An Endogenous Dynamic General Equilibrium Model of Fertility for England 1881-1931

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Senior Honors Thesis

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2007

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Abstract

Using the basic structure of Galor & Weil (1996), I construct an overlapping-generations dynamic general equilibrium model to consider a new mechanism to explain the demographic transition (fertility rates) in England. The mechanism I consider is an increase in female labor market opportunities through the relaxation of social norms following the industrialization of England.

Assuming the existence of a disparity in labor market opportunities for both men and women, the opportunity cost of childrearing is less than that of the husband within the household. As social norms relax, the opportunity costs of childrearing diminish. The wife substitutes her labor away from childrearing towards formal work, thereby lowering the total household labor allocated to childrearing. This reduction leads to a lowering of fertility. Over time, the decrease in fertility provides a positive feedback loop for growth as the decrease in population growth expands capital per worker further.

I calibrate the parameter values of the model to ensure that the model's prediction correspond to key data moments in England from 1881 to 1931, the period associated with England's demographic transition. I find that an increase in female labor market opportunities accounts for roughly one-third of the drop in surviving fertility per household from 1881 to 1931 in England.

*Acknowledgements: I am grateful for the advice and support of Donna Gilleskie and Oksana Leukhina.

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1. Introduction

1.1 Social Norms within Economic Theory

In his 2007 Presidential Address to the American Economic Association, Akerlof (2007) argues that the inclusion of social norms provides a missing motivation to explain recent controversies in macroeconomics.[1] In a discipline like modern economics built upon the individual rational behavior, the idea of external social constraints influencing one's behavior may seem foreign to some economists. Yet incorporating social norms into the paradigm would not necessarily be incompatible with modern economic theory. Within an incorporation of social norms with the assumption of *homo economicus*, an individual is free to maximize her own well-being while constrained by social limitations. As scarcity fundamentally constrains individual utility maximization, so too may social norms as a byproduct of the aggregation of individuals' decisions. Following Akerlof's proposition, if such considerations may be taken to explain short run economic activity, what prevents the application of social norms in the context of long run economic growth? Further, as other disciplines like sociology and history incorporate social norms to explain gender roles and fertility decisions, can economics add to this literature to help us better understand such processes?

I consider economic growth from a demographic perspective, specifically from the vantage point of fertility choice constrained by social norms. As investment decisions are essential to capital accumulation, household fertility decisions determine the future labor stock. First to examine the effect of a relaxation of such social norms on fertility, I assume that fertility is a rational decision in which a household calculates the expected value versus the expected costs of having each offspring. Using a rational decision model, I follow Becker (1960) by incorporating the standard economic approach to fertility choice in which the opportunity cost for each child born is the income that parent(s) must forgo to raise the child. However, assuming a household consisting of more than one parent, this idea begs the question to which parent does income matter in fertility choice?

[1] By social norms, Akerlof referred to social constraints on individual behavior defining how one should act.

Most of the economic literature which considers household fertility choice ignores the potential existence of differential opportunity costs of fertility between a husband and wife. What matters in this methodology is the total household income rather than any discrepancies between a husband and wife. However, the intrinsic nature of the child rearing process towards the wife would seem to place extra costs of fertility on females. I argue that a consideration for the specific female costs of fertility is essential to developing a deeper understanding of household fertility decisions.

I examine to what extent did social norms constrain female labor opportunities which then affected long run fertility. Following the relaxation of such social norms through industrialization, the opportunity cost of fertility increases for women. During this process, within the household the wife substitutes away from child rearing and towards entering the labor force as the husband continues to supply his labor to the economy. This drop in household labor pushes fertility rates down, diminishes population and creates higher capital per household. As capital per household gains, per capita output rises as the drop in population creates a positive feedback loop that further accelerates growth, lowers fertility rates and leading to the modern period of sustained long run growth.

The structure of this work is as follows: in the first section, I discuss a brief landscape of literature surrounding the transition from the Malthusian Period to the modern period of sustained economic growth, in particular with considerations for the demographic transition. In the same section, I outline literature from history, sociology and economics incorporating the role of social norms constraining female labor opportunities within England and explain its connection with English industrialization. In the next section, I introduce my model, solve it and consider the dynamics within the model. In the third section, I calibrate the model to ensure that the model's predictions correspond to key data moments in England from 1881 to 1931.

1.2 Malthus to Sustained Growth

Modern economic growth theory is largely derived from the work of Thomas Malthus during the late 18th Century. Malthus (1798) hypothesized that fertility rates were predominately a product of instinctive behavior, stimulated by periods of prosperity and halted by periods of famine. The result of such behavior results in a positive relationship between income and fertility rates. Under circumstances of high income levels and largely positive living conditions, fertility rates would increase exponentially as households would have the means to support larger families. However, as fertility rates would increase geometrically, Malthus noted that resources (largely food supplies) could only grow arithmetically. Therefore, any large fertility booms would eventually be met by shortages of food and resources as population growth outpaced growth of food and similar resources. In the process, starvation and disease would become more prevalent as resources per capita begin to shrink as instinctive mechanisms push population growth further. Eventually as the effects of poverty spread, households can no longer support larger families as resources per capita become scarcer. Fertility rates then drop back to their original equilibrium values which then raise resources per capita.

Yet shortly after Malthus' thesis, most Western economies started a transformation into modern industrialized economies, aptly named the Industrial Revolution. The Industrial Revolution represents a sharp transition from agricultural-based economies towards industrial based economies. During industrialization, new innovations in production processes yielded higher aggregate output levels. This increase in aggregate output yielded increases in income and consumption levels. Unlike earlier periods where short run gains in output equilibrate back to long run values due to the Malthusian population mechanism, the tremendous gains in technology outpaced population gains which yielded higher output per capita. Instead, these nations were moving into a period of sustained economic growth, as it appears today.

Modern economic literature provides a myriad of potential mechanisms to explain the transition from the Malthusian period to sustainability of economic growth. Several such mechanisms include: opening of trade within Europe (O'Rourke & Williamson, 2005); decrease in relative capital accounts (Fernandez-Villaverde, 2001); and institutional change of technology (Mokyr, 2003).[2]

Following Malthus, other economists have considered the role of demographics within the transition to sustained growth. Around the eighteenth and nineteenth centuries, change within western economies consisted more than the production and technological changes associated with an industrial revolution. Considering demographics, the decades

[2] There does exist a wealth of literature considering fertility and growth outside of the three period transition model, for example see Lucas (2002); Kremer (1993); Jones (2002).

following the industrial revolution of Europe and the United States, these economies experienced decreasing birth rates and death rates. While there is evidence of historical short run fluctuations in such indicators, this decrease in birth and death rates failed to equilibrate back to original levels. Instead, as income in developed nations continued to rise, birth and death rates continued to decline with little variance. Demographers call this sustained drop in birth and death rates following industrialization the demographic transition.

Most economic growth literature that considers growth from a demographics perspective largely centers on the role the demographic transition helped to sustain an environment appropriate for sustained economic growth.[3] Galor (2005) provides a brief review research that links long run sustained growth with the demographic transition. Galor first reinforces the argument of Galor & Weil (2000) that a rise in the demand for human capital shifted household's fertility decision to focus on the quality (or level of human capital investment) of offspring rather than their quantity of offspring. Then Galor presents reinforcing mechanisms that prolong the effect of a transition to the focus of human capital investment including: restrictions on child labor (Hazan & Berdugo, 2002; Galor & Moav, 2003, 2004; Doepke & Zilibotti, 2003); a rise in life expectancy; natural selection and the evolution of preference for offspring's quality (Galor & Moav, 2002); a decline in the gender gap (Galor & Weil, 1996). Other demographic and behavioral considerations exist for the shift to sustained growth that is not directly linked to the demographic transition. Gould, Moav & Simhon (2002) consider the effect of a behavior shift of adults towards monogamous relationships. Gills (2002) highlights changes in human evolution triggered by the effect of markets on individual decision-making.

My motivation for this paper is two-fold. First, I want to consider a new mechanism within the process of the demographic transition: an increase in labor opportunities for women through a relaxation of social norms. Ultimately, this mechanism may be a part of several significant parts of a complex, general process. Therefore, such a mechanism could play a supplemental role to other theories surrounding the demographic transition

[3] Most economic literature that considers the causes of the demographic transition largely are concerned with fertility (birth) rates rather than mortality (death) rates. The reasoning is that it is assumed that fertility rates are inherently the product of individual decision-making while mortality rates are largely the product of one's environment.

to better understand the demographic transition. Secondly, this paper attempts to apply concepts encapsulated in the disciplines of history and sociology and incorporate these ideas with economy theory within a quantitative model. Through such interdisciplinary research, a more well-rounded understanding of basic social science phenomenon is possible.

1.3 Social Norms Constraining Female Labor Opportunities

Modern economic literature surrounding the existence of gender discrimination and the historical gender wage gap largely centers on Becker (1991). Becker attributes gender discrimination to the comparative advantage each gender has on relative tasks. For example, men received a higher wage in agricultural economies because their labor was physical intensive and therefore more productive in agriculture and other physically intensive production. Women on the other hand have a comparative advantage in child rearing and home production. Such specialization in labor is manifested in the gender wage gap.[4] Contrary to Becker, Galbi (1994) reaches a different conclusion. Galbi finds that gender differences in physical and supervisory abilities and the tasks of childrearing were insufficient to account for observed gender differences in the factory labor market. Instead, Galbi suggests other factors like male unions, male interests in limiting female labor opportunities and patriarchal ideology by employers were instead more important. I take the view of Galbi and extend Galbi's consideration by incorporating social norms within a general equilibrium framework.

Examining historical literature, there is not a clear consensus of the effect the English industrialization had on female labor participation and female labor opportunities.[5] Roberts (1988) outlines two broad views on the premise. On one side, the 'optimistic' view holds that industrialization created labor opportunities for women which then led to an increase in female labor participation rates. On the other hand, the view of 'pessimistic' scholars is unenthusiastic towards the effect industrialization had on female

[4] This idea is the basis for Galor & Weil (1996).

[5] The remaining evidence will explore the example of English industrialization. While this does not necessarily mean that other countries did not undergo a similar process, more extensive research on each individual country is needed before one may apply the model that will be presented later in the text.

labor opportunities.[6] In consideration of changes with labor opportunities for women, the fact that such a change could or could not be quantified, proving such a claim is very difficult.[7] Mokyr (1999) notes that historical labor force participation rates in general represent something of an anachronism since it is impossible to know the exact situation around those individuals not included within the labor force. In the context of women, women's work was intrinsically linked to the household (whether it be their own or household work for others). For a woman to be considered in and/or participating in the labor force in 1850, what were the conditions? Although industrialization corresponded to a structural shift in female participation into textiles and other factory work, the most common occupation for working-class women during the late nineteenth century was as domestic servants (Roberts 1988). Considerations for such work may have been easily neglected and/or difficult to quantify within labor force participation rates; therefore, yielding the potential for an underestimation of women's role in the economy.[8] Wages and labor conditions could also indicate the existence and the subsequent relaxation of social norms hindering female labor opportunities.[9] However, for the purpose of this work I will focus on female labor force participation as an indicator of female labor opportunities. To strengthen my claim that female labor force opportunities expanded following the Industrial revolution, I will now present further qualitative evidence from historical writings.

1.3.1 Historical Evidence

Evidence from sociology provides that the social perception of women has historically constrained labor opportunities for women in England (Holloway, 2005; Hill, 1994; Jordan, 1999). Such authors contend that the dominant historical social perception was that woman's labor should be constrained to household labor (e.g. child rearing,

[6] However, even within this perspective there is variation between how industrialization affected female labor participation rates.

[7] Therefore I uphold the optimistic view that industrialization increased labor opportunities for women; however, whether or not such an effect increased female labor participation is not definitive and not directly addressed in this paper.

[8] See the Quantitative Analysis section for more discussion on the specific data used in simulating the model.

[9] A key distinction made in this paper is that social norms only hindered how much labor women would supply to the economy versus the household; therefore, affecting the supply of labor. However, another interpretation of such social norms could be a demand effect. In this perspective, firms could discriminate against women to avoid potential social costs associated with hiring women outside of the household.

cooking, etc.). Individuals within society would have an incentive to adhere to this social norm in order to prevent the costs of stigmatization. Holloway (2005) elaborates that the costs of social perception constrained the behavior of women as well. Holloway argues that

> "working women's behavior and demeanour was measured against norms expected of an idealized, leisured middle-class lady so any behavior which was not passive or docile was deemed unfeminine, roundly condemned and used as a pretext to either limit or exclude women's participation in the occupation" (17).

Moreover, dominance of female work inside of the home reinforces this social norm during the early nineteenth century in which most women worked in either agriculture, domestic service, textiles, needlework or teaching. Although the potential manifestation of the social norm of a woman's work could vary, I briefly examine literature considering social norms in educational and legal institutions which constrained female labor opportunities in England.

Historical social norms constrained education opportunities for women within pre-industrialized English society (Jordan, 1999). In the process, education served as an indirect constraint on women's labor opportunities by failing to prepare women for occupations that males were to dominate. Part of this constraint may come from a lack of the number of years young girls were educated relative to males of similar class and status may receive. However, a second part was that even for those girls who were fortunate enough to attend years of schooling, the curriculum on average differed immeasurably for girls than for boys. Instead of learning practical skills to enter the work force, most girls that did attend school were taught homemaking skills that reinforced the perception of women as a mother, a wife and a homemaker. Even if firms were willing to hire women as a professional, most women were thus ill equipped to hold such a position relative to more educated male laborers.

Holloway (2005) notes that most children, regardless of gender or household background, faced constraints on educating children in the form of explicit (payments to private schools) and implicit costs (forgone contributions to family economy). In particular, Holloway highlights the increasing importance female children had in raising siblings, in particular to reinforce the training of young girls to a motherly role later in life. Jordan (1999) argues on similar lines that the social norm of a woman's place in society constrained female education more than the explicit cost constraint. Financial constraints did not overwhelm parents from educating their daughters as much as did the concern of parents over the content of their daughter's education. Jordan attributes such an argument to the rise in women's colleges, emphasized increase in arts education specifically for young women and the continued training of a governess in the mid-to-late nineteenth century. Therefore education for women progressed gradually but slowly. For women educated under such constraints, they would be prepared for the expanding demand of teachers as education rates as a whole rose. Similarly, Jordan connects the female perception of philanthropy with the increasing demand for nursing to provide another industry in which demands for women outside of the household increased dramatically with industrialization and gains in technology and medicine.

Legal barriers on entering certain trades, limitation of work hours and/or regulating wages of women made up a more explicit constraint on female labor opportunities in England. Society's perception of women was a major factor in the implementation of such legal constraints on female labor opportunities. For example, social scrutiny of the perception of women hampered women from working in occupations that appeared to society preventing them from their expected, domicile work. One major manifestation of such social norms was the implementation of legal constraints hindering the hours of labor women may work. For example, Holloway (2005) notes that factory work typically was on average the highest pay a woman could receive in the mid to late nineteenth century. However, the implementation of the various Factory Acts, including the 1844 Factory Act which limited hours for women to twelve and then again to ten in the following act in 1847. Instead of being implemented based on ideas to improve working conditions, Holloway links these acts being implications of patriarchal pressures that implied that the factory would detract from a woman's socially perceived domicile obligations. Instead, the firm (and society) would have little incentive to limit labor, as such an artificial constraint would create surplus of labor.

Moreover, Roberts (1988) contends that some male workers may have also felt the emergence of female workers would threaten their employments. To protect their jobs, a possible solution for males would be to lobby for stricter laws and policies forbidding the female work at all and/or limiting the wages and the type of jobs women may hold. She points out the example of the 1842 Yorkshire Short Time Committee which "demanded not merely that women should work only 'short' time but also that all married women should be banned from paid work while their husbands were employed" (58). Roberts notes that other trades tried to ban women from employment in industries indirectly by banning them from trade membership.[10]

1.4 Explaining the Demographic Transition in England

I will now examine the demographic transition in England following its Industrial Revolution closely to consider data during this period. England is an ideal choice to study first because of its association as a standard example first nation to undergo the transition from agriculture to an industrial state. Moreover, Wrigley & Schofield (1981) provide a wealth of data estimating most pertinent growth and demographic statistics between 1541 and 1871.

As was typical in most nations before their shift into industrialization, England's long run population and growth rate was largely stagnant before its Industrial Revolution. Figure 1 shows the development of such statistics from the middle of the sixteenth century to the late nineteenth century. Over this period population remained constrained between around three to five million inhabitants within England. On the other hand growth rates varied around .5%, but stayed largely positive as a whole.[11] However, around the period roughly associated with England's first Industrial Revolution (mid eighteenth century), population and growth rates change from their historical trends. Population begins to increase dramatically, nearly exponentially. In 1751, England's population was a little over five million. By 1811 population had doubled and would double once again by 1856. Growth rates rose, now hovering above one percent annual growth, in essence a doubling of the growth rate.

[10] Roberts notes that the last quarter of the nineteenth century saw a dramatic rise in female labor union participation, especially relative to previous years.

[11] One large exception to this would be associated with the period of the mid 1500s in which the Bubonic Plaque ravaged England's population and thus its work force.

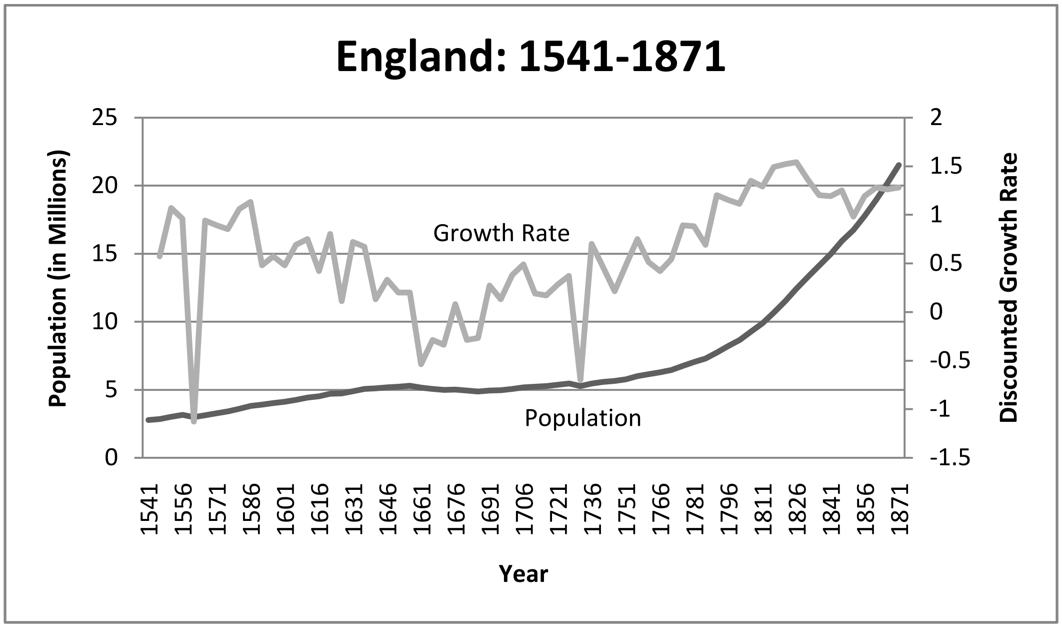

Figure 1: England 1541-1871 Growth Rate & Population
Source: Wrigley & Schofield (1981)

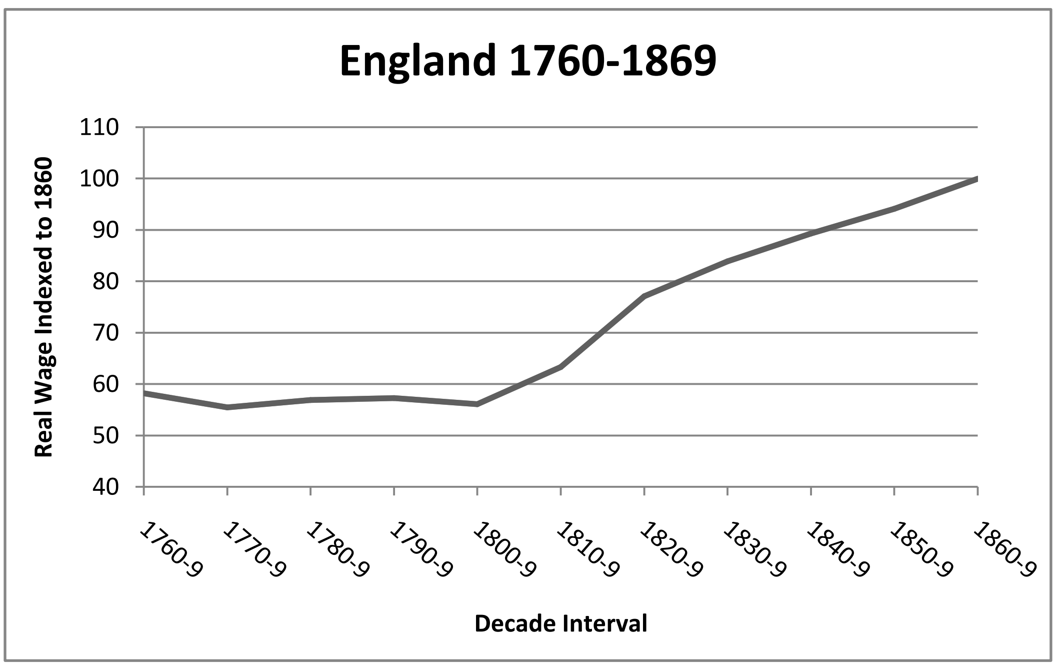

Figure 2: England 1760-1869 Real Wages
Source: Clark (2001)

Considering these factors from an individual perspective, Figure 2 shows the increase in real wages during the Industrial Revolution. While real wages nearly remained stagnate between 1760 until the 1800s, by 1860 real wages had nearly doubled. Moreover, demographic phenomenon appears elevated for the decades following the

Industrial Revolution. Figure 3 represents crude birth and death rates in England from 1541 to 1986. Between the sixteenth to the eighteenth centuries, the variation in crude birth and death rates in England was rather low with the exception of a few outliers. Nevertheless, the nineteenth century represented the beginnings of a sweeping change in English demographics. Roughly around this period, crude death rates started to decline and continued until the levels started to stagnate in the 1920s. Crude birth rates, on the other hand, start to increase but later stagnate briefly before England begins to undergo a

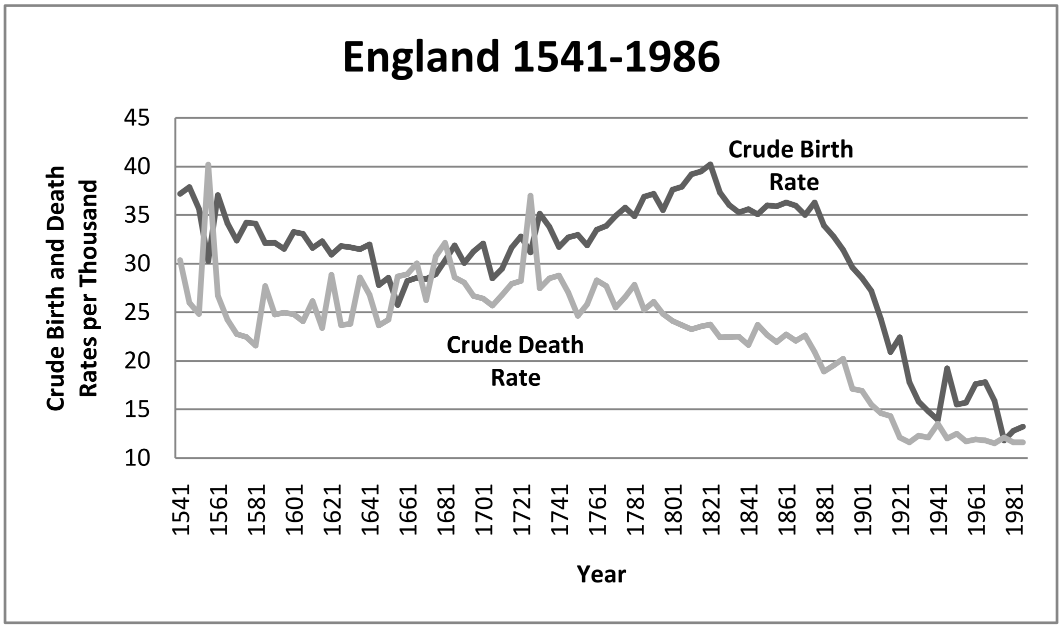

Figure 3: England 1541-1986 Birth & Mortality Rates
Source: Wrigley and Schofield (1981); Mitchell (1975)

dramatic drop in fertility rates around 1881.[12] The long run aggregate population effects of such changes are somewhat reflected in Figure 1. The initial drop in death rates while fertility rates start to increase in the early 1800s stimulates total population. Unlike the reinforcement of the Malthusian mechanism to eventually push down aggregate income gains, growth continues as population additionally increases. However, the eventual drop in fertility rates begins to reduce population growth. As technological and industrialization continues to increase, real per capita output continues to rise despite of population levels. In fact, as shown in Figure 4, from the early 1900s to the present

[12] This drop in fertility rates is the key data moment I calibrate the model to exhibit in Section IV.

output per capita continues to grow exponentially as population rises but somewhat through a linear process, with some recent signs of slowing down.[13]

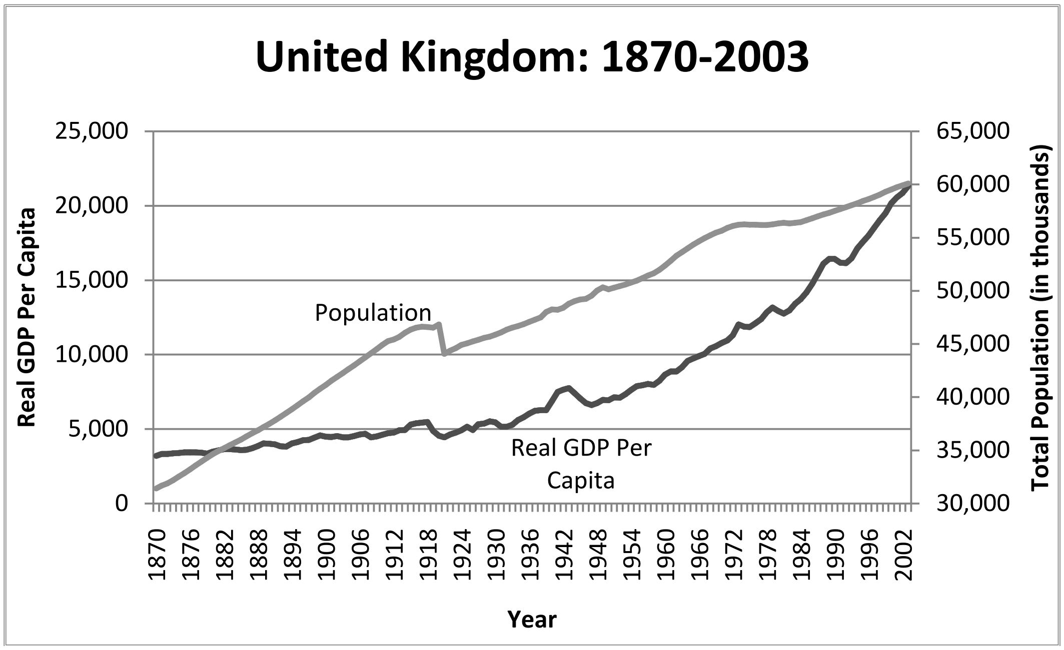

Figure 4: United Kingdom 1870-2003 Population & Real GDP Per Capita
Source: Maddison (2001)

2 Model

2.1 The Model

I develop an overlapping-generations dynamic general equilibrium model[14] to explain a mechanism affecting household fertility during the English demographic transition. I examine the role of a relaxation of social norms which limit labor market opportunities for women.[15] The model uses the basic framework of Galor & Weil (1996).[16]

[13] Note that Figure 4 represents the United Kingdom and not just England. However, the similarities between the growth of output per capita and population between England and the rest of the U.K. seem minute.

[14] See de la Croix & Michel (2002) for considerations for overlapping-generations general equilibrium models.

[15] As the model considers one possible explanation for the demographic transition, by no means is this only possible cause. In fact, there may be a myriad of causes. However, to examine fully the possibility of such a mechanism at work, the model simplifies other potential factors like increases in life expectancy, decreases in infant mortality and a shift towards quality versus quantity of offspring. I do this simplification to better evaluate the potential role the mechanism played in the demographic transition.

[16] The model differs from Galor & Weil as the change in women's income is assumed to be a result of social norms. Galor & Weil posit that an increase in capital following the Industrial Revolution raises women's relative income higher than male's income since their labor is assumed to be more complementary to such gains in capital.

Time t is discrete and goes from 0 to ∞. t belongs to the set of natural numbers such that $t = 0,1,2\ldots$.

2.2 Household Sector

Each household consists of a male (husband) and a female (wife). Each household lives for three periods: childhood (t - 1), adulthood (t) and retirement ($t+1$). During childhood and retirement, households do not work and are not endowed with labor. In an adult household, the husband and wife is endowed each with one unit of labor that he/she may allocate either to working in the economy or working in the household (i.e. raising offspring). The amount of labor spent raising offspring for the husband is l_t^m and l_t^f for the wife in period t so that $l_t^m, l_t^f \in [0,1]$.

Each household produces n_t actual young households for period t. So n_t is the number of young households (offspring) who will survive until the next period (until adulthood).

The household's utility during period t is a function of the number of young households in period t, the consumption of the household in period $t+1$ (c_{t+1}) and the amount of household labor allocated per each child in period t (q_t).The preferences of the household are described by

$$U_t = (1-\gamma)\ln n_t + \gamma \ln c_{t+1}. \tag{1}$$

The parameters $1-\gamma$ and γ are the utility weights for n_t and c_{t+1} respectively such that $\gamma \in (0,1)$. The model does not include a bequest motive. As in Galor & Weil (1996), instead I assume that the adult household will not consume their income at period t. Rather each adult household will save its entire income in order to consume its total savings plus accrued interest in the following period ("retirement"). I make this assumption to ensure that households do not consume all of their income immediately. If each household would consume its entire income without savings, the economy would not accumulate capital over time. Alternatively, parents could derive utility from future consumption of their offspring.

Home production consists only of child rearing. The total amount of labor the parents allocate to working in the household at period t ($l_t^m + l_t^f$) is equal to the parents' labor spent raising their offspring at period t $(n_t q_t)$. The number of offspring per household is given by

$$n_t = \frac{1}{q_t}\left(l_t^m + l_t^f\right), \tag{2}$$

where q_t is the required time cost of raising one surviving offspring.

The income of the household is the sum of the parents' incomes. The husband's income is $w_t\left(1-l_t^m\right)$. The wife's income is $w_t\left(1-l_t^f\right)\left(1-\tau_t\right)$. Parameter τ_t is a wedge representing social norms that constrain the wife's income during period *t*, where $\tau_t \in (0,1)$. The household's income goes directly into savings because the adult household defers consumption until the following (retirement) period. The budget constraint faced by an adult household in period *t* is

$$s_t = w_t\left[\left(1-l_t^m\right)+\left(1-\tau_t\right)\left(1-l_t^f\right)\right]. \tag{3}$$

Retired households consume the household's discounted savings, period $t+1$, and do not allocate any consumption directly to their offspring. The budget constraint faced by an older household is

$$c_{t+1} = r_{t+1}s_t, \tag{4}$$

where r_{t+1} is the rental rate of capital paid in period $t + 1$.[17] Capital depreciates entirely at the end of each period.[18]

As there are three stages for households, there are also three types of households: a young household living during period (t – 1), an adult household living during period (t), and a retired household living during (t + 1). The total number of households (N_t) can be divided into three distinct sub-populations at period *t*: the number of young households (N_t^c), the number of adult households (N_t^a),[19] and number of retired households (N_t^e). So the economy's total household population is

[17] Implicitly, I assume that households have perfect foresight i.e., household's expectation for the rental rate of capital for the next period (*t +1*) is exactly what the rental rate of capital will be.
[18] This assumption is plausible considering the length (25 years) of each period.
[19] Note that while only adults are endowed with labor, the total population of adults (N_t^a) does not necessarily equal the total economy labor force since individuals may allocate labor to the household and they may be endowed with more than one unit of labor.

$$N_t = N_t^c + N_t^a + N_t^e. \tag{5}$$

As each household transitions into the next for the following period, the total number of households in each life stage transitions from one to the next at the same rate. The number of young households in period *t* is the product of the number of offspring per household and the number of adult households from period *t* ($N_t^c = n_t N_t^a$). The population of adult households at period *t* is the number of young households from period *t* – 1 ($N_t^a = N_{t-1}^c$). The number of retired households at period *t* is the number of adult households from period *t* – 1 ($N_t^e = N_{t-1}^a$). Following the period of retirement, households leave the population. The change in the household population is represented by

$$\frac{N_{t+1}^c + N_{t+1}^a + N_{t+1}^e}{N_t^c + N_t^a + N_t^e}.$$

Upon simplification (see Appendix 6.1), the population growth rate is

$$\frac{n_{t-1}\left(n_t\left(n_{t+1}+1\right)+1\right)}{\left(n_{t-1}\left(n_t+1\right)+1\right)}. \tag{6}$$

The labor supply of the economy is an aggregation of the individual household's optimization problem, allocating labor either to the economy or to the household. The economy's total labor supply is given by

$$L_t = N_t^a\left(2 - l_t^m - l_t^f\right). \tag{7}$$

2.3 Production Sector

Assume a standard neoclassical production technology that exhibits constant returns to scale. Because of constant returns to scale, the number of firms is irrelevant. There exists one aggregate firm whose total output (Y_t) at period *t* is represented by following the Cobb-Douglas production function

$$Y_t = Z_t K_t^\alpha L_t^{1-\alpha},\ 0 < \alpha < 1, \tag{8}$$

where Z_t is the level of technology at period *t*, K_t is the capital stock at period *t* and L_t is the labor supply at period *t*.

The firm's profit is its total revenue minus its total costs. The firm's total revenue is the product of its output and the price level for period *t*. The firm's total cost is the sum of the market wage times the number of laborers and the rental rate of capital times the capital stock during period *t*. The firm's profit function for period *t* is[20]

$$\Pi_t = Y_t - w_t L_t - r_t K_t . \tag{9}$$

To reflect the competitive nature of production, the firm takes the market wage (w_t) and the rental rate of capital (r_t) as given by the market. Under this knowledge, the firm's choice variables are (L_t) and (K_t) as the firm maximizes its profit. Since there exists only one firm in the economy, the variable L_t makes up the labor market's demand for labor and K_t represents the demand for loanable funds.

2.4 Competitive Equilibrium

I may now define a competitive equilibrium within the model.

Definition 2.3.1 Given an initial endowment $\{s_0, Z_0, K_0, N_0^a, N_0^e\} \in \mathbb{R}_{++}$, a competitive equilibrium is given by allocations $\{l_t^{f*}, l_t^{m*}, c_t^*, s_t^*, K_t^*, L_t^*\}_{t=0}^{\infty}$, prices $\{w_t^*, r_t^*\}_{t=0}^{\infty}$, population variables $\{n_t^*, N_t^{c*}, N_t^{a*}, N_t^{e*}\}_{t=0}^{\infty}$ such that:

1. The household maximizes its utility (1) subject to (2), (3) and (4)

$$\max_{l_t^f, l_t^m, n_t, c_{t+1}} (1-\gamma)\ln n_t + \gamma \ln c_{t+1}$$

subject to

$$s_t = w_t\left(1-l_t^m\right) + w_t\left(1-l_t^f\right)\left(1-\tau_t\right)$$

$$c_{t+1} = s_t r_{t+1}$$

$$n_t = \frac{1}{q_t}\left(l_t^f + l_t^m\right).$$

2. The firm maximizes profit (11)

$$\max_{K_t, L_t} Z_t K_t^{\alpha} L_t^{1-\alpha} - r_t K_t - w_t L_t .$$

[20] Note that the firm pays each gender the same wage. To the firm, female and male labor are perfect substitutes. The wedge in female wages is loss to the household following the distribution of income to the wife. Therefore, the firm will still pay the wife her full wage; however, she will only collect her income minus the wedge. The difference in this transaction is assumed to be lost for society.

3. The labor market clears when the demand of labor from the firm equals the supply of labor from households (7). This condition holds when

$$L_t = N_t^a\left(2 - l_t^f - l_t^m\right).$$

4. The asset market clears when the supply of loanable funds from the households ($s_{t-1}N_{t-1}^a$) equals the demand for loanable funds by the firm (K_t),

$$K_t = s_{t-1}N_{t-1}^a.$$

5. Feasibility occurs when total output ($ZK_t^\alpha L_t^{1-\alpha}$) equals households' consumption ($c_t N_t^e$) plus investment ($s_t N_t^a$) minus the total wedge on all female wages ($\tau_t w_t\left(1-l_t^f\right)N_t^a$) such that

$$s_t N_t^a + c_t N_t^e - \tau_t w_t\left(1-l_t^f\right)N_t^a = Z_t K_t^\alpha L_t^{1-\alpha}.$$

6. The population dynamics hold such that

$$N_{t+1}^a = N_t^c,\ N_{t+1}^e = N_t^a, \text{ and } N_t^c = n_t N_t^a,\ \forall t.$$

2.5 Household's Optimization Problem

Given r_{t+1} and w_t, an adult household living in period t solves

$$\max_{n_t, s_t, c_{t+1}} (1-\gamma)\ln n_t + \gamma \ln c_{t+1}$$

subject to

$$\begin{aligned} s_t &= w_t\left[\left(1-l_t^m\right)+\left(1-\tau_t\right)\left(1-l_t^f\right)\right], \\ c_{t+1} &= s_t r_{t+1}, \\ n_t &= \frac{1}{q_t}\left(l_t^f + l_t^m\right), \end{aligned}$$

where n_t is the number of surviving children. Here q_t measures the amount of time needed to raise each surviving child. The variable q_t can be thought of as a decreasing function of the child survival rate. Therefore, a decline in child mortality would make the household production sector more productive.

The household's decision simplifies to

$$\max_{n_t, s_t, c_{t+1}} \left(1-\gamma\right)\ln\left(\frac{1}{q_t}\left(l_t^f + l_t^m\right)\right) + \gamma \ln r_{t+1} w_t\left[\left(1-l_t^m\right)+\left(1-\tau_t\right)\left(1-l_t^f\right)\right], \text{ or}$$

$$\max_{n_t, s_t, c_{t+1}} (1-\gamma)\ln\left(l_t^f + l_t^m\right) + \gamma \ln w_t\left[\left(1-l_t^m\right) + \left(1-\tau_t\right)\left(1-l_t^f\right)\right].$$

Lemma 2.5.1 *In equilibrium,* $l_t^f + l_t^m > 0$ *and* $\left(1-l_t^m\right) + \left(1-\tau_t\right)\left(1-l_t^f\right) > 0$.

Proof Otherwise, $U = -\infty$ ■

Proposition 2.5.2 *An interior solution with* $l_t^m, l_t^f \in (0,1)$ *does not exist.*

Proof For the sake of contradiction, suppose that an interior solution exists. Suppose $\bar{l}_t^m, \bar{l}_t^f \in (0,1)$ maximizes the objective function. Let $\varepsilon > 0$, where ε is arbitrarily small. Then consider an alternative solution $\hat{l}_t^m, \hat{l}_t^f = \left(\bar{l}_t^m - \varepsilon, \bar{l}_t^f + \varepsilon\right) > 0$. Then

$$\hat{l}_t^m + \hat{l}_t^f = \bar{l}_t^m + \bar{l}_t^f$$

so the first term in the objective function is the same. But the second term,

$$\begin{aligned}\left(1-\hat{l}_t^m\right) + \left(1-\tau_t\right)\left(1-\hat{l}_t^f\right) &= \left(1-\bar{l}_t^m + \varepsilon\right) + \left(1-\tau_t\right)\left(1-\bar{l}_t^m - \varepsilon\right) \\ &= \left(1-\bar{l}_t^m\right) + \left(1-\tau_t\right)\left(1-\bar{l}_t^m\right) + \varepsilon - \left(1-\tau_t\right)\varepsilon,\end{aligned}$$

is larger. Contradiction. ■

Corollary 2.5.3 *Assume* $\tau_t \neq 0$, $\forall t$. *Either (Case 1)* $l_t^f = 1$ *and* $0 < 1 - l_t^m < 1$; *(Case 2)* $l_t^f = 0$ *and* $l_t^m = 0$; *(Case 3)* $0 < 1 - l_t^f < 1$, $l_t^m = 0$.

Hence, one of the following three cases is established in equilibrium.

Case 1 Wife allocates zero labor to the market. Husband allocates positive labor to both the economy and the household.

$$l_t^f = 1$$
$$0 < 1 - l_t^m < 1$$

Maximization problem then reduces to

$$\max_{n_t, s_t, c_{t+1}} (1-\gamma)\ln\left(l_t^m + 1\right) + \gamma \ln\left(1 - l_t^m\right).$$

The solution is

$$l_t^m = 1 - 2\gamma .$$

This case arises when $1 - 2\gamma > 0$.

Case 2 Full specialization.

$$l_t^m = 0$$
$$l_t^f = 1$$

Case 3 Husband allocates zero labor to home production. Wife allocates positive labor to both the economy and the household.

$$0 < 1 - l_t^f < 1$$
$$l_t^m = 0$$

Maximization problem then reduces to

$$\max_{n_t, s_t, c_{t+1}} (1-\gamma)\ln\left(l_t^f + 1\right) + \gamma \ln\left[1 + (1-\tau_t)\left(1 - l_t^f\right)\right].$$

The solution is

$$l_t^f = \frac{(2-\tau_t)(1-\gamma)}{1-\tau_t} .$$

This case arises when $0 < \dfrac{(2-\tau_t)(1-\gamma)}{1-\tau_t} < 1$.

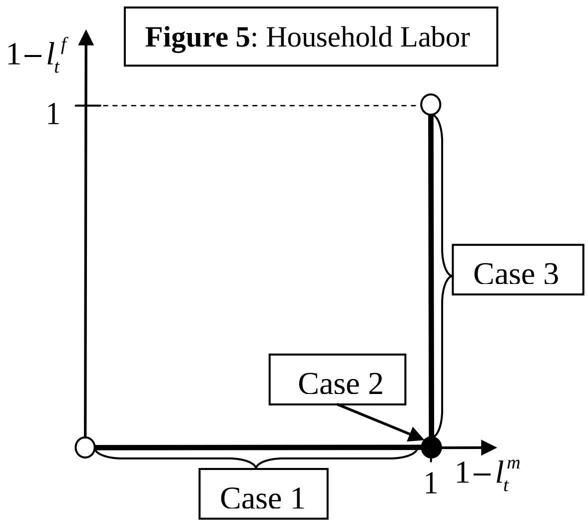

Figure 5: Household Labor

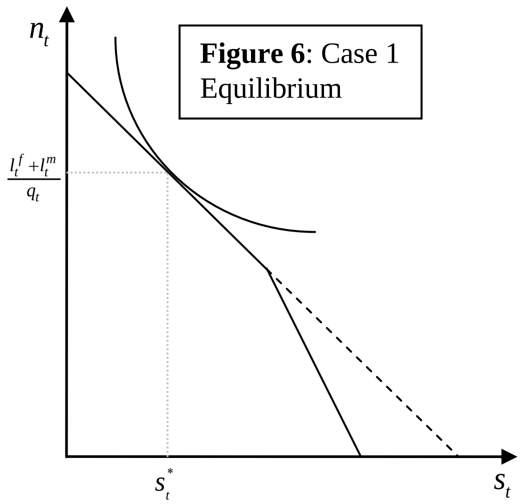

Figure 6: Case 1 Equilibrium

Considering the three different cases, only one case may occur given an equilibrium. Figure 5 reveals this conclusion graphically and shows the possible labor allocations to household production for each parent.

When the household allocates between one and two units of labor in period t (Case 1), both parents will allocate at least some of their labor to the household. If $\tau \neq 0$, the wife will allocate her full labor endowment to the household as the husband will allocate part of his labor allocation to the household. The sum of their household allocation will equal the total household time spent rearing the household's offspring. The husband will, be the only parent to work in the economy; therefore he will act as the breadwinner and his income will equal the household's income (and savings).

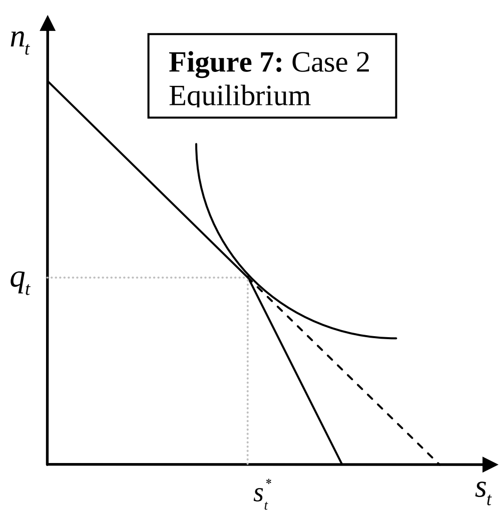

Figure 7: Case 2 Equilibrium

When the household allocates exactly one unit of labor in period t (Case 2), the parents will specialize their labor. That is, the wife will allocate all of her labor to the household and the husband will allocate all of his labor to the economy. So the household income will be exactly the income of the husband as the wife does not work in the economy

When the household allocates between zero and one unit of labor in period *t* (Case 3) the wife will be the only parent to work in the household (assuming that $\tau \neq 0$). Therefore, the husband will work only in the economy and the wife will work in both the economy and in the household. Under this equilibrium, the wife's labor allocation to the household will equal the total household labor spent raising the household's offspring, as the husband does not allocate any labor to the household.

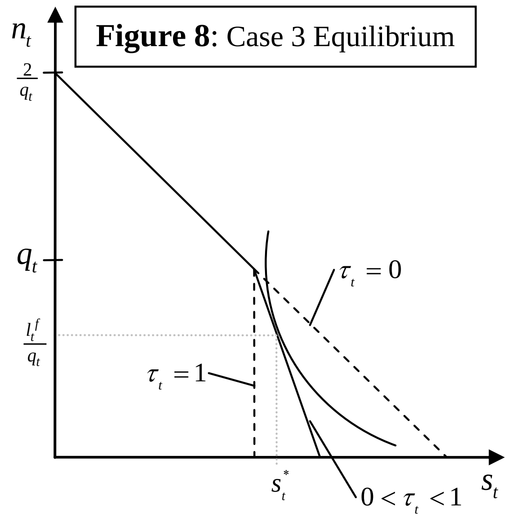

Figure 8: Case 3 Equilibrium

To evaluate the basic mechanisms at work within the household's optimization problem, consider the comparative statics within the household's decision.

Proposition 2.5.4 *Assume an equilibrium exists (either Case 1, Case 2 or Case 3). An exogenous change in* w_t *does not affect* n_t. (See Appendix A.3)

Proposition 2.5.4 is shown graphically in Figure 5. An exogenous change in w_t (*ceteris paribus*) will shift out the household's budget constraint. However, the end result of the change will only translate into an increase in household income (or savings). As n_t is a function of household labor, since n_t stays constant, there is no change in the household (or economy) labor allocation. This result implies that the household labor supply is perfectly inelastic to changes in w_t. Therefore, the Malthusian positive relationship between income and fertility may exist, but only through an increase in household labor allocated to the economy. If the household works more in the economy and less in the household, household income will rise holding constant w_t and fertility will drop as parents spend less labor towards childrearing.

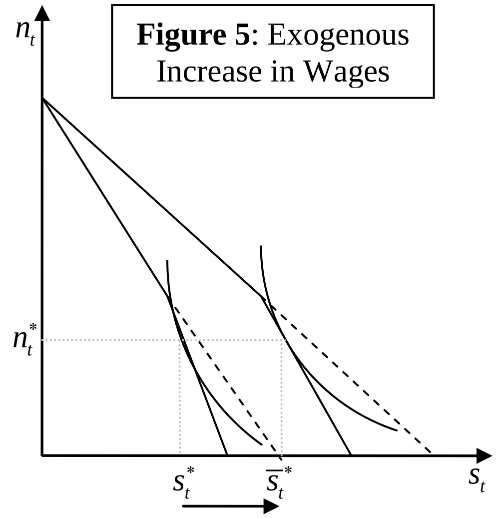

Figure 5: Exogenous Increase in Wages

Proposition 2.5.5 *Assume a Case 3 equilibrium exists. An exogenous change in* τ_t *will result in a positive change in* n_t. (See Appendix A.4)

Figure 6 depicts the mechanisms of Proposition 2.5.5 graphically. An exogenous decrease in τ_t under a Case 3 equilibrium will, holding all other variables constant, shift out the 'kinked' portion of the household's budget constraint. This shift reflects the decrease in the opportunity costs to enter the workforce for the wife. As such, the wife will substitute her labor away from the household and towards working in the economy. As the wife is the only parent to supply labor to household production, the household will spend less labor on childrearing and thus n_t will decrease.

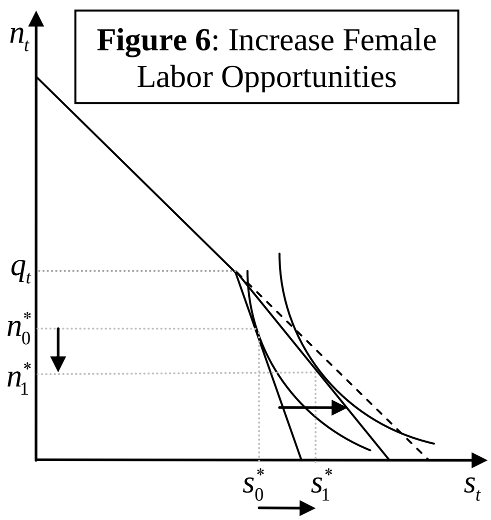

Figure 6: Increase Female Labor Opportunities

2.6 Firm's Optimization Problem

As constant returns to scale exists, perfect competition characterizes the economy in which firms make zero long run economic profit. Under such conditions, firms' pay the real market wage (w_t) that is equal to the marginal product of labor[21] and the rental rate of capital (r_t) that is equal to the marginal product of capital.[22]

The firm's maximization problem is

$$\max_{K_t, L_t} Z_t K_t^{\alpha} L_t^{1-\alpha} - r_t K_t - w_t L_t \tag{12}$$

subject to

$$\frac{\partial \Pi_t}{\partial K_t} = r_t, \ \frac{\partial \Pi_t}{\partial L_t} = w_t . \tag{13}$$

Definition 2.5.1 An equilibrium for the firm's maximization problem is at $\mathbf{p}_t^* = (w_t^*, r_t^*)$ in which the firm maximizes its profit as

$$\max_{K_t, L_t} Z_t K_t^{\alpha} L_t^{1-\alpha} - \alpha Z_t K_t \left(\frac{K_t}{L_t}\right)^{\alpha-1} - (1-\alpha) Z_t L_t \left(\frac{K_t}{L_t}\right)^{\alpha} . \tag{14}$$

2.7 Dynamics of the Model

2.7.1 Steady State Equilibria

Let us introduce the definition and the conditions for the existence of a steady state equilibrium of a linear function.

Definition 2.7.1 *A steady state equilibrium of the linear difference equation* $x_{t+1} = ax_t + b$ *is* $\bar{x} \in \mathbb{R}$ *such that* $\bar{x} = a\bar{x} + b$.

Proposition 2.7.1 *(Existence of Steady State Equilibrium) A steady-state equilibrium of the difference equation* $x_{t+1} = ax_t + b$ *exists if and only if*

$$\{a \neq 1\} \text{ or } \{a = 1 \text{ and } b = 0\} .$$

Proof For $\bar{x} \in \mathbb{R}$, rearranging the definition in 2.7.1[23] yields

[21] An important note is that while the firm pays the market wage (w_t), assuming $\tau \neq 0$, women do not receive the full market wage. Instead, the difference between the market wage and the wage paid to the wife is a deadweight loss. Therefore, considering that the labor for males and females are perfect substitutes, the firm is indifferent to hiring either women or men.

[22] We assume the economy is closed, restricting any trade, migration or capital flows.

$$\bar{x} = \begin{cases} \frac{b}{1-a} & \text{if} \quad a \neq 1 \\ x_0 & \text{if} \quad a = 1 \text{ and } b = 0. \end{cases} \quad \blacksquare$$

Let us derive the adult household law of motion. Recall that $N^a_{t+1} = N^a_t n_t$. Plugging in (2) of the household decision yields

$$N^a_{t+1} = \tfrac{1}{q_t}\left(l^f_t + l^m_t\right) N^a_t .$$

Under a Case 3 equilibrium, only the wife will allocate labor to household production. Substituting the solution to the Case 3 equilibrium of the household decision yields

$$N^a_{t+1} = \tfrac{1}{q_t}\left(\tfrac{(1-\gamma)(2-\tau_t)}{1-\tau_t}\right) N^a_t .$$

Proposition 2.7.2 *Assume a Case 3 equilibrium and initial conditions* $N^a_0 > 0$. *Then* $\lim_{t\to\infty} N^a_t = \infty$.

Proof For the sake of contradiction, suppose $\exists \bar{N}^a \in \mathbb{R}$, where $\bar{N}^a$ is the steady state. Recall $N^a_{t+1} = \tfrac{1}{q_t}\left(\tfrac{(1-\gamma)(2-\tau_t)}{1-\tau_t}\right) N^a_t$, where $a = \tfrac{1}{q_t}\left(\tfrac{(1-\gamma)(2-\tau_t)}{1-\tau_t}\right)$ and $b = 0$. Since $\tfrac{1}{q_t}\left(\tfrac{(1-\gamma)(2-\tau_t)}{1-\tau_t}\right) > 1$, for $\bar{N}^a \in \mathbb{R}$, $\bar{N}^a = \tfrac{b}{1-a}$. As $b = 0$, then $\bar{N}^a = 0$. However, $\bar{N}^a = N_0 > 0$. Contradiction. $\blacksquare$

Proposition 2.7.2 is shown graphically in Figure 7. As $N^a_0 > 0$, notice that N^a_{t+1} continues to increase by as $t \to \infty$.

[23] See Galor (2007).

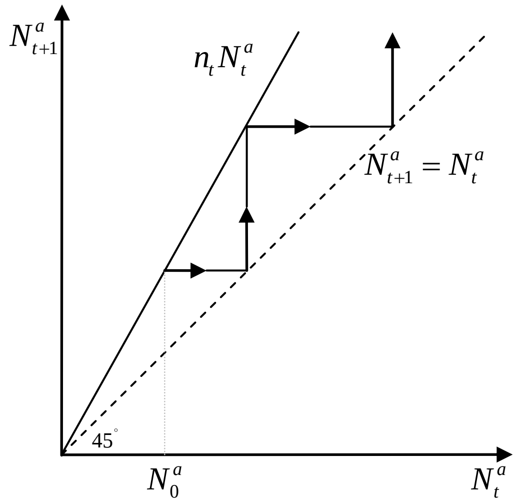

Figure 7: Adult Household

From (5), the total number of households is

$$N_t = N_t^c + N_t^a + N_t^e \ ,$$

N_t diverges as well (i.e. $\lim_{t\to\infty} N_t = \infty$).

The determinant of knowledge accumulation is a function of a steady growth rate (where $0 < g < 1$) given an initial knowledge stock $Z_0 > 0$. The function for knowledge accumulation is

$$Z_t = Z_0 \left(1+g\right)^t . \tag{15}$$

Let $\frac{Z_t}{N_t^a} \equiv z_t$, where z_t is the stock of knowledge per adult household. Substituting this equation yields

$$\frac{Z_{t+1}}{N_{t+1}^a} = \frac{\left(1+g\right)}{n_t} \frac{Z_t}{N_t^a} ,$$

which implies

$$z_{t+1} = z_t \frac{\left(1+g\right)}{n_t} .$$

Proposition 2.7.3 *Assume* $Z_0 > 0$. *Then* $\lim_{t\to\infty} z_t = \infty$.

Proof As $t \to \infty$, z_t increases without bound by $\frac{\left(1+g\right)}{n_t} > 0$. ■

The determinant for capital accumulation is a function of the previous period investment with full depreciation each period. Assume a competitive equilibrium in which a Case 3 equilibrium characterizes the household problem. Because of the assumption of a closed economy,[24] the capital stock at period $t+1$ is

$$K_{t+1} = N_t^a s_t . \tag{16}$$

To consider the existence of a steady state, let us examine such a case under a Case 3 equilibrium to the household problem. Substituting $s_t = w_t\left[1+\left(1-\tau_t\right)\left(1-l_t^f\right)\right]$ into (16) yields

$$K_{t+1} = N_t^a w_t\left(1+\left(1-\tau_t\right)\left[1-\tfrac{(1-\gamma)(2-\tau_t)}{1-\tau_t}\right]\right).$$

Substituting $w_t = \left(1-\alpha\right)Z_t\left(\frac{K_t}{L_t}\right)^{\alpha}$ and condition 3 of a competitive equilibrium with a Case 3 equilibrium, $L_t = N_t^a\left(2-\frac{(1-\gamma)(2-\tau_t)}{1-\tau_t}\right)$, yields

$$K_{t+1} = \left(1-\alpha\right)Z_t K_t^{\alpha}\left(N_t^a\right)^{1-\alpha}\left[2-\tfrac{(1-\gamma)(2-\tau_t)}{1-\tau_t}\right]^{-\alpha} . \tag{17}$$

Substituting and dividing both sides by N_{t+1}^a yields $\frac{K_{t+1}}{N_{t+1}^a} = f\left(\frac{K_t}{N_t^a}\right)$ so that

$$k_{t+1} = \gamma q Z_t \frac{\left(1-\alpha\right)\left(1-\tau\right)}{\left(1-\gamma\right)\left[2-\frac{(1-\gamma)(2-\tau_t)}{1-\tau_t}\right]^{\alpha}}\left(k_t\right)^{\alpha},$$

where $k_t \equiv \frac{K_t}{N_t^a}$. Let $\Phi = \frac{\gamma}{1-\gamma}\frac{\left(1-\alpha\right)\left(1-\tau\right)}{\left[2-\frac{(1-\gamma)(2-\tau_t)}{1-\tau_t}\right]^{\alpha}}$, were Φ is some constant, then the capital per household transition equation is

$$k_{t+1} = q Z_t \Phi\left(k_t\right)^{\alpha} . \tag{18}$$

Definition 2.7.2 *For a nonlinear function, a steady state equilibrium of the difference equation* $x_{t+1} = f\left(x_t\right)$ *is* $\bar{x} \in \mathbb{R}$ such that $\bar{x} = f\left(\bar{x}\right)$.

[24] Therefore, total savings equals total investment during period *t*.

Using this definition, let us show that capital per household converges to a steady state as $t \to \infty$.

Proposition 2.7.4 *Assume a competitive equilibrium with a Case 3 equilibrium. Let* $k_{t+1} = f\left(k_t\right)$*. It follows that* $\bar{k} = f\left(\bar{k}\right)$*, where* $\bar{k} \in \mathbb{R}$*.*

Proof Since $N_0^a, K_0 > 0$, k_{t+1} evolves to $\bar{k}$ such that $\lim_{t\to\infty} k_t = \bar{k}$. ■

Now I solve for the steady state level of capital per household. Because $\bar{k} = f\left(\bar{k}\right)$, rearranging the terms yields

$$\bar{k} = \left(qZ_t\Phi\right)^{\frac{1}{1-\alpha}}. \tag{19}$$

Now consider the evolution of labor per household over time. By condition 3 of a competitive equilibrium, the following equation holds:

$$L_t = N_t^a\left(2 - l_t^f - l_t^m\right).$$

Dividing both sides by N_t^a and letting $l_t \equiv \frac{L_t}{N_t^a}$, where l_t is defined as labor per household, we find that

$$l_t = \left(2 - l_t^f - l_t^m\right).$$

Since this is a constant, labor per household is constant based on the household labor allocations to the economy. Therefore, l_t is a continuum of steady state equilibria.

Now consider output per household. Let $\frac{Y_t}{N_t^a} = y_t$, where we define y_t as output per household at period *t*. From condition 5 of a competitive equilibrium, dividing both sides by the number of adult households yields

$$\frac{Y_t}{N_t^a} = \frac{Z_t K_t^\alpha L_t^{1-\alpha}}{N_t^a}.$$

Multiplying the right hand side by $\frac{\left(N_t^a\right)^{2\alpha}}{\left(N_t^a\right)^{2\alpha}}$, we can reduce the previous equation to

$$y_t = Z_t \frac{K_t^\alpha}{\left(N_t^a\right)^\alpha}\left(N_t^a\right)^\alpha \frac{L_t^{1-\alpha}}{\left(N_t^a\right)^{1-\alpha}}\left(N_t^a\right)^{-\alpha},$$

which yields

$$y_t = Z_t k_t^{\alpha} l_t^{1-\alpha}, \tag{20}$$

where $k_t = \frac{K_t}{N_t^a}, l_t = \frac{L_t}{N_t^a}$. Since we know that $\lim_{t\to\infty} Z_t = \infty$, output per worker only will increase in Z_t once k_t and l_t reach their steady states (i.e. $\lim_{t\to\infty} y_t = \infty$). Therefore, output per worker does not have a steady state.

To consider the steady state equilibrium for savings (income) per household, recall the steady state of capital per household. Assume a competitive equilibrium in which a Case 3 equilibrium characterizes the household problem. Condition (4) of a competitive equilibrium requires

$$K_t = s_{t-1} N_{t-1}^a$$

must hold. Dividing both sides by the number of adult households in period *t* yields

$$\frac{K_t}{N_t^a} = s_{t-1} \frac{N_{t-1}^a}{N_t^a}. \tag{21}$$

Recall that the population dynamics of a competitive equilibrium hold so that $N_t^a = n_{t-1} N_{t-1}^a$. Substituting this equation and $k_t = \frac{K_t}{N_t^a}$ into (21) yields

$$k_t = s_{t-1} \frac{N_{t-1}^a}{n_{t-1} N_{t-1}^a}.$$

Solving for s_{t-1} yields

$$s_{t-1} = n_{t-1} k_t. \tag{22}$$

As $t \to \infty$, both capital per household and surviving fertility per household approaches their steady state, $\bar{k}$ and $\bar{n}$ respectively. Therefore, as $t \to \infty$, savings per household will also converge to a steady state.

Let us now consider the steady state equilibrium for consumption per household. Assume a competitive equilibrium in which a Case 3 equilibrium characterizes the household problem. For period *t +1*, consumption per household is

$$c_t = r_t s_{t-1}. \tag{23}$$

Considering $t \to \infty$, savings per household and the rental rate of capital will both be constants. Therefore, as $t \to \infty$, consumption per household will converge to a steady state.

2.7.2 Dynamic Implications of Change in Social Norms

Let us now consider how capital per household transitions towards the steady state level of capital per household when it begins in disequilibrium.[25] Figure 9a represents equation (17) graphically. If the economy starts at a capital per household level of k_0, capital per household will increase until it converges to the steady state level $\bar{k}$ as it is globally stable. Figure 9b represents the transition towards $\bar{k}$. At first, k_t increases rapidly to "catch up" to its steady state level. However, as k_t begins to approach $\bar{k}$, it begins to slow down its progression until it finally converges.

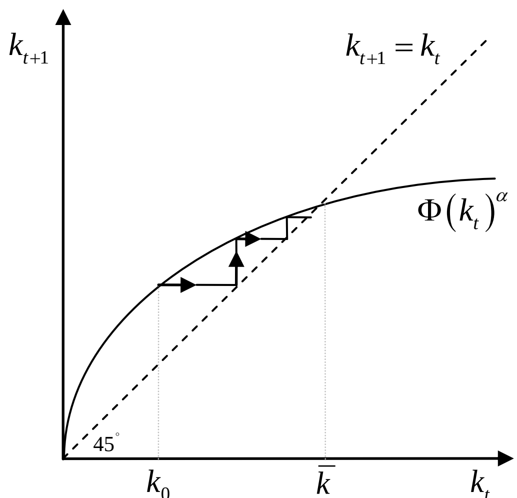

Figure 9: Capital Per Household

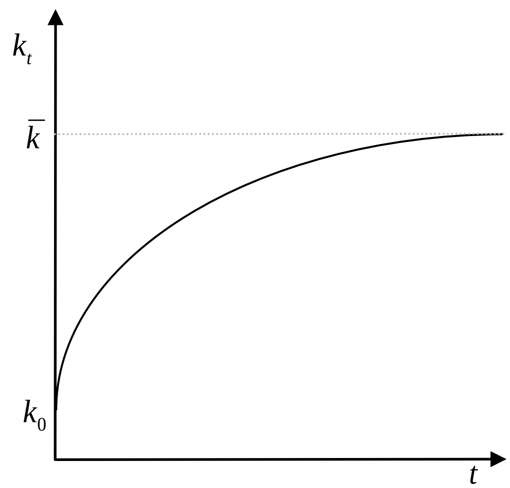

Figure 9: Transition Towards $\bar{k}$

I will now consider the implications with the introduction of an exogenous shock to τ_t within the model. Figure 10 represents the law of motion for capital per household when we reduce τ_t as more labor opportunities are available to women. At first, the economy is at the steady state level $\bar{k}_1$. A reduction of τ_t will shift the curve up and, in the process, the steady state will transition from $\bar{k}_1$ to $\bar{k}_2$. However, k_t does not to its new steady state. It will rise gradually and at a decreasing rate until it reaches its new steady state.

[25] Although not explicitly addressed in this work, see Galor (2007) for the definition and conditions for stability and uniqueness of steady state equilibria. As the transition equation of capital per household is an exponential function, it is assumed that there exists a unique and locally stable steady state equilibrium for capital per household.

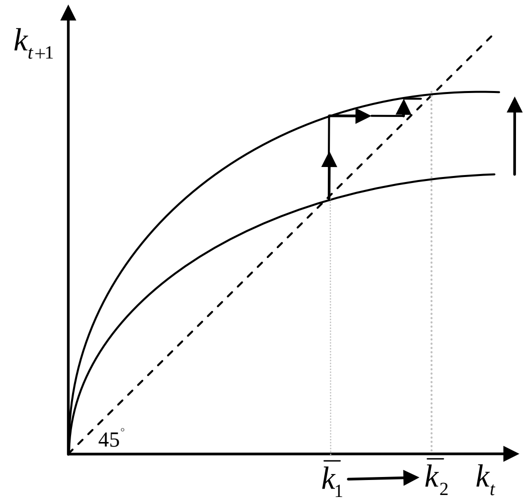

Figure 10: Decrease τ

The change in τ_t, however, has a larger, dynamical effect on other variables within the economy. Figure 11 represents such changes at period t_0. The second panel of Figure 11 represents the decrease in the long run level of fertility from $\bar{n}_1$ to $\bar{n}_2$ as the wife substitutes her labor away from household production and towards working in the economy. The third panel shows how capital per household will increase following the change, but in the long run, such a change will only have a level effect and not a growth effect. Similarly, the fourth and fifth panels represent the effect on output per household. Panel four is the growth rate of output per household. Notice that the change in τ_t will at first increase the growth rate of output per household. However, as time continues, the change will affect the economy's balanced growth path and increase the level of output per household over time. However, this change will not affect the growth rate of output per household on the balanced growth path. Panel five illustrates further that the decrease in τ_t will only have a level effect on output per household.

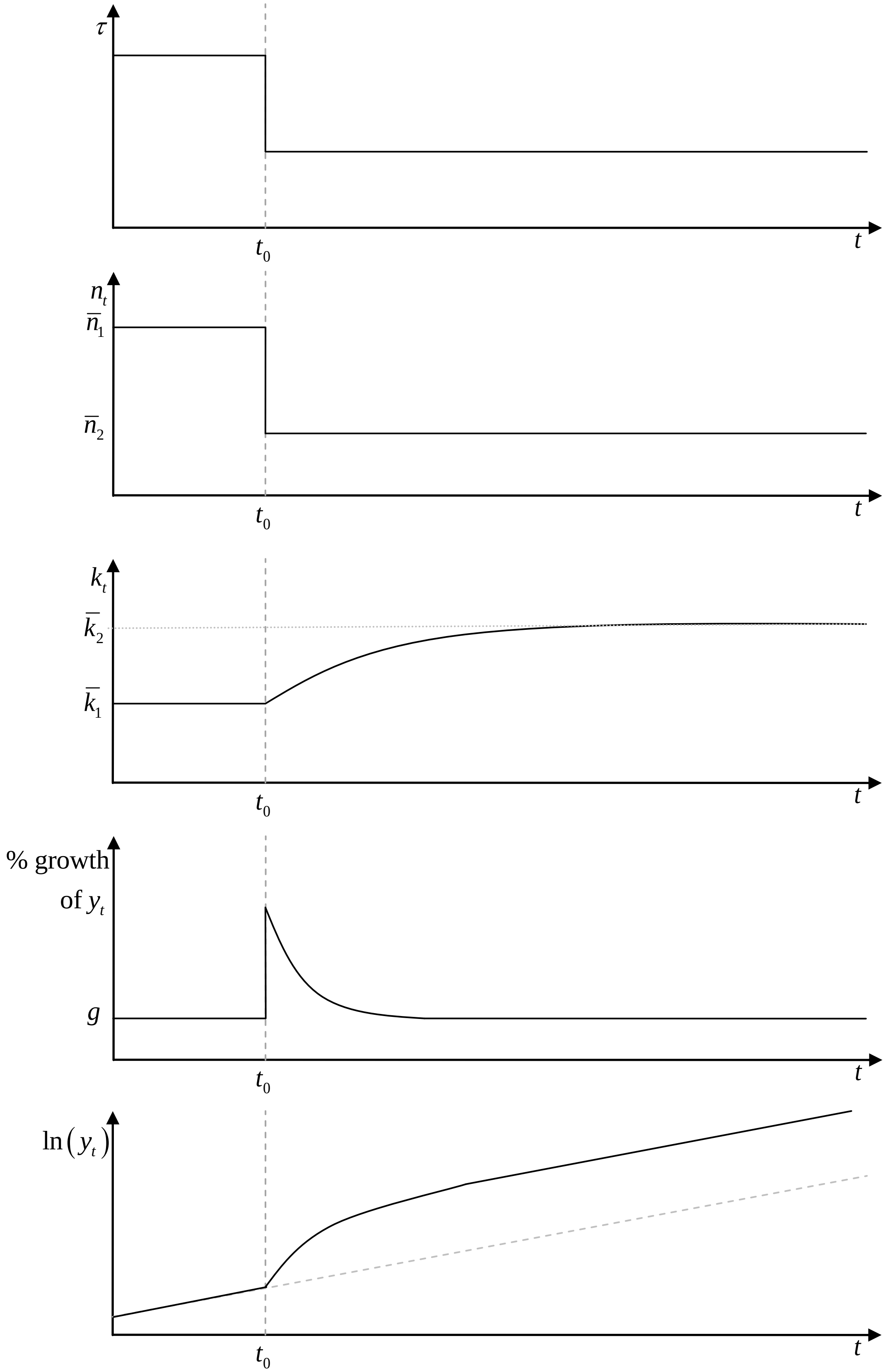

Figure 11: Dynamic Effect of Decrease in τ

3 Quantitative Analysis

3.1 Model Simulation

One objective of this paper is to calibrate and to simulate the model to correspond to key data moments from England 1881-1931, which is the period associated with the English demographic transition. To do so, I first establish the conditions for a balanced growth path in the model. I then establish efficiency variables to use as tools so that I can set the first period of the model (corresponding to 1881) existing with a balanced growth path behavior. Next I calibrate for key parameter values in the model using historical data from England. Finally, I simulate the model and compare the results of the model to historical data to establish the effectiveness of the model to explain the English demographic transition.

3.2 Balanced Growth Path

Let us first assume there exists a competitive equilibrium characterized by a Case 3 equilibrium in household production. We find that:

$$
\begin{aligned}
&l_t^m = 0,\\
&l_t^f = \tfrac{(1-\gamma)(2-\tau_t)}{1-\tau_t},\\
&n_t = \tfrac{1}{q}\left(\tfrac{(1-\gamma)(2-\tau_t)}{1-\tau_t}\right),\\
&s_t = w_t\left(1+\left(1-\tau_t\right)\left(1-\tfrac{(1-\gamma)(2-\tau_t)}{1-\tau_t}\right)\right),\\
&w_t = \left(1-\alpha\right)Z_t\left(\tfrac{K_t}{L_t}\right)^{\alpha},\\
&r_t = \alpha Z_t\left(\tfrac{K_t}{L_t}\right)^{1-\alpha},\\
&L_t = \left(2-\tfrac{(1-\gamma)(2-\tau_t)}{1-\tau_t}\right)N_t^a,\\
&Y_t = Z_t K_t^{\alpha} L_t^{1-\alpha},\\
&K_{t+1} = s_t N_t^a,\\
&N_{t+1}^a = N_t^a n_t.
\end{aligned}
$$

Now let us define the growth for a given variable within an economy.

Definition 3.2.1 The balanced growth rate of a given variable x_t, such that $x_{t+1} = f\left(x_t\right)$, is defined as

$$\gamma_x = \frac{x_{t+1} - x_t}{x_t}, \forall t,$$

where γ_x is some constant such that $\gamma_x \in \mathbb{R}$.

Considering this definition, let us now conjecture to the conditions for the economy to exhibit a balanced growth path (BGP).

Proposition 2.7.3 If the system exhibits BGP behavior, i.e. all variables grow at a constant although not necessarily equal rate, then

$$\gamma_k = \gamma_w = \gamma_y = \gamma_s = (1+g)^{\frac{1}{1-\alpha}},$$

$$\gamma_L = \gamma_{N^a} = n,$$

$$\gamma_l = \gamma_r = 1.$$

Now consider equation (18), which is defined as

$$k_{t+1} = \gamma q Z_t \frac{(1-\alpha)(1-\tau)}{(1-\gamma)\left[2 - \frac{(1-\gamma)(2-\tau_t)}{1-\tau_t}\right]^{\alpha}} (k_t)^{\alpha}.$$

Let us first defined the balanced growth rates for the state variables (k_t, N_t^a). Knowing that γ, q, α and τ are parameters, capital per adult household is growing at a constant rate in period *t +1* (defined as γ_k), this growth will be the product of a constant growth rate of Z_t and $\gamma_k{}^{\alpha}$ (since under k_t is growing at the same rate $\forall t$). Equation (15) yields $\gamma_Z = 1+g$, where γ_Z is a constant growth rate for Z_t . Substituting shows that

$$\gamma_k = (1+g)\gamma_k{}^{\alpha},$$

where γ_k can be solved as

$$\gamma_k = (1+g)^{\frac{1}{1-\alpha}}.$$

Because N_{t+1}^{a} grows at a constant rate of n_t (Recall the population dynamics such that $N_{t+1}^{a} = N_t^a n_t$), then the balanced growth rate of adult households is defined as $\gamma_N = n$.[26] From $L_t = \left(2 - \frac{(1-\gamma)(2-\tau_t)}{1-\tau_t}\right) N_t^a$, we find that $\gamma_L = n$.

Similarly we may find for γ_w, γ_y, and γ_s,

$$\gamma_s = \gamma_y = \gamma_w = (1+g)\gamma_k^{\alpha} = \gamma_k.$$

3.3 Efficiency Variables

Before we may calibrate the model, we need to define the efficiency variable for k_t, defined as $\hat{k}_t$:[27]

$$\hat{k}_t = \frac{k_t}{\left((1+g)^{\frac{1}{1-\alpha}}\right)^t}.$$

Considering now period *t +1* and solving for k_{t+1} yields

$$k_{t+1} = \hat{k}_{t+1}\left((1+g)^{\frac{1}{1-\alpha}}\right)^{t+1}. \tag{24}$$

Substituting the transition equation of capital per household (18) for k_{t+1}, then

$$k_{t+1} = qZ_0\Phi\left(k_t\right)^{\alpha}$$

$$\hat{k}_{t+1}\left((1+g)^{\frac{1}{1-\alpha}}\right)^{t+1} = qZ_0\Phi(1+g)^t\left(\hat{k}_t(1+g)^{\frac{1}{1-\alpha}}\right)^{\alpha}$$

$$\hat{k}_{t+1}\left((1+g)^{\frac{1}{1-\alpha}}\right)^{t}(1+g)^{\frac{1}{1-\alpha}} = qZ_0\Phi(1+g)^t$$

$$\hat{k}_{t+1}\left((1+g)^{\frac{1}{1-\alpha}}\right)^{t+1} = qZ_0\Phi\left(\hat{k}_t\left((1+g)^{\frac{1}{1-\alpha}}\right)^{t}\right)^{\alpha}$$

$$\hat{k}_{t+1}(1+g)^{\frac{1}{1-\alpha}} = qZ_0\Phi\hat{k}_t^{\alpha},$$

---

[26] We assume here that n_t is at a constant (steady state) level.

[27] I will only present the efficiency variable for k_t ; however, as k_t is the state variable, the other efficiency variables are implied by deriving $\hat{k}_t$.

and assuming the efficiency capital per adult household is at its steady state ($k_t = \hat{k}_{ss}, \forall t$), I find that:

$$\hat{k}_{ss} = \left(\frac{qZ_0\Phi}{(1+g)^{1/(1-\alpha)}} \right)^{\frac{1}{1-\alpha}}.$$

By the initial assumption that the economy grows on a BGP at period t = 0, I assume $\hat{k}_t = \hat{k}_{ss}$. We can now find capital per household along a BGP by substituting (24):

$$\begin{aligned} k_t^{BGP} &= \hat{k}_{ss} \left((1+g)^{\frac{1}{1-\alpha}} \right)^t \\ &= \left(\frac{qZ_0\Phi}{(1+g)^{1/(1-\alpha)}} \right)^{\frac{1}{1-\alpha}} \left((1+g)^{\frac{1}{1-\alpha}} \right)^t \\ &= \left(\frac{qZ_t\Phi}{(1+g)^{1/(1-\alpha)}} \right)^{\frac{1}{1-\alpha}}. \end{aligned}$$

Figure 12 represents the relationship between the evolution of k_t and $\hat{k}_t$. The graph on the left represents the evolution of $\hat{k}_t$ with respect to k_t. The balanced growth rate for an economy is one in which the economy starts with capital per household equal to $\hat{k}_{ss}$. If

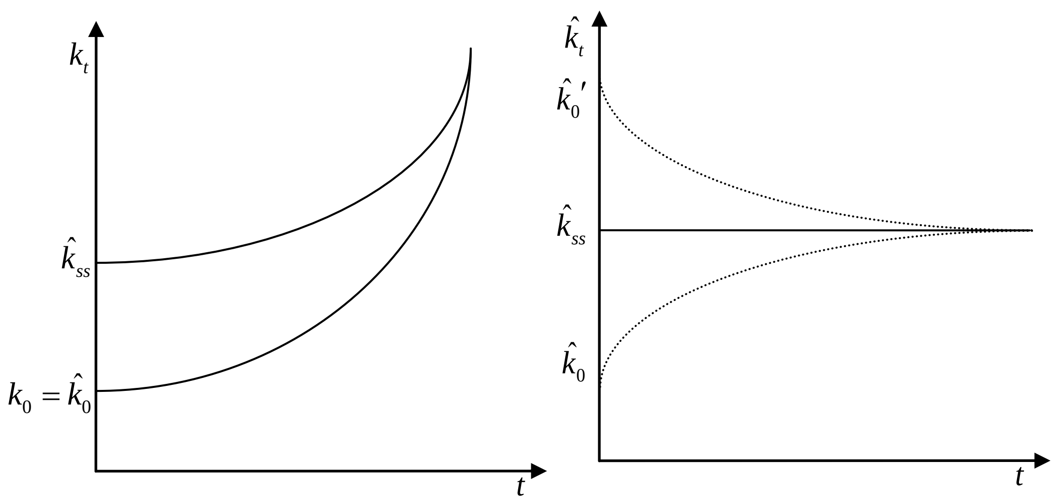

Figure 12: Evolution of Capital per Household and Efficiency Variable

an economy starts with a level of capital below $\hat{k}_{ss}$ (e.g. $k_0 = \hat{k}_0$), then at first the economy will “catch up” and growth for k_t will be quicker than as t evolves. The graph on the right shows how an initial efficiency capital stock will eventually converge to the steady state efficiency capital stock. Again, if $k_0 = \hat{k}_0$, then as t evolves it will gradually approach $\hat{k}_t$.

To ensure that the economy exists at a steady for capital per household, we need to solve for the quotient of capital per household and income per household such that:

$$\begin{aligned}\frac{k^{BGP}}{y^{BGP}} &= \frac{k^{BGP}}{Z_t k_t^{\alpha}\left(\frac{L_t}{N_t}\right)^{1-\alpha}} \\ &= \frac{k_t^{1-\alpha}}{Z_0\left(1+g\right)^t\left(\frac{L_t}{N_t}\right)^{1-\alpha}} \\ &= \frac{\left(\hat{k}^{ss}\left(\left(1+g\right)^{\frac{1}{1-\alpha}}\right)^t\right)^{1-\alpha}}{Z_0\left(1+g\right)^t\left(\frac{L_t}{N_t}\right)^{1-\alpha}} \\ &= \left(\frac{qZ_0\Phi}{\left(1+g\right)^{1/(1-\alpha)}}\right)^{1-\alpha}\frac{\left(2-l_t^f\right)^{\frac{1}{1-\alpha}}}{Z_0}.\end{aligned}$$

3.4 Parameter Choice

To simulate the model, certain parameters must be calibrated to correspond to key data moments around 1881. Under calibration, I assume for the model that the economy

Table 1: Parameter Choice

Data Moment	Description
$\gamma = .7615$	Utility Weight of c_{t+1}.
$q = .1886$	Average Household Labor Required of Surviving Offspring.
g = 0.209	Growth of Knowledge Accumulation.
$\alpha = 0.3$	Share of Income to Capital.

exhibits BGP behavior around 1881. Each time period represents 25 years. I also assume that the household equilibrium is in Case 3.[28]

3.4.1 Initial Conditions

I use several data sources to find my initial conditions. From Joshi, Layard & Owen (1985), I use values of female labor force participation between certain age groups in England during 1881 to 1931. Table 2 displays a breakdown of female labor force participation from 1871 to 1931. Joshi, Layard & Owen (1985) provides data on the labor force participation of each of the groups. However, for this model, the labor force participation for all females between the ages of 20-44 was calculated through aggregate population statistics in Mitchell (1975).[29]

Table 2: Female Labor Force Participation

Year	Age Group			
	20-24	25-34	35-44	20-44*
1871	60	40.4	36.3	
1881	55.9	29	29	0.39
1891	58.1	33	25.1	
1901	56.5	31.5	25.8	
1911	62	33.8	24.1	
1921	62.2	33.5	22.9	
1931	65.1	36.3	24.5	0.46

Source: Joshi, Layard & Owen (1985); Mitchell (1975)

I set $l_0^f = .6047$.

To calculate the estimates for surviving fertility per household, I used data from Joshi, Layard & Owen (1985) of aggregate household fertility times the average rate of childhood survival until adulthood. Table 3 represents the data on surviving fertility per household. Fertility is the aggregate fertility levels per household; pi indicates the percentage of children who survive to adulthood; and pi*TFR is the number of surviving offspring per household. The percentage of children who

[28] I make this assumption in order to ensure that a change in τ_t will affect surviving fertility per household.

[29] I choose to cover ages 20-44 because (1) this represents roughly the ages in which women can give birth and (2) it corresponds to the rough approximation that each period is 25 years in length.

survived until age 25 in 1881 to 1931 (pi) is from Bar & Leukhina (2006). I set $n_0 = 3.206$.

Table 3: Surviving Fertility per Household

Year	Fertility	pi	pi*TFR
1860-65	4.66		
1881-85	4.58	0.7	3.206
1901-05	3.46		
1921-25	2.39	0.89	2.1271

Source: Joshi, Layard & Owen (1985); Bar & Leukhina (2006)

For the values of the capital stock, adult and elderly population, I set these values so that the economy starts on a balanced growth path (i.e. at period 0). Table 4 represents these values.

Table 4: Initial Conditions

Data Moment	**Description**
$K_0 = .326$	Initial Capital Stock
$N_0^a = 3.50$	Initial Adult Household Population
$N_0^e = 1.01$	Initial Retired Household Population
$l_0^f = .6047$	Initial Female Labor Allocated to the Household
$l_0^m = 0$	Initial Male Labor Allocated to the Household
$n_0 = 3.206$	Initial Surviving Fertility Rate per Household
$s_{-1} = .3228$	Previous Period Savings per Household
$Z_0 = 1$	Initial Stock of Knowledge
$\tau_0 = .3485$	Initial Wedge on Female Labor Market Opportunities

3.4.2 Parameters α, g, q

From Clark (2001), I use 0.3 for α during each period. To find *g*, I use GDP per capita data from Maddison (2001) to calculate the percent change in GDP growth from 1881 to 1931 ($k^{1881} = 249, k^{1931} = 364$). So to find *g*:

$$\begin{aligned} 1+g &= \sqrt{\tfrac{364}{249}} \\ &= 1.209. \end{aligned}$$

Clark (2001) calculates that the annual rental rate of capital is 4% per annum. Since we know that,

$$\alpha = \frac{r_t k_t}{y_t} = 0.3,\ r_t = (1.04)^{25},$$

solving for $\frac{k_t}{y_t}$ yields,[30]

$$\begin{aligned} \frac{k_t}{y_t} &= \frac{0.3}{(1.04)^{25}} \\ &\cong 0.1. \end{aligned}$$

To find q¸ I use equation (2) from a Case 3 household equilibrium with the initial conditions so that:

$$\begin{aligned} n_0 &= \tfrac{l_0^f}{q} \\ q &= \tfrac{l_0^f}{n_0} \\ q &= .1886 = \tfrac{.6047}{3.206}. \end{aligned}$$

3.4.3 Parameters τ_t, γ

To solve for the parameters τ_t, γ, I used two equations. As I know the value of l_0^f, using the solution of a Case 3 equilibrium, I find that:

$$.6047 = \frac{(1-\gamma)(2-\tau_0)}{1-\tau_0}.$$

The second equation I use is the quotient of capital and income per household to ensure that the economy starts at a steady at period 0, such that:

---

[30] I estimate that the balanced growth path of $\frac{k_0}{y_0}$ is slightly below the actual estimate of .11. I do this to ensure that a Case 3 equilibrium exists. Otherwise, the model's parameters cannot find a solution and the model cannot be calibrated to yield a Case 3 equilibrium.

$$\frac{k_0}{y_0} = \frac{k^{BGP}}{y^{BGP}}$$
$$\cong .1$$

From this equation, I use the efficiency variables to find $.1 = f(\gamma, \tau_0)$. Solving for these values, I find

$$\tau_0 = .3485, \gamma = .7615.$$

3.5 Simulation

In the simulation of the model, I conduct two experiments. In the first experiment, I increase labor opportunities for women, through the variable (τ_t), to see what effect this changes has on fertility from 1881-1931. In the second experiment, I use the same values of τ_t but I also increase the values of q_t for each of the following periods to estimate what effect an increase in the amount of household labor per child decreased fertility rates.

In the first experiment, I found that an increase in female labor force participation (implicitly female labor opportunities) contributed to roughly one third of the decline of fertility rates during the English demographic transition. In the second experiment, I found that two-thirds of the decline of fertility rates can be attributed to an increase in the costs per surviving child.

Table 5 shows the results from the first experiment. To correspond to the drop in l_t^f, τ_t decreases by 56.0%. This drop in τ_t, corresponding to an increase in female labor market opportunities, drops n_t by 11.3%. Relative to historical results, this drop in τ_t accounts for roughly one-third of the drop in surviving household fertility rates between 1881 and 1931.

Table 5: Experiment I Results

	Model Output		Historical Data	
Year	τ_t	n_t	pi*TFR	l_t^f
1881	0.3485	3.206	3.206	.6047
1931	0.2036	2.844	2.127	.5365
Change	-41.6%	-11.3%	-33.7%	-11.3%

From the results of the first experiment, it is clear that an increase in female labor market opportunities only accounted for a part of the drop in n_t. Considering the mainstream literature view that the drop in n_t during this period is largely due to higher costs per child (i.e. a transition from the quantity of children to the quality), in the second experiment I consider how large does the values of q_t must increase to account for the remaining two-thirds drop in n_t.

Table 6 shows the results from the second experiment. Keeping the same τ_t variables, I find that q_t must be raised by one-third in order to account for the remaining drop in surviving fertility per household.

Table 6: Experiment II Results

	Model Output			Historical Data
Year	τ_t	n_t	q_t	l_t^f
1881	0.3485	3.206	0.189	.6047
1931	0.2036	2.127	0.252	.5365
Change	-41.6%	-33.7%	33.3%	-11.3%

4. Conclusion

I develop a general equilibrium model to consider that an increase in female labor opportunities lowered fertility rates in England during the demographic transition. I find that such a change contributed partially to the decrease in fertility rates, but this factor is not the sole cause of the English demographic transition.[31] My results imply that other factors which increased the amount of labor spent on each surviving child additionally contributed to the English demographic transition.

The results of my simulation reiterate the presence of a Becker quantity-quality trade-off occurring during England's demographic transition. Such a trade-off raised

[31] However, as mentioned in the introduction, the role of an increase in female labor opportunities could have been higher due to low estimates used in calibration of female labor participation. Some historical authors, e.g. Mokyr (1993), note the difficulty in considering any historical labor force participation estimates since it is difficult to know what the standard for being considered within the labor force was at that time. In particular, as women's work during this period still largely revolved around the household, it seems probable that female labor force participation statistics could have been underestimated.

the costs for each household to raise a child, which promoted households to have fewer children but to focus more labor on their offspring. One manifestation of the transition from the quantity to the quality of offspring is through an increase in child labor laws. Such an increase in child labor laws will lead households to invest more in the education (quality) of their children as such laws prohibit children from working in the economy (see Doepke & Zilibotti (2003)). A second manifestation could be through Unified Growth Theory, as posited by Galor & Weil (2000). This hypothesis concludes that technological increases raise the returns to human capital which then promotes the transition of an emphasis from the quantity to the quality of offspring.

There are two main contribution of this work to the literature pertaining to long run fertility choice. First, this work introduces a new theory to fertility rates within a demographic transition. By understanding more about the conditions suitable for economic growth (including perhaps low fertility rates), better policy recommendations for undeveloped countries may result to help promote long growth for these economies. Secondly, this work attempts to introduce a quantitative model relating to the concept of social norms in which most economic literature has omitted. The cross disciplinary emphasis within this model highlights the need for inter-disciplinary studies. Such inter-disciplinary research represents a viable process to refine and to advance economic theory.

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Appendix

A.1 Household Population Growth

Given that the rate of household population growth is $\frac{N_{t+1}^c + N_{t+1}^a + N_{t+1}^e}{N_t^c + N_t^a + N_t^e}$, we may simplify this function as a function of $\{n_t\}\ \forall t$. (Recall $N_t^c = n_t N_t^a$, $N_t^a = N_{t-1}^c$, $N_t^e = N_{t-1}^a$)

Replace $N_t^a = N_{t-1}^c$ so that

$$\frac{N_{t+1}^c + N_{t+1}^a + N_{t+1}^e}{N_t^c + N_t^a + N_t^e} = \frac{N_t^c\left(n_{t+1}+1\right) + N_t^a}{N_{t-1}^c\left(n_t+1\right) + N_{t-1}^a}.$$

Now replace $N_t^c \equiv n_t N_t^a$ so that this implies

$$= \frac{N_t^a n_t (n_{t+1}+1) + N_t^a}{N_{t-1}^a n_{t-1}(n_t+1) + N_{t-1}^a}.$$

Factor out adult household populations for t and $t-1$ and consider that

$N_t^a = N_{t-1}^c = n_{t-1} N_{t-1}^a$ so that

$$= \frac{n_{t-1} N_{t-1}^a \left(n_t (n_{t+1}+1) + 1\right)}{N_{t-1}^a \left(n_{t-1}(n_t+1)+1\right)}.$$

If we know divide through by the adult household population for period t -1, we get

$$\frac{N_{t+1}^c + N_{t+1}^a + N_{t+1}^e}{N_t^c + N_t^a + N_t^e} = \frac{n_{t-1}\left(n_t (n_{t+1}+1)+1\right)}{\left(n_{t-1}(n_t+1)+1\right)}$$

A.2 Comparative Statics of the Household's Decision

Proposition 2.5.4 *Assume an equilibrium exists (either Case 1, Case 2 or Case 3). An exogenous change in* w_t *does not affect* n_t.

Proof First, consider a Case 3 household equilibrium. Let f be defined as

$$f\left(l_t^f, w_t\right) = \frac{1-\gamma}{l_t^f} + \frac{\gamma(1-\tau_t)}{1+(1-\tau_t)\left(1-l_t^f\right)} = 0$$

(f is the partial derivative with respect to female household labor). Since we are in a Case 3 equilibrium, $l_t^m = 0$. f is continuous and differentiable when

$l_t^f, \tau_t \in (0,1), w_t > 0$. By the implicit function theorem, we find that

$$\frac{dl_t^f}{dw_t} = -\frac{\partial f / \partial w_t}{\partial f / \partial l_t^f}.$$

By taking each partial derivative, we find that

$$\frac{\partial f}{\partial w_t} = 0, \frac{\partial f}{\partial l_t^f} = -\frac{1-\gamma}{l_t^{f\,2}} - \frac{\gamma(1-\tau_t)^2}{\left(1+(1-\tau_t)\left(1-l_t^f\right)\right)^2},$$

which implies:

$$\frac{dl_t^f}{dw_t} = 0.$$

Proposition 2.5.5 *Assume a Case 3 equilibrium exists. An exogenous change in* τ_t *will result in a positive change in* n_t.

Proof Let f be defined as $f\left(l_t^f, \tau_t\right) = \frac{1-\gamma}{l_t^f} + \frac{\gamma\left(1-\tau_t\right)}{1+\left(1-\tau_t\right)\left(1-l_t^f\right)} = 0$ (f is the partial derivative with respect to female household labor). f is continuous and differentiable when $l_t^f, \tau_t \in \left(0,1\right)$. By the implicit function theorem, we find that

$$\frac{dl_t^f}{d\tau_t} = -\frac{\partial f / \partial \tau_t}{\partial f / \partial l_t^f}.$$

By taking each partial derivative, we find that

$$\frac{\partial f}{\partial \tau_t} = \frac{\gamma}{\left(1+\left(1-\tau_t\right)\left(1-l_t^f\right)\right)^2}, \quad \frac{\partial f}{\partial l_t^f} = -\frac{1-\gamma}{l_t^{f\,2}} - \frac{\gamma\left(1-\tau_t\right)^2}{\left(1+\left(1-\tau_t\right)\left(1-l_t^f\right)\right)^2}.$$

We may substitute these equations in to find that $\frac{dl_t^f}{d\tau_t} > 0$. Therefore, as the wife is the only parent to work in the household (Case 3 household equilibrium), a decrease in τ_t will decrease l_t^f. As $n_t = \frac{l_t^f}{q_t}$, where q_t is constant, then such a decrease in l_t^f will also decrease n_t.