Finding the Optimal Tax with Pollution Haven Effects

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In this paper, I seek a solution to the government’s problem of imposing an optimal pollution tax to control emissions. I use a model with two countries and foreign direct investment. Firms leave the country if their costs of production become prohibitively expensive. This mimics the real world, in which industries may flee and produce abroad if regulation is too strict. In my model, the government derives utility from consumption, which is increasing in the number of industries that stay at home and a decreasing in temperature. I combine Levinson and Taylor’s (2008) theoretical framework for pollution abatement and trade with Weitzman’s (2009) global warming utility functions. Then, I seek to answer the questions, “To what extent should governments regulate? How does this depend on country size?”

1. Introduction

The Pollution Haven Hypothesis (PHH) posits that environmental regulation changes trade flows because production of “dirty” goods is outsourced to countries with less stringent environmental regulation. (Copeland and Taylor, 2004) Stricter environmental regulations cause a country’s import demand for a given dirty good to increase, so for any world price, the country imports more of that good, causing emissions of trade partners to rise.

The PHH has important implications for environmental regulation and trade for two reasons. First, where it occurs, it undermines unilateral carbon dioxide emissions policy. When one country decides to more strictly regulate CO₂ emissions, it hopes to decrease total world emissions. Regrettably, if the PHH holds, production is outsourced and thus emissions merely shift to another country. Hence, world emissions do not change. Second, the pollution haven effect creates economic distortions. Consider for a moment two countries, A and B. Assume that country A has a comparative advantage in production of dirty goods. When country A implements environmental regulations, production of dirty goods shifts to country B, the country that did not have a comparative advantage in production of dirty goods. In sum, the pollution haven effect undermines unilateral environmental regulation and distorts efficient trade patterns.

Some previous empirical studies support the PHH. Aichele and Felbermayr (2010) find that the Kyoto protocol led to substantial carbon leakage. They also find that Kyoto members had more incentives and subsidy programs targeted at emission reduction. Grether and Mathys (2008) note that production has shifted faster eastwards than

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economic activity, and this finding supports the carbon leakage hypothesis. Kellenberg and Levinson (2010) explore international waste trade and find that the Ban Amendment to the Basel Convention on the Control of Transboundary Movements of Hazardous Waste and Their Disposal was influential in changing trade patterns. Caves (1982) and Brainard (z ) show that all else equal, foreign investors will locate in countries where factors they use in high proportions are relatively inexpensive. Xing and Kolstad (2002) undertake an empirical study of US FDI in dirty industries and find a positive relationship between FDI and pollution intensity.

Many views exist on why the pollution haven effect is observed. One important explanation is firm response to policy in the form of abatement. Cole, Elliot, and Frederickson (2007) and Swee (2003) explain that changes in factor returns are due to increased abatement activity. Abatement action that firms undertake in response to taxes leads to an ambiguous outcome when it comes to the effect of environmental taxes on investment and output decisions. Should an industry choose to invest more capital in the home country in order to make abatement requirements less expensive to comply with, the industry may increase output at home to take advantage of lower marginal costs.

Other empirical studies seem to contradict the findings on the PHH. Eskelanda and Harrison (2003) question whether the pollution haven hypothesis is upheld in the US and find that the evidence is weak. Ederington et al (2004) undertakes a study of US trade liberalization and find that there is no evidence that pollution intensive industries are disproportionately affected by tariff changes.

There are also plausible explanations as to why we would not observe the PHH. Ederington et al (2005) show that dirty industries are less mobile than other industries, supporting the assertion that liberalization itself would not cause a pollution haven effect. Brainard (1997) explains that tariff barriers and transport costs mean that even if conditions are favorable, industries may not relocate (or more plausibly, fewer will relocate). Economies of scale also play a part in a relocation decision.

Interestingly, some authors have argued that the relationship between government stringency (or lack thereof) and FDI could go both ways. Cole and Elliot (2006) and Cole, Elliot, and Frederickson (2007) comment on a “reverse Pollution Haven Hypothesis,” that is, an effect of FDI on the stringency of environmental policy. Foreign firms lobby and bribe host country governments in order to weaken environmental regulation.

Most studies of the PHH discuss how, if, and why it occurs. I am interested in calculating an optimal tax for one government, taking the pollution haven effect as given. I do this because the preponderance of evidence is in favor of the PHH holding, and where it does not hold, there are often specific and anomalous reasons, such as immobility of firms.

A related paper by Light (1999) considers the problem of all countries taken together and calculates an optimal tax given a target reduction in global carbon emission using a numerical technique called Mathematical Programs with Equilibrium Constraints (MPEC). He finds that the optimal tax would need to balance the benefits of equal marginal
abatement costs against the costs of additional carbon abatement as a result of carbon leakage to other countries. In the case that the second effect is stronger, Light paradoxically finds that it may be best to exempt some polluters from the tax.

One recent paper by Wu (2004) asks a similar question to ours but goes about the analysis in a very different way. Wu establishes a game theoretic framework for the problem in order to explore the interactions between governments and polluting multinationals. Wu makes the assumption that the firm has private information about the pollution intensity of its production process thus informational rents exist. In one particularly interesting case, this causes the home government, which is privy to the private information, to encourage firms to cheat the foreign government so that the foreign government must pay a higher information rent to firms, since in the end this becomes the home government’s tax revenue. Wu finds that cooperation between governments increases welfare even as the dirty industry moves toward the foreign country with more pollution. Moreover, Wu explains that an efficiency-seeking government needs to design a regulatory regime that induces a firm to reveal its technology.

II. Definition of the Pollution Haven Hypothesis

For simplicity, I define the pollution haven hypothesis as the hypothesis that higher domestic environmental standards will cause industries to move to foreign countries (which have lower environmental standards). We will assume that standards take the form of a tax \( \tau \) on emissions of composite gas object \( z \).

III. The Model

Following Levinson and Taylor (2008), production in a representative industry is given by

\[
y = [1 - \theta]f(K, L)
\]  

(1)

where \( y \) is production, \( \theta \) is the fraction of the factors used for abatement, \( K \) is capital, and \( L \) is labor. We assume constant returns to scale. Pollution of a representative industry is a function of intensity of abatement:

\[
z = \varphi[\theta]f(K, L)
\]  

(2)

We define \( \varphi[\theta] = (1 - \theta)^{\frac{1}{\alpha}} \), where \( \alpha \) is between 0 and 1 (and hence \( 1/\alpha > 1 \)), since this is a decreasing function of \( \theta \), so when \( \theta \) is high, we have less pollution. Rearranging, we see that \( (1 - \theta) = \left(\frac{z}{f(K, L)}\right)^{\alpha} \)  

Hence, we can rewrite \( y \) as:

\[
y = z^{\alpha}f(K, L)^{1-\alpha}
\]  

(3)

Below, I calculate the first and second derivatives of \( \varphi[\theta] \):
\[
\frac{\partial \varphi[\theta]}{\partial \theta} = \frac{-1}{\alpha} (1 - \theta)^{\frac{1-\alpha}{\alpha}} < 0
\]  

(4)

\[
\frac{\partial^2 \varphi[\theta]}{\partial \theta^2} = \frac{1 - \alpha}{\alpha^2} (1 - \theta)^{\frac{1-2\alpha}{\alpha}} > 0
\]  

(5)

which implies that \( \varphi[\theta] \) is decreasing in \( \theta \) at an decreasing rate. This assumes that the more pollution we have abated, the harder it is to further abate, i.e. the marginal impact of the abatement on the pollution is smaller. This is consistent with what we would expect, because we run out of low-hanging fruit in our production process after some point.

Abatement costs and regulation differ across industries. As Levinson points out, \( \alpha \) (pollution intensity of a production process) is different for each industry. We can express this as \( \alpha(\eta) \), where \( \eta \), the sector index, is a variable on a continuum from 0 to 1, where \( \eta=0 \) is the least pollution-intensive firm and \( \eta=1 \) is the most pollution-intensive firm.\(^2\) Then, \( y(\eta) = z(\eta)^{\alpha(\eta)} f(K(\eta), L(\eta))^{1-\alpha(\eta)} \) and \( \alpha'(\eta) > 0 \), so that \( \alpha(\eta) \) is increasing in \( \eta \). In the above expression, \( z(\eta) \) is the amount of greenhouse gas emitted in sector \( \eta \), \( y(\eta) \) is production in sector \( \eta \), and \( K(\eta) \) and \( L(\eta) \) are factor use in sector \( \eta \). To interpret the equation for \( y(\eta) \), note that we have written output as a Cobb-Douglas function of pollution and other factors of production.

**IV. Simplifying Assumptions:**

I adopt the following simplifying assumptions:

1. Assume that the Pollution Haven Hypothesis is only a foreign direct investment effect.\(^3\) This means that firms from the home country decide whether to pollute at home or abroad. They do not go out of business, but rather simply move their production to a foreign country. They still sell their goods to the same market.
2. The only pollutant is a composite gas good \( z \). We define this to be a gas that contributes to global warming and affects all citizens of the world equally. Thus, the detrimental effects of a unit of \( z \) emitted in one country are shared among all countries uniformly.\(^4\) All countries have the same disutility of \( z \) and no countries dispute the detrimental effects of \( z \); further, \( z \) follows a deterministic process, so there is no uncertainty over the effects of \( z \).
3. Intermediate goods and re-exports do not exist. All goods are final goods.
4. There are no shipment costs.

\(^2\)From now on, I depart from Levinson significantly.

\(^3\) Winchester points out that since climate policies in developed countries that raise fossil fuel prices simultaneously lower fossil fuel prices in countries without fossil fuel price restrictions. This results in increased energy consumption in other countries.

\(^4\) This is a very simplistic assumption. Consider the island nations, which may completely disappear when the ocean level rises. This assumption says that the residents of these nations care the same about global warming as do the farmers of the US Midwest, where crop yields may actually increase.

\(^5\) Note that this assumption implies that the United States believes in global warming!
5. Following Levinson, the model is partial equilibrium, in that factor prices and environmental taxes are taken as constants. Consumers spend a constant fraction of their income on goods from each firm, which means that demand for each good is perfectly inelastic.

6. Products sell for the same price in every country. Factors of production $K$ and $L$ are homogeneous and not mobile. $L$ is the total population in the home country. This assumption means that comparative advantage in the $f$-good is driven by initial endowments in the countries. This assumption builds on the work of Heckscher (1919) and Ohlin (1933). We will see in section XI why this is a necessary assumption.

V. Production Costs and firm location

Trade is determined by production costs in a country. Because the marginal cost of production must equal the marginal product, the unit cost of producing $y$ is:

$$ c = \alpha^{-\alpha}(1 - \alpha)^{-(1- \alpha)} \tau^\alpha \left(c^f\right)^{1-\alpha} $$

(6)

Where $\tau$ is a pollution tax and $c^f$ is the cost of producing $f$.

I have assumed above, unlike Levinson, that each industry contains one firm and produces one good. This is not a particularly restrictive assumption since if we wanted to model an industry that produced more than one good, we could just use two different $\eta$'s for the two goods (of course, this ignores considerations of economies of scope).

Now, to make things concrete, we will posit two players: home and foreign. I use these players because size of country will have a bearing on the optimal amount of abatement that a country chooses. We will denote the costs of producing $y$ in industry $\eta$ at home by $c(\eta)$

$$ c(\eta) = \alpha(\eta)^{-\alpha(\eta)}(1 - \alpha(\eta))^{-(1- \alpha(\eta))} \tau^{\alpha(\eta)} \left(c^f\right)^{1-\alpha(\eta)} $$

(7)

Similarly, the cost of producing $y$ abroad is:

$$ c(\eta)^* = \alpha(\eta)^{-\alpha(\eta)}(1 - \alpha(\eta))^{-(1- \alpha(\eta))} \tau^{\alpha(\eta)} \left(c^f\right)^{1-\alpha(\eta)} $$

(8)

An industry is located at home if

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6 This is a standard result that can be found in Varian (1992)

7 Dividing $c(\eta)$ by $c(\eta)^*$ gives us the ratio of production costs including the taxes:

$$ \frac{c(\eta)}{c(\eta)^*} = \frac{\alpha(\eta)^{-\alpha(\eta)}(1 - \alpha(\eta))^{-(1- \alpha(\eta))}}{\alpha(\eta)^{-\alpha(\eta)}(1 - \alpha(\eta))^{-(1- \alpha(\eta))}} \frac{\tau^{\alpha(\eta)}}{\tau^{\alpha(\eta)}} \frac{\left(c^f\right)^{1-\alpha(\eta)}}{\left(c^f\right)^{1-\alpha(\eta)}} $$

Which means that

$$ \frac{c(\eta)}{c(\eta)^*} = \left(\frac{\tau}{\tau^*}\right)^{\alpha(\eta)} \left(\frac{c^f}{c^f}\right)^{1-\alpha(\eta)} $$
\[
\frac{c_f}{c^*f} \leq \left(\frac{\tau^*}{\tau}\right)^{\frac{\alpha(\eta)}{1-\alpha(\eta)}} \equiv \gamma
\]

Where \(c_f\) is the unit cost of producing one unit of \(f\) using labor and capital, \(c^*f\) is the unit cost of producing one unit of \(f\) in the foreign country, and \(\tau\) and \(\tau^*\) are the cost of emitting one unit of \(g\) in the home and foreign countries, respectively. Notice that firms care only about cost differentials, and that, as Levinson points out, \(\frac{c_f}{c^*f}\) is completely independent of \(\eta\), since it is a ratio of costs of non-polluting factors in the industry. We have assumed that for a particular industry \(\eta\), \(\alpha(\eta)\) does not vary between the two countries. That is, while the costs of production of \(f\) may vary in the country (wage and rent to capital may vary, because in the Cobb-Douglas case they depend on the amount of labor and capital), the intensity of pollution in the industry does not.

We can say this because we know that items are shipped effortlessly and sell for the same price in every country, so production costs are the only relevant variable. This is consistent with the PHH in that an increase in \(\alpha\) makes a sector more pollution intensive and decreases \(\gamma\), thus increasing the likelihood that the good will be produced abroad.

Below we graph \(\gamma\) and \(\eta\):\(^{8}\)

This ratio = 1 when \(\frac{c_f}{c^*f} = \left(\frac{\tau^*}{\tau}\right)^{\frac{\alpha(\eta)}{1-\alpha(\eta)}} \equiv \gamma\), which means the ratio is <1 (and thus it makes sense to produce at home) when: \(\frac{c_f}{c^*f} \leq \left(\frac{\tau^*}{\tau}\right)^{\frac{\alpha(\eta)}{1-\alpha(\eta)}} \equiv \gamma\).

\(^{8}\) The \(\gamma\) function meets the x-axis when the foreign tax is 0, since in that case, all business flocks to the foreign country. It touches the y-axis when the foreign and domestic taxes are equal, since in that case, the foreign country cannot attract any business.
We solve for the last firm that will leave and obtain the efficiency equilibrium condition:

\[
\alpha(j) = \frac{\ln\left(\frac{c^j}{c^{j+1}}\right)}{\ln\left(\frac{c^j}{c^{j+1}}\right) + \ln\left(\frac{T^*}{T}\right)}
\]  

(10)

For simplification, assume that \(\alpha(j) = j\). This is plausible since \(\alpha'(\eta) > 0\), i.e. the higher the pollution intensity of the firm, the higher the Cobb-Douglas coefficient for pollution in the production process.

So, an expression for \(j\) is given by:

\[
j = \frac{\ln\left(\frac{c^j}{c^{j+1}}\right)}{\ln\left(\frac{c^j}{c^{j+1}}\right) + \ln\left(\frac{T^*}{T}\right)}
\]  

(11)

VI. Weitzman’s Utility function

Martin Weitzman (2009) defines country utility for global warming as follows:

\[
U = U(C) = U(C^*, T)
\]  

(12)

Where:

- \(C^*\): potential consumption (consumption under scenario of no global warming, \(T=0\))
- \(C\): realized consumption
- \(T\): Temperature

The signs of the variables and partial derivatives are:

\(C^* > 0, U > 0, U_C > 0, U_{CC} < 0, U_T < 0, U_{TT} < 0\)

Weitzman explains that \(T\) influences \(U\) for a given \(C^*\) by diminishing the “effectiveness” of \(C^*\) in producing a “welfare equivalent” \(C\). In other words, positive \(T\) negatively affects the transmission of potential consumption into consumption \(C\).
VII. My Utility function:

Since the utility function is the government’s utility function, my utility function departs significantly from Weitzman’s. The government, wishing to be re-elected and controlled by powerful political lobbyists, cares only about domestic output. Politics puts pressure on the government to keep production in the country, and thus the government wants to minimize $1-j$, which is the number of firms/industries that are outsourced.

I depart from Weitzman by introducing a disutility of temperature function, $\zeta$. I model a country’s utility as

$$U = U(Y, \zeta(T))$$

(13)

Where:

$Y > 0, U > 0, U_Y > 0, U_{YT} < 0, \zeta(T) > 0$

I express $\zeta$ in utility units, and hence write this as:

$$U = U(Y) - \zeta(T)$$

(14)

(Note that I am not specifying the derivatives of $\zeta(T)$ since I will explore two cases.)

Assume that government utility is linear in output. Then, we can express this as:

$$U = Y(j) - \zeta(T)$$

(15)

The disutility function is now in units of output. It could be the case that, as in Weitzman’s model, temperature has a negative effect on output, thus dampening potential output. If we express some “realized output” value by $\bar{Y}(j)$, we can write:

$$\bar{Y}(j) = Y(j) - \zeta(T)$$

(16)

And thus

$$U = \bar{Y}(j) = Y(j) - \zeta(T)$$

(17)

Note that, similar to Weitzman’s approach, $Y(j)$ gives output of industries if there is no disutility from pollution. $Y(1)$ gives output if $j = 1$, that is, if there is no industry flight due to taxation.

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9 My utility function is regarded as the ruling agent’s maximand, rather than the principal’s social welfare function.

10 I have implicitly used a strong separability assumption. Intuitively, I am eliminating the possibility that disutility from global warming $\zeta(T)$ interacts with output. I have done this to simplify the problem, but many cases could arise in which this assumption invalidates my analysis. One important case would be that, instead of “dampening” output directly, a change in $z$ affects capital only, i.e. an increase in $z$ causes immediate capital depreciation. This is not implausible in the least: it is commonly accepted that coastal cities will be physically damaged by global warming. Thankfully, politicians generally operate within a short political career and thus mostly take $T$ as given.
Another interpretation of our disutility function is that the government is somehow motivated to reduce emissions through political pressures. In this case, $\zeta(T)$ does not have to decrease output directly, and can be any function of $T$.

**VIII. Temperature function**

Now, we analyze $T$, which is a function of emissions. The relationship between temperature and pollution emissions is crucial to our model. Liffman (2010) plotted the observed linear correlation between atmospheric $CO_2$ concentration in parts per million (NAO0Web) and average Global Temperature in degrees Celsius (NASAWeb) between 1959 and 2009 which I have reproduced below.

This motivates us to model temperature ($T$) and emissions ($Z$) in a linear fashion:

$$T = \alpha + \beta Z$$

We will normalize, simplifying this to:

$$T = Z$$

Our country is concerned with *global* emissions and thus cares about emissions taking place in both Home and Foreign, so when firms move abroad, their emissions still contribute to $T$ in the same way that the emissions of the firms that stay at home do. We denote $T = Z = \int_0^1 z \, d\eta$ as the total pollution of all firms that could possibly locate in the home country. Then, we can write $Z = Z_1 + Z_2$ where $Z_1 = \int_0^1 z(\tau, \eta) \, d\eta$ is the pollution from the industries that stay at home and $Z_2 = \int_1^\infty z(\tau^*, \eta) \, d\eta$ is the pollution from the industries whose products are produced abroad because of tax policy.
IX. Solving for the Optimal Tax

The government’s problem is to maximize a social welfare function $U = Y - \zeta$ using a tax that will decrease emissions, $Z$. As we will see, this is a complicated endeavor, and the problem needs to be recalculated frequently because the maximization problem will be different when the ratios $\frac{e^y}{c^y}$ and $\frac{r^t}{r}$ change. Thus, the government exhausts all of its tax revenue in hiring economists to figure out the optimal tax.

To solve for the optimal tax, we differentiate the utility function:

$$\frac{\partial U}{\partial \tau} = \frac{\partial Y}{\partial \tau} - \frac{\partial \zeta}{\partial \tau} \quad (20)$$

Where $Y = \int_0^j y \, d\eta$ is domestic output, $\zeta = BT$ is a disutility function with parameter $B$.

Now, for the first term,

$$\frac{\partial Y}{\partial \tau} = \frac{\partial}{\partial \tau} \int_0^j y \, d\eta$$

$$= \left( \frac{\partial}{\partial j} \int_0^j y \, d\eta \right) \frac{\partial j}{\partial \tau}$$

$$= \left( \frac{\partial}{\partial j} \int_0^j z^a f^{1-a} \, d\eta \right) \frac{\partial j}{\partial \tau}$$

By Leibnitz’s Rule,

$$= \left( \int_0^j \frac{\partial}{\partial j} z^a f^{1-a} \, d\eta + z^j f^{-j} \frac{\partial j}{\partial j} - z^0 f^1 * 0 \right) \frac{\partial j}{\partial \tau}$$

And since $z^a f^{1-a}$ does not depend on $j$, this becomes:

$$= \left( 0 + z^j f^{-j} * 1 - 0 \right) \frac{\partial j}{\partial \tau}$$

$$= \left( z^j f^{-j} \right) \frac{\partial j}{\partial \tau} \quad (21)$$
And for the second term,

\[ \frac{\partial \zeta}{\partial \tau} = \frac{\partial BT}{\partial \tau} = B \frac{\partial Z}{\partial \tau} \]

\[ = B \frac{\partial Z \partial j}{\partial j \partial \tau} \]

\[ = B \frac{\partial (\int_0^1 z(\tau, \eta) \, d\eta + \int_j^1 z(\tau^*, \eta) \, d\eta) \, \partial j}{\partial \tau} \]

By the Second Fundamental Theorem of Calculus, (21) becomes:

\[ \frac{\partial \zeta}{\partial \tau} = B [z(\tau, j) - z(\tau^*, j)] \frac{\partial j}{\partial \tau} \]  

(23)

Now note that because of the fact that for every \( \eta \) the production function is the same in the two countries, it is proven in the appendix that \( z(\tau^*, \eta) \phi = z(\tau, \eta) \), where \( \phi = \frac{\tau^*}{\tau} \).

So, we have that:

\[ \frac{\partial \zeta}{\partial \tau} = B z(\tau, j) [1 - z(\tau^*, j)] \frac{\partial j}{\partial \tau} \]

\[ = B z(\tau, j) \left[ 1 - \frac{\tau}{\tau^*} \right] \frac{\partial j}{\partial \tau} \]

\[ = B z(\tau, j) \left[ 1 - \frac{1}{\phi} \right] \frac{\partial j}{\partial \tau} \]

(24)

Then,

\[ \frac{\partial U}{\partial \tau} = \left( z^j f^{1-j} \right) \frac{\partial j}{\partial \tau} - B z(\tau, j) \left[ 1 - \frac{1}{\phi} \right] \frac{\partial j}{\partial \tau} \]

So social welfare is maximized when
Which implies that

\[0 = (z^j f^{1-j}) \frac{\partial j}{\partial \tau} - Bz(\tau, j) \left[ 1 - \frac{1}{\phi} \right] \frac{\partial j}{\partial \tau}\]  \hspace{1cm} (25)

And now, note that for the tax to be optimal, the government must take into account the firm’s reaction to the tax. That is, the government must recognize that the firm will set the tax equal to the marginal product of pollution. Thus, for firms under perfect competition, we have that:

\[(1 - \alpha)z = \tau z\]
\[\alpha z = fc^f\]

Then, \[\frac{f}{z} = \frac{\alpha z}{\tau z} = \frac{\alpha}{1 - \alpha}\]

Therefore,

\[0 = \left( \frac{j}{z} \right)^{1-j} - B \left[ 1 - \frac{1}{\phi} \right]
= \left( \frac{\alpha}{(1 - \alpha) c^f} \right)^{1-j} - B \left[ 1 - \frac{1}{\phi} \right]\]

Thus,

\[\frac{\alpha}{(1 - \alpha) c^f} = \left( B \left[ 1 - \frac{1}{\phi} \right] \right)^{\frac{1}{1-j}}\]
And so,

\[
\tau = \frac{(1 - j) c^f}{j} \left( B \left[ 1 - \frac{1}{\phi} \right] \right)^{\frac{1}{1-j}}
\]  

(27)

We can see already that our optimal tax is going to depend negatively on \( j \), the number of firms that will leave the country when the tax is erected. This makes sense because the government will want to lower tax if it knows that many firms will leave the country. The tax will depend positively on our disutility coefficient, which makes sense because if the disutility coefficient is very high, the government will prefer to tax at a higher rate to prevent the global temperature from rising.

To simplify this expression, we need to rewrite (11) to get:

\[
\ln \left( \frac{\frac{c^f}{c^{f^*}}}{\phi} \right) = \ln \left( \frac{\frac{c^f}{c^{f^*}}}{\phi} \right) - \ln \left( \frac{\frac{c^f}{c^{f^*}}}{\phi} \right)
\]

We will refer to \( \frac{c^f}{c^{f^*}} \) as \( \lambda \). Then, \( j = \frac{\ln(\lambda)}{\ln(\lambda) - \ln(\phi)} \)

And note that

\[
\frac{1}{1 - j} = \frac{1}{1 - \ln \left( \frac{c^f}{c^{f^*}} \right)} \left( \frac{c^f}{c^{f^*}} \right) - \ln(\phi)
\]

\[
= \frac{\ln \left( \frac{c^f}{c^{f^*}} \right) - \ln(\phi)}{-\ln(\phi)}
\]

\[
= \frac{\ln(\phi) - \ln(\lambda)}{\ln(\phi)}
\]

(29)

And

\[
\frac{j}{1 - j} = \frac{\ln(\lambda)}{\ln(\lambda) - \ln(\phi)} \frac{\ln(\phi) - \ln(\lambda)}{\ln(\phi)} = \frac{-\ln(\lambda)}{\ln(\phi)}
\]

(30)
Hence the tax that maximizes utility is given by:

\[
\tau_{\text{optimal}} = \left( \frac{-\ln(\phi)}{\ln(\lambda)} \right) c_f \left( B \left[ 1 - \frac{1}{\phi} \right] \right) \frac{\ln(\phi) - \ln(\lambda)}{\ln(\phi)}
\]  (31)

It is trivial to show that the second order condition holds.\(^{11}\)

1. **Classic case**
   \[
   \phi = \frac{r^*}{r} < 1 \quad \text{and} \quad \lambda = \frac{c_f}{c_f^*} > 1
   \]

   This is the typical case where the home country is an advanced economy and the foreign economy is a more competitive economy with lower unit costs; thus, under no taxes, the foreign economy has a comparative advantage in producing the \(f\) good.

   We assume all costs and taxes are positive.\(^{12}\) We see that \(\ln(\phi) < 0, 1 - \frac{1}{\phi} < 0, \phi < \lambda, 0 < \frac{\ln(\phi) - \ln(\lambda)}{\ln(\phi)} < 1\) and so all else equal, we have that \(\frac{\partial \tau_{\text{optimal}}}{\partial B} > 0\) and \(\frac{\partial \tau_{\text{optimal}}}{\partial c_f} < 0\). This means that if the “disutility” that the country gets from temperature, \(B\), increases, then the government should increase the tax. Also, if the costs of producing the \(f\) good go up, we should decrease the tax. This is very non-intuitive: it is saying that we must tax our most efficient sectors! Another way to put this is that if costs are relatively high, relative inefficiency in production is doing the taxing job already. Note that crucially, we are holding \(\lambda\) fixed in this analysis. So, if the cost of producing the \(f\) good goes up, we need to decrease the tax to make sure that domestic producers are still competitive on world markets.

2. **Case of country with low taxes and low production costs**
   \[
   \phi = \frac{r^*}{r} > 1 \quad \text{and} \quad \lambda = \frac{c_f}{c_f^*} < 1
   \]

   In this case we see that

   \[
   \frac{\partial \tau_{\text{optimal}}}{\partial B} > 0 \quad \text{and} \quad \frac{\partial \tau_{\text{optimal}}}{\partial c_f} > 0.
   \]

   That is, the government should tax pollution more when the disutility parameter \(B\) goes up, and should tax pollution more when costs of producing the \(f\) good increase, so long as costs are still below that of the foreign country. This makes sense because if the costs are still below those of the foreign country, the home country needs to tax as much as it can

\(^{11}\) Just kidding! See Appendix for the derivation.

\(^{12}\) This is an unreasonable assumption if governments care about the environment very much, and thus want to subsidize (use a negative tax for) low pollution industries.
because of the disutility effect from pollution, but not so much that it loses its comparative advantage.

3. Case of country with low taxes and high production costs of the f good

Assume \( \phi = \frac{r^*}{T} > 1 \) and \( \lambda = \frac{c_f}{c_T} > 1 \). Then, \( \frac{-\ln(\phi)}{\ln(\lambda)} < 0 \), \( 1 - \frac{1}{\phi} > 0 \). Thus, we have two subcases, both with the same conclusion:

1. \( 0 < \frac{\ln(\phi) - \ln(\lambda)}{\ln(\phi)} < 1 \) if \( \lambda < \phi \)

In this case, \( \frac{\partial \tau_{optimal}}{\partial B} > 0 \) and \( \frac{\partial \tau_{optimal}}{\partial c_f} < 0 \)

2. \( \frac{\ln(\phi) - \ln(\lambda)}{\ln(\phi)} = 1 - \frac{\ln(\lambda)}{\ln(\phi)} < 0 \) if \( \lambda \geq \phi \)

In this case, \( \frac{\partial \tau_{optimal}}{\partial B} > 0 \) and \( \frac{\partial \tau_{optimal}}{\partial c_f} < 0 \)

The analysis in these two cases is the same as that of the classic case.

4. Case of country with high taxes and low production costs of the f good

Assume \( \phi = \frac{r^*}{T} < 1 \) and \( \lambda = \frac{c_f}{c_T} < 1 \). Then, note that \( \ln(\phi) < 0 \) and \( 1 - \frac{1}{\phi} < 0 \).

Again, we have two sub-cases:

1. \( \frac{\ln(\phi) - \ln(\lambda)}{\ln(\phi)} = 1 - \frac{\ln(\lambda)}{\ln(\phi)} < -1 \) if \( \lambda < \phi \)

In this case, since \( \frac{-\ln(\phi)}{\ln(\lambda)} < 0 \) and \( 1 - \frac{1}{\phi} < 0 \), the tax depends positively on \( B \) and \( c_f \). This tells us that when our comparative advantage in producing the f good is strong enough, we can tax without worry, as long as we are still in this range. The SOC are not met for this case; this is a corner solution, as we would expect. An interesting question is whether or not there are goods or countries in the real world for which this would hold. A possible candidate is oil production: oil production taxes are sometimes levied on producers, but do not seem to cause firms to relocate, since not every location is a viable production site.

2. \( \frac{\ln(\phi) - \ln(\lambda)}{\ln(\phi)} = 1 - \frac{\ln(\lambda)}{\ln(\phi)} \epsilon (-1, 0) \) if \( \lambda \geq \phi \)

In this case, we get an imaginary solution for \( \tau_{optimal} \), and again we have a case in which the SOC are not met.

X. Does size matter?

Now we move to the question of whether or not country size will have an effect on our results. Define a variable \( \theta = \frac{L}{L^* + L} \) where \( L \) is the population of the home country and \( L^* \)
is the population of the foreign country (As explained in the assumptions, these are both exogenous and not mobile between countries). Then, $\theta$ is a country’s fraction of the world. If the government cares about the people living in its country, and presumably does not care about the people living in the other country, it should be more concerned with its pollution when the country size is large than when it is small, because the disutility per person $\frac{BZ}{L}$ will be high. Thus, we see the need for a new formulation for our disutility function:

$$\zeta(T, \theta) = \theta BZ$$

(25)

Because $\theta$ is exogenous, this can be written as

$$\zeta = \tilde{B} Z$$

(26)

where $\tilde{B} = \theta B$.

Now, our optimal tax is:

$$\tau_{optimal} = \left( -\frac{\ln(\phi)}{\ln(\lambda)} \right) c^f \left( \frac{\ln(\phi) - \ln(\lambda)}{\ln(\phi)} \right)$$

(27)

which is less than the one we found before for cases 1, 2, and 3 since $\theta < 1$. For all three of these cases, $\frac{\partial \tau_{optimal}}{\partial \tilde{B}} > 0$. This means that as the population of a country grows compared with the size of the world population, the optimal tax grows because the government cares more about emissions. This is consistent with the Coase Theorem, since when the country is larger, it internalizes more of the pollution and thus has an incentive to pollute less.

**XI. Relaxing the immobility assumption**

The assumption that $K$ and $L$ are immobile (Assumption 6) is necessary for our analysis because it means that the cost of producing the $f$ good differs in the two countries. If the cost of producing the $f$ good in the two countries did not differ, then no country would have a comparative advantage in producing the $f$ good. Therefore, comparative advantages in production would come only from differences in the pollution tax across countries, and all industries would produce in either one country or the other (specifically, the country with the lower pollution tax).

Importantly, the assumption that $K$ is mobile and $L$ is immobile is not enough. For a concrete example, suppose $K$ is mobile and $L$ is immobile. Then, since production of the $f$ good is CRS, and thus competitive, we see that profits are 0 for the $f$ good, and thus the revenues equal costs for the $f$ good for any industry in both countries. Since a producer could produce in either country, if he wants to produce $f$ units that are sold at a cost of $c^f$ he could do so in the domestic country:
\[ fc^f = rK + wL \]

Or in the foreign country:
\[ fc^*f = r^*K^* + w^*L^* \]

So
\[
\begin{align*}
  c^f &= \frac{rK + wL}{f} \\
  c^*f &= \frac{r^*K^* + w^*L^*}{f}
\end{align*}
\]  

(28)

Let's examine the cost ratio \( \frac{c^f}{c^*f} \). Canceling out the \( f \) terms, we get:
\[
\frac{c^f}{c^*f} = \frac{rK + wL}{r^*K^* + w^*L^*}
\]

Dividing the numerator and denominator by \( rK \) we see that
\[
\begin{align*}
  \frac{c^f}{c^*f} &= \frac{1 + \frac{wL}{rK}}{\frac{r^*K^* + w^*L^*}{rK} + \frac{r^*K^*}{rK}} \\
  &= \frac{1 + \frac{wL}{rK}}{1 + \frac{wL}{rK} \frac{r^*K^*}{rK}} \\
  &= \frac{r^*K^*}{rK} \left( 1 + \frac{w^*L^*}{r^*K^*} \right)
\end{align*}
\]

Since the production technology is the same for a given industry in the two countries, the factor share ratios are equal:
\[
\frac{wL}{rK} = \frac{w^*L^*}{r^*K^*} = 1
\]

(29)

So,
\[
\frac{c^f}{c^*f} = \frac{rK}{r^*K^*}
\]

Since capital is mobile, capital will flock to the country that pays the highest rent, and thus it will be the case that eventually either all of the capital is located in one country, or:
\[ rK = r^*K^* \text{ with } r = r^* \]
So, combining (28) and (29),

\[
\frac{c^f}{c^{*f}} = \frac{rK}{r^*K^*} = 1
\]

Then, we have shown that no country has a comparative advantage in production of the \( f \) good if capital is mobile. Hence, a firm will produce at home if and only if:

\[
1 \leq \left( \frac{\tau^*}{\tau} \right)^{\frac{\alpha(\eta)}{1-\alpha(\eta)}} \equiv \gamma
\]  

(30)

Which is equivalent to the condition that

\[
\tau \leq \tau^*
\]  

(31)

Notice that this is independent of pollution intensity \( \alpha(\eta) \) of the firm. This means that all firms produce in the country with the lowest tax. In this case, the government’s problem is simple: the government always does best to set its tax less than the foreign tax. This is the typical case of a “race to the bottom” in environmental standards, because the foreign government must respond by setting its tax even lower.\(^{13}\)

**XII. Conclusion**

I have found the optimal tax that a country should set given the various cost ratios and tax ratios that it faces. Taxes should increase when disutility from pollution increases. A poor country with low taxes can afford to tax more when its production costs of the \( f \) good increase, whereas a rich country where the costs of producing the \( f \) good are already high cannot. Size does matter: a country with a larger population compared with the world population has a higher tax, other things equal. This is consistent with the Coase Theorem. Lastly, one mobile factor could cause a situation in which it is always optimal for the domestic government to set its tax lower than that of the foreign government, which would likely generate the classic “race to the bottom” in environmental standards.

**XIII. Appendix:**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha(\eta) )</td>
<td>Pollution intensity of a production process</td>
</tr>
<tr>
<td>( B, \hat{B} )</td>
<td>Temperature function coefficients</td>
</tr>
<tr>
<td>( c^f, c^{*f} )</td>
<td>Cost of producing one unit of ( f ) using labor</td>
</tr>
</tbody>
</table>

\(^{13}\) One key assumption behind this result is that factor prices are fixed in this model. Were they not fixed, both factors would need to be mobile in order to achieve this result.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C^*(j)$</td>
<td>Potential consumption</td>
</tr>
<tr>
<td>$f$</td>
<td>CRS function of capital and labor</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Nonlinear function of tax ratio</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Industry, firm, or sector index</td>
</tr>
<tr>
<td>$j$</td>
<td>The most pollution-intensive industry that stays at home</td>
</tr>
<tr>
<td>$K(\eta), L(\eta)$</td>
<td>Factors used in sector $\eta$</td>
</tr>
<tr>
<td>$K^<em>, L^</em>$</td>
<td>Foreign capital and labor</td>
</tr>
<tr>
<td>$K, L$</td>
<td>Home capital and labor</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Production cost ratio</td>
</tr>
<tr>
<td>$n$</td>
<td>The total number of industries that would exist in a country in the case of no abatement. Also denotes the index for the most pollution-intensive industry or firm. In our model, we normalize this to 1.</td>
</tr>
<tr>
<td>$\varphi[\theta]$</td>
<td>Functional form of intensity of abatement</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Pollution tax ratio</td>
</tr>
<tr>
<td>$r, r^*$</td>
<td>Rent to capital in home and foreign country, respectively</td>
</tr>
<tr>
<td>$S$</td>
<td>Size of country as a fraction of the world</td>
</tr>
<tr>
<td>$T$</td>
<td>Temperature</td>
</tr>
<tr>
<td>$\tau$</td>
<td>Pollution tax</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Fraction of the factors used for abatement</td>
</tr>
<tr>
<td>$U$</td>
<td>Utility of a country</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>Disutility from temperature increases in consumption units</td>
</tr>
<tr>
<td>$Y$</td>
<td>Output</td>
</tr>
<tr>
<td>$y(\eta)$</td>
<td>Output in industry $\eta$</td>
</tr>
<tr>
<td>$\mathcal{Y}(j)$</td>
<td>Realized output</td>
</tr>
<tr>
<td>$z(\eta)$</td>
<td>Greenhouse gas pollution emitted in sector $\eta$</td>
</tr>
<tr>
<td>$Z$</td>
<td>Greenhouse gas pollution in all sectors</td>
</tr>
</tbody>
</table>

**Second Order conditions:**

First, we calculate:

$$
\frac{\partial j}{\partial \tau} = \left( \frac{- \ln(\lambda)}{\ln(\lambda) - \ln(\phi)} \right)^{-1} \frac{\tau^*}{\phi \tau^2} 
$$

$$
= \left( \frac{j}{\ln(\lambda) - \ln(\phi)} \right)^{1} \frac{1}{\tau}
$$
\[
\frac{-j}{\tau} = \left( \frac{-1}{\ln(\phi) - \ln(\lambda)} \right) \quad (32)
\]

Note that since \( j \) and \( \tau \) are always positive, in the classic case (\( = \frac{\tau^*}{\tau} < 1 \) and \( \lambda = \frac{c^f}{c^{f^*}} > 1 \)), we have that this is positive. Keep in mind that this is evaluated at the optimal \( \tau \), which is why if \( j \) goes up, we should increase \( \tau \) to maximize utility. In essence, when the government implements the optimal tax they are indirectly choosing an optimal \( j \); when \( j \) increases, the government has the liberty to tax more while still above the optimal \( j \).

Now,

\[
\frac{\partial^2 j}{\partial \tau^2} = -\frac{j}{\tau} \left( \frac{-1}{(\ln(\phi) - \ln(\lambda))^2} \right) + \left( \frac{j}{\tau^2} \right) \left( \frac{1}{(\ln(\phi) - \ln(\lambda))} - 1 \right)
\]

\[
= \frac{\partial j}{\partial \tau} \frac{1}{\tau} \left( \frac{1}{(\ln(\phi) - \ln(\lambda))} - 1 \right) \quad (33)
\]

Also, note that \( \frac{\partial^2 (\tau, j)}{\partial \tau} < 0 \), since for any firm \( z \) with a pollution intensity \( j \), when the tax rises, in order to maximize profits the firm will set

\[
\tau z(\tau, j) = jy \quad (34)
\]

This means that:

\[
z(\tau, j) = \frac{jy}{\tau} \quad (35)
\]

Since we hold \( j \) constant when taking the derivative,

\[
\frac{\partial z(\tau, j)}{\partial \tau} = - \frac{jy(\tau, j)}{\tau^2} + \frac{j \frac{\partial y(\tau, j)}{\partial \tau}}{\tau} \quad (36)
\]

And we know that \( \frac{\partial y(\tau, j)}{\partial \tau} < 0 \), so \( \frac{\partial^2 (\tau, j)}{\partial \tau} < 0 \)
Now, our task is to differentiate:

$$\frac{\partial U}{\partial \tau} = (z^j f^{1-j}) \frac{\partial j}{\partial \tau} - Bz(\tau, j) \left[ 1 - \frac{1}{\phi} \right] \frac{\partial j}{\partial \tau}$$  \hspace{1cm} \text{(37)}$$

Which we rewrite as:

$$\frac{\partial U}{\partial \tau} = \frac{\partial j}{\partial \tau} \left\{ (z^j f^{1-j}) - Bz(\tau, j) \left[ 1 - \frac{1}{\phi} \right] \right\}$$

So

$$\frac{\partial^2 U}{\partial \tau^2} = \frac{\partial j}{\partial \tau} \frac{\partial}{\partial \tau} \left\{ (z^j f^{1-j}) - Bz(\tau, j) \left[ 1 - \frac{1}{\phi} \right] \right\} + \frac{\partial^2 j}{\partial \tau^2} \left\{ (z^j f^{1-j}) - Bz(\tau, j) \left[ 1 - \frac{1}{\phi} \right] \right\}$$  \hspace{1cm} \text{(38)}$$

We plug in our expression for $\frac{\partial^2 j}{\partial \tau^2}$ to get

$$\frac{\partial^2 U}{\partial \tau^2} = \frac{\partial j}{\partial \tau} \left[ \frac{\partial}{\partial \tau} \left\{ (z^j f^{1-j}) - Bz(\tau, j) \left[ 1 - \frac{1}{\phi} \right] \right\} \right]$$

$$+ \frac{1}{\tau} \frac{1}{\ln(\phi) - \ln(\lambda)} \left\{ (z^j f^{1-j}) - Bz(\tau, j) \left[ 1 - \frac{1}{\phi} \right] \right\}$$

$$= \frac{\partial j}{\partial \tau} \left[ \frac{\partial}{\partial \tau} \left\{ (z^j f^{1-j}) - Bz(\tau, j) \left[ 1 - \frac{1}{\phi} \right] \right\} \right]$$

$$+ \frac{1}{\tau} \frac{1}{\ln(\phi) - \ln(\lambda)} \left\{ (z^j f^{1-j}) - Bz(\tau, j) \left[ 1 - \frac{1}{\phi} \right] \right\}$$

$$= \frac{\partial j}{\partial \tau} \left[ \frac{\partial}{\partial \tau} \left\{ (z^j f^{1-j}) - Bz(\tau, j) \left[ 1 - \frac{1}{\phi} \right] \right\} \right]$$

$$+ \frac{1}{\tau} \frac{1}{\ln(\phi) - \ln(\lambda)} \left\{ (z^j f^{1-j}) - Bz(\tau, j) \left[ 1 - \frac{1}{\phi} \right] \right\}$$

$$= \frac{\partial j}{\partial \tau} \left[ \frac{\partial}{\partial \tau} \left\{ (z^j f^{1-j}) - Bz(\tau, j) \left[ 1 - \frac{1}{\phi} \right] \right\} \right]$$

$$+ \frac{1}{\tau} \frac{1}{\ln(\phi) - \ln(\lambda)} \left\{ (z^j f^{1-j}) - Bz(\tau, j) \left[ 1 - \frac{1}{\phi} \right] \right\}$$

$$= \frac{\partial j}{\partial \tau} \left[ \frac{\partial}{\partial \tau} \left\{ (z^j f^{1-j}) - Bz(\tau, j) \left[ 1 - \frac{1}{\phi} \right] \right\} \right]$$

$$+ \frac{1}{\tau} \frac{1}{\ln(\phi) - \ln(\lambda)} \left\{ (z^j f^{1-j}) - Bz(\tau, j) \left[ 1 - \frac{1}{\phi} \right] \right\}$$

$$= \frac{\partial j}{\partial \tau} \left[ \frac{\partial}{\partial \tau} \left\{ (z^j f^{1-j}) - Bz(\tau, j) \left[ 1 - \frac{1}{\phi} \right] \right\} \right]$$

$$+ \frac{1}{\tau} \frac{1}{\ln(\phi) - \ln(\lambda)} \left\{ (z^j f^{1-j}) - Bz(\tau, j) \left[ 1 - \frac{1}{\phi} \right] \right\}$$
And notice that \( j(z^{l-1}f^{1-j}) \) is simply a partial derivative of our production function with respect to one of its inputs, and hence by definition is positive.

Rewriting, we have that

\[
\frac{\partial^2 U}{\partial \tau^2} = \frac{\partial j}{\partial \tau} \left[ j(z^{l-1}f^{1-j}) - B \left[ 1 - \frac{1}{\phi} \right] \frac{\partial z}{\partial \tau} + \frac{1}{\tau} \left( \frac{1}{\ln(\phi) - \ln(\lambda)} - 1 \right) \left( z(\tau, j) \left( (z^{l-1}f^{1-j}) - B \left[ 1 - \frac{1}{\phi} \right] \right) \right) \right] \tag{39}
\]

**Classic Case:** \( \phi = \frac{r^*}{\tau} < 1 \) and \( \lambda = \frac{e^f}{\sigma} > 1 \):

\[
\frac{\partial j}{\partial \tau} > 0, \frac{\partial z(\tau, j)}{\partial \tau} < 0, \left[ 1 - \frac{1}{\phi} \right] > 0, \left( j(z^{l-1}f^{1-j}) - B \left[ 1 - \frac{1}{\phi} \right] \right) > 0 \Rightarrow \text{the first term in the brackets is negative}
\]

Notice that

\[
\left( \frac{1}{\ln(\phi) - \ln(\lambda)} - 1 \right) < 0, \frac{1}{\tau} > 0, (z^{l-1}f^{1-j}) > 0, (z^{l-1}f^{1-j}) - B \left[ 1 - \frac{1}{\phi} \right] > 0, \text{ and therefore the second term in brackets is negative. Hence, we have a maximum.}
\]

**Case of country with low taxes and low production costs:** \( \phi = \frac{r^*}{\tau} > 1 \) and \( \lambda = \frac{e^f}{\sigma} < 1 \)

\[
\frac{\partial j}{\partial \tau} < 0, \frac{\partial z(\tau, j)}{\partial \tau} < 0, \left[ 1 - \frac{1}{\phi} \right] > 0, \left( j(z^{l-1}f^{1-j}) - B \left[ 1 - \frac{1}{\phi} \right] \right) > 0 \Rightarrow \text{the first term in the brackets in equation (39) is positive if } j(z^{l-1}f^{1-j}) - B \left[ 1 - \frac{1}{\phi} \right] > 0, \text{ and the second term in brackets is positive if } (z^{l-1}f^{1-j}) > B \left[ 1 - \frac{1}{\phi} \right]. \text{ Notice that this all is equivalent to the condition } Bz > y. \text{ If } Bz > y \text{ for all } \eta, \text{ then we have:}
\]

\[
Bz > z^{l}f^{1-j} \tag{40}
\]

And

\[
B > z^{l-1}f^{1-j}
\]

Also, taking the derivative of both sides of (40), we get:

\[
B > jz^{l-1}f^{1-j}
\]
Clearly this is problematic: we can rewrite the SOC as:

\[
0 = \frac{\partial j}{\partial \tau} \left\{ j \left( z^{j-1} f^{1-j} \right) - B + \frac{1}{\phi} \frac{\partial z}{\partial \tau} \right. \\
\left. + \frac{1}{\tau} \left( \frac{1}{\ln(\phi) - \ln(\lambda)} - 1 \right) \left( z(\tau, j) \left( (z^{j-1} f^{1-j}) - B + \frac{1}{\phi} \right) \right) \right\} 
\]

(41)

Then, if \( Bz > y \) for all \( \eta \), then the optimal \( z = 0 \) and the optimal \( j = 0 \), so this needs to be ruled out.

If we assume \( Bz \leq y \) for \( \eta \leq j \), then we see that the SOC are satisfied.

Case 3 can be analyzed in a similar way to Case 1.

**Proof that the ratio of amounts of pollution used in the two countries is the inverse of the ratio of taxes in the two countries**

Our assumption that consumers spend a fraction of their income on goods from each firm means that a given firm \( \eta \) will produce exactly \( y(\eta) \) units of good \( \eta \) to satisfy demand.

Then, if the industry is located at home,

\[
y(\eta) = z^{\eta} f^{1-\eta} 
\]

(42)

And if the industry is located abroad,

\[
y(\eta) = z^{*\eta} f^{*1-\eta} 
\]

(43)

Then, no matter where the industry is located, it will set the marginal product of pollution equal to the amount it must pay for emitting a unit of pollution:

\[
\eta y(\eta) = \tau z 
\]

(44)

And

\[
\eta y(\eta) = \tau^* z^* 
\]

(45)

Then we see that if industry \( \eta \) needs to produce \( y(\eta) \) units, it will be indifferent between producing in the two countries if

\[
\tau^* z^* = \tau z 
\]

(46)
Which means that

\[
\frac{z}{z^*} = \frac{\tau}{\tau^*} = \phi
\]

And so we have shown that:

\[
z(\tau^*, \eta)\phi = z(\tau, \eta). \tag{47}
\]

**XIV. References**


Grether, J. & N. Mathys, 2008. "Is the World’s Economic Center of Gravity Already in Asia?," Cahiers de Recherches Economiques du DÃ©partement d'EconomÃ trie et d'Economie politique (DEEP) 08.03, Université de Lausanne, Faculté des HEC, DEEP.


