Applied Mechanism Design: Solving for the Optimal Collusive Mechanism at First-Price Auctions

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Honors thesis submitted in partial fulfillment of the requirement for Graduation with Distinction in Economics in Trinity College of Duke University.

Duke University
Durham, North Carolina
2010

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Acknowledgements

I am indebted to my thesis advisor, Professor Leslie Marx, for her unwavering guidance, mentorship, and willingness to answer any question. I am also grateful to my thesis seminar professor, Michelle Connolly, who not only provided valuable feedback but also pushed me to gain a full understanding of this topic through her inquisitiveness and eagerness to learn. Finally, I want to thank my seminar classmates for their patience, constructive criticisms, and encouragement. Without all their support, this paper would not have been possible.

Abstract

I examine the role of collusion in first-price sealed-bid auctions in the simple environment of an all-inclusive cartel of two symmetric bidders. A cartel that cannot control its members' bids at a first-price auction with two symmetric bidders cannot do better than non-cooperative payoffs as the bid increment approaches zero. In this paper, I analytically solve for the optimal collusive mechanism at an auction with discrete bid increments that achieves the maximum payoffs for the cartel.

1 Introduction

Mechanism design theory, also called reverse game theory, provides a framework for analyzing and creating institutions that produce desired outcomes, such as efficient outcomes or revenue-maximizing outcomes, in an environment where agents have private information and are self-interested. Auction theory is an applied branch of game theory and mechanism design. In an auction setting, buyers and sellers have opposing motives: sellers have an incentive to seek the highest possible sale price, while buyers desire the lowest purchase price, but face the chance that they do not win the auction. With competing interests, the interaction of these two parties at the auction will likely result in an inefficient outcome. Mechanism design techniques demonstrate which mechanisms are optimal for different participants. In this paper, I investigate the optimal collusive behavior of bidders in a cartel colluding at auctions.

Bidder collusion may harm non-colluding bidders in two ways. First, upon detecting collusive behavior, sellers may become more aggressive in their use of reserve prices. A higher reserve price in an auction not only results in reduced utility for all the bidders but may even also result in non-cartel bidders being denied an object they may have won in the absence of the cartel. Second, "bidder collusion may create a better-informed bidder relative to the non-colluding bidders" (Marshall & Meurer, 2004, p. 87). Therefore, non-colluding bidders may adopt a more conservative bidding strategy, thinking that winning against better-informed colluding bidders might result in buyer's remorse (Marshall & Meurer, 2004).

For cartel members, colluding can be quite advantageous. Marshall and Marx show that strategic bidding at auctions can distort prices, reduce seller surplus, and reduce competition (Marshall & Marx, 2007). Thus, bidder collusion is a common form of price fixing (Marshall & Meurer, 2004). According to a 1990 General Accounting Office Report, from 1982 to 1988, over half of the criminal restraint of trade cases filed by the U.S. Department of Justice's Antitrust Division involved auction markets (GAO, 1990, p. 43). In response, recent U.S. policy has become more "antagonistic" to antitrust cases; competition authorities have more actively pursued the detection and prosecution of bidder collusions (Athey & Bagwell, 2001, p. 1).

Consequently, "understanding the mechanisms by which bidders support collusive agreements can be valuable for informing the choice of auction format, as well as detecting and prosecuting cartels," (Lopomo, Marx, and Sun, 2009, p. 2). This paper focuses on bidder collusion at a one-shot single-object first-price sealed-bid auction in which bidders simultaneously submit bids, with the high bidder winning and paying the highest bid price. I simplify the auction environment by examining an all-inclusive cartel involving two symmetric bidders at an auction with discrete bid increments. I analytically solve for the optimal collusive mechanism for a cartel that cannot control its bidders' actions. My findings reveal how the optimal cartel operates in maximizing its expected payoffs and collusive gains.

1.1 General Auction Theory and Concepts

There are four common auction types: (i) First-price sealed-bid auctions; (ii) Second-price sealed-bid auctions (Vickrey auctions); (iii) Open Ascending-bid auctions (English auctions); and (iv) Open Descending-bid auctions (Dutch auctions). At a first-price sealed-bid auction, the bidders place their bid privately to the auctioneer. The individual with the highest bid wins, paying the exact price of the submitted bid. Second-price sealed-bid auctions, also called Vickrey auctions, operate like first-price sealed-bid auctions, except the individual with the highest bid wins, paying the price of the second-highest bid. The open ascending-bid auction, also called the English auction, is where the auctioneer steadily raises the price with bidders dropping out once the price becomes too high. This continues until there is only one bidder who wins the auction at the current price. The open descending-bid auction, otherwise the Dutch auction, starts off at price high enough to deter all bidders, and is progressively lowered until a bidder is willing to pay the current price.

A central finding in auction theory is the Revenue Equivalence Theorem which states that given the following conditions, any auction format will lead to the same revenue for the seller: (i) the winning bidder has the highest valuation of the object; (ii) the bidder with the lowest valuation expects zero surplus; (iii) all bidders are risk neutral; and (iv) all bidders are drawn from a monotonically increasing distribution function. In reality, these conditions cannot always be met, resulting in different outcomes depending on the format (Krishna, 2002).

Auction theory dictates that the bidding strategy and results of first-price sealed-bid and open descending-bid auction formats are strategically equivalent and likewise in the case of second-price sealed-bid and open ascending-bid formats. First-price and open descending-bid auctions are designed such that the winning bidder pays the highest bid price. Here, the payoff from winning equals the gap between a bidder's valuation of winning and the bid. Since the bidder is trying to maximize surplus, bid shading, the practice of placing a bid below valuation, is necessary for the bidder to gain utility from winning the auction. As a result, these formats are not ex post efficient: the bidder with the highest valuation may not necessarily win. A bidder with a lower valuation may simply have submitted the highest bid, willing to take a smaller surplus from winning the auction. Additionally, the winner is unaware of how the second strongest bidder values the object, thus, leaving the possibility that the winner may have bid too much for the auctioned item (Krishna, 2002).

In second-price and open ascending-bid auctions, the winning bidder pays the second-highest bid price. The dominant strategy for bidders is to bid their valuation. By doing so, they maximize their odds of winning the auction without the risk of overbidding; if they win, since the winner pays the second-highest bid price, there is a guarantee that they gain some surplus. The results at these auctions are ex post efficient because the bidder with the highest valuation is the winner. Since the second highest bid will have been less than the highest, the winner still gains a surplus. However, these auction formats are subject to shill bidding, where bidders collude with the seller to artificially increase the price for the seller (Krishna, 2002).

Auction design literature generally characterizes second-price and ascending bid auctions to be more susceptible to collusion than first-price auctions (Marshall & Marx, 2008). For first-price auctions, the presence of bid shading makes collusion more difficult (Marshall & Meurer, 2004). Bid shading describes the practice of bidders placing a bid below the price at which they value the item. For a cartel to profitably collude, it must shade its bids below their non-cooperative levels. However, this increased shading makes the bid of the highest valuation cartel member more vulnerable to deviation by another cartel member. In contrast, at an ascending bid auction, the winner of the auction is willing to bid up to his/her valuation (Marshall & Meurer, 2004). At a second-price auction, the dominant bidding strategy is to bid one's valuation (Krishna, 2002). Thus, there is no room for profitable deviation by another cartel member (Marshall & Meurer, 2004).

1.2 Cartel Operations and Collusive Mechanisms

I examine cartels that operate prior to the auction by enabling communication and enforcing ex ante transfer payments among members. These cartels usually hold a pre-auction only for cartel members on the rights to bid at the actual auction. Therefore, the cartel can reduce within-cartel rivalry and reduce prices by eliminating some competition. In this paper, the cartel is all-inclusive, meaning that every bidder at the auction can be a member of the cartel and participate in the auction. I also assume that the cartel gathers information about its bidders through a third-party incentiveless mechanism agent, known as the bid recommendation center, which I refer to simply as the center. Each cartel member makes a report regarding the object's valuation to the center. Based on these reports, the center recommends a bid to be made by each cartel member and requires ex ante transfer payments between the cartel members.

There are two types of collusive mechanism frameworks under which a cartel can operate. The first is the "bid submission mechanism" (BSM). A cartel operating under the BSM can control the bids of its members. The other mechanism is the "bid coordination mechanism" (BCM). The BCM dictates that the cartel can make recommendations, but cannot enforce the bidding activities of its members. Because cartel members can deviate from the recommendations when colluding under the BCM, this paper requires that the incentive compatibility constraint be satisfied. This means that bidders must have incentive and find it favorable in terms of expected payoffs to report their valuations truthfully and obey the center's recommendations. The incentive compatibility constraint dictates that the center will suggest optimal bidding behavior such that a bidder cannot gain more utility by deviating from the cartel's recommendations (Lopomo, Marx, & Sun, 2009).

This paper proceeds as follows. Section 2 provides a background on bidder collusion by reviewing relevant auction theory literature. Section 3 introduces the model and the mechanism design problem of the cartel. Section 4 presents the results and solutions of solving for the optimal collusive mechanism with two symmetric bidders, each with two possible types. Section 5 concludes by providing a discussion of the results and potential avenues for future research.

2 Related literature on bidder collusion

Auction theory literature has grown considerably in recent years as game theory and its applications have become a popular branch of economics. When auction theory was less developed and much of the literature investigated non-cooperative play, McAfee and McMillan (1992) produced a seminal paper in literature on the theory of collusion at first-price auctions. In this paper, cartel members first bid for the right to represent the cartel at the auction. The highest bidding cartel member is sent to the main auction, while other cartel members are forbidden from participating in the auction. The paper assumes symmetric bidders and the implementation of the BSM, where the cartel can enforce its members' bid behavior. McAfee and McMillan demonstrate that a cartel can maximize its collusive gains if it successfully suppresses within-cartel rivalry and sends only the highest-valuing ring member to the auction and bid solely against the non-cartel bidders. The model also requires that transfer payments be made between cartel members to attain efficiency. While it may be possible in some cases for a cartel to implement a BSM, as assumed in McAfee and McMillan, it may be difficult for a cartel to actually monitor and control the bids of its members. In such cases, the cartel may be limited to using a BCM.

Marshall and Marx's (2007) "Bidder Collusion" paper explores collusion generally, investigating both the BSM and BCM at first-price, second-price, and open-ascending auction formats. The paper assumes an environment of a general n-bidder cartel with the application of incentive compatibility constraints and the ex-ante individual rationality principle. The ex-ante individual rationality principal requires that a bidder's expected payoff from cartel participation must be greater or equal than that of non-cooperative play. Marshall and Marx conclude that even with a BCM, a cartel can successfully suppress all within-cartel rivalry and thus, achieve the first-best collusive outcome at a second-price or open ascending-bid auction as the threat of cheating by cartel members not selected to win the auction is not present in these auction types (Marshall & Marx, 2007).

Lopomo, Marx, and Sun's "Bidder Collusion at First-Price Auctions" (2009) examines cartels at first-price auctions under both the BCM and BSM. Without solving for the optimal collusive mechanism, the paper demonstrates that given an all-inclusive two symmetric bidder environment, a bidding ring operating at a first-price auction cannot profitably collude and realize an increase in gains compared to non-cooperative bidding in the limit as the bid increments approach zero. The paper leaves open the possibility of successful collusion with discrete bid increments. This paper proposes to serve as an extension to Lopomo, Marx, and Sun (2009) by analytically solving for the optimal collusive mechanism of an all-inclusive cartel of two symmetric bidders under the BCM at an auction with discrete bid increments. As no one has yet analytically solved for the optimal BCM for first-price sealed-bid auctions, this finding would help to confirm prevailing understanding of bidder collusion.[1]

[1] The optimal BSM solution is the all-weak strategy for every situation, since both bidders bidding the lowest possible bid value maximizes the cartel's expected surplus. Under the BSM, the cartel does not have to pay any attention to the possibility of bidder deviation. Therefore, the center can expect that both bidders will follow the weak bidding strategy recommendation.

3 Model

3.1 Setup

I focus on a setting of a single-object first-price auction with two symmetric bidders. Each risk neutral bidder i □ {1, 2} independently draws a private value v_i from a distribution F_i with finite support $V_i \equiv V_1$ x V_2. Since bidders are symmetric, F_1=F_2 meaning that bidders obtain their values from the same probability distribution, independently and identically distributed. Let $f_i(v_i)$ be the probability that bidder i's value is v_i. Let β denote the set of feasible bids.

I assume that the sellers know the distributions from which the bidders draw their values, but not the value themselves. I also assume that the seller's reserve price is equal to the lower bound of the bidders' value distributions such that the seller's reserve price equals the lower bound of the bidders' value distributions; thus, the seller is "non-strategic" (Lopomo, Marx, & Sun, 2009, p. 5).

At a first-price auction, the each bidder submits a bid, and the high bidder wins, paying the amount of its bid. In the case of a tie, I assume the object is randomly allocated to one of the bidders with the high bid. Let u_i (b_1, b_2, v_i) be bidder i's surplus in a first-price auction, given bidder i's value v_i and bid vector (b_1, b_2); in the case of a tie, I assume the object is randomly allocated to one of the (two) high bidders:

$$u_i(b_1, b_2, v_i) \equiv \left\{ \begin{array}{cc} v_i - b_i, & \text{if } \mathrm{b}_i > b_{-i} \\ 0, & \text{if } \mathrm{b}_i < b_{-i} \\ (v_i - b_i)/2, & \text{otherwise.} \end{array} \right\}$$

Since the cartel is all-inclusive, both bidders are eligible to participate in the cartel. Even in environments with more than two bidders, the assumption of an all-inclusive cartel is common in auction theory literature, particularly because the study of non-all-inclusive cartels at first-price auctions is made difficult by the lack of analytic bid functions (Lopomo, Marx, & Sun, 2009).[2]

In this model, I focus on incentive compatible collusive mechanisms, which has two dimensions in this environment: truthtelling and obedience. First, it must be incentive compatible for each cartel member to report its value truthfully. Second, it must be incentive compatible for each cartel member to follow the bid recommendation of the center. In addition, I require ex-ante individual rationality, so that a bidder's expected payoff from participation in the cartel is at least as great as its expected payoffs from non-cooperative play. Thus, cartel members must commit to participation in the cartel prior to learning their values and that the failure of either bidder to join results in the complete dissolution of the cartel. I assume the cartel can compel its members to make their required payments, but that it cannot prevent cartel members from participating in the auction (Lopomo, Marx, & Sun, 2009).

3.2 Mechanism Design Problem

By the revelation principle, I restrict attention (without loss of generality) to the class of all incentive compatible and individually rational direct revelation mechanisms. Any incentive compatible and individually rational direct revelation mechanism specifies for each pair of reports $(v_1, v_2) \in V$, (i) a probability distribution over all feasible bid pairs

[2] Also see (Krishna, 2002).

$p\,(\cdot \mid v_1, v_2)$ □$\Delta_{ß\times ß}$ represents the probability simplex with each pair of bids as the vertex; and (ii) monetary transfers $m_1\,(v_1, v_2)$ and $m_2\,(v_1, v_2)$. The vector (p, m_1, m_2) defines the cartel mechanism. Incentive compatibility for truthtelling implies that ring members have no incentive to misreport their values to the center. The probability distribution p specifies the center's bid recommendations to the two bidders–each bidder observes only its own recommended bid and not that of its rival. Incentive compatibility for obedience implies no bidder has an incentive to deviate from the bid recommendation it receives from the center. The payments m_1 and m_2 are the transfer payments from the bidders to the center and may be positive or negative. These payments are constrained to sum to zero in expectation in the ex-ante budget balance condition below (Lopomo, Marx, & Sun, p. 6).

The mechanism design problem that the Lopomo, Marx, and Sun (2009) paper uses is on the following page:

The center's problem of maximizing expected cartel surplus can be written as follows: maximize the total expected surplus to the two bidders,

$$\max_{p,m_1,m_2} \sum_{i\{1,2\},\ (v_1,v_2)\in V, b1,b2\in\beta} p(b_1,b_2 \mid v_1,v_2) u_i(b_1,b_2,v_i) f_1(v_1) f_2(v_2), \tag{1}$$

subject to p representing a probability distribution, $\forall (v_1,v_2)\in V$,

$$p(\cdot \mid v_1,v_2) \in \Delta_{\beta\times\beta}, \tag{2}$$

ex-ante budget balance,

$$\sum_{i\in\{1,2\},\ (v_1,v_2)\in V} m_i(v_1,v_2) f_1(v_1) f_2(v_2) = 0, \tag{3}$$

and incentive compatibility, $\forall \mathrm{i} \in \{1,2\}, \forall v_i, v_i^{'} \in V_i, \forall \delta_i : \beta \to \beta,$

$$U_i^*(p,\delta_i,v_i^{'} \mid v_i) - U_i(p \mid v_i) \le 0, \tag{4}$$

where $U_i(p \mid v_i)$ is bidder i's expected surplus under truthtelling and obedience,

$$\begin{aligned} U_i(p \mid v_i) \equiv & \sum_{v_{-i}\in V_{-i}, b_i, b_{-i}\in\beta} p(b_1,b_2 \mid v_1,v_2) u_i(b_1,b_2,v_i) f_{-i}(v_{-i}) \\ & - \sum_{v_{-i}\in V_{-i}} m_i(v_1,v_2) f_i(v_i), \end{aligned} \tag{5}$$

and $U_i^*(p,\delta_i,v_i^{'} \mid v_i)$ is bidder i's expected surplus when bidder i has value v_i, reports $v_i^{'}$, and bids according to a deviation function $\delta_i(b_i)$ depending on the center's recommendation b_i,

$$\begin{aligned} U_1^*(p,\delta_1,v_1^{'} \mid v_1) \equiv & \sum_{v_2\in V_2, b_1, b_2\in\beta} p(b_1,b_2 \mid v_1^{'},v_2) u_1(\delta_1(b_1),b_2,v_1) f_2(v_2) \\ & - \sum_{v_2\in V_2} m_1(v_1^{'},v_2) f_2(v_2) \end{aligned} \tag{6}$$

and

$$\begin{aligned} U_2^*(p,\delta_2,v_2^{'} \mid v_2) \equiv & \sum_{v_1\in V_1, b_1, b_2\in\beta} p(b_1,b_2 \mid v_1,v_2^{'}) u_2(b_1,\delta_2(b_2),v_2) f_1(v_1) \\ & - \sum_{v_1\in V_1} m_2(v_1,v_2^{'}) f_1(v_1) \end{aligned} \tag{7}$$

Equation (1) describes the expected surplus of the two bidders, which is maximized subject to constraint equations (2) – (7). Equation (2) refers to the constraints on the probability distribution of the bid center, the set of the 16 possible combinations of weak and strong bids given weak and strong valuations for the two bidders.[3] Equation (3) characterizes the ex-ante budget balance such that the expectation of the transfer payments sum to 0 where $m_1(v_1, v_2)$ and $m_2(v_1,v_2)$ represent the monetary transfers of the bidders. Equation (4) symbolizes the incentive compatibility mechanism: the expected cartel surplus under truthtelling and obedience is at least as great as the expected surplus when there is deviation from the center's recommendation. Equation (5) defines the expected cartel surplus under truthtelling and obedience. Equations (6) and (7) define the expected surplus of bidder 1 and 2, respectively, when the bid recommendation is not necessarily followed.

While most of the components of the constrained optimization problem above are straightforward, the incentive compatibility constraint of (4) deserves additional commentary. Because bidders receive their bid recommendations from the center prior to submitting their bids at the auction, bidders can potentially make use of the inferences from the recommendation regarding the bidding behavior of its rival, and adjust their bids, thereby deviating. Thus, I allow the deviation function $\delta_i (b_i)$ in (4) to be a function mapping the bidder *i's* recommended bid from the center to its actual bid. For any of the center's strategies to be incentive compatible, truthful or not, there is no deviation function δ_i that improves upon $\delta_i (b_i) = b_i$ (Lopomo, Marx, & Sun, 2009, p. 8).

[3] See Appendix: Mathematica Code.

Moreover, while the above formulation accounts for possible inferences made by a bidder regarding its rival's recommended bid based on the recommendation it receives, the formulation does not explicitly allow a bidder to make inferences regarding its rival based on the transfer payment required by the center. One might think a bidder's ex ante required payment is informative to the rival's report. However, only bidder i's expected payment given its own report enters the problem (Lopomo, Marx, and Sun, 2009, p. 8).

To solve this problem, I work with two symmetric bidders of two types: I define two possible values for bids and valuations, respectively: a weak bid (denoted as b_w), a strong bid (b_s), a weak valuation (v_w), and a strong valuation (v_s). We can consider having only two bid values to be equivalent to examining auctions with discrete bid increments at a single increment level. The weak bid value reflects the bid price of the auctioned item at a given time while the strong bid value represents the next bid increment price.

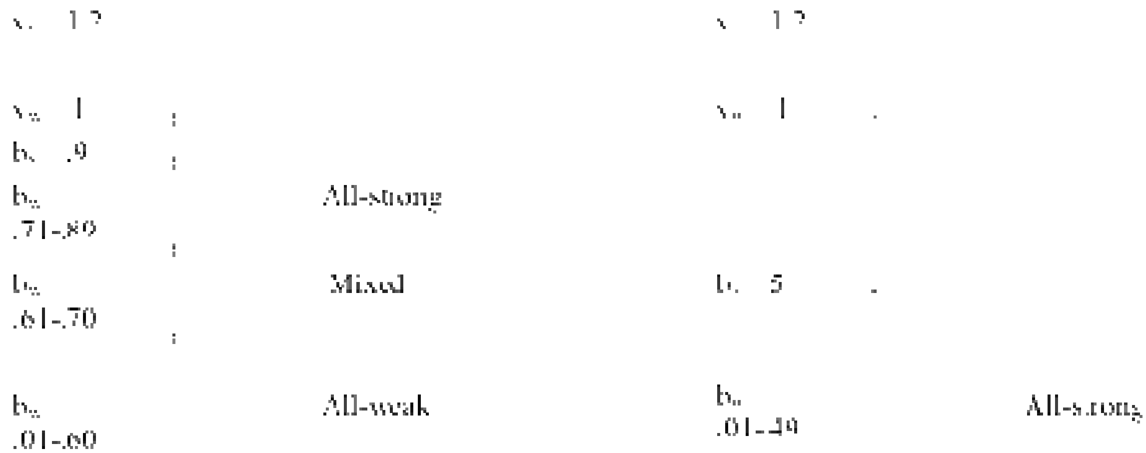
4 Analytic results for two symmetric bidders of two types

In this section, I demonstrate the results of solving for the optimal collusive mechanism under the BCM. There are two general cases that I examine. The first is the "bids below valuations" case ($b_w < b_s < v_w < v_s$), where the weak and strong bid values are both less than the weak and strong valuation values. The other case is the "bids between valuations" case ($b_w < v_w < b_s < v_s$), where the weak bid is less than the weak valuation, the strong bid is less than the strong valuation, but the strong bid is between the weak and strong valuations. For each type of case, I vary the bid and valuation values to evaluate the maximization problem in different auction environments.

To maximize its payoffs, the center employs three strategies: "all-strong", "mixed," and "all-weak." The all-strong strategy suggests that both bidders bid strong. The "mixed" strategy means that each bidder bids based on how each values the item: if the bidder has a weak valuation, the bidder bids the weak bid; if the bidder values the item strongly, the bidder bids strong. Finally, the all-weak recommendation implies that both bidders bid the weak value. Figures 1-4 demonstrate the results for the different scenarios. Then, Table 5 outlines the results demonstrating in what cases the optimal collusive mechanism recommends a certain type of bid recommendation.

4.1 Bids Below Close Valuations

Figure 1

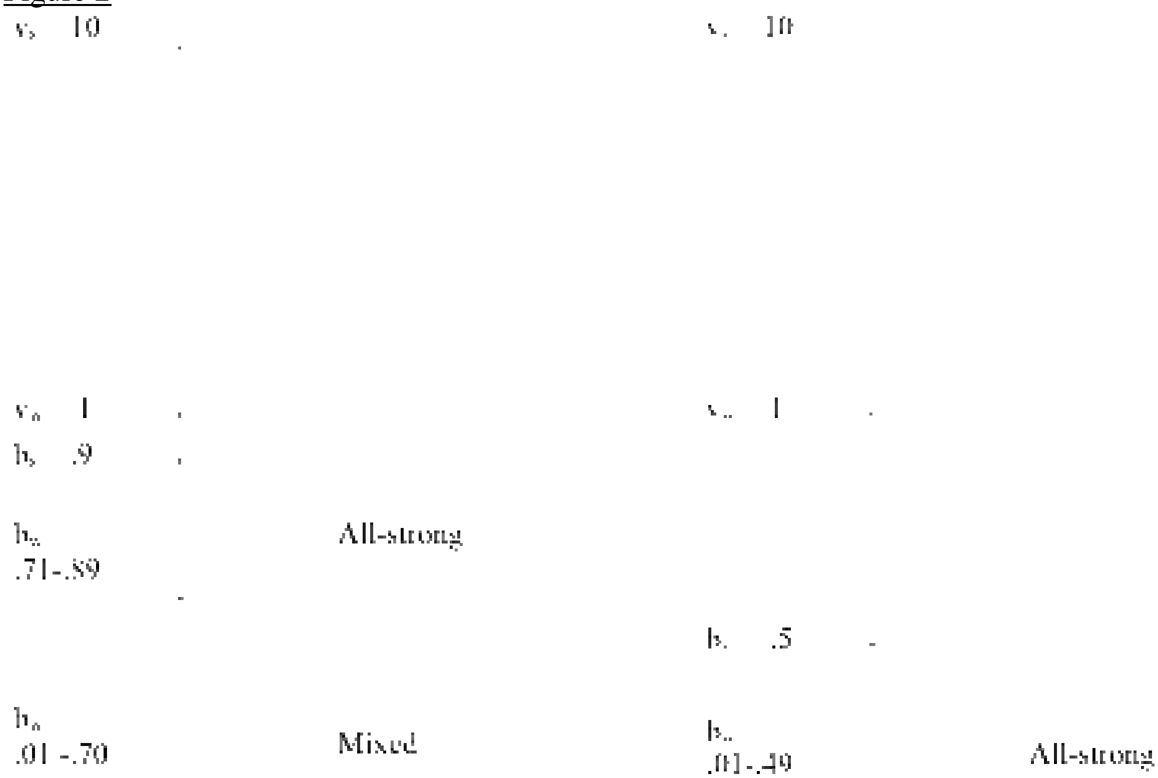


In the case of "bids below close valuations," both bid values are lower than both valuations, which are close to each other. On the left-hand side of the figure, with the strong bid value close to the weak valuation value, the cartel employs different strategies depending on the weak bid value. When the weak bid is close to the strong bid value (and hence, the weak valuation), the center recommends that both bidders bid strong. There is an intermediate range of weak bid values where it is optimal for the cartel to utilize the mixed strategy. Finally, for weak bid values sufficiently far from both the strong bid and weak valuation values, the cartel tells its bidders bid all-weak. On the right side, the strong bid value is far from the weak valuation and we see that that the cartel will employ the all-weak strategy for all weak bid values lower than that strong bid.

In every case, the cartel would like both of its members to bid weak so that it pays less to the auctioneer. Figure 1 shows that when bid increments are large enough, the cartel is able to effectively collude and suppress the sale price by having both its members bid the weak value. On the other hand, when the weak bid is close to the strong bid, the bid increment is small, and thus, the bidders both bid strong. This is a bad outcome for the cartel and a good outcome for auctioneers, since the seller is now guaranteed to receive the strong bid price. Therefore, this result suggests that that auctioneers should consider allowing smaller bid increments. By allowing for smaller bid increments, a ring member's incentive to cheat on the cartel becomes stronger. Therefore, the cartel is more limited in what it can coordinate without violating the incentive compatibility constraints.

4.2 Bids Below Distant Valuations

<u>Figure 2</u>

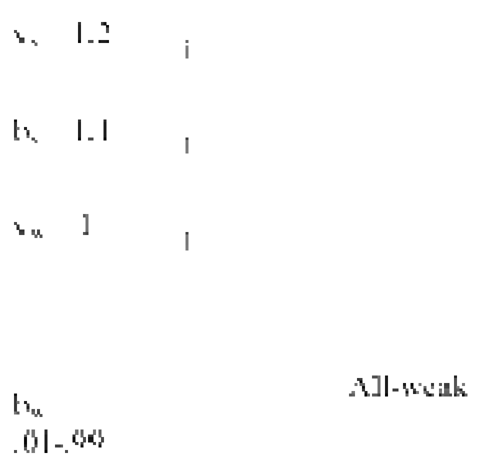


In Figure 2, I examine the case of "bids below distant valuations" and the result is similar to that of Figure 1. The difference in this case is that the weak and strong valuations are quite far apart from each other. Like the results shown in Figure 1, we see that with the strong bid close to weak valuation, a weak bid value close to the strong bid and weak valuation results in the mixed strategy; and for weak bid values far from the strong bid and weak valuation numbers, the cartel recommends all-strong.

Parallel in intuition, Figure 2 reveals that with a small enough bid increment, the seller will be able to maximize its surplus as cartel members have an incentive to bid stronger to increase their probability of winning the auction. In this case, because the strong valuation is much greater than the weak valuation, there is an even greater incentive for the bidder to cheat; thus, there is no all-weak outcome in this environment. As Figures 1 and 2 show that the "bids below valuations" case is quite susceptible to bidder deviation and thus, the all-strong outcome is prevalent in these cases, which is what sellers desire.

4.3 Bids Between Close Valuations

Figure 3



In Figure 3, I examine the case where the bids are between the valuation values and the two valuations are close to each other. Regardless of what the weak bid value is relative to the other three values, both bidders bid weak to maximize their expected utility. With the difference between the weak and strong valuations small, the strong bid is placed in between those two values such that bidding strong results in very small utility gain. While bidding strong would improve their chances of winning the auction, it is not advantageous in terms of expectation. Bidders would rather stick with their weak bid, hoping that the other did not bid strong, and have a 50% chance of gaining a large amount of utility. Because bidders have little incentive to bid strong, the cartel is able to collude successfully, a scenario the auctioneer would like to avoid.

4.4 Bids Between Distant Valuations

<u>Figure 4</u>

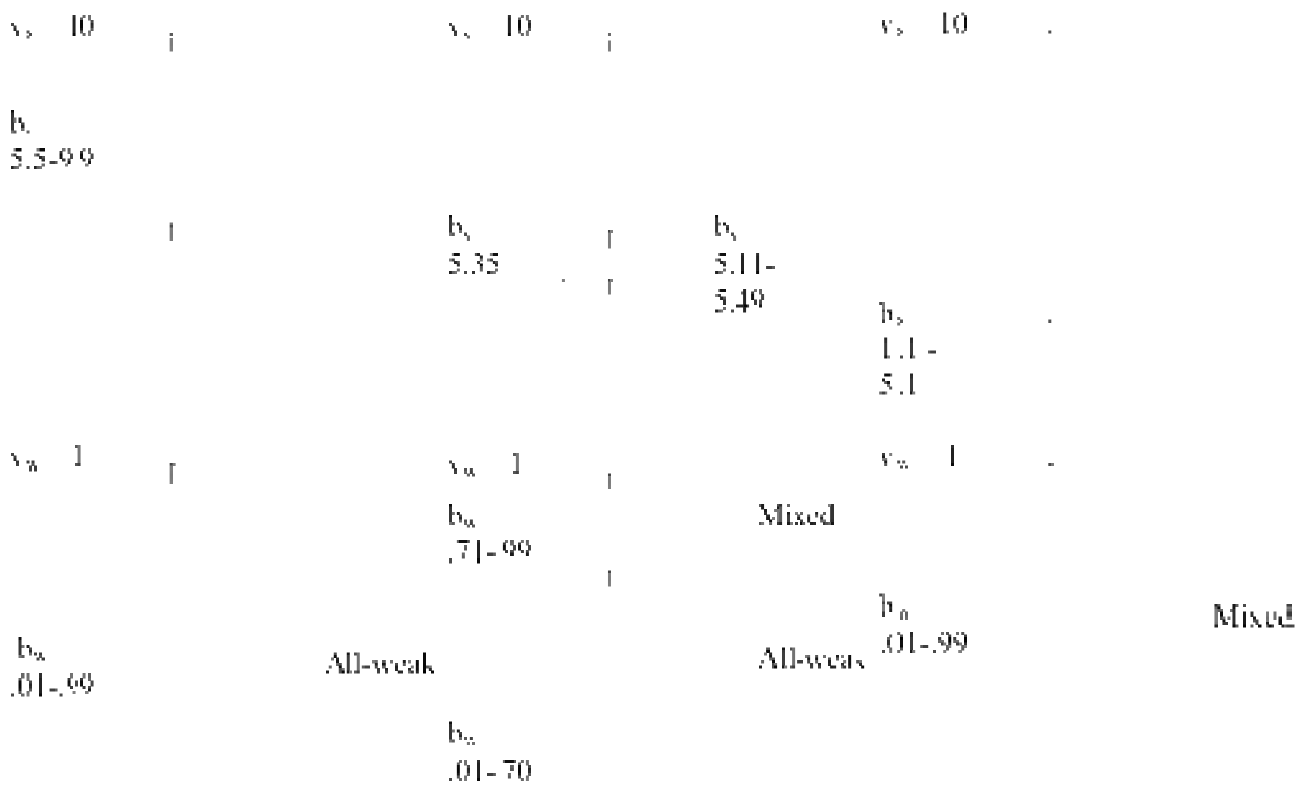


Figure 4 delineates the solution to the "bids between distant valuations" case. We can see that since the weak and strong valuations are far apart, there is a large range of possible strong bid values between the valuations. As a result, we see different results depending on where the strong bid is located in that range.

On the left, there is a large range of strong bid values close to the strong valuation value such that all weak bid values below the weak valuation number result in the all-weak bid. Similarly, on the right, there is a large array of strong bid values far from the strong valuation number and closer to the weak valuation value where the cartel will use the mixed strategy. Finally, in the middle, there is a small range of strong bid values (b_s = 5.11 – 5.49) that are intermediately spaced between the weak and strong valuation values such that depending on the weak bid value determination, the cartel uses either the all-weak or the mixed bids. I choose a strong bid value in that range (b_s = 5.35), to show specific ranges of weak bid values that produce different outcomes. When the weak bid is set to be close to the weak valuation, the center recommends the mixed strategy. When the weak bid is not close to the weak valuation, the center recommends all-weak.

Figure 4 helps to strengthen the intuition behind Figure 3's results. Figure 4 demonstrates that bidders have almost no incentive to bid strong in the general "bids between valuations case." In this case, when the difference between the weak and strong valuations is large, there is an even stronger desire for bidders to cheat and bid strong because their expected utility remains high. Thus, this situation produces the mixed outcome in addition to the all-weak result. A bidder with a strong value for the item will be willing to take the trade-off of losing a bit of utility for a significant probability increase in winning the auction. For both bidders and auctioneers, this is neither a bad or good outcome; in fact, the mixed result is the most economically efficient outcome. In these cases, the auction environments are able to distinguish the weak valuers from the strong. As a result, the cartel is forced to reveal its bidders' valuation information by employing the mixed strategy. Ultimately, Figures 3 and 4 make clear that "bids between valuations" environments are not particularly conducive to bidder deviation. Hence, we do not see any all-strong outcomes.

4.5 Summarizing Results

Table 5 breaks down the results by describing what situations produce a particular outcome:

Table 5	All-strong	Mixed	All-Weak
Bids Below Valuations ($b_w < b_s < v_w < v_s$)			
Close Valuations	b_s far from v_w With b_s close to v_w, b_w close to b_s	With b_s close to v_w, intermediate range of b_w, neither close nor far from b_s	With b_s close to v_w, b_w far from b_s
Distant Valuations	b_s far from v_w With b_s close to v_w, b_w close to b_s	With b_s close to v_w, b_w far from b_s	
Bids Between Valuations ($b_w < v_w < b_s < v_s$)			
Close Valuations			b_s close to v_w
Distant Valuations		With b_s far from v_s & v_w (intermediate range of b_s), b_w close to b_s b_s far from v_s & close to v_w	b_s is far from v_w, b_s b_s is close to v_w

Revenue-maximizing sellers are looking to replicate the environments where the cartel recommends all-strong bids for both bidders because bidders have an incentive to deviate from the center's recommendations. When the cartel must employ this strategy, the cartel colludes inefficiently and sub-optimally, and produces the best result for the auctioneer. We observe four such scenarios in the "bids below valuations" case, which we note from earlier that it is susceptible to bidder deviation. For both close and distant valuations, we can have the strong bid and weak valuation be far apart. Alternatively, with the strong bid close the weak valuation, we can have the weak bid be close to the strong bid. In these cases, by deviating, bidders maximize their expected utility by significantly improving their chances of winning the auction for a small reduction in utility gain because the probability gain of winning far outweighs the difference in utility between submitting a weak and strong bid.

The cases that auctioneers want to avoid are those where the cartel colludes most effectively by having both bidders bid weak and suppress the sale price. Given "bids below close valuations," the strong bid is close to the weak valuation while the weak bid is far from the strong bid value. In this situation, bidders do not have an incentive to deviate and bid more strongly because the strong bid is close to the weak valuation. As the weak bid value is far less than the strong bid value, there is a significant difference between utility levels having bid weakly versus strongly. This difference evidently outweighs the probability component of winning the auction, as both bidders are willing to bid weakly, risking having a smaller chance of winning the auction in exchange for huge utility gains.

There are also two scenarios where the "bids are between valuations." The strong bid can be close to the weak valuation for both close and distant valuations. Otherwise, the weak bid can be far away from the weak valuation and the strong bid given valuations that are intermediately spaced. The bids between valuations cases produce vastly different results from the bids below valuations because bidders lose incentive to deviate. Since the strong bid value is now greater than the weak valuation, bidders with weak valuations will not bid more (bid strongly) than what they value the item. Unconcerned about bidder deviation, cartels can utilize the all-weak strategy without the center having issues making recommendations that are incentive-compatible. Hence, the cartel does not recommend all-strong bids in bids between valuation situations.

Finally, there are four conditions in which a cartel will suggest the mixed strategy. These environments promote economically efficient allocations by being able to distinguish among bidders with different valuations. There are two situations where the bids are below valuations. When the valuations are close together and the strong bid is close to the weak valuation, there is a range of weak bid values which are intermediately distanced from the strong bid value. When the valuations are far apart, the strong bid must be close to the weak valuation while the weak bid value is significantly lower than the strong bid value. There are also two situations where the bids are between valuations where they are distant from each other. First, there is a small intermediate range of strong bid values such that the strong bid value is far from both weak and strong valuations; with the strong bid in that small range, the weak bid must be close to the strong bid. Otherwise, the strong bid must be far from the strong valuation and close to the weak valuation.

Auctioneers who are looking for an economic efficient outcome should study these four environments, which are able to distinguish between weak and strong valuers such that the auction is ex post efficient. All four of these situations are cases where bidders have to make a tradeoff in maximizing their expected utility. Because the weight of the probability component is comparable to that of the utility component, there is no dominant strategy that both bidders can employ. Therefore, these auction conditions prevent the cartel's members from making a joint-decision on how they are going to bid, and instead, promotes competition among cartel members.

5 Conclusion

In this paper, I set out to determine the strategy of an all-inclusive cartel with 2 symmetric bidders of two types at a single first-price sealed bid auction where the cartel can only make non-binding recommendations to coordinate among its members. My initial hypothesis was that the center's bid recommendations would be stochastic in order for the incentive-compatibility constraint to hold. By randomizing recommendations, the bidders would not be able to guess the center's strategy and try to maximize its expected surplus by choosing not to follow the recommendation and deviate. After solving the maximization problem, we see that the optimal cartel bases its recommendations solely on expected utility.

Building upon Lopomo, Marx, and Sun (2009), this paper's determination of the optimal collusive mechanism for discrete bid increments reveals how the cartel can achieve profits above the non-cooperative level for discrete increments. These results demonstrate that collusion is effective in ensuring that colluding bidders work together and not bid more than one bid increment below the lowest value among ring members. For the majority of scenarios, we see that it is optimal for the cartel to submit two weak or strong bids.

This finding confirms existing intuition on the role of collusion at auctions. The U.S. Department of Justice published an antitrust primer on price fixing and bid rigging ascertaining that "identical prices may indicate a price-fixing conspiracy," (U.S. Department of Justice [DOJ], n.d., p. 3). However, the primer does not explain the reason why identical prices are a strong sign of collusion. Through this paper, I have analytically shown that a major part of a cartel's optimal collusive mechanism is for bidders to submit identical bids, whether they be all-weak or all-strong.

My results also show that collusion can result in inefficient auction outcomes. Because bidders have submitted identical bids, the auctioneer has to essentially randomly choose the winner. Since the center is recommending this strategy regardless of how the bidder values the object, the winning bidder may not have been the highest valuer of the object, and thus, not gain as much surplus as the other cartel member, leading to an ex post inefficient result.

Along the same lines, the results demonstrate that it is very difficult to design an economically efficient auction. We see that there are only a few cases where the cartel has to use the mixed strategy. Only when the seller sets precise bid increments in specific situations do we see that the auction is able to separate a weak valuer and strong bidder and thus, efficiently allocate the item to the highest valuing buyer. While private sellers are seeking to maximize revenue, an auctioneer like the government may desire to maximize revenue *and* have an economically efficient outcome. For example, when the FCC is auctioning spectrum licenses, the FCC wants to award the licenses to corporations who will use them most effectively. The government has a responsibility to the public that the telecommunications company that wins will properly use those spectrum licenses in serving the public's interest and needs, in addition to maximizing firm profit.

A limitation of my model is the use of a simple environment in which there are only two symmetric bidders of two types. The n=2 symmetric case is a special case in mathematics. It would be interesting to see how these results may change with the introduction of asymmetry among bidders, adding more bidder types, or including more bidders with n=3 or greater. A logical extension to this paper is to rewrite and optimize the problem for a cartel with more bidders and more bidder types. In addition, we could develop a model that supports asymmetry among bidders. Lastly, it would be interesting to build upon this research by examining cartels that operate in multiple auctions. Repeated interaction significantly changes the operation of both a cartel and its members, especially if bidder deviation results in the expulsion of that bidder from future cartel participation. Further studies on this topic would continue to advance our knowledge on collusive behavior and mechanisms in auction theory, as we strive to determine how cartels can maximize their payoffs. Continuing this line of research will expand our knowledge on how effective cartels operate. We can use this understanding to design auctions that are more robust to bidder collusion and develop methods to better detect the presence of cartels.

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Appendix: Mathematica Code

This is the Mathematica code used to solve the maximization problem discussed in Section 3:

```
v1s=?;v2s=?;v1w=?;v2w=?;b1s=?;b2s=?;b1w=?;b2w=?;
f1[v1w]=.5;f1[v1s]=.5;f2[v2w]=.5;f2[v2s]=.5;

u1[b1w,b2w,v1w]=(v1w-b1w)/2;
u1[b1w,b2w,v1s]=(v1s-b1w)/2;
u1[b1w,b2s,v1w]=0;
u1[b1w,b2s,v1s]=0;
u1[b1s,b2w,v1w]=v1w-b1s;
u1[b1s,b2w,v1s]=v1s-b1s;
u1[b1s,b2s,v1w]=(v1w-b1s)/2;
u1[b1s,b2s,v1s]=(v1s-b1s)/2;
u2[b1w,b2w,v2w]=(v2w-b2w)/2;
u2[b1w,b2w,v2s]=(v2s-b2w)/2;
u2[b1s,b2w,v2w]=0;
u2[b1s,b2w,v2s]=0;
u2[b1w,b2s,v2w]=v2w-b2s;
u2[b1w,b2s,v2s]=v2s-b2s;
u2[b1s,b2s,v2w]=(v2w-b2s)/2;
u2[b1s,b2s,v2s]=(v2s-b2s)/2;

maximand= p[b1w,b2w,v1w,v2w]u1[b1w,b2w,v1w]f1[v1w]f2[v2w]+
p[b1s,b2w,v1w,v2w]u1[b1s,b2w,v1w]f1[v1w]f2[v2w]+
p[b1w,b2s,v1w,v2w]u1[b1w,b2s,v1w]f1[v1w]f2[v2w]+
p[b1s,b2s,v1w,v2w]u1[b1s,b2s,v1w]f1[v1w]f2[v2w]+
p[b1w,b2w,v1s,v2w]u1[b1w,b2w,v1s]f1[v1s]f2[v2w]+
p[b1s,b2w,v1s,v2w]u1[b1s,b2w,v1s]f1[v1s]f2[v2w]+
```

p[b1w,b2s,v1s,v2w]u1[b1w,b2s,v1s]f1[v1s]f2[v2w]+
p[b1s,b2s,v1s,v2w]u1[b1s,b2s,v1s]f1[v1s]f2[v2w]+
p[b1w,b2w,v1w,v2s]u1[b1w,b2w,v1w]f1[v1w]f2[v2s]+
p[b1s,b2w,v1w,v2s]u1[b1s,b2w,v1w]f1[v1w]f2[v2s]+
p[b1w,b2s,v1w,v2s]u1[b1w,b2s,v1w]f1[v1w]f2[v2s]+
p[b1s,b2s,v1w,v2s]u1[b1s,b2s,v1w]f1[v1w]f2[v2s]+
p[b1w,b2w,v1s,v2s]u1[b1w,b2w,v1s]f1[v1s]f2[v2s]+
p[b1s,b2w,v1s,v2s]u1[b1s,b2w,v1s]f1[v1s]f2[v2s]+
p[b1w,b2s,v1s,v2s]u1[b1w,b2s,v1s]f1[v1s]f2[v2s]+
p[b1s,b2s,v1s,v2s]u1[b1s,b2s,v1s]f1[v1s]f2[v2s]+
p[b1w,b2w,v1w,v2w]u2[b1w,b2w,v2w]f1[v1w]f2[v2w]+
p[b1s,b2w,v1w,v2w]u2[b1s,b2w,v2w]f1[v1w]f2[v2w]+
p[b1w,b2s,v1w,v2w]u2[b1w,b2s,v2w]f1[v1w]f2[v2w]+
p[b1s,b2s,v1w,v2w]u2[b1s,b2s,v2w]f1[v1w]f2[v2w]+
p[b1w,b2w,v1s,v2w]u2[b1w,b2w,v2w]f1[v1s]f2[v2w]+
p[b1s,b2w,v1s,v2w]u2[b1s,b2w,v2w]f1[v1s]f2[v2w]+
p[b1w,b2s,v1s,v2w]u2[b1w,b2s,v2w]f1[v1s]f2[v2w]+
p[b1s,b2s,v1s,v2w]u2[b1s,b2s,v2w]f1[v1s]f2[v2w]+
p[b1w,b2w,v1w,v2s]u2[b1w,b2w,v2s]f1[v1w]f2[v2s]+
p[b1s,b2w,v1w,v2s]u2[b1s,b2w,v2s]f1[v1w]f2[v2s]+
p[b1w,b2s,v1w,v2s]u2[b1w,b2s,v2s]f1[v1w]f2[v2s]+
p[b1s,b2s,v1w,v2s]u2[b1s,b2s,v2s]f1[v1w]f2[v2s]+
p[b1w,b2w,v1s,v2s]u2[b1w,b2w,v2s]f1[v1s]f2[v2s]+
p[b1s,b2w,v1s,v2s]u2[b1s,b2w,v2s]f1[v1s]f2[v2s]+
p[b1w,b2s,v1s,v2s]u2[b1w,b2s,v2s]f1[v1s]f2[v2s]+
p[b1s,b2s,v1s,v2s]u2[b1s,b2s,v2s]f1[v1s]f2[v2s]

delta1a[b1w]=b1w;delta1a[b1s]=b1s;
delta1a[b2w]=b2w;delta1a[b2s]=b2s;
delta1b[b1w]=b1w;delta1b[b1s]=b1w;
delta1b[b2w]=b2w;delta1b[b2s]=b2w;

```
delta1c[b1w]=b1s;delta1c[b1s]=b1w;
delta1c[b2w]=b2s;delta1c[b2s]=b2w;
delta1d[b1w]=b1s;delta1d[b1s]=b1s;
delta1d[b2w]=b2s;delta1d[b2s]=b2s;

con2a = p[b1w,b2w, v1w, v2w]>=0;
con2b=p[b1s, b2w, v1w, v2w]>=0;
con2c=p[b1w,b2s,v1w,v2w]>=0;
con2d=p[b1s,b2s,v1w,v2w]>=0;
con2e=p[b1w,b2w,v1s,v2w]>=0;
con2f=p[b1s,b2w,v1s,v2w]>=0;
con2g=p[b1w,b2s,v1s,v2w]>=0;
con2h=p[b1s,b2s,v1s, v2w]>=0;
con2i=p[b1w,b2w,v1w,v2s]>=0;
con2j=p[b1s,b2w,v1w,v2s]>=0;
con2k=p[b1w,b2s,v1w,v2s]>=0;
con2l=p[b1s,b2s,v1w,v2s]>=0;
con2m=p[b1w,b2w,v1s,v2s]>=0;
con2n=p[b1s,b2w,v1s,v2s]>=0;
con2o=p[b1w,b2s,v1s,v2s]>=0; con2p=p[b1s,b2s,v1s,v2s]>=0;
con2q = p[b1w,b2w, v1w, v2w]+p[b1s, b2w, v1w,v2w]
+p[b1w,b2s,v1w,v2w]+p[b1s,b2s,v1w,v2w] ==1;
con2r =p[b1w,b2w,v1s,v2w]+p[b1s,b2w,v1s,v2w]+
p[b1w,b2s,v1s,v2w]+p[b1s,b2s,v1s, v2w]==1;
con2s=p[b1w,b2w,v1w,v2s]+p[b1s,b2w,v1w,v2s]+
p[b1w,b2s,v1w,v2s]+p[b1s,b2s,v1w,v2s]==1;
con2t=p[b1w,b2w,v1s,v2s]+p[b1s,b2w,v1s,v2s]+
p[b1w,b2s,v1s,v2s]+p[b1s,b2s,v1s,v2s]==1;
```

```
con3a=m1[v1w,v2w]f1[v1w]f2[v2w]+m2[v1w,v2w]f1[v1w]f2[v2w]+m1[v1s,v2w]f1[v1s]f2[v2w]+m2[v1s,v2w]f1[v1s]f2[v2w]+m1[v1w,v2s]f1[v1w]f2[v2s]+m2[v1w,v2s]f1[v1w]f2[v2s]+m1[v1s,v2s]f1[v1s]f2[v2s]+m2[v1s,v2s]f1[v1s]f2[v2s]==0;

U1star[px_,deltax_,v1primex_,v1x_]:=
px[b1w,b2w,v1primex,v2w]u1[deltax[b1w],b2w,v1x]f2[v2w]+px[b1s,b2w,v1primex,v2w]u1[deltax[b1s],b2w,v1x]f2[v2w]+px[b1w,b2s,v1primex,v2w]u1[deltax[b1w],b2s,v1x]f2[v2w]+px[b1s,b2s,v1primex,v2w]u1[deltax[b1s],b2s,v1x]f2[v2w]+px[b1w,b2w,v1primex,v2s]u1[deltax[b1w],b2w,v1x]f2[v2s]+px[b1s,b2w,v1primex,v2s]u1[deltax[b1s],b2w,v1x]f2[v2s]+px[b1w,b2s,v1primex,v2s]u1[deltax[b1w],b2s,v1x]f2[v2s]+px[b1s,b2s,v1primex,v2s]u1[deltax[b1s],b2s,v1x]f2[v2s]-m1[v1primex,v2w]f2[v2w]-
m1[v1primex,v2s]f2[v2s];

U2star[px_,deltax_,v2primex_,v2x_]:=px[b1w,b2w,v1w,
v2primex]u2[b1w,deltax[b2w],v2x]f1[v1w]+px[b1s,b2w,v1w,v2primex]u2[b1s,deltax[b2w],v2x]f1[v1w]+px[b1w,b2s,v1w,v2primex]u2[b1w,deltax[b2s],v2x]f1[v1w]+px[b1s,b2s,v1w,v2primex]u2[b1s,deltax[b2s],v2x]f1[v1w]+px[b1w,b2w,v1s,v2primex]u2[b1w,deltax[b2w],v2x]f1[v1s]+px[b1s,b2w,v1s,v2primex]u2[b1s,deltax[b2w],v2x]f1[v1s]+px[b1w,b2s,v1s,v2primex]u2[b1w,deltax[b2s],v2x]f1[v1s]+px[b1s,b2s,v1s,v2primex]u2[b1s,deltax[b2s],v2x]f1[v1s]-m2[v1w,v2primex]f1[v1w]-m2[v1s,v2primex]f1[v1s];

cons4a=U1star[p,delta1a,v1w,v1w]-U1[p,v1w]<=0;
cons4b=U1star[p,delta1a,v1s,v1w]-U1[p,v1w]<=0;
cons4c=U1star[p,delta1a,v1w,v1s]-U1[p,v1s]<=0;
cons4d=U1star[p,delta1a,v1s,v1s]-U1[p,v1s]<=0;

cons4e=U1star[p,delta1b,v1w,v1w]-U1[p,v1w]<=0;
cons4f=U1star[p,delta1b,v1s,v1w]-U1[p,v1w]<=0;
cons4g=U1star[p,delta1b,v1w,v1s]-U1[p,v1s]<=0;
cons4h=U1star[p,delta1b,v1s,v1s]-U1[p,v1s]<=0;
```

```
cons4i=U1star[p,delta1c,v1w,v1w]-U1[p,v1w]<=0;
cons4j=U1star[p,delta1c,v1s,v1w]-U1[p,v1w]<=0;
cons4k=U1star[p,delta1c,v1w,v1s]-U1[p,v1s]<=0;
cons4l=U1star[p,delta1c,v1s,v1s]-U1[p,v1s]<=0;

cons4m=U1star[p,delta1d,v1w,v1w]-U1[p,v1w]<=0;
cons4n=U1star[p,delta1d,v1s,v1w]-U1[p,v1w]<=0;
cons4o=U1star[p,delta1d,v1w,v1s]-U1[p,v1s]<=0;
cons4p=U1star[p,delta1d,v1s,v1s]-U1[p,v1s]<=0;

cons4pa=U2star[p,delta1a,v2w,v2w]-U2[p,v2w]<=0;
cons4pb=U2star[p,delta1a,v2s,v2w]-U2[p,v2w]<=0;
cons4pc=U2star[p,delta1a,v2w,v2s]-U2[p,v2s]<=0;
cons4pd=U2star[p,delta1a,v2s,v2s]-U2[p,v2s]<=0;

cons4pe=U2star[p,delta1b,v2w,v2w]-U2[p,v2w]<=0;
cons4pf=U2star[p,delta1b,v2s,v2w]-U2[p,v2w]<=0;
cons4pg=U2star[p,delta1b,v2w,v2s]-U2[p,v2s]<=0;
cons4ph=U2star[p,delta1b,v2s,v2s]-U2[p,v2s]<=0;

cons4pi=U2star[p,delta1c,v2w,v2w]-U2[p,v2w]<=0;
cons4pj=U2star[p,delta1c,v2s,v2w]-U2[p,v2w]<=0;
cons4pk=U2star[p,delta1c,v2w,v2s]-U2[p,v2s]<=0;
cons4pl=U2star[p,delta1c,v2s,v2s]-U2[p,v2s]<=0;

cons4pm=U2star[p,delta1d,v2w,v2w]-U2[p,v2w]<=0;
cons4pn=U2star[p,delta1d,v2s,v2w]-U2[p,v2w]<=0;
cons4po=U2star[p,delta1d,v2w,v2s]-U2[p,v2s]<=0;
cons4pp=U2star[p,delta1d,v2s,v2s]-U2[p,v2s]<=0;
```

```
U1[px_,v1x_]:=
px[b1w,b2w,v1x,v2w]u1[b1w,b2w,v1x]f2[v2w]+px[b1w,b2w,v1x,v2s]u1[b1w,b2w,v1x]
f2[v2s]+px[b1s,b2w,v1x,v2w]u1[b1s,b2w,v1x]f2[v2w]+px[b1s,b2w,v1x,v2s]u1[b1s,b2w
,v1x]f2[v2s]+px[b1w,b2s,v1x,v2w]u1[b1w,b2s,v1x]f2[v2w]+px[b1w,b2s,v1x,v2s]u1[b1
w,b2s,v1x]f2[v2s]+px[b1s,b2s,v1x,v2w]u1[b1s,b2s,v1x]f2[v2w]+px[b1s,b2s,v1x,v2s]u1[
b1s,b2s,v1x]f2[v2s]-m1[v1x,v2w]f2[v2w]-m1[v1x,v2s]f2[v2s];

U2[px_,v2x_]:=
px[b1w,b2w,v1w,v2x]u2[b1w,b2w,v2x]f1[v1w]+px[b1w,b2w,v1s,v2x]u2[b1w,b2w,v2x]
f1[v1s]+px[b1s,b2w,v1w,v2x]u2[b1s,b2w,v2x]f1[v1w]+px[b1s,b2w,v1s,v2x]u2[b1s,b2w
,v2x]f1[v1s]+px[b1w,b2s,v1w,v2x]u2[b1w,b2s,v2x]f1[v1w]+px[b1w,b2s,v1s,v2x]u2[b1
w,b2s,v2x]f1[v1s]+px[b1s,b2s,v1w,v2x]u2[b1s,b2s,v2x]f1[v1w]+px[b1s,b2s,v1s,v2x]u2[
b1s,b2s,v2x]f1[v1s]-m2[v1w,v2x]f1[v1w]-m2[v1s,v2x]f1[v1s];

soln=Maximize[{maximand,constraints},{p[b1w,b2w,v1w,v2w],p[b1s,b2w,v1w,v2w],p[
b1w,b2s,v1w,v2w],p[b1s,b2s,v1w,v2w],p[b1w,b2w,v1s,v2w],p[b1s,b2w,v1s,v2w],p[b1w
,b2s,v1s,v2w],p[b1s,b2s,v1s,v2w],p[b1w,b2w,v1w,v2s],p[b1s,b2w,v1w,v2s],p[b1w,b2s,v
1w,v2s],p[b1s,b2s,v1w,v2s],p[b1w,b2w,v1s,v2s],p[b1s,b2w,v1s,v2s],p[b1w,b2s,v1s,v2s],
p[b1s,b2s,v1s,v2s],m1[v1w,v2w],m1[v1s,v2w],m1[v1w,v2s],m1[v1s,v2s],m2[v1w,v2w],
m2[v1s,v2w],m2[v1w,v2s],m2[v1s,v2s]}]
```