OPTIMAL ORDERING IN SEQUENTIAL ENGLISH AUCTIONS:

A REVENUE-COMPARISON MODEL FOR 18TH CENTURY ART AUCTIONS IN LONDON AND PARIS

AMAAN AMJAD MITHA

PROFESSOR NEIL DE MARCHI, FACULTY ADVISOR

Honors Thesis submitted in partial fulfillment of the requirements for Graduation with Distinction in Economics in Trinity College of Duke University.

Duke University
Durham, North Carolina
2012

ACKNOWLEDGEMENTS

Professor De Marchi, thank you for all of your help, support, and guidance. You've been a fantastic advisor, and your enthusiasm has been inspiring. Professor Van Miegroet, thank you for all of your advice and support, and for introducing me to DALMI and the trailblazing work you're doing; I'm happy to have been a part of it, and I'm excited to see what's in store. Ms. Benedicte Miyamoto-Pavot and Ms. Hilary Coe Smith, thank you both for providing me your data; I know how closely-guarded PhD data is, and I appreciate all that you did to get me access. Professor Roberts, thank you for all of your helpful insights and suggestions, you helped me focus my study and my attention to specific issues. Finally, to my family and my friends: thank you for putting up with the absurd hours and the cantankerousness that naturally accompany an undertaking of this magnitude.

TABLE OF CONTENTS

ABSTRACT

We develop a model based on several auction parameters to test the widely held notion that in a sequential English auction, it is optimal for the seller to arrange the lots in order of decreasing value. We test this model against two datasets of 18th century auctions, one of various auctions from Paris and the other from Christie's sales in London. We find that the Paris data support the claim, while the Christie's data seem to refute the optimal strategy. We also find a rationale for bidders in the Christie's auctions to alter their strategies, accounting for the discrepancy.

JEL Classifications: D44, Z11

Keywords: Auctions, Lot Ordering, Optimal Auction Strategy, English Auction, Sequential Auctions

I. INTRODUCTION

The concept of the auction has existed for several millennia. The first recorded mention of the auction comes from Herodotus, who tells us that auctions occurred as long ago as the 5th century BC. Historically, auctions have been used to sell all manner of objects, from people (wives and slaves) to entire empires; at one point the Praetorian Guard killed the sitting emperor and auctioned off the Roman Empire (McAfee & McMillan, 1987, p. 701; Krishna, 2002, p. 1). Today, auctions are more widely used than ever before, from the sale of physical commodities to art to public resources to debt. Trading securities can even be modeled as a continuous sequential auction of common value goods (Kyle, 1985), hence the use of the term "bid" for a price offer. The benefit of using an auction is that it is an efficient way of determining the value of an object with an inherently indeterminate value. A seller wants to get rid of some object, but does not know how to appropriately assign a price to it. So, if he devises a suitable auction mechanism, he can sell the object to the person who values it most, benefitting both the seller and the buyer.

Today, there are four generally recognized auction formats. These are the following:

(1) Open Ascending Price Auction (English);

(2) Open Descending Price Auction (Dutch);

(3) Sealed-Bid First-Price Auction;

(4) Sealed-Bid Second-Price Auction.

There are several variants on each of these, but they are widely considered the four standard formats.

The English auction is by far the most common format in use today (McAfee & McMillan, 1987, p. 702); when most people imagine an auction, they are thinking of the English form. Essentially, an English auction can be described with the following scenario. An object is put up for auction, and bidders openly offer competing bids. These bids must increase, and the object is finally sold to the last remaining bidder. He must pay the amount at which his final competitor dropped out (Krishna, 2002, p. 2). This is the format generally used in art auctions today, and it is the format of the auctions studied in this paper.

The Dutch auction is the descending price foil to the English auction. The auctioneer starts the bidding at a value far higher than any bidder's valuation, and he steadily decreases the bid until a bidder agrees to buy at that price. This bidder wins the object, and must pay the agreed-upon price. This format is rarely used today, but was used to sell cut flowers in the Netherlands, hence the name. It has also been used in sales of perishable commodities, like fish and tobacco (McAfee & McMillan, 1987, p. 702).

The sealed-bid first-price auction operates as follows: bidders offer their competing bids in sealed envelopes, and whoever offers the highest bid wins; he must then pay his bid (or in the case of a contract, the lowest bid wins). The sealed-bid second-price auction is set up similarly, but instead of paying his own bid, the winner pays the second-highest bid. Sealed-bid first-price auctions are used for such objects as government contracts and mineral rights on public lands (McAfee & McMillan, 1987, p. 702). The sealed-bid second-price auction was originally a theoretical development published by Vickrey (1961), but it has come into use in modern times. For example, on eBay, a bidder can submit his maximum bid, and if it is the highest valuation at the end of the auction, he must pay only $0.01 more than the second-highest bid.

One of the issues that must be considered in auction theory is the way in which a bidder values an object at auction. There are two distinct possibilities, private versus common value. In a private values context, each bidder assigns his own valuation to the object independent of the other bidders' valuations. Private values makes sense when modeling objects that are used solely for consumption. In a common values (or "interdependent" values) setting, the object has some unique value; the reason that an auction is necessary is that nobody actually knows this true value, though each bidder has his own estimate. This is a useful model for goods that can be resold, like securities or plots of land with unknown mineral content (Krishna, 2002, pp. 3-4). In this situation, the bidders' valuations represent estimates of the true value. One result of a common values auction with incomplete information is the so-called winner's curse, in which the bidder with the highest valuation overestimates the value of the object. It can, however, be avoided by using an equilibrium bidding strategy (Krishna, 2002, p. 85).

Unfortunately, art is neither a consumption good nor an investment good; it falls in that murky space somewhere in between. This makes modeling art auctions somewhat more difficult, as theorists must choose ex ante which valuation model they deem appropriate. In the case of this study, it turns out that either valuation model (private or common) offers the same hypothesis: the seller's optimal strategy is to sell objects in a sequential English auction in order of decreasing value. This is a notion supported by both theory and empirical study (see *II. Background and Literature Review*). Tests of this claim have focused solely on recent auctions, despite the fact that institutional auctions have been used in sales of artwork since the 18^{th} century. This study offers a test of the optimal hypothesis using historical records. This allows us to discern whether the claim holds up in auctions that occurred prior to the theoretical developments, confirming that modern empirical agreement with the theory is not a result of auction houses utilizing that same theory. We do this by developing a model that allows us to compare the auction revenues with the degree to which values declined over the course of a specific auction.

II. BACKGROUND AND LITERATURE REVIEW

Since Vickrey's seminal work, "Counterspeculation, Auctions, and Competitive Sealed Tenders, " there has been a steady stream of research in auctions, in the context of both theory and practice. Vickrey's (1961) groundbreaking discovery was the fact that under his model, the Dutch auction was strategically equivalent to the sealed-bid first-price auction, and the English auction was weakly equivalent to the sealed-bid second price auction. This led him to discover that the auctioneer's expected revenue for a first-price sealed-bid auction was the same as the expected revenue for second-price sealed-bid auction, given independent and identically distributed (iid) private-valuations among symmetric bidders facing uniform distributions. He later generalized this finding for symmetric bidders receiving signals from any continuous probability distribution function (Krishna, 2002, p. 28), and this fundamental rule is called the "revenue equivalence" principle.

Riley and Samuelson (1981) further analyzed auctions featuring iid private values for symmetric bidders. They studied a more general set of auctions, what Krishna likes to call a "standard" auction. A standard auction is one in which the bidder with the highest bid wins the object and in which the bidder with the lowest possible valuation must pay nothing. More specifically, given N bidders, valuation x_i and the corresponding expected payment $m(x_i)$ for a bidder i, a standard auction is one in which $m(0) = 0$, and bidder j must win when $\max\{m(x_1), \ldots, m(x_j), \ldots, m(x_N)\} = m(x_j)$. So, the first-price and second-price auctions that Vickrey studied were both standard auctions; the theoretical third-price auction, studied by Kagel and Levin (1993), is also standard, although it is unheard of in practice. Riley and Samuelson (1981) and Myerson (1981), through a different derivation involving auction mechanisms, found that for all of these possible auction types and any others that can be classified as a standard auction, Vickrey's revenue equivalence principle holds. Thus, given the conditions established by Vickrey, an auctioneer earns the same expected revenue from any standard format. An example of a nonstandard auction would be something like a lottery, in which the person who "bids" the most has the highest probability of winning but is not guaranteed to win (Krishna, 2002, p. 29). Here, the revenue equivalence principle breaks down and is no longer consistent.

Since then, there have been several theoretical works relaxing some of Vickrey's original restrictions. In the same work as above, Riley and Samuelson (1981) consider the effects of reserve prices on optimal auction structure. Krishna specifically defines the prerequisites of revenue equivalence in private value models to be independence, risk neutrality, no budget constraints, and symmetry (Krishna, 2002, p. 37). He explores the results of relaxing each constraint one by one, comparing the revenues from different auction types under each possible variation (Krishna, 2002, pp. 38-58).

One limiting issue with Vickrey's work (and all of its derivatives) was that it only considered private value models. Milgrom and Weber (1982) developed a model for interdependent valuations and affiliated signals. Essentially, they defined a bidder's value V_i, a vector $\boldsymbol{X} = (X_1, \ldots, X_N)$ whose components are the N bidders' value signals, and a vector $\boldsymbol{S} = (S_1, \ldots, S_M)$, whose components are M pieces of information that could adjust the bidders' valuations. In their model, V is a function of all the possible valuations and pieces of information, so we have $V_i = U(\mathbf{S}, \boldsymbol{X})$, where $U: \mathbb{R}^{M+N} \to \mathbb{R}_{\geq 0}$. Then, the common value model and the private value model are just specific cases of their more general interdependent-affiliated value model. The common value model is the one for which $M = 1$ and $V_i = S_1 \forall i$, while the private value model occurs when $M = 0$ and $V_i = X_i \forall i$. They showed that the English auction generates at least as much revenue as the second-price sealed-bid auction; the second-price auction in turn generates at least as much revenue as the first-price sealed-bid auction, which is still strategically equivalent to the Dutch auction. According to McAfee and McMillan (1987), the reason the English auction has the highest expected revenue when values are interdependent is because bidders can see not only that other bidders have dropped out, but the specific bids at which they drop out, so the remaining bidders can divine the approximate signal distribution. This tempers the effects of the winner's curse, and thus the expected revenue increases relative to both sealed-bid auction formats.

Since the English auction is the most widely used format today, there have been several studies on its particular properties. Maskin (1992) showed that the English auction, given two bidders and under certain conditions, resulted in efficient allocations. Birulin and Izmalkov (2009) extend Maskin's (1992) findings for the efficiency of English auctions. Whereas Maskin showed that the single-crossing condition was sufficient for an efficient outcome given two bidders, Birulin and Izmalkov developed a generalized single-crossing condition and show that it is a necessary and sufficient condition for an efficient equilibrium with N bidders. The majority of work done by theorists considers only rational agency, i.e. all equilibria are assumed to be Bayesian Nash equilibria. Gonçalves (2008) considers how irrationality could affect the equilibrium in a common value English auction between two bidders. He finds that if only one bidder acts irrationally, the expected price could be either higher or lower than the symmetric equilibrium. If, however, both bidders match irrational strategies, the sale price is always at least as high as the symmetric equilibrium. This is an important new field for theoretical research, as auction houses tend to consider irrational influences when ordering lots in sequential auctions. For example, one important aspect in any auction is the sense of excitement generated by sales of highly valued paintings. For this reason, auction houses tend to put higher valued pieces in the middle of an auction in order to build enthusiasm and anticipation (Beggs & Graddy, 1993, p. 547).

Since the auctions I study are all multi-unit auctions, the effects of holding a sequential auction compared to a single-unit auction are paramount. According to Pitchik (2006), lot order affects the competition for each good being sold, thus in turn affecting the auction's overall revenue. Benoît and Krishna (2001) showed that if heterogeneous common value objects are sold in an open ascending sequential auction with budget-constrained bidders, it is optimal to order them from highest-valued to lowest-valued. Elmaghraby (2003) studied the private value sequential auction of heterogeneous goods and found that under several different cases, an efficient equilibrium can be reached so long as the goods are ordered according to a specific algorithm. Pitchik (2006) studied budget-constrained sealed-bid sequential auctions with private values. She found that if a particular piece is allocated to a strong bidder regardless of lot order, then the auction revenue increases if that piece is placed earlier in the auction. Elkind and Fatima (2007) started with the premise that it would require exponential time to find an algorithm to determine the optimal ordering of lots in a sealed-bid second-price sequential auction. They found, however, that such an algorithm can be derived in polynomial time. This algorithm depends only on the first and second-order statistics of the signal distribution from which values are derived. They also showed that dynamic canceling, i.e. removing objects from auction, can increase the seller's revenue.

We can see that in general, the theoretical models suggest that the optimal strategy on the part of the seller is to order lots in sequential English auctions from highest-valued to lowest-valued, and there have been several empirical studies that support this model. The law of one price suggests that homogeneous goods sold sequentially should fetch the same price. Ashenfelter (1989) studied sequential sales of identical bottles of wine and found that when prices changed over the course of a single auction, they were twice as likely to decrease as they were to increase, an effect he called the "declining price anomaly." He claims that "it is common knowledge among auctioneers that, when identical lots of wine are sold in a single auction, prices are more likely to decline than to increase with later lots" (p. 29). Ashenfelter and Genesove (1992) studied identical condominium sales at auction and found a similar result; lots placed earlier sold for significantly more than later lots. Zulehner (2009) also found support for the declining price anomaly in cattle auctions. Ashenfelter (1989) suggests that it may be due to risk aversion on the part of the bidders, since there is a limited quantity of the homogeneous good (p. 31). McAfee and Vincent (1993) found support for this notion, and established that earlier bids are equal to the expected value of later lots plus a risk premium. Beggs and Graddy (1997), however, disagreed with the notion that the declining price anomaly could be chalked up to risk-averse behavior in all auctions. They claim that there is nothing to stop a bidder from paying more for a later lot should he be risk-averse, and in the case of an open ascending auction, bids are observable and bidders can estimate the value distribution, mitigating the risk (p. 561). They do permit that in the case of a sealed-bid auction, where there is uncertainty in other bidders' valuations, risk aversion could have a significant influence, but it makes no sense in the case of an English auction. They further studied a series of auctions of heterogeneous goods, and they found that the price declined over the course of each auction. Moreover, the sale price relative to the estimated sale price also declined, demonstrating that the declining price anomaly was not limited to sequential auctions of homogeneous goods. Hong *et al.* (2009) studied sequential auctions in what they call a "natural experiment." During auction week in New York City, Christie's and Sotheby's alternate who goes first. Hong *et al.* (2009) found that when the house with more expensive paintings goes first, the sale premium is 25% higher on average than the unconditional mean sale premium. They also found support for revenue-maximization through declining-value lot ordering in sequential sales of heterogeneous goods.

There is clearly a firm theoretical basis supporting an optimal auction design featuring declining prices over the course of a sequential auction, and empirical studies support the notion as well.

III. MODEL

We start with the assumption, laid out in *II. Background and Literature Review*, that ordering lots by declining values is the optimal strategy in a sequential English auction of heterogeneous goods (c.f. Ashenfelter, 1989; Ashenfelter & Genesove, 1992; McAfee & Vincent, 1993; Beggs & Graddy, 1997; Benoît & Krishna, 2001; Elmaghraby, 2003; Pitchik, 2006; Elkind & Fatima, 2007; Zulehner, 2009; Hong *et al.*, 2009). Here, we develop a rather simple model to test the relationship between the relative decline of value in paintings with respect to lot order and the revenues generated in the auction. This is most certainly not a model that includes every possible variable, but merely adjusts for specific factors that are common to all auctions.

Consider an auction A. In any auction, there are several parameters that differentiate it from the other auctions. In this model, we consider the following parameters: the value of the pieces sold, for which we assign a characteristic value μ; the number of lots sold, Λ (this is distinct from the largest lot number, as we shall see later); the year in which the auction took place, Y; the rate, β, at which the values declined with respect to lot position; and the revenue generated, R. So, an auction A can be defined by the vector

$$\boldsymbol{A} = (\mu, \Lambda, Y, \beta, R).$$

In order to compare any two auctions, we have to account for each of these parameters. The features we want to study are auction revenue R with respect to the declination of the values β, so we must first normalize R and β with respect to the other parameters; this way, we are in effect holding all parameters other than R and β constant.

A. Revenue Normalization

First we need to consider how to compare the revenues of different auctions. Clearly, an auction A_l will generate higher nominal revenues than auction A_k under the following conditions, constraining all other parameters to be constant:

(1) The values of the objects sold is higher on average in A_l than in A_k: $\mu_l > \mu_k$;

(2) There are more objects sold in A_l than in A_k: $\Lambda_l > \Lambda_k$;

(3) The price index for the year of A_l is higher than the index for A_k.

The characteristic value is determined from the quality of the items sold in the auction, and it is a value that can be compared linearly between auctions. So, normalizing with respect to value, we can define our value-adjusted revenue $R_{(\mu)}$ as

$$R_{(\mu)} = \frac{R}{\mu}.$$

We can think of this as a relative measure of revenue holding the value of pieces sold constant. (N.B. with regards to notation, a subscript without parentheses refers to a counting index, typically the letters i and j when comparing objects sold within a given auction and k and l when comparing auctions and their parameters; a subscript in parentheses refers to a variable or parameter that is held constant).

Next, let's consider the number of lots sold in the auction. Again, this is a measure that can be compared linearly between auctions, so we normalize for auction size in the same way as we did for value:

$$R_{(\Lambda)} = \frac{R}{\Lambda},$$

so, to hold both value and auction size constant, we have

$$R_{(\mu,\Lambda)} = \frac{R}{\mu\Lambda}.$$

This gives us a measure of revenue per item, holding value constant.

Now, let's consider the year in which the auction took place. The price index for a given year gives us an aggregate measure for any effects that could influence prices in a given year. Thus, we define a function $\psi(Y)$ with a one-to-one mapping that provides the CPI given an input year Y (from here, for the sake of simplicity in notation we drop the argument Y and simply consider the output ψ). Again, ψ can be compared between auctions in the same way that μ and Λ can, so we simply divide the revenue by ψ to compensate for changing price levels from year to year:

$$R_{(\mu,\Lambda,\psi)} = \frac{R}{\mu\Lambda\psi}.$$

Since this is the adjusted revenue measure we can compare between auctions, we call it $R^* \equiv R_{(\mu,\Lambda,\psi)}$.

B. Value Declination Normalization

First, we must find a proxy measure for the relative decline in values. If we linearly regress value on lot position, the slope gives us a general measure of the rate of change of value with respect to order. Consider an auction $A = (\mu, \Lambda, Y, \beta, R)$. Define the highest valuation (i.e., the valuation of the bidder who values it the most) of a given piece i to be v_i, and its corresponding lot position as λ_i; then the lot number of the last piece sold is λ_Λ. The reason we have to differentiate between λ_Λ and Λ is because other objects (not under study) may be sold in the same auction, or in case an object with a lot number is not sold in the auction. In either case, $\lambda_\Lambda \neq \Lambda$, unless by sheer coincidence.

If we perform a standard ordinary-least-squares (OLS) linear regression of value v on lot position λ in auction A, we get the relationship

$$\beta = \frac{\partial v}{\partial \lambda} = \frac{\sum_{i=1}^{\Lambda} \lambda_i v_i - \frac{1}{\Lambda}\sum_{i=1}^{\Lambda} \lambda_i \sum_{i=1}^{\Lambda} v_i}{\sum_{i=1}^{\Lambda} \lambda_i^2 - \frac{1}{\Lambda}\left(\sum_{i=1}^{\Lambda} \lambda_i\right)^2} = \frac{\mathrm{cov}[\lambda, v]}{\mathrm{var}[\lambda]}$$

This β is inherently a measure of the relationship between v and λ. We can split this into three cases:

(1) $\beta < 0 \Rightarrow$ values decline over the course of the auction;

(2) $\beta > 0 \Rightarrow$ values increase over the course of the auction;

(3) $\beta = 0 \Rightarrow$ there is no general trend in the values over the course of the auction.

From the magnitude of β, we can gather the rate at which this decline/increase is occurring. If $\beta_l, \beta_k \in \mathbb{R}_-$ and $|\beta_l| > |\beta_k|$, then we can infer that the objects in A_l were organized more coherently in a declining order, i.e. the distribution of values is closer to monotonic in lot order. Then, we expect that the effects of having declining values should be stronger in A_l than in A_k.

Clearly, β depends on several parameters within a given auction. These include the lot position of the last object, the year in which the auction occurred, and the characteristic value of the objects sold. These can be adjusted in the measurements we use to calculate β, namely λ and v. We know

$$\beta = \frac{\text{cov}[\lambda, v]}{\text{var}[\lambda]},$$

so if we normalize the lot numbers with respect to the last lot sold, we have a measure that can be compared between auctions. Let's call this measure λ'. Then, we have

$$\lambda' \equiv \frac{\lambda}{\lambda_\Lambda},$$

which gives us

$$\beta_{(\lambda_\Lambda)} = \frac{\text{cov}[\lambda', v]}{\text{var}[\lambda']}.$$

This new measure λ' can be interpreted as a percent value for the lot position of our object.

Next, we adjust with respect to the price index. Again, we can do this by defining a new variable in terms of our value v. Here, we put

$$v' \equiv \frac{v}{\psi}$$

as our new value measurement. This simply adjusts for inter-annual effects and gives us a value in a base year of our choosing, the same way we adjusted revenue. So, our slope coefficient adjusted for both the maximum lot position and the price index is

$$\beta_{(\lambda_\Lambda, \psi)} = \frac{\text{cov}[\lambda', v']}{\text{var}[\lambda']}.$$

Now, the only remaining effect to adjust for is the characteristic value. We can adjust for μ again by creating a new relative value variable:

$$v'' \equiv \frac{v'}{\mu} = \frac{v}{\mu\psi}$$

Then, we finally have our adjusted slope coefficient measure:

$$\beta^* \equiv \beta_{(\mu,\Lambda,\psi)} = \frac{\text{cov}[\lambda', v'']}{\text{var}[\lambda']}.$$

In terms of our original variables, we have

$$\beta^* = \frac{\text{cov}\left[\frac{\lambda}{\lambda_\Lambda}, \frac{v}{\mu\psi}\right]}{\text{var}\left[\frac{\lambda}{\lambda_\Lambda}\right]}$$

$$= \frac{\frac{1}{\mu\lambda_\Lambda\psi}\text{cov}[\lambda, v]}{\frac{1}{\lambda_\Lambda^2}\text{var}[\lambda]}$$

So for a given auction A_k, we have our final parameter-adjusted slope coefficient

$$\beta_k^* = \frac{\lambda_{\Lambda_\mathrm{k}}}{\mu_k\psi_k}\beta_k.$$

C. Testable Model

For a set of auctions $\{A_1, \dots, A_\Omega\}$, we can regress R^* on β^*. Performing a standard linear OLS regression, we have

$$R_i^* = \mathfrak{a} + \mathfrak{b}\upbeta_\mathrm{i}^* + \mathfrak{e}_i.$$

Our best linear unbiased estimator of $\mathfrak{b}$ is

$$\hat{\mathfrak{b}} = \frac{\text{cov}[\beta^*, R^*]}{\text{var}[\beta^*]}.$$

Once we run this regression, we expect to find a statistically significant $\hat{\mathfrak{b}} < 0$. This would imply that our adjusted revenue R^* is inversely related to our measure of value declination, in which a greater decline corresponds to a lower β^*.

IV. DATA

This study uses two primary data sources, both of which I received through contacts at the Duke Art, Law, and Markets Initiative (DALMI). For my Paris auction data, I used a collection of auction records consolidated by Ms. Hilary Coe Smith, a PhD candidate at Duke University's Art, Art History, and Visual Studies department (AAHVS). For the Christie's auction data, I received permission from Christie's Archives to use a database compiled by Ms. Bénédicte Miyamoto-Pavot, a PhD candidate at the Université Paris Diderot VII.

A. PARIS DATABASE

For my Paris auction data, I used Ms. Smith's archives, which are now available at Duke through the AAHVS server. These archives contain auction records from 1675 to 1814, and since I wanted to compare contemporary auctions, the first limiting factor on my Paris dataset was the timeframe established by the Christie's database (1767 to 1789). The second problem I faced was the lack of information on annual price levels in Paris. I found two studies, but both stopped in 1786, so that became the second limiting factor in which records I could use. The third issue was that the catalogs were not organized by lot order, but by academic notions of prestige. For example, Italian paintings were always placed first in the catalogs, regardless of the day of sale or the lot number; these were followed by other schools of paintings (French, Spanish, Dutch/Flemish, etc.), gouaches and miniatures, bronze and marble statues, designs, prints, busts, and vases. Some catalogs, however, were accompanied by *feuilles de vacation* (session sheets), which specified the day of sale and the lot position of each piece. So, I restricted my study exclusively to those catalogs that included these supplementary *feuilles*.

I did not have access to a database of these records. Instead, Ms. Smith had collected images of individual pages from auction catalogs. She had not yet begun transcribing the data for the time horizon that interested me, so a significant portion of my background research involved recording the information from these catalogs in a usable spreadsheet (see *Appendix A*). Each datum I logged had the following entries: auction, day of sale, year of auction, lot position, catalog number, and sale price (in livres, sous, and deniers). I then converted the sale price to a decimalized value (denominated in livres), and adjusted these values for inflation using 1767 as my base year.

Because I was mining data from original auction catalogs, there were several issues with the figures. One particular problem was that some catalogs combined several paintings into a single catalog number, which had a single sale price associated with it. If the paintings were all sold on the same day, they had a single lot number and I had no problems. If, however, the paintings were split up and sold over several days of sale, I had multiple lot numbers on different days corresponding to a single catalog number, and hence one total price. In this case, since the paintings were grouped because they were often identical, or at least very similar, I assumed that on average the paintings sold for about the same price each day and divided the total sale price by the number of days over which the catalog entry was sold. Another issue was that the prices listed in the catalogs were handwritten by auction attendees. I had no way of double-checking these sale prices, and I had to assume that the recorded sale prices were correct. Since the sale prices are dependent variables in my model, I can assume that any recording error was random, and thus my findings remain unbiased from this error under Gauss-Markov assumptions (Wooldridge, 2009, p. 316). Again, since these records were kept by audience members present at the auction, I also had to assume that if a catalog entry had no sale price next to it, the piece was retired and not sold. In the end, my Paris database comprised information on the sales of 1,485 paintings over 98 days of sale from 1767 to 1779.

Table 1: Paris Data Summary

Number of Auctions	98
Number of Lots	1484
Average Lots per Auction	15.143

Table 2: Paris Revenues Summary

Max	75%	Median	25%	Min	IQR
68096.740	27191.486	15089.212	6450.532	73.812	20740.955

(N.B. Because lot values vary widely within a given auction, I used median statistics rather than mean statistics to modulate the effects of tail behavior; see *V. Results*)

B. CHRISTIE'S DATABASE

For her dissertation, Ms. Miyamoto-Pavot assembled a database of Christie's auction sales from 1767 to 1789. She physically copied the data from handwritten catalogs in Christie's London archives to a spreadsheet that included the following information for each piece: artist, painting description, lot number, auction year, day of sale (within a particular auction), and sale price (in pounds, schillings, and pence). My first step was to convert all the given sale prices into a decimalized notation, denominated in pounds. I then adjusted the prices for inflation using 1767 as my base year.

This dataset originally included 3,611 entries. There were several paintings for which the painter's name was for some reason unknown, but since this was irrelevant to my study I included these data, as the lot positions and sale prices were given. There were also four pieces of data that did not include sale price; these may have been paintings that were removed from the auction after the catalog's printing, or the sale price may not have been recorded. I removed these four, and so my final database included data from the sales of 3,607 paintings over 45 days of sale from 1767 to 1789.

Table 3: Christie's Data Summary

Number of Auctions	45
Number of Lots	3607
Average Lots per Auction	80.156

Table 4: Christie's Revenue Summary

Max	75%	Median	25%	Min	IQR
3801.617	1439.876	854.634	354.339	94.338	1085.537

V. RESULTS

The data were organized in two separate spreadsheets, one for Paris auctions and the other for Christie's auctions. Since our information was limited, we had to make certain assumptions in order to continue. We have no details on the number of bidders present at a given auction, so we assume that the number remains constant throughout the course of a single day of sale. We cannot be sure that the same bidders will be present every day of a given auction, so we treat separate days of sale as individual auctions.

The second issue regards the value of the objects sold. Since there are no figures on value estimates for the pieces sold, I assume that sale price can act as a proxy for value. In *Appendix B*, we show that this assumption produces a bias, but as it affects our entire dataset, the only result of this bias is that all of our estimates for β^* are lower than the true values of β^*. This does not affect our study, as we only care about the relative differentiation between values of β^*, not the specific values themselves. This bias will further cause our estimate of $\mathfrak{b}$ to be lower than the true value, but again all we are looking for is the sign of $\mathfrak{b}$, so it is irrelevant to our study. So although the bias exists, it does not alter the findings.

The third thing we must consider is the characteristic value. Since art prices tend to have extreme right-tail events, skewing the distribution, the mean is not an accurate indicator of the average. We want to reduce the effects of tail events, and the median provides a more robust estimator of location than the mean. So, we set μ to be the median sale price in a given auction. Since the standard deviation is a measure of variability corresponding to the mean, we use the interquartile range (IQR) defined as $IQR = Q_3 - Q_1$, or the range of the middle 50% of values, as our robust measure of statistical dispersion rather than the standard deviation. IQR is analogous to 2σ for a distribution $\mathcal{F}$ determined by mean and variance, $\mathcal{F}(\mu, \sigma^2)$.

A. Paris Results

From our Paris data, we have the following summary statistics:

Table 5: Paris R* and β* Statistics

	Min	25%	Median	75%	Max	IQR
R*	0.669	1.198	1.585	2.491	11.153	1.293
β*	-37.943	-1.790	-0.260	0.946	5.716	2.736

We run the following OLS regression of R^* on β^*:

$$R_l^* = \mathfrak{a} + \mathfrak{b}\beta_l^* + \mathfrak{e}_l$$

This gives us the following results:

Table 6: Paris Coefficient Estimates

Coefficient	Estimate	Std Error	t Ratio	Prob > |t|
$\mathfrak{a}$	1.171	0.111	10.51	<.0001
$\mathfrak{b}$	-0.312	0.017	-18.47	<.0001

So we have the following regression:

$$R^* = 1.171 - 0.312\beta^*$$

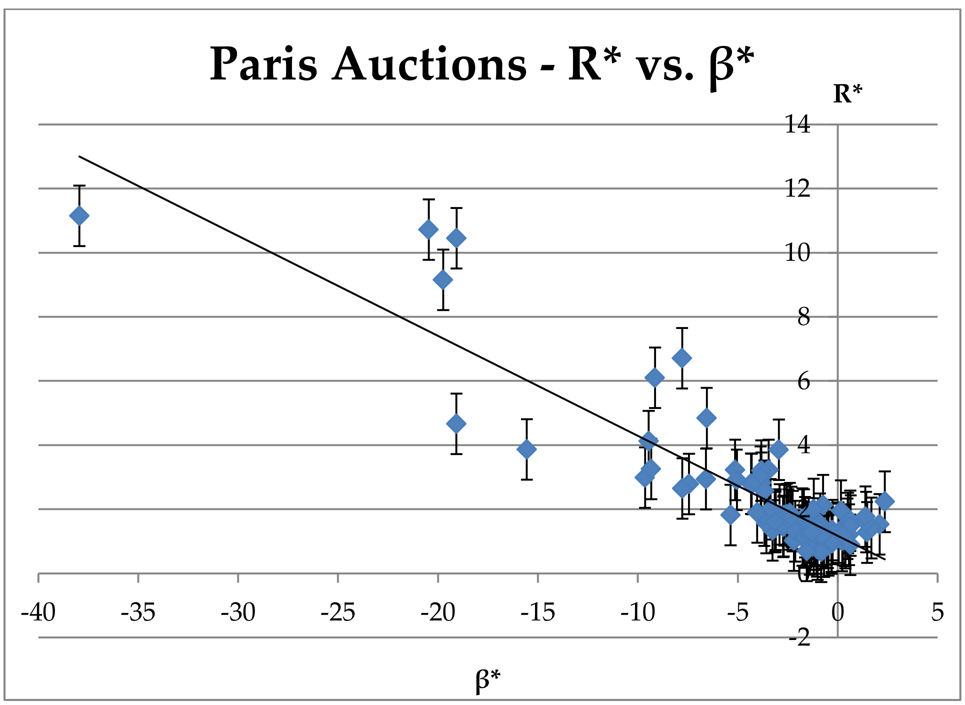


Figure 1: Paris Auctions – R* vs. β*

We can see from the t-ratios that our estimates for $\mathfrak{a}$ and $\mathfrak{b}$ are statistically significant. As expected, $\mathfrak{b} < 0$, confirming the hypothesis that arranging lots in order of decreasing value results in higher revenues.

Table 7: Paris Summary of Fit

R^2	0.780
Adj R^2	0.778
RMS Error	0.949
Mean of Response	2.219
Observations (or Sum Wgts)	98

Table 8: Paris Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Ratio	Prob > F
Model	1	307.619	307.619	341.265	<.0001
Error	96	86.535	0.901		
C. Total	97	394.154			

Moreover, with an R^2 value of 0.78 and an F-statistic of 341.265, the fit is fairly accurate and captures a significant portion of the effect.

B. Christie's Results

For our Christie's data, we have the following summary statistics:

Table 9: Christie's R* and β* Statistics

	Min	25%	Median	75%	Max	IQR
R*	1.041	1.543	1.772	2.096	3.322	0.553
β*	0.387	3.065	4.246	6.558	11.910	3.493

We run the same OLS regression of R^* on β^*:

$$R_i^* = \mathfrak{a} + \mathfrak{b}\beta_i^* + \mathfrak{e}_i$$

This gives us the following results:

Table 10: Christie's Coefficient Estimates

Coefficient	Estimate	Std. Error	t Ratio	Prob > |t|
$\mathfrak{a}$	0.993	0.059	16.83	<.0001
$\mathfrak{b}$	0.187	0.011	17.49	<.0001

So we have the estimated regression

$$R^* = 0.993 + 0.187\beta^*$$

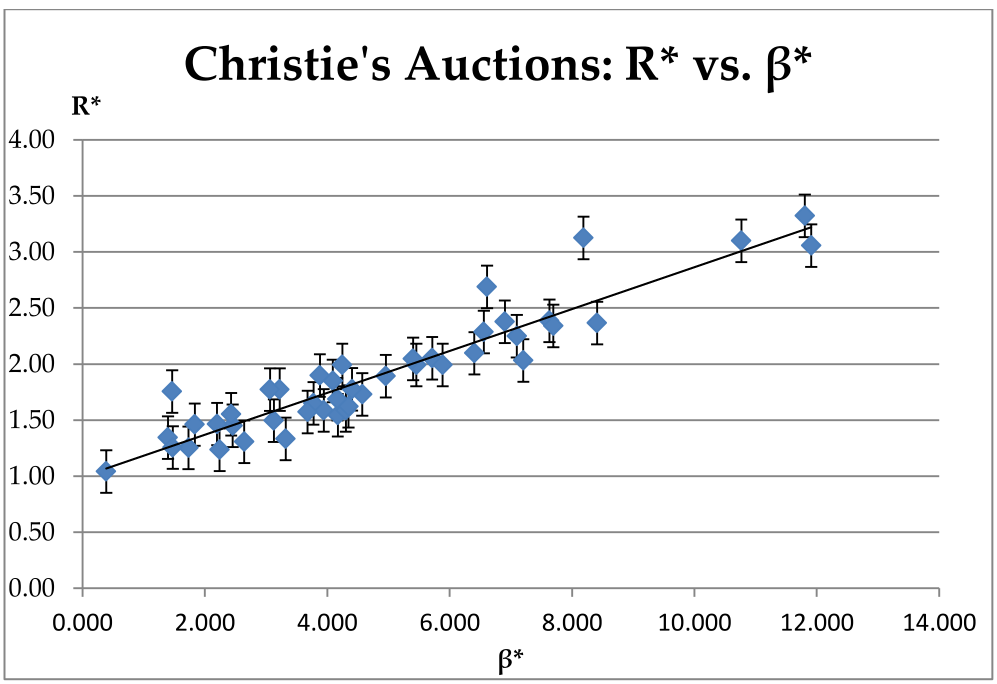


Figure 2: Christie's Auctions - R* vs. β*

According to the t-ratios, this is a statistically significant result. Here, though, we have $\mathfrak{b} > 0$, which was not expected. Moreover, from the fit statistics, this is clearly a solid model for the Christie's dataset:

Table 11: Christie's Summary of Fit

R^2	0.877
Adj R^2	0.874
RMS Error	0.192
Mean of Response	1.896
Observations (or Sum Wgts)	45

Table 12: Christie's Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Ratio	Prob > F
Model	1	11.216	11.216	305.775	<.0001
Error	43	1.577	0.037		
C. Total	44	12.794			

With an R^2 value of 0.877 and an F ratio of 305.775, this is a highly predictive model suggesting that revenues increase as objects are placed in order of ascending value, counter to both the theoretical standard and previous empirical analysis.

According to Ms. Miyamoto-Pavot (2011), the Christie's auctions were structured in a consistent manner throughout the period. These auctions typically started at noon and lasted three hours. Relatively low-valued paintings were auctioned for the first half. Then, there was a short break from paintings as other types of objects were sold, while wealthier patrons arrived. Around an hour later, auctions of paintings resumed, but now the pieces were significantly more valuable. The auctions ended around 3:00, after which dinner was served. This was the standard format for auctions in 18th century London. It may make sense, then, that if such a pattern were institutional, bidders would adjust their strategies to account for the change in the value of the paintings from the first part of the auction to the second part.

In *Appendix C*, we run a simple test to see if there was a specific pattern followed in the ordering of lots at Christie's auctions compared to those in Paris. Based on the analysis, there was considerable variability in the value-ordering at the Paris auctions, and relatively little variation in the pattern at Christie's. This implies that there was in fact a very regular process at Christie's, and bidders could adjust their strategies to account for this, explaining the discrepancy in our results for the Christie's data.

VI. CONCLUSIONS

Using our data from Paris auctions, we find that the model developed in this paper supports the claim that ordering lots in decreasing value generates higher revenues for the seller. The data from Christie's does not support the hypothesis, but as demonstrated in *Appendix C*, this is due to structural praxes stemming from an institutionally regular pattern at Christie's during this period.

For our set of Paris auctions, we find that the relationship between the adjusted revenue and the adjusted slope coefficient is proportional (disregarding the constant) with a factor of $\mathfrak{b} = -0.312$, and this measure has a t-statistic of $t = -18.47$, so it is a statistically significant relationship.

Modern empirical studies generally focus on recent auctions that have occurred since the theory was developed, and until now nobody has studied historical auctions. This study provides some evidence that this optimal ordering is not related to modern practices by auction houses. There is, however, definite need to expand this study. There remains a substantial supply of unexamined auction records from Paris, available through the AAHVS server at Duke. This study meant to compare the information about contemporary Paris and Christie's auctions, but a new study could focus solely on Paris auctions over centuries, providing further insights into our assumed optimality conditions for sequential auctions. Another study could consider variations on the basic model outlined here, either by incorporating more details about individual pieces of art or the auctions themselves. A further extension could consider using a modified characteristic value, $\mu^* = \frac{\mu}{IQR}$, for each auction, using an analog of the Sharpe ratio for median statistics.

References

Ashenfelter, O. (1989). How Auctions Work for Wine and Art. *The Journal of Economic Perspectives*, 23-36.

Ashenfelter, O., & Genesove, D. (1992). Testing for Price Anomalies in Real-Estate Auctions. *The American Economic Review*, 501-505.

Ashenfelter, O., & Graddy, K. (2003). Auctions and the Price of Art. *Journal of Economic Literature*, 763-787.

Ausubel, L. (1997). On Generalizing the English Auction. 1-14.

Beggs, A., & Graddy, K. (1997). Declining Values and the Afternoon Effect: Evidence from Art Auctions. *The RAND Journal of Economics*, 544-565.

Benoit, J.-P., & Krishna, V. (2001). Multiple-Object Auctions with Budget Constrained Bidders. *Review of Economic Studies*, 155-179.

Birulin, O., & Izmalkovy, S. (2009). On Efficiency of the English Auction. *CEFIR/NES Working Paper Series*, 1-28.

Elkind, E., & Fatima, S. (2007). Maximizing Revenue in Sequential Auctions. *WINE*, 491-502.

Elmaghraby, W. (2003). The Importance of Ordering in Sequential Auctions. *Management Science*, 673-682.

Ginsburgh, V. (1998). Bidders and the Declining Price Anomaly in Wine Auctions. *Journal of Political Economy*, 1302-1319.

Goncalves, R. (2008). Irrationality in English Auctions. *Journal of Economic Behavior & Organization*, 180-192.

Hong, H., Kremer, I., Kubik, J., Mei, J., & Moses, M. (2009). Who's on First? Ordering and Revenue in Art Auctions. 1-41.

Kagel, J. H., & Levin, D. (1993). Independent Private Value Auctions: Bidder Behaviour in First-, Second- and Third-Price Auctions with Varying Numbers of Bidders. *The Economic Journal*, 868-879.

Krishna, V. (2002). *Auction Theory.* San Diego: Elsevier Science.

Kyle, A. S. (1985). Continuous Auctions and Insider Trading. *Econometrica*, 1315-1335.

Li, H., & Riley, J. G. (2007). Auction Choice. *International Journal of Industrial Organization*, 1269-1298.

Maskin, E., & Riley, J. (2000). Equilibrium in Sealed High Bid Auctions. *Review of Economic Studies*, 439-454.

McAfee, R. P., & McMillan, J. (1987). Auctions and Bidding. *Journal of Economic Literature*, 699-738.

McAfee, R. P., & Vincent, D. (1993). The Declining Price Anomaly. *Journal of Economic Theory*, 191-212.

Milgrom, P. R., & Weber, R. J. (1982). A Theory of Auctions and Competitive Bidding. *Econometrica*, 1089-1122.

Milgrom, P., & Weber, R. J. (1982). The Value of Information in a Sealed-Bid Auction. *Journal of Mathematical Economics*, 105-114.

Miyamoto-Pavot, B. (2011, November 2). Christie's 18th Century Auctions. (A. Mitha, Interviewer)

Myerson, R. B. (1981). Optimal Auction Design. *Mathematics of Operations Research*, 58-73.

Neeman, Z. (2003). The Effectiveness of English Auctions. *Games and Economic Behavior*, 214-238.

Pitchik, C. (2006). Budget-Constrained Sequential Auctions with Incomplete Information. 1-35.

Riley, J. G., & Samuelson, W. F. (1981). Optimal Auctions. *The American Economic Review*, 381-392.

Twigger, R. (1999). Inflation: the Value of the Pound 1750-1998. *House of Commons Library*, 1-20.

Vickrey, W. (1961). Counterspeculation, Auctions, and Competitive Sealed Tenders. *The Journal of Finance*, 8-37.

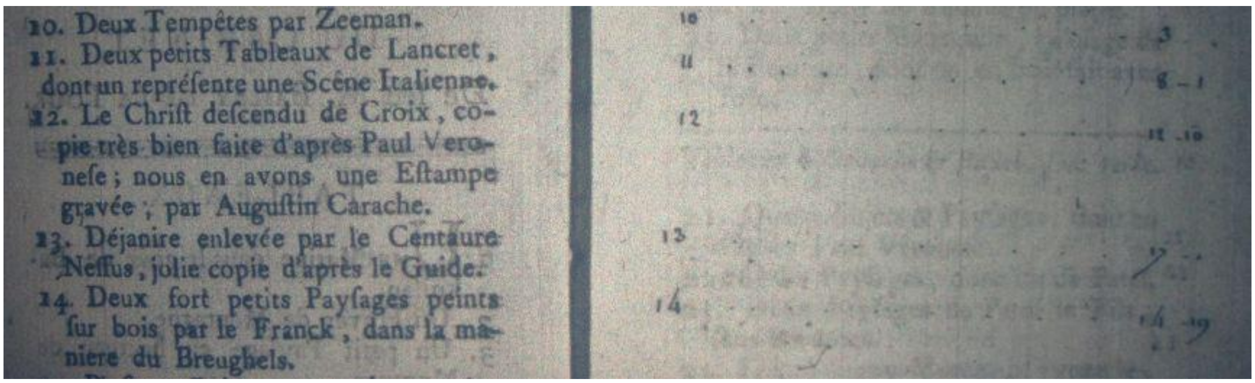
Wooldrige, J. M. (2009). *Introductory Econometrics: A Modern Approach.* Mason, OH: South-Western Cengage Learning.

Zulehner, C. (2009). Bidding Behavior in Sequential Cattle Auctions. *International Journal of Industrial Organization*, 33-42.

APPENDICES

A. COLLECTION OF PARIS AUCTION DATA

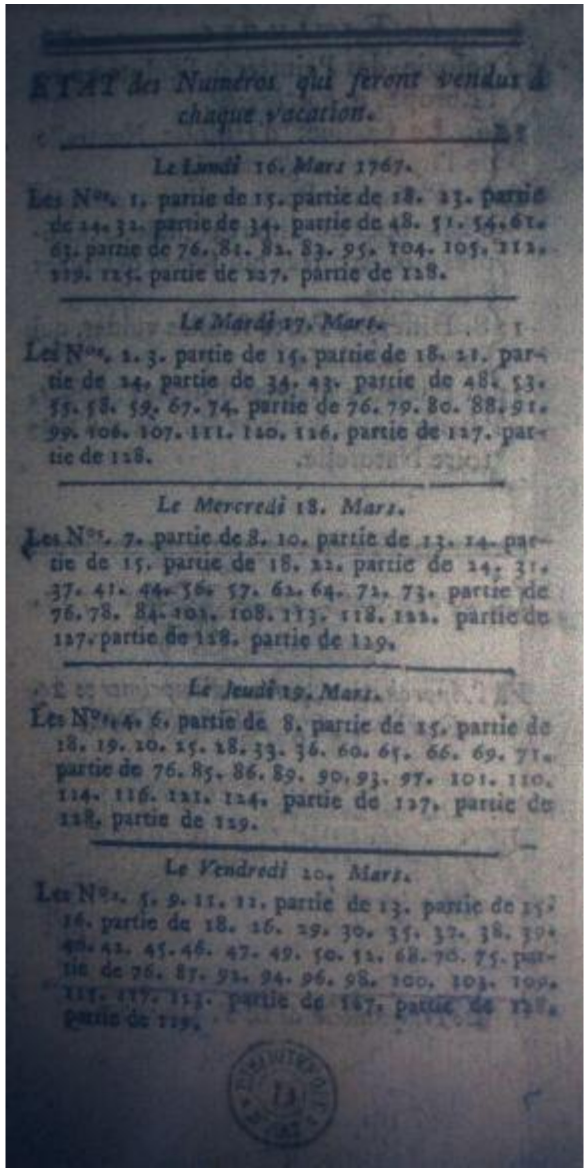
For the Paris data, I had access to a set of PDF files that had images of auction catalogs, like the sample image below:



20. Deux Tempêtes par Zeeman.
21. Deux petits Tableaux de Lancret, dont un repréſente une Scène Italienne.
22. Le Chriſt deſcendu de Croix, copie très bien faite d'après Paul Veroneſe ; nous en avons une Eſtampe gravée, par Auguſtin Carache.
23. Déjanire enlevée par le Centaure Neſſus, jolie copie d'après le Guide.
24. Deux fort petits Payſages peints ſur bois par le Franck, dans la maniere du Breughels.

Figure 3: Auction Catalog

Note, there is no information on the day or order of sale here. For that, we must turn to the *feuilles de vacation*, which order the catalog numbers by day and lot position:



ETAT des Numéros qui seront vendus à chaque vacation.

Le Lundi 16. Mars 1767.

Les Nos. 1. partie de 15. partie de 18. 23. partie de 24. 32. partie de 34. partie de 48. 51. 54. 61. 63. partie de 76. 81. 82. 83. 95. 104. 105. 112. 119. 125. partie de 127. partie de 128.

Le Mardi 17. Mars.

Les Nos. 2. 3. partie de 15. partie de 18. 21. partie de 24. partie de 34. 43. partie de 48. 53. 55. 58. 59. 67. 74. partie de 76. 79. 80. 88. 91. 99. 106. 107. 111. 120. 126. partie de 127. partie de 128.

Le Mercredi 18. Mars.

Les Nos. 7. partie de 8. 10. partie de 13. 14. partie de 15. partie de 18. 22. partie de 24. 31. 37. 41. 44. 56. 57. 62. 64. 72. 73. partie de 76. 78. 84. 102. 108. 113. 118. 122. partie de 127. partie de 128. partie de 129.

Le Jeudi 19. Mars.

Les Nos. 4. 6. partie de 8. partie de 15. partie de 18. 19. 20. 25. 28. 33. 36. 60. 65. 66. 69. 71. partie de 76. 85. 86. 89. 90. 93. 97. 101. 110. 114. 116. 121. 124. partie de 127. partie de 128. partie de 129.

Le Vendredi 20. Mars.

Les Nos. 5. 9. 11. 12. partie de 13. partie de 15. 16. partie de 18. 26. 29. 30. 35. 37. 38. 39. 40. 42. 45. 46. 47. 49. 50. 52. 68. 70. 75. partie de 76. 87. 92. 94. 96. 98. 100. 103. 109. 115. 117. 123. partie de 127. partie de 128. partie de 129.

Figure 4: *Feuille de Vacation*

The corresponding entry for these data in my spreadsheet is as follows:

Table 13: Sample Data from Paris Auctions

Auction	Year	Day of Sale	Lot	Norm Lot	Catalog #	Livre	Sous	Denier	Total (L)	Adj Total	1767 Livres
1	1767	3	3	0.23	10	3	0	0	3.00	3.00	3.00
1	1767	5	3	0.14	11	8	1	0	8.05	8.05	8.05
1	1767	5	4	0.18	12	12	10	0	12.50	12.50	12.50
1	1767	3	4	0.31	13	17	0	0	17.00	8.50	8.50
1	1767	5	5	0.23	13	17	0	0	17.00	8.50	8.50
1	1767	3	5	0.38	14	14	19	0	14.95	14.95	14.95

Note that lot 13 is entered twice; in the *feuille*, it appeared under day 3, lot 4 and day 5, lot 5; the given sale price was 17 livres, so the model splits this value into equal parts for each day.

B. Bias in Price as a Proxy for Value

Consider a set of n symmetric bidders, and bidder i has valuation $X_i \in [0, w]$, where w is the maximum possible value. Each X_i value is iid, and is distributed on some increasing distribution F with continuous density $f = F'$. Bidder i knows his realization x_i of X_i. According to Krishna (2003), in the case of an English auction, the equilibrium strategy ξ_i for a bidder i is to bid up to his true value, i.e. $\xi_i(x_i) = x_i$. The realized valuations among the n bidders are given by the set $\{x_1, \dots, x_n\}$. Then, the highest valuation v is the largest order statistic of the set, $v = x_{(n)} = \max\{x_1, \dots, x_n\}$. The sale price, however, will be marginally more than the second order statistic – the winner will stop increasing his bid after he sees his last remaining competitor drop out.

Without loss of generality, assume $v = x_1$, and $x_2 = x_{(n-1)} = \max\{x_2, \ldots, x_n\}$. If $x_1 = x_2$, we have the sale price $p = v$. Otherwise, $p = x_2 + 1$ in the marginal unit. If $x_2 + 1 = x_1$, we again have $p = v$. Otherwise, $x_2 + 1 < x_1$, so $p < v$. Therefore, we must have the relationship $p \leq v$, and the winning bidder must have a nonnegative surplus, i.e. $v - p \geq 0$.

This implies that using sale price p as a proxy for value v will bias our estimate for the value of an object slightly below its true value. This will artificially dampen our estimates of β^* for individual auctions. The actual values for β^* are, however, irrelevant; all we care about is the general relationship between R^* and β^*. Thus, since the bias is in the same direction for all our data points, we can use p as a proxy for v.

C. Test for Regularity in Value vs. Lot Order

We want to test the data to see if Christie's ran a specific program in ordering lots by value regularly over each auction. We can do this by considering a regression of value v on lot position λ. Here, we use sale price p as a proxy for v (see *Appendix B*). First, we must normalize each measure. As stated earlier, an auction can be characterized by the vector $A = (\mu, \Lambda, Y, \beta, R)$ (see *III. Model*). We can normalize the lot position by using the variable $\lambda' = \frac{\lambda}{\lambda_\Lambda}$ (see *III.B Value Declination Normalization*). Similarly, we can normalize the price by the characteristic value; consider the variable

$p' = \frac{p}{\mu}$. This is analogous to our original adjustment of the revenue by the characteristic value, where we found $R_{(\mu)} = \frac{R}{\mu}$ (see *III.A Revenue Normalization*). Clearly, revenue is the sum of the prices in an auction, so $R = \sum_{i=1}^{\Lambda} p_i$. It follows that

$$R_{(\mu)} = \frac{1}{\mu}\sum_{i=1}^{\Lambda} p_i = \sum_{i=1}^{\Lambda} \frac{p_i}{\mu} = \sum_{i=1}^{\Lambda} p_i',$$

so our price transformation $p' = \frac{p}{\mu}$ preserves our original revenue transformation.

Now, since we want to look at the trends with respect to the lot position, we'll study the relative values for each decile of the normalized lot variable. For the Paris data, we have the following statistics:

Table 14: Aggregate Paris Decile Statistics

Decile	Q_3	Q_1	Q_2	QCoD	CV
1	0.963	0.095	0.265	0.820	163.5%
2	1.579	0.141	0.420	0.836	171.1%
3	1.160	0.029	0.257	0.951	220.4%
4	2.393	0.148	0.485	0.884	231.5%
5	4.738	0.300	0.739	0.881	300.2%
6	2.181	0.225	0.746	0.813	131.1%
7	2.396	0.161	0.677	0.874	165.0%
8	1.918	0.154	0.503	0.851	175.4%
9	2.139	0.118	0.544	0.895	185.7%
10	0.814	0.029	0.191	0.932	206.1%

Again, since our distribution is skewed, we want a measure that does not heavily weigh outliers, so we use quartile statistics. Here, the deciles each represent one-tenth of the lots, i.e. the first decile corresponds to the interval $\lambda' \in (0.0, 0.1]$, the second decile to $\lambda' \in (0.1,0.2]$, and so on. Q_2 is the median value of the adjusted price p_i in a given decile, Q_3 corresponds to the 75th percentile, and Q_1 to the 25th percentile. We calculate two different measures of the relative statistical dispersion inherent in the measure, the quartile coefficient of dispersion ($QCoD$) and the coefficient of variation (CV). We define $QCoD$ and CV as

$$QCoD = \frac{Q_3 - Q_1}{Q_3 + Q_1},$$

$$CV = \frac{(Q_3 - Q_1)/2}{\mu}.$$

These are both measures that can be compared between different sets of data, and both depend on the interquartile range ($IQR = Q_3 - Q_1$), a relatively robust measure of absolute statistical dispersion. Similarly, for our Christie's dataset, we have the following statistics:

Table 15: Aggregate Christie's Decile Statistics

Decile	Q_3	Q_1	Q_2	QCoD	CV
1	0.463	0.171	0.295	0.461	49.5%
2	0.735	0.290	0.463	0.434	48.1%
3	0.995	0.411	0.630	0.416	46.4%
4	1.218	0.522	0.802	0.400	43.4%
5	1.492	0.613	0.925	0.418	47.5%
6	1.930	0.772	1.199	0.428	48.3%
7	2.362	0.882	1.434	0.456	51.6%
8	2.606	1.038	1.623	0.430	48.3%
9	3.372	1.259	2.067	0.456	51.1%
10	6.434	1.905	3.422	0.543	66.2%

We can see that both measures of variation are significantly higher for each decile of the Paris data than for the Christie's data, which implies that there was little variation in the trend for Christie's auctions relative to the Paris auctions. Comparing just the relative dispersion statistics, we get the following results:

Table 16: Relative Dispersion Statistics

	QCoD		CV	
Decile	Paris	Christie's	Paris	Christie's
1	0.820	0.461	163.5%	49.5%
2	0.836	0.434	171.1%	48.1%
3	0.951	0.416	220.4%	46.4%
4	0.884	0.400	231.5%	43.4%
5	0.881	0.418	300.2%	47.5%
6	0.813	0.428	131.1%	48.3%
7	0.874	0.456	165.0%	51.6%
8	0.851	0.430	175.4%	48.3%
9	0.895	0.456	185.7%	51.1%
10	0.932	0.543	206.1%	66.2%

On average, the Paris $QCoD$ was 1.97 times higher than the Christie's $QCoD$, and the Paris CV was 3.95 times higher than the Christie's CV, suggesting that there was far greater variability in the Paris auctions than in the Christie's auctions. We can see this graphically below. In both figures, the points represent the Q_2 value in each decile, while the vertical lines correspond to the interquartile range for each decile.

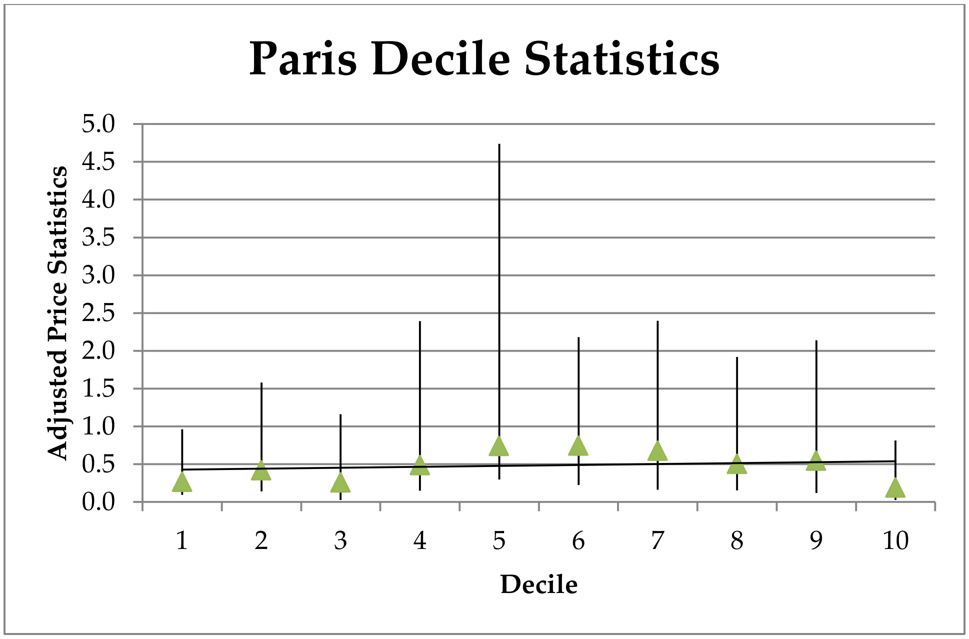


Figure 5: Paris Decile Statistics

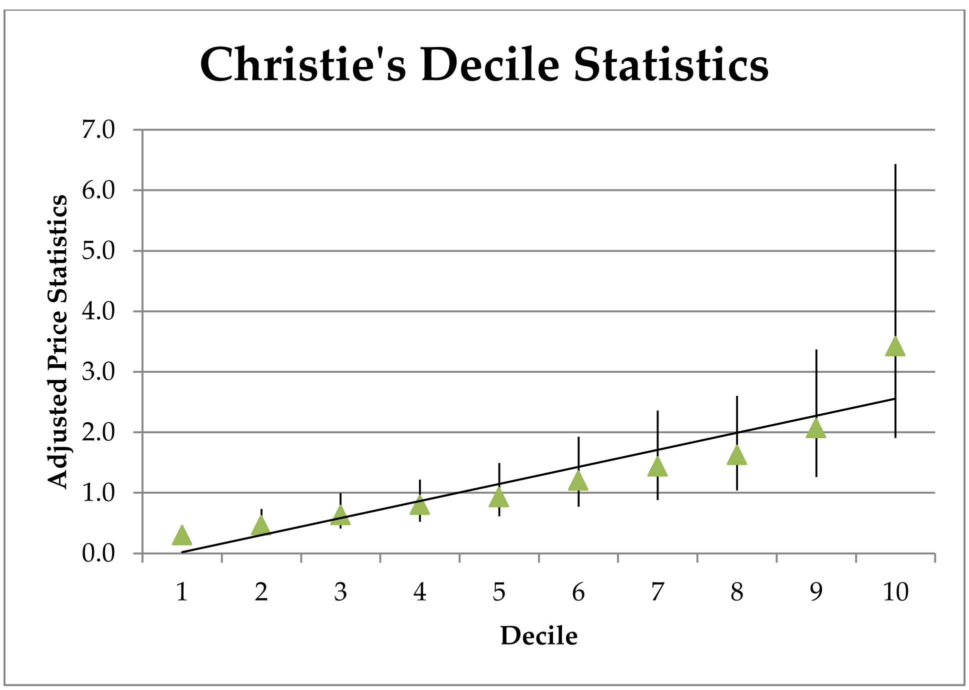


Figure 6: Christie's Decile Statistics

We can see that the Christie's distribution fits the description provided by Ms. Miyamoto-Pavot, with a general upward trend in value as the relative lot number increases, while the Paris distribution seems to be relatively flat. If we run an OLS regression of Q_2 on λ' for each dataset, we get the following results:

Table 17: Paris Decile Statistics Regression Analysis

Summary of Fit					
R^2	0.034				
Adj R^2	-0.087				
Root Mean Square Error	0.210				
Mean of Response	0.483				
Observations (or Sum Wgts)	10.000				
Analysis of Variance					
Source	DF	Sum of Squares	Mean Square	F Ratio	Prob > F
Model	1	0.012	0.012	0.280	0.6113
Error	8	0.352	0.044		
C. Total	9	0.364			
Parameter Estimates					
Term	Estimate	Std Error	t Ratio	Prob > |t|	
Intercept	0.416	0.143	2.9	0.0199	
Coefficient	0.122	0.231	0.53	0.6113	

Table 18: Christie's Decile Statistics Regression Analysis

Summary of Fit					
R^2	0.844				
Adj R^2	0.824				
RMS Error	0.390				
Mean of Response	1.286				
Observations (or Sum Wgts)	10.000				
Analysis of Variance					
Source	DF	Sum of Squares	Mean Square	F Ratio	Prob > F
Model	1	6.554	6.554	43.123	0.0002
Error	8	1.216	0.152		
C. Total	9	7.770			
Parameter Estimates					
Term	Estimate	Std Error	t Ratio	Prob > |t|	
Intercept	-0.264	0.266	-0.99	0.3502	
Coefficient	2.819	0.429	6.57	0.0002	

For Christie's, we have a $F = 43.123$ and $R^2 = 0.844$, confirming a high statistical correlation between an object's value and its normalized lot number. For our Paris auctions, $F = 0.280$ and $R^2 = 0.034$, a statistically insignificant correlation. This further supports our claim that bidders could easily, and probably did, adjust their strategies in Christie's auctions given the high regularity in value-ordering, which is why we did not see the effects we expected.