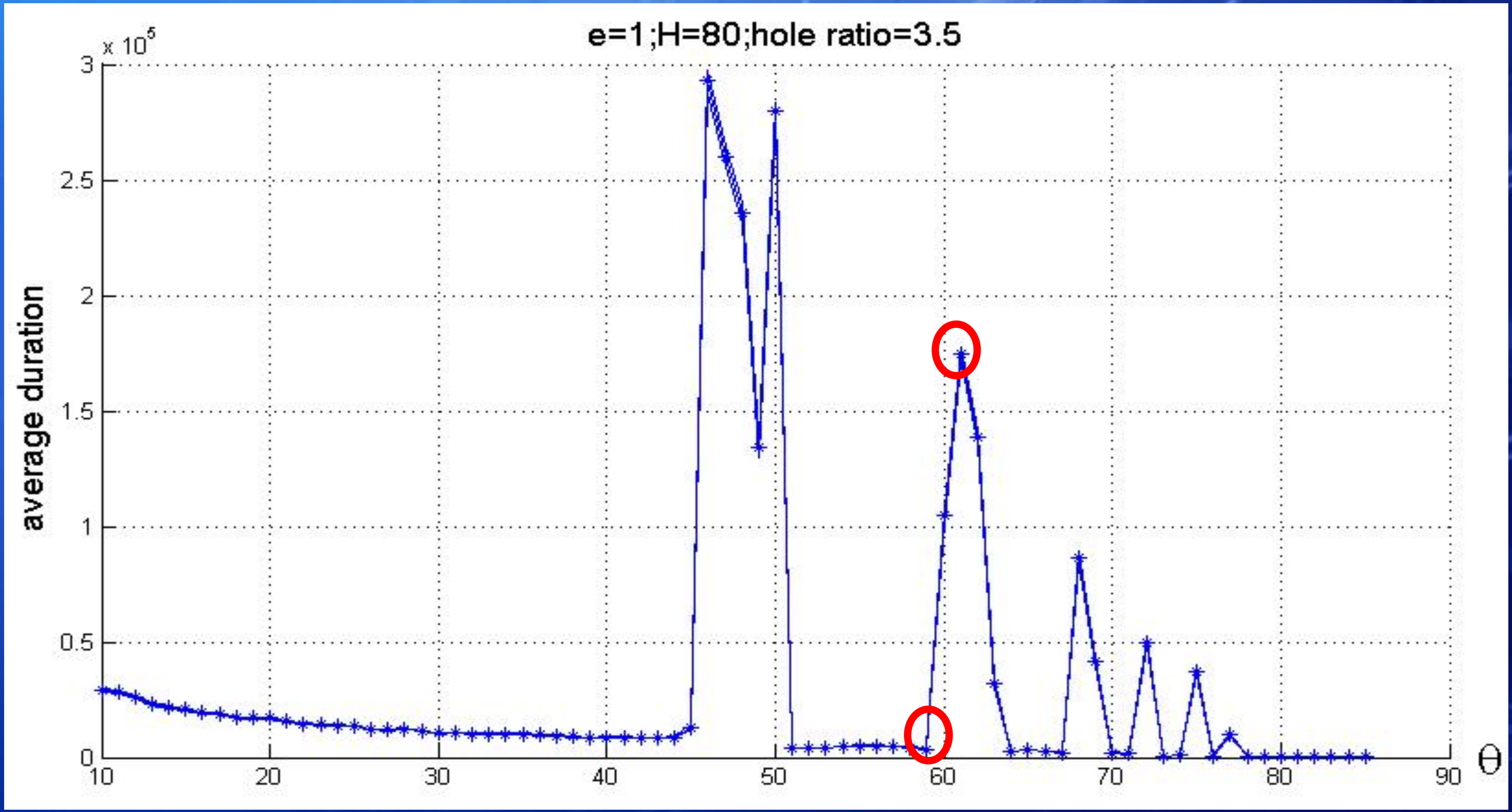




Samples of trajectories



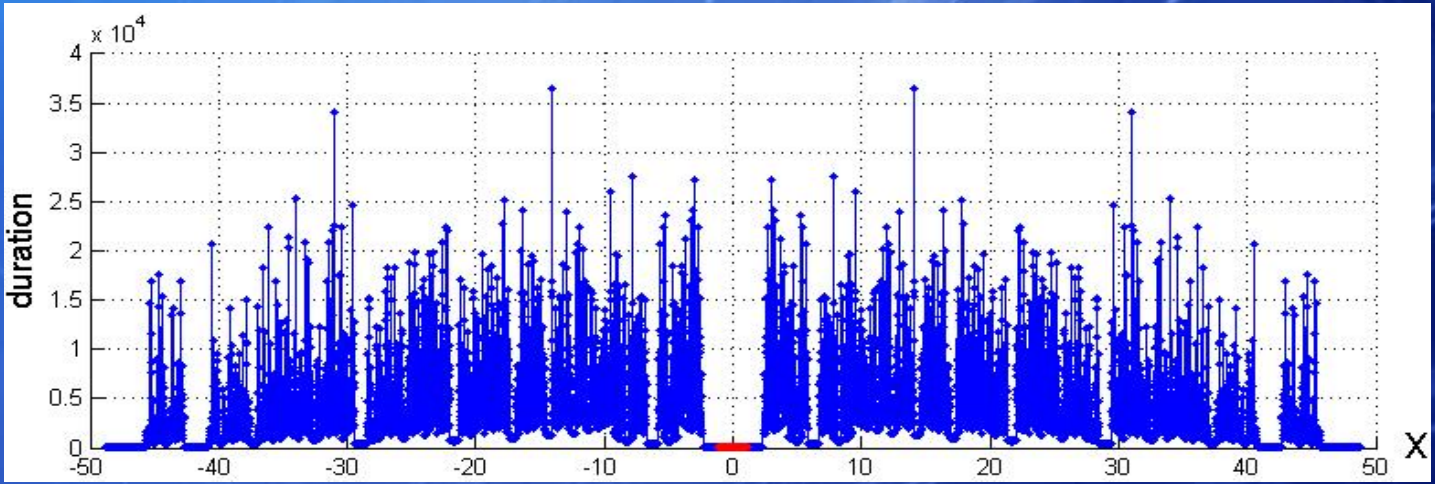
e=1



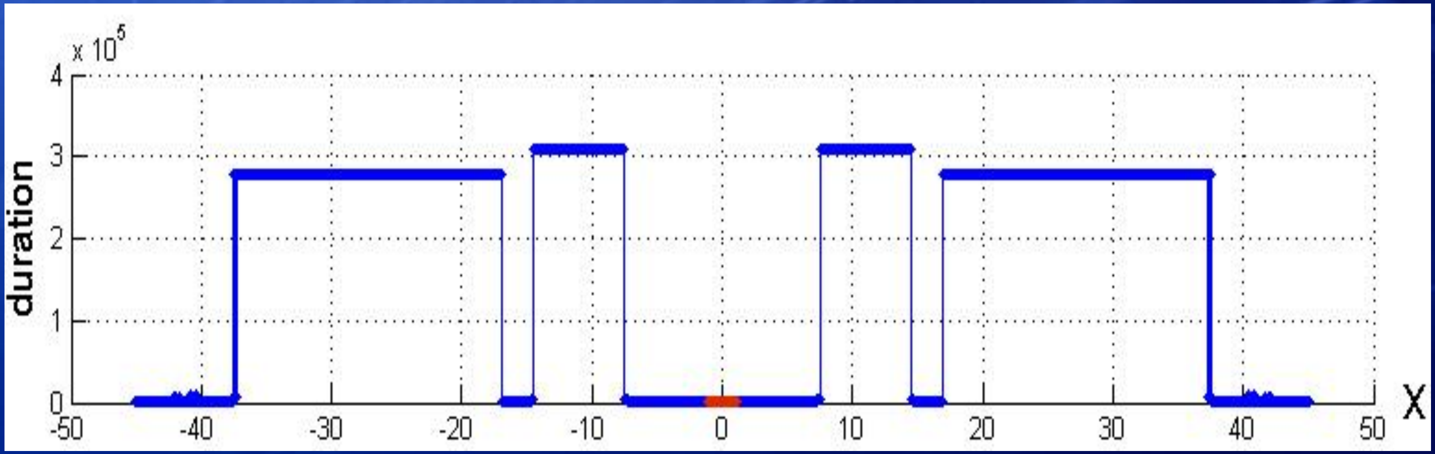
e=1;H=80;hole ratio=3.5
x 10^5
average duration
3
2.5
2
1.5
1
0.5
0
10
20
30
40
50
60
70
80
90
θ

Duration versus injection location

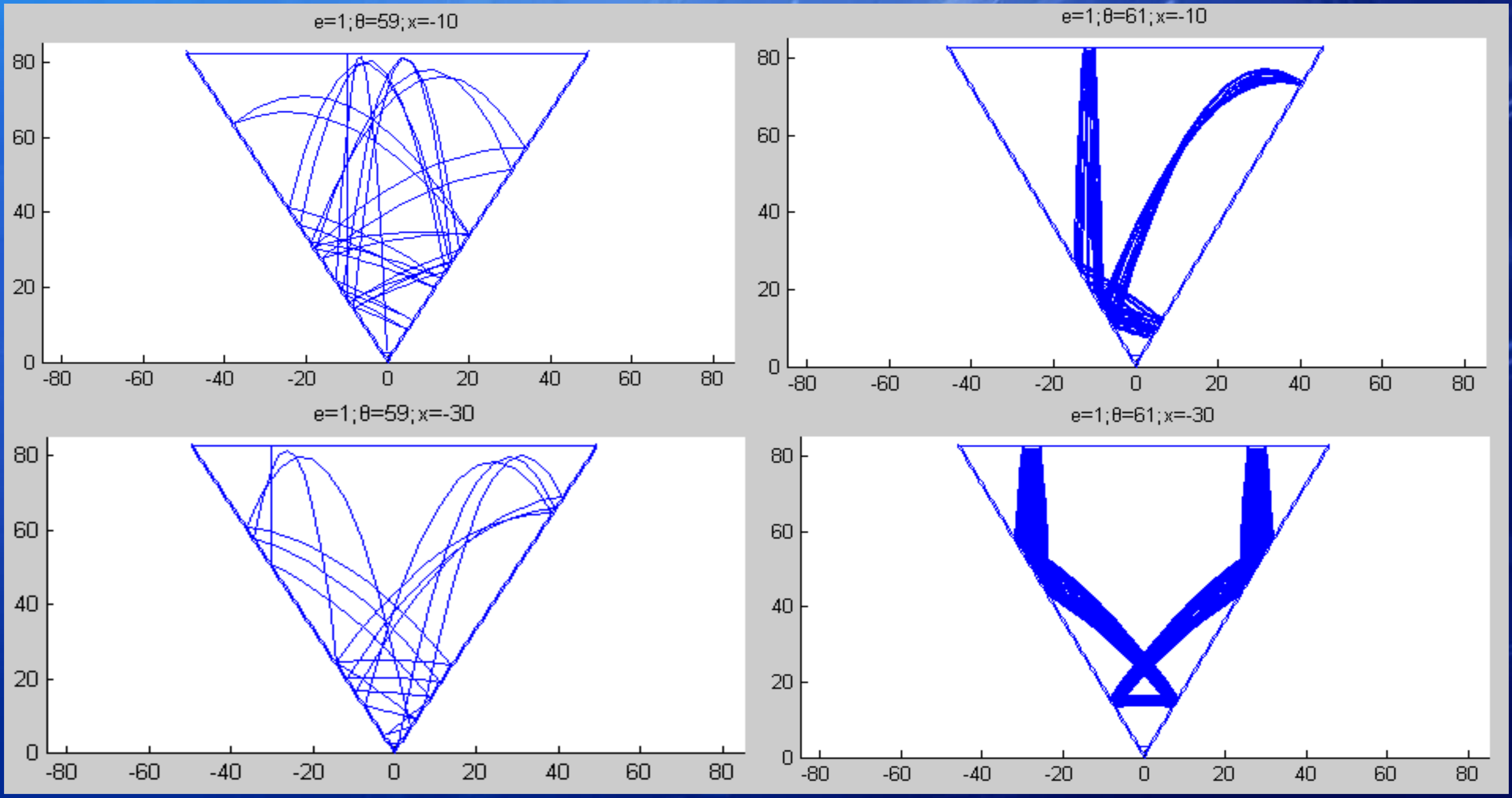
e=1, θ=59



e=1, θ=61



Samples of trajectories



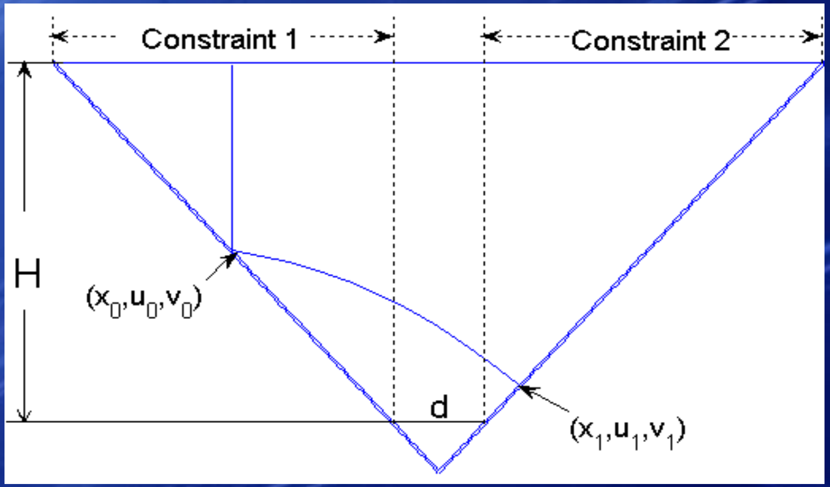
Simplest periodic orbit

Location and velocities of this orbit:

$$x^* = \frac{H(3-\tan^2\theta)(5\tan^2\theta+1)}{3\tan^5\theta - 18\tan^3\theta - 5\tan\theta}$$

$$u^* = \sqrt{2g(H + x^*\tan\theta)}\sin 2\theta$$

$$v^* = \sqrt{2g(H + x^*\tan\theta)}\cos 2\theta$$



Constrains

constrain 1: $x_0 \leq -\frac{d}{2}$

constrain 2: $\frac{d}{2} \leq x_1$

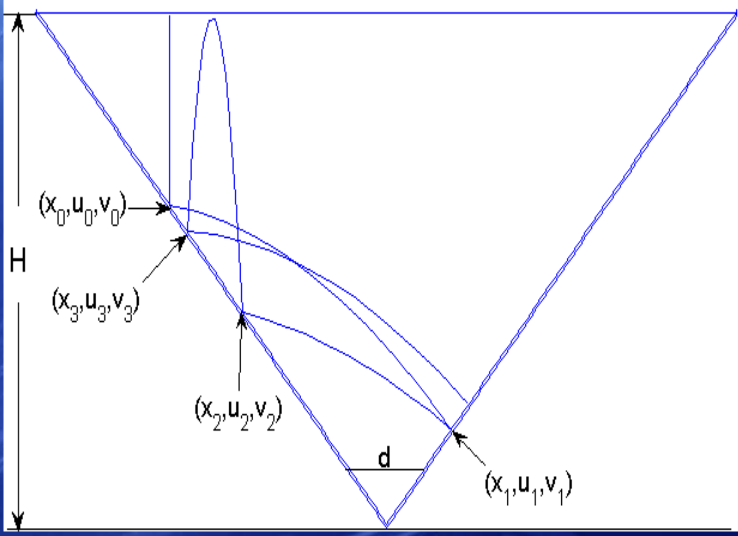
$$\frac{\pi}{6} + \frac{1}{6}\frac{d}{2H} \leq \theta \leq \frac{\pi}{3} - \frac{1}{2}\frac{d}{2H}$$

Stability analysis

There are three variables: (x, u, v)

Eliminate x from energy conservation, we only need to focus on

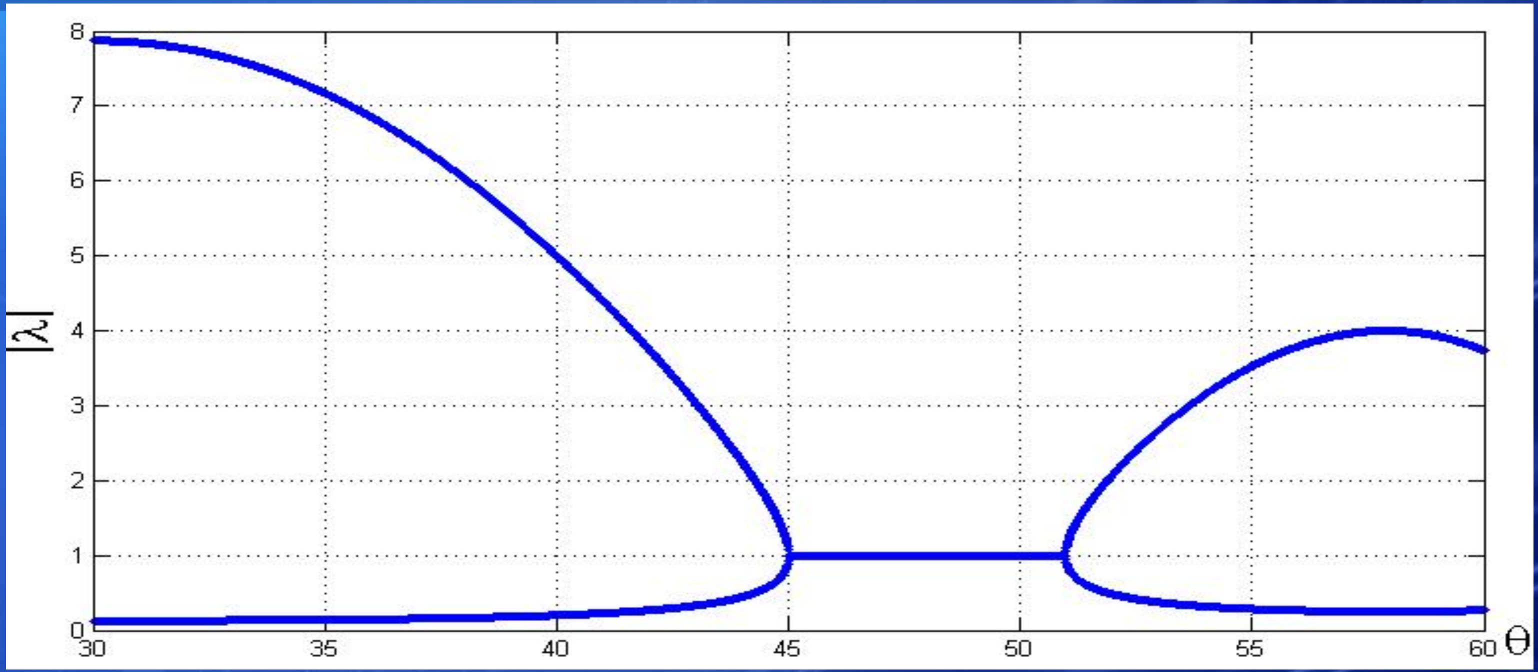
(u, v)

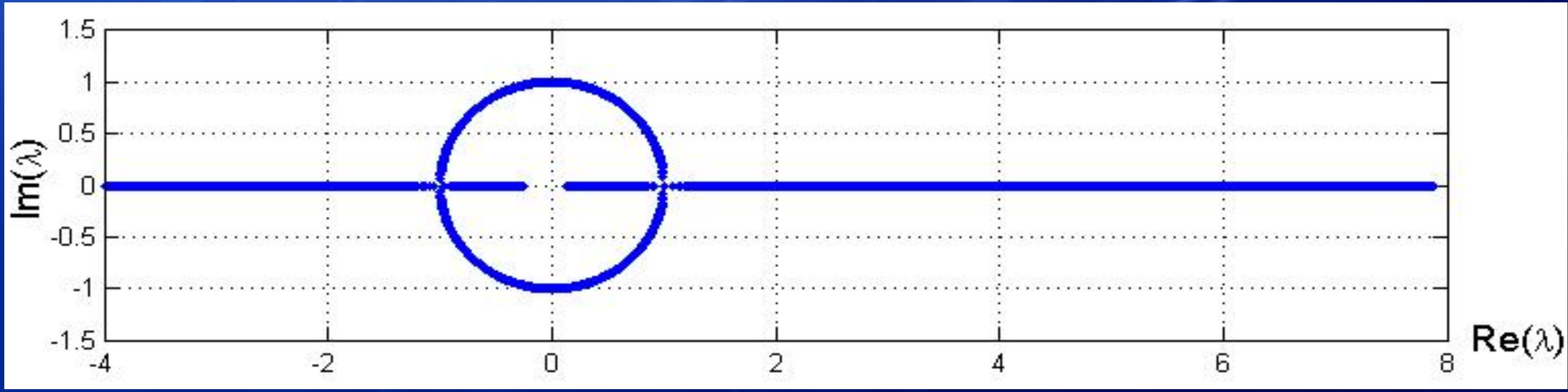


$$\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} = F\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \Rightarrow \begin{pmatrix} u_3^* + \triangle u_3 \\ v_3^* + \triangle v_3 \end{pmatrix} = F\begin{pmatrix} u_0^* + \triangle u_0 \\ v_0^* + \triangle v_0 \end{pmatrix}$$

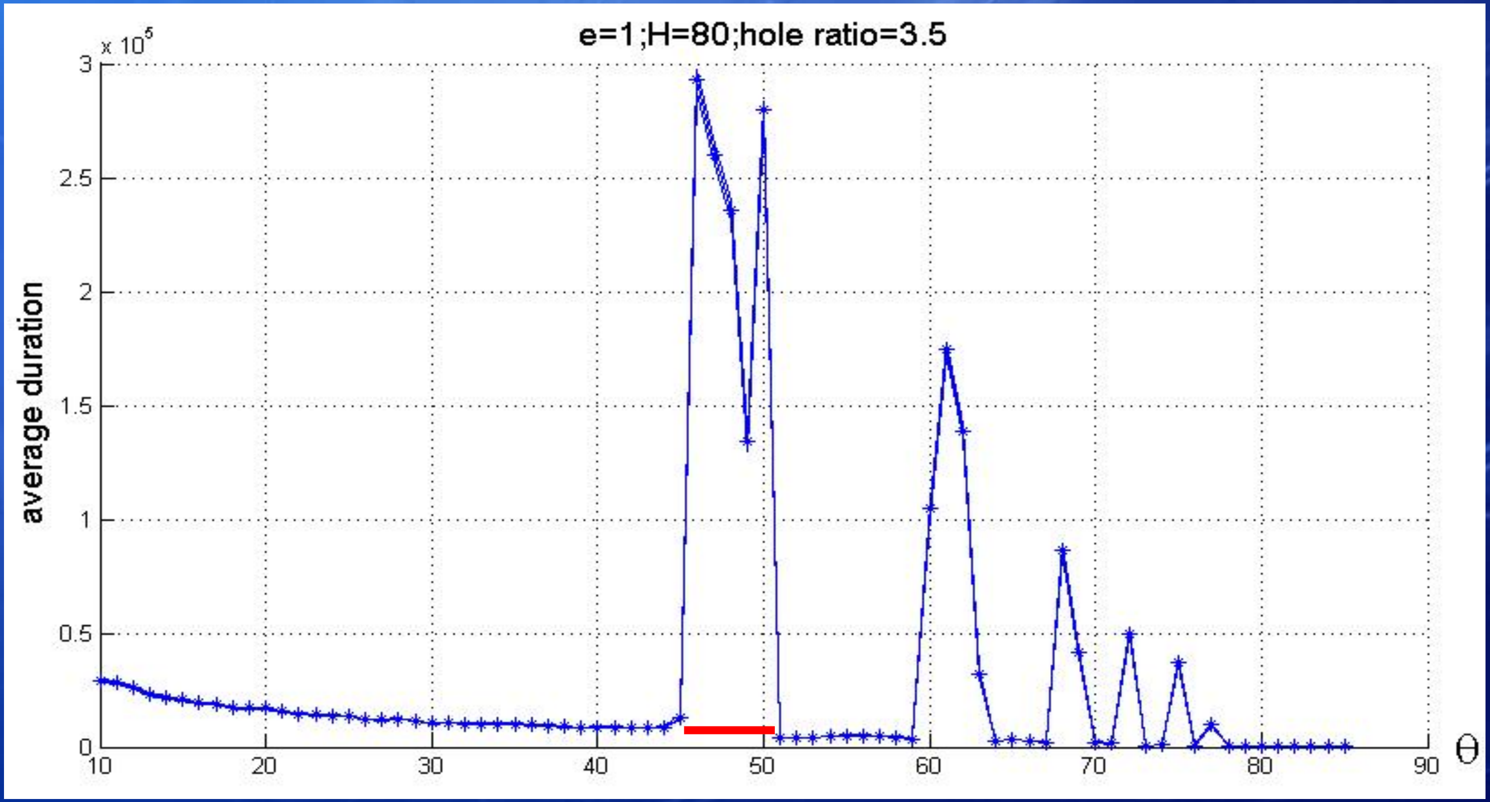
$$\Rightarrow \begin{pmatrix} \triangle u_3 \\ \triangle v_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial u_3}{\partial u_0} & \frac{\partial v_3}{\partial u_0} \\ \frac{\partial v_3}{\partial u_0} & \frac{\partial v_3}{\partial v_0} \end{pmatrix} \begin{pmatrix} \triangle u_0 \\ \triangle v_0 \end{pmatrix} \stackrel{\text{def}}{=} J \begin{pmatrix} \triangle u_0 \\ \triangle v_0 \end{pmatrix}$$

Eigenvalues of J



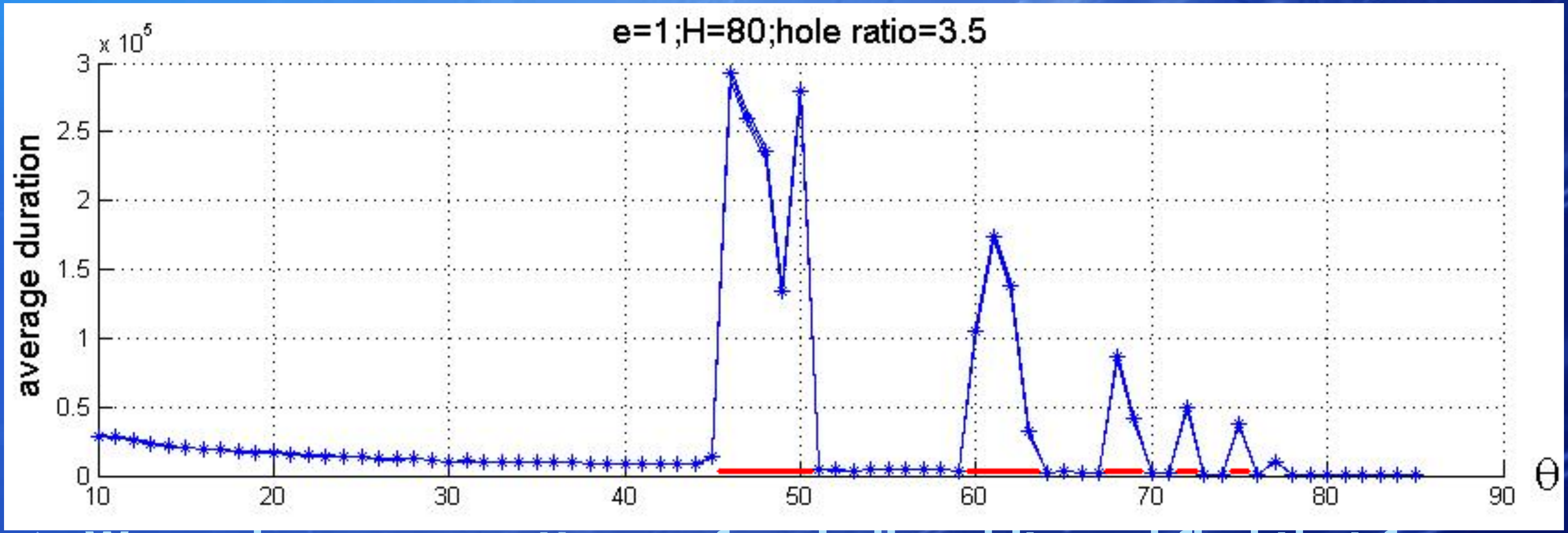


To insure stability, $|\lambda_1|$ and $|\lambda_2|$ must be less than or equal to 1.



$$|256\cos^8\theta - 768\cos^6\theta + 752\cos^4\theta - 264\cos^2\theta + 26| < 2.$$

$$\theta \in \left(\frac{\pi}{4}, \arccos\sqrt{\frac{3-\sqrt{2}}{4}}\right) = (45°, 50.97°)$$



- **We analyze more patterns of periodic orbits and find that for every θ in the ranges marked with red, there exists at least one stable periodic orbit**

Conclusion

- Single inelastic particle falling through a hopper has highly nontrivial behavior
- For small e, the residence time decreases uniformly with wall angle (intuitive)
- For larger e, there are small windows of wall angles that have anomalously long residence times
- Developed a analytical theory that predicts anomalous behavior and shows the underlying mechanism is the existence of stable periodic orbits