Collusion with Three Bidders at First-Price Auctions

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Honors thesis submitted in partial fulfillment of the requirements for Graduation with Distinction in Economics in Trinity College of Duke University

Duke University
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2010

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Acknowledgments

I am indebted to my Faculty Advisor, Professor Leslie Marx, for her patience, guidance, and unwavering support throughout the year. I would also like to thank Professor Michelle Connolly and my Econ 198S and 199S classmates for their helpful feedback and insight at each step of the process. Special thanks also to Peter Franklin and Peng Sun for their invaluable computer programming assistance. This project would not have been possible without these individuals' generous donation of time and expertise. Lastly, I would like to thank the National Science Foundation for support under grant SES-0849349.

Abstract

Lopomo, Marx & Sun (2009) show that, given a specified environment, profitable collusion is not possible for a two-person bidding ring operating at a first-price sealed-bid auction. This research investigates the rigor of their result by expanding the theoretical framework to the case of a three-bidder cartel. The output generated from the linear programming model confirms the authors' earlier result. This is a key finding as it is the first to establish a basis for comparison of equilibrium surplus scenarios among multiple-bidder auction formats. The analytic and numerical results pave the way for future research examining the effect of cartel size on profitability and have many real-world implications for both private and public policy alike.

1 Introduction

Over the last decade, a string of high-profile cases of collusion have drawn increasing awareness to the issue of illegal cartels operating in both public and private auctions. This attention has garnered stricter legal enforcement as well as a stronger desire among academics to understand how such cartels operate. A number of studies have examined better ways to detect collusive behavior. Other economics literature has examined what precautionary measures can be taken in order to create a bidding framework that is robust to collusion. The possibility for (and susceptibility *to*) collusion in real-life auctions motivates research on this topic. Lopomo, Marx & Sun (2009) previously determined that profitable collusion was impossible in a simplified first-price sealed-bid auction format. This paper builds upon their existing theoretical model. More specifically, the research investigates whether the framework remains immune to collusive behavior when the scenario is extended to a more relaxed case with three symmetric bidders. The findings have implications for realistic cases mentioned above.

Auction markets move billions of dollars in goods and services each day. The enormous growth of e-commerce has encouraged business to construct auctions for goods whose trade was previously negotiated bilaterally (Klemperer, 2002). Auctions have a significant impact on the economic landscape of the public sector through procurement processes of government contracts and public goods. This year alone, international governments are expected to bring in tens of billions of dollars by auctioning off third-generation (3G) radio spectrum and hundreds of billions of dollars through the sale of CO2 Cap-and-Trade permits (Batten et al., 2008 and Budget, 2009). While these auctions are an effective means of allocating public and private goods, they have recently faced pervasive episodes of bid rigging by illegal cartels as evidenced by guilty verdicts in antitrust lawsuits spanning multiple countries and sectors (Kovacic et al., 2006).

Collusion is undesirable in the sense that it can decrease the revenue obtained by sellers and lead to an inefficient final allotment of goods from the auction (Burtraw et al., 2008). Given this fact, research on the causal mechanisms by which bidders support collusive agreements is both useful and important for auctions; by understanding the preconditions that enable collusion, sellers may suitably design their auctions in order to anticipate and avoid such circumstances. In this way, sellers can *preemptively* eliminate collusion or successfully detract would-be cartels from attempting it. Such precautionary auction design is a more valuable approach to combating collusive behavior than ex-post detection and prosecution, which can prove very difficult to accomplish.

Various authors have shown that environmental factors play a large role in how profitable collusion can be (Skrzypacz & Hopenhayn, 2004). It has also been found that not all bidding formats are equally immune to collusion. More specifically, Mailath & Zemsky (1991) wrote a seminal paper which concluded that bidders can successfully and profitably[2] collude at second-price auctions when there is a mechanism in place to enforce compliance among ring members during the legitimate auction. Another study by McAfee & McMillan (1992) demonstrated that bidders can successfully and profitably collude at first-price auctions within a similar submissive framework. Marshall & Marx (2007) took this finding one step further and proved that cartels are less profitable when facing a first-price auction than other common auction formats such as open-ascending-bid and second-price auctions.

A current working paper (Lopomo, Marx & Sun, 2009) considers a first-price sealed-bid (FPSB) auction with two bidders operating within a theoretical context whereby coordinated behavior cannot be enforced. Contrary to the aforementioned research, which found that cartels can achieve some degree of collusive gain in various auction formats, Lopomo et al. determined that profitable collusion is not possible *at all.* Numerical computations and a linear programming duality argument are used to complete their proof. The authors' novel approach enables them to draw much stronger conclusions than previous research. However, their analytic results are only applicable for a simplified model involving an all-inclusive cartel with two symmetric bidders. As a result, a general robustness issue with their work is the extent to which the results from this two-bidder setting can be extrapolated to cases in which more than two bidders are active. In view of this gap in literature, this research extends the theoretical model used by Lopomo et. al (2009) by relaxing a constraint. Specifically, the research introduces an additional bidder to the model. It tests the strength of the authors' previous result[3] when the model is expanded to a three-bidder environment.

The extension to $n = 3$ is important because if the model holds for $n = 3$, there is reason to believe that it will work for all cases of $n > 3$ as well. The notation and computer code is rewritten and retested. The results prove that the framework holds for this particular extension, despite the relaxed constraint. These findings encourage future

[2] "Successful" collusion occurs when a cartel is able to support the collusive agreement. This is a necessary *but not sufficient* requirement for a cartel to be "profitable," in which case the members generate a surplus that is greater than the non-cooperative outcome.

[3] Lopomo et. al's (2009) claim "*...in simple environments, a bidding ring operating at a first-price sealed-bid auction cannot achieve any gains relative to non-cooperative bidding if the ring is unable to control the bids that its members submit at the auction.*" (Abstract).

research involving scenarios with additional variables and relaxed assumptions.

The paper proceeds as follows. Section 2 provides a literature review of related research and places this particular extension within its proper context. Section 3 discusses the theoretical framework of auction theory and lays out the specific model for this study. Section 4 presents the analytical results of the research by solving the optimization problem for a cartel using a bid coordination mechanism and benchmarking this outcome against other equilibrium scenarios. The final section concludes the paper with a discussion on the real-world implications of the findings and suggestions for other possible extensions to this research.

2 Literature Review

There have been numerous studies related to bidder collusion at auctions in the past. Empirical literature has generally focused on finding ways to detect the presence of collusion *ex-post*. Porter & Zona (1993, 1997), Zona (1986) and Baldwin, Marshall & Richard (1997) each propose tests to detect collusion in auction markets. Specifically, Porter and Zona (1993, 1997) analyze collusion in the context of Long Island highway construction data and Ohio school milk markets, respectively. Baldwin, Marshall & Richard (1997) examine collusive and competitive behavior using forest timber sales data. Howard & Kasermann (1989) and Nelson (1993) numerically calculate damages for cases where explicit bid-rigging was known to have occurred. McMillan (1991) performs a similar study for Japanese public works contracts. Kovacic, Marshall & Raiff (2007) investigate tacit and explicit collusion in the vitamins industry during the 1990s while Pesendorfer (2000) uses data for school milk contracts at first-price sealed-bid auctions in Florida and Texas during the 1980s as a case study for collusive behavior in auctions. Empirical work on collusion generally suffers from the difficulty in acquiring extensive data on what is an illegal and secretive activity.

Theoretical literature has researched the causal mechanisms driving collusion for various single-object auctions. Graham & Marshall (1987) and Mailath & Zemsky (1991) consider collusive agreements whereby cartels use side payments (often categorized as "strong" cartels) and assess the payments' resulting impact on efficiency. Klemperer (2002) lists and gives examples of critical pitfalls in auction design and discusses feasible resolutions for them. He concludes that the two most important features of an auction are its robustness to collusion and its attractiveness to potential bidders and that there are considerable tradeoffs associated with each. A separate body of work examines the ways in which bidding rings allocate bids and transfers to their members in a way that is incentive-compatible and generates the maximum expected surplus, given a set of institutional features (Asker, 2009a). McAfee & McMillan (1992), Marshall & Marx (2007), and Lopomo, Marx & Sun (2009) characterize the optimal collusive mechanism for cartels given certain environments and auction formats. These papers stand out among the others due to their acute relevance to this particular research and will now be discussed in further detail based on chronological order.

2.1 McAfee & McMillan (1992)

McAfee & McMillan (1992) consider a first-price, sealed-bid auction with symmetric bidders and an all-inclusive cartel. The cartel members communicate and bid for the right to be the sole cartel member to attend the main auction. The highest bidding cartel member is sent to the main auction, while other cartel members are prevented from bidding. Several assumptions are made regarding the theoretical framework. First, it is assumed that in event of a tie, a winner is randomly chosen among the highest bidding cartel members. Second, the identity and bid of the winning bidder becomes known. Finally, a re-auction function is enabled which allows cartel members to make transfer payments amongst themselves both before and after the auction. This possibility for resale facilitates self-enforcing collusion amongst the cartel members in what is known as a "bid submission mechanism" (BSM).

The paper concludes that bidders can successfully and profitably collude at first-price auctions when such a mechanism is in place to enforce ring members to comply with the cartel's bid recommendations during the auction. In equilibrium, the cartel member who won the right to attend the legitimate auction is the one with the highest value for the object, and that cartel member wins the object at a price equal to the seller's reserve price.[4] The cartel members then share a sum of money equal to the difference between the price reached in the cartel's own auction and the price reached in the main auction. Therefore, the mechanism is efficient and extracts the entire surplus from the seller and the side payments feature achieves efficiency within the bidding ring.

2.2 Marshall & Marx (2007)

Another study (Marshall & Marx, 2007) expands existing literature by creating a direct comparison of collusion at first-price versus second-price auctions. The work con-

[4]The reserve price is the minimum price the seller is willing to accept, below which the good in question will not be sold. The reserve price is usually a positive value (otherwise the seller would gain nothing from auctioning off the good), although there are certain cases where the reserve price may be pegged to zero (Asker, 2009b).

siders a FPSB scenario with no resale when pre-auction side payments between the bidders are allowed. Thus, the bid submission mechanism from the framework presented by McAfee & McMillan (1992) does not exist. Rather, cartel members must rely on a "bid coordination mechanism" (BCM) whereby the cartel can make recommendations as to how its members should bid, but has no means of enforcing those suggestions.

Marshall & Marx (2007) find that profitable collusion is much more difficult to achieve in a first-price auction than in a second-price auction when no enforcement mechanism is incorporated. It is the first research to establish that, in some environments, first-price auctions are robust to certain types of bidder collusion. Furthermore, the paper concludes that cartel payoff is lower than in the framework of McAfee & McMillan but remains higher than in a non-cooperative setting.

The intuition behind the different result from prior research has to do with the replacement of the bid submission mechanism with a bid coordination mechanism. In addition to providing incentives for truthful revelation of private information, the cartel must also provide incentives for its members to follow instructions and not deviate from the agreed upon plan. Therefore, the problems presented by game theory models such as adverse selection and moral hazard make the task of profitable collusion more challenging and consequently decrease the expected cartel payoff.

2.3 Lopomo, Marx & Sun (2009)

Although Marshall & Marx (2007) formalize the notion that collusive viability and profitability may vary based on auction format, they do not show the extent to which this is, in fact, the case. A recent study by Lopomo, Marx & Sun (2009) addresses this issue.[5] The authors use a very similar model to the abovementioned paper, with two symmetric bidders operating under a bid coordination mechanism in a FPSB auction.

In contrast to previous studies of scenarios involving BSMs, which found that a cartel can achieve some degree of collusive gain, this paper determines that a cartel using a BCM does *no* better than non-cooperative bidding. In fact, at least in some simple settings, profitable collusion is not possible *at all* when there is no enforcement mechanism in place to control the bids of the cartel members (For a graphical representation of non-cooperative and collusive outcomes for BSM vs. BCM models, please see Figure 2 in Appendix A). This is a seminal finding in the sense that the researchers are able to reformulate a game theoretic cartel problem into a linear optimization model. Their

[5]This research is a direct extension of Lopomo, Marx & Sun's working paper, "Bidder Collusion at First-Price Auctions" (2009).

study extends the model of collusion in the first-price auctions in a discretized framework, and incorporates linear programming along with duality arguments to prove their result.

This analytic approach allows Lopomo, Marx & Sun (2009) to determine a much stronger finding than previous literature has been able to achieve. Nevertheless, their results are only valid within the context of the highly-specified framework which they construct. Said differently, there is still room for further improvement upon the robustness of their model. For instance, it may be the case that their finding is strictly reserved to the special instance of two symmetric bidders. However, this paper strengthens the robustness of their framework and affirms the validity of their findings for a more relaxed environment of $n = 3$ bidders.

3 Theoretical Framework

3.1 Overview of Auctions

3.1.1 Auction Theory

Auction theory falls under a branch of game theory commonly referred to as "mechanism design" (Adams, 2007). Mechanism design involves games of private information in which the participants are directed to choose from different payoff structures. Such games are ultimately solved by motivating the agents to disclose their private information and reach a collective decision in what is known as an "Incentive-Compatible Mechanism" (Chung & Ely, 2002). According to Tabarrok (2007), the goal of mechanism design is "to create institutions that produce a desirable outcome while respecting the fact that agents have private information and are self-interested."[6] However, constructing such efficient institutions is difficult to achieve in practice due to the adverse selection problems associated with information asymmetry.

The study of frameworks which provide ample incentives for truthful revelation of information has particular usefulness in the design of auctions. Auctions are considered to be well-treated by such game theoretic techniques because the behavior and expectations of each bidder depends on the strategic environment in which he operates, including the behavior and expectations of the other bidders as well as the rules of bidding set forth by the seller (Alexandrova, 2009). In these games of incomplete and imperfect information, rational players attempt to bid in a way that maximizes their expected utility given their own set of beliefs about their competitors' strategies, which ultimately results in a Bayesian Nash equilibrium (BNE) outcome. Academic research often attempts to isolate

[6] Tabarrok (2007) pg. 1.

the essential elements of auction design and analyze equilibrium scenarios in idealized models in order to better understand and explain actual real-world phenomena.

3.1.2 Auction Formats

In order to effectively grasp the theoretical framework of this study, one must first have a basic understanding of how auctions operate. The bidding model in an auction may differ in a number of ways. Examples of variations to the basic form include time constraints, limits on bid prices, private or public participation, and et cetera. However, the four most widely used auction structures are the open-ascending (or English auction), the open-descending (or Dutch auction), the first-price sealed-bid (hereafter referred to as "FPSB"), and the second-price sealed-bid (or Vickrey Auction) (Marshall & Marx, 2008).

This paper examines a static FPSB format. The model is distinct from open-ascending auctions in that there is only one round of bidding and the identity of each bidder remains private. The sealed first-price auction format is typically used for the procurement of government contracts and public goods.

Consider a simplified framework for a FPSB auction with n bidders. Each bidder $i \in \{1, 2, ..., n\}$ has a valuation v_i for the good being auctioned.[7] Let the probability choice set of the buyer's valuations for the good, v_{low},...v_{high} be independently and uniformly distributed on $[0, 1]$ and assume that player i submits a bid b_i based on his particular valuation. Each bidder simultaneously submit his "best and final" offer[8], with the highest bidder winning the object and paying an amount equal to the higher of (i) his bid or (ii) the reserve price.

3.2 Introduction to Collusion

3.2.1 How it Works & Impact on Efficiency

Most developed countries contain a framework of laws that explicitly prohibit the abuse of market power. For instance, price fixing, bid rigging, market allocation, and other forms of collusion are prohibited by Section 1 of the Sherman Act and are subject to criminal prosecution by the Antitrust Division of the United States Department of Justice (U.S. Department of Justice, 2005). Violation of the Sherman Act is a felony, punishable

[7] To simplify notation, it is assumed that there is only one good for sale. This restriction is without loss of generality since valuations are independently distributed across multiple goods. Thus, the equilibrium for a single object first-price auction is equivalent to the equilibrium of any multiple-object auction (Pesendorfer 2000).

[8] Language used in (Klemperer, 2002).

by a maximum fine of $100 million for corporations, $1 million for individuals, and a jail sentence of 10 years.[9] While the legal status of many kinds of collusive behavior remains ambiguous, antitrust violations have been found to be unambiguously harmful; that is, such violations are *per se* illegal (Klemperer, 2002). These serious crimes contribute to inflation, destroy public confidence, and undermine the free enterprise system, which all have potentially devastating effects on the global economy. Price fixing, bid rigging, and market division have also been shown to defraud consumers and unquestionably raise prices or restrict output without creating any plausible positive externalities to offset the damages incurred.

Bid rigging is the most relevant form of collusion for this research. Such conspiracies are usually categorized as bid suppression, complementary bidding, bid rotation, and/or subcontracting schemes. At an elemental level, bidding rings use private information to manipulate the auction result and the final price level of a particular good in their favor. The bidders are aware the identity of the other bidders, form a cartel, and then weaken competition by agreeing to only bid against outsiders and not against themselves, thereby increasing their individual and collective surplus (Asker, 2009a). This collusion may be done prior to, during, and after the legitimate bidding process. Real-world examples of pre-auction meetings include U.S. v. Addyston Pipe & Steel Co. (cast iron pipe) and Finnegan v. Campeau Corp (bidding for Federated Department Stores). Well-known instances of post-auction meetings include U.S. v. Ronald Pook (antiques), U.S. v. Seville Industrial Machinery (used machinery), and District of Columbia v. George Basiliko (real estate).[10]

McAfee & McMillan (1992) investigate the major obstacles which cartels must overcome in order to be successful. The ring must first develop a mechanism for dividing the surplus as well as a mechanism for enforcing the collusive agreement. Thus, they face internal issues related to adverse selection and misaligned incentives. From an external perspective, profitable cartels must also beat out new entrants to a colluding industry and ward off potential retributive actions by prior victims of collusion to destabilize the ring. The Department of Justice has also conducted research on the industry and auction-specific conditions which are more favorable to collusion. These factors include

[9] In some circumstances, maximum fines and jail sentences may be increased, depending on the offense committed.

[10] Associated citations: U.S. v. A-A-A Elec. Co., Inc., 788 F.2d 242 (4th Cir. 1986); U.S. v. Addyston Pipe & Steel Co. et al., 1897 LEXIS 2499 (E.D. Tenn. Feb. 5, 1897); U.S. v. Raymond J. Lyons, No. 81-1287, 1982 U.S. App. LEXIS 22194 (Feb. 1, 1982); U.S. v. Ronald Pook, No. 87-274, 1988 U.S. Dist. LEXIS 3398 (E.D. Pa. April 18, 1988); U.S. v. Seville Industrial Machinery Corp., 696 F.Supp. 986 (D.N.J. 1988); U.S. v. W.F. Brinkley & Son Construction Company, Inc., 783 F.2d 1157 (4th Cir. 1986).

the number of sellers, the substitutability or standardization of the good(s) in question, repetitive purchases, as well as the social and geographic proximity of the bidders to one another (U.S. Department of Justice, 2005).

Successful collusion can result in lower revenue for the auctioneer and give an unfair advantage to cartel members. It may also impede an efficient outcome in the sense that colluders generate false price signals and/or increase price volatility by submitting bids that do not fully reflect their genuine valuations (Burtraw et al., 2008). Thus, there is a gap between the bidders' true values for the goods being sold and the bids that are actually made at auction. However, with regard to allocative efficiency, collusion has no *real* impact since the good will always be given to the highest bidder (a prerequisite for equilibrium) and any remaining surplus will be captured by either the buyer or the seller.

Bid rigging and other forms of collusion can be very difficult to detect (and even harder to prove) since collusive agreements are by their nature secret, with only the participating parties having knowledge of the scheme (U.S. Department of Justice, 2005). Indeed, the cumbersome rules required to consummately combat behavior classified as tacit collusion may in of themselves result in extreme market inefficiencies (Klemperer, 2002). In the United States, the Antitrust Division of the Department of Justice is specifically responsible for detecting, investigating, and prosecuting these crimes once they are thought to have occurred. A proactive approach to the problem is more desirable than these limited attempts to uncover collusion in the sense that preventative measures could be taken to *preemptively* eliminate collusive agreements before they are even reached. For example, a straightforward and effective solution that directly addresses big rigging behavior is better auction designs. However, premeditated auction design is rarely the method of choice in practice.

3.2.2 Simplified Model

With the aforementioned background on mechanism design, auction formats, and bidder collusion in mind, the theoretical framework for this research will be examined more closely. This paper considers collusion in static auctions where coordinated behavior cannot be enforced among members of the all-inclusive cartel during the legitimate auction - an identical model to that of Lopomo, Marx & Sun (2009), with the exception that this research introduces a third bidder into the framework.

"Static" indicates that the game is played once, without repeated interactions by the players. "All-inclusive" refers to the fact that the cartel members are the only participants in the legitimate (also referred to as the "target") auction, where the good is actually transferred from the seller to the buyer.[11] To understand why this is a sufficient condition, note that it is an optimal environment for collusion when the bidding ring must only control the bids of its members in order to manipulate the auction and need not worry about outside bidders. Thus, if the cartel cannot generate a surplus greater than or equal to the non-cooperative outcome in a best-case scenario (i.e. an all-inclusive auction) then it would be even harder to profitably collude in a suboptimal setting where non-cartel members are incorporated. The following assumptions are made in developing the framework.

First, there is an *ex-ante* individual rationality assumption which states that each bidder's expected payoff from participation in the cartel is at least as great as the expected payoff from non-cooperative play.[12] An emphasis must be placed on the term "expected" as there is no guarantee that the bidder's *actual* payoff from participation will be greater than or equal to his non-cooperative outcome. Said differently, cases may arise where a bidder would have generated a higher surplus had he simply acted on his own accord and without the help of the cartel. On average, however, the payoff results will favor collusive behavior to non-collusive behavior. Indeed, if this *ex-ante* rationality constraint did not hold, there would be no incentive to join the cartel in the first place.

Second, each of the three risk-neutral bidders in the cartel independently draws a private value v_i from a probability distribution F_i with finite support $\mathcal{V}_i$. Let $\mathcal{V} \equiv \mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{V}_3$. Therefore, $f_i(v_i)$ is the probability that bidder i's value is v_i. It is assumed that cartel members are made aware of their respective values *only after* they have committed to participation in the cartel (in accordance with the *ex-ante* individual rationality constraint). It is also assumed that the seller knows the distributions from which bidders draw their values but not the values themselves. The seller is considered to be "non-strategic," meaning he sets the reserve price equal to the lower bound of the support of the bidders' value distributions.[13]

Third, each bidder may choose his bid b_i from the entire feasible bid set, denoted

[11] Even in environments with more than three bidders, the assumption of an all-inclusive cartel is common in the literature, particularly because the study of non-all-inclusive cartels at first-price auctions is made difficult by the lack of analytic bid functions.

[12] Generally speaking, there are three classes of rationality constraints: (i) *ex-ante*, or before each bidder knows his value or any other bidders' value, (ii) *interim*, where each bidder knows his own value but has no information on the others' values, and (iii) *ex-post*, where all values are known in hindsight. This paper only considers *ex-ante* individual rationality, defined to be "an assumption that cartel members must commit to participation in the cartel prior to learning their values and that the failure of either bidder to join results in the complete dissolution of the cartel. (Lopomo, Marx, & Sun (2009) pg. 6)

[13] The results discussed later in the paper, when appropriately adjusted, continue to hold if the auctioneer sets a higher fixed reserve price.

$\mathcal{B}$. The set of feasible bids is assumed to be discrete but with a vanishingly small bid increment. (For an example of a hypothetical feasible bid set, please see Table 1 in Appendix C, Section 6.3).

Fourth, each bidder's strategy is a best-reply to the other bidder's strategy given their beliefs about the other bidders. In the case of a tie, it is assumed that the object is randomly allocated to one of the equally-high bidders. The winning bidder keeps the entire payoff from the auction to himself; no surplus is shared among bidders *ex-post*.

Finally, let $u_i(b_1, b_2, b_3, v_i)$ be bidder i's surplus in a first-price auction, given bidder i's value v_i and bid vector (b_1, b_2, b_3). Thus, the *expected* payoff function is as follows:

$$u_1(b_1, b_2, b_3, v_1) \equiv \begin{cases} v_1 - b_1, & \text{if } b_1 > b_2, b_3 \\ 0, & \text{if } b_1 < \max\{b_2, b_3\} \\ (v_1 - b_1)/2, & \begin{array}{l} \text{if } b_1 = b_2 > b_3 \\ \text{or } b_1 = b_3 > b_2 \end{array} \\ (v_1 - b_1)/3, & \text{if } b_1 = b_2 = b_3 \end{cases}$$

$$u_2(b_1, b_2, b_3, v_2) \equiv \begin{cases} v_2 - b_2, & \text{if } b_2 > b_1, b_3 \\ 0, & \text{if } b_2 < \max\{b_1, b_3\} \\ (v_2 - b_2)/2, & \begin{array}{l} \text{if } b_2 = b_1 > b_3 \\ \text{or } b_2 = b_3 > b_1 \end{array} \\ (v_2 - b_2)/3, & \text{if } b_1 = b_2 = b_3 \end{cases}$$

$$u_3(b_1, b_2, b_3, v_3) \equiv \begin{cases} v_3 - b_3, & \text{if } b_3 > b_1, b_2 \\ 0, & \text{if } b_3 < \max\{b_1, b_2\} \\ (v_3 - b_3)/2, & \begin{array}{l} \text{if } b_3 = b_1 > b_2 \\ \text{or } b_3 = b_2 > b_1 \end{array} \\ (v_3 - b_3)/3, & \text{if } b_1 = b_2 = b_3 \end{cases}$$

As indicated in the above function, expected surplus is equal to the difference between a bidder's value and his bid for a good, multiplied by his probability of winning (based on the bids submitted by the other two cartel members). The highest expected payoff $(v_i - b_i)$ occurs when a bidder submits the single highest bid. Conversely, the lowest expected payoff (0) occurs when a bidder submits a bid that is lower than at least one of the other cartel member's bids.[14]

[14] Note that in the case of a two-way [or three-way] tie, the division by 2 [3] is an indication that each high-valuing bidder has a probability of $\frac{1}{2}$ [$\frac{1}{3}$] of winning the good and gaining the associated surplus (since it is randomly allocated in such instances). The respective fractions do *not* signify that the surplus is evenly shared between the highest bidders.

The pre-auction collusion mechanism operates in the following way: Each ring member sends his respective valuation to the "center," a central incentiveless mechanism agent (Myerson, 1983). All bidders take this collusive mechanism as given. Based on these value reports, the center (i) makes a bid recommendation for each member and (ii) collects their required payments.[15] These required payments are used as a means of sharing potential spoils, *ex-ante*, among cartel members in the case that the ring wins the good at the target auction. Intuitively, the higher-valuing bidders pay into the center and the lower-valuing members receive money from the center.[16] It is necessary that the center's budget be balanced in expectation. That is, the payments flowing into and out of the center are constrained to sum to zero. Lastly, the central payments are considered sunk costs for each member, meaning they are made prior to the auction and do not factor into his strategy once the bidding process begins.

Cartel members observe only their own bid recommendations and required payments; they are *not* privy to the other bidders' valuations, payments, or suggested bids. Additionally, it is assumed that each bidder cannot make any inferences about the other bidders' valuations based on his own required payment, although possible inferences may be made by a bidder regarding his rivals' recommended bids based on the recommendation he receives from the center.

In the environment considered, there are two dimensions to the incentive-compatible collusive mechanism: truth-telling and obedience. The first component suggests that it must be incentive-compatible for each cartel member to report his value truthfully. The incorporation of required payments to the center is successful in achieving this truthful revelation of values. The second constraint requires that there exists an incentive for each cartel member to follow the bid recommendations from the center and not deviate from the agreed upon path. It is assumed that the cartel can compel its members to make their required payments, but that it cannot prevent members from participating in the official auction. To reiterate, this framework is classified as a bid coordination mechanism.

During the legitimate auction, each ring member's strategy is a report to the ring (as a function of his value) and a bid (as a function of his value and the ring's recommendation

[15]Interestingly, in the case of a bidding ring of North American stamp dealers in the 1990s, the central incentiveless agent took the form of a New York taxicab driver employed by the cartel. This real-world designated auction agent operated in the same manner as the mechanism described above (Asker, 2009b).

[16]For example, the bidder with the highest report could be asked to pay the difference between his report and the second-highest report. This payment would then be distributed among lower-valuing members. As a result, the higher-valuing member gets to keep the good (and a portion of the surplus), but the total cartel surplus is more evenly distributed among ring members, thereby rewarding all participants in the collusive scheme.

and required transfer). If the plan is successful, the cartel member with the highest value for the good will win it and pay a price that is less than what the equilibrium price would have been in an otherwise competitive setting. This consequently generates a surplus for the individual bidder and the cartel as a whole.

3.3 BCM Optimization Problem

3.3.1 Basic Linear Program

This section will investigate the effectiveness of a cartel using a bid coordination mechanism in extracting surplus from the seller; *this is the central focus of the paper.* The analytical framework used to test the hypothesis concentrates on an underlying mechanism design problem at work. It returns to the collusive model discussed in Section 3.2.2 and considers the class of all incentive-compatible and individually-rational direct revelation mechanisms.

Any incentive-compatible and individually-rational direct revelation mechanism specifies for each vector of reports $(v_1, v_{2,} v_3) \in \mathcal{V}$; (i) a probability distribution over all feasible bid vectors $p\,(\cdot \mid v_1, v_2, v_3) \in \triangle_{\mathcal{B}\times\mathcal{B}\times\mathcal{B}}$, where $\triangle_{\mathcal{B}\times\mathcal{B}\times\mathcal{B}}$ represents the probability simplex over bid vectors; and (ii) monetary transfers $m_i(v_1, v_2, v_3)$ for $i \in \{1, 2, 3\}$. The probability distribution p specifies the center's bid recommendations to the three bidders. The payments m_1, m_2, and m_3 are the transfer payments from the bidders to the center and may be positive or negative. These payments are constrained to sum to zero in expectation in the ex-ante budget balance condition (3) below. Therefore, the vector (p, m_1, m_2, m_3) defines the cartel mechanism.

The center's problem of maximizing expected cartel surplus (thereby creating the optimal BCM) can be written as follows[17]: *maximize the total expected surplus to the three bidders,*

$$\max_{p,m_1,m_2,m_3} \sum_{i\in\{1,2,3\},(v_1,v_2,v_3)\in\mathcal{V},b_1,b_2,b_3\in\mathcal{B}} p(b_1, b_2, b_3 \mid v_1, v_2, v_3) u_i(b_1, b_2, b_3, v_i) f_1(v_1) f_2(v_2) f_3(v_3), \tag{1}$$

subject to p representing a probability distribution, $\forall (v_1, v_2, v_3) \in \mathcal{V}$,

$$p\,(\cdot \mid v_1, v_2, v_3) \in \triangle_{\mathcal{B}\times\mathcal{B}\times\mathcal{B}}, \tag{2}$$

[17] The notation parallels that of Myerson (1985).

ex-ante budget balance,

$$\sum_{i\in\{1,2,3\},(v_1,v_2,v_3)\in\mathcal{V}} m_i\left(v_1,v_2,v_3\right) f_1(v_1)f_2(v_2)f_3(v_3)=0, \tag{3}$$

and incentive compatibility, $\forall i\in\{1,2,3\}$, $\forall v_i,v_i'\in\mathcal{V}_i$, $\forall\delta_i:\mathcal{B}\to\mathcal{B}$,

$$U_i^*\left(p,\delta_i,v_i'\mid v_i\right)-U_i\left(p\mid v_i\right)\leq 0, \tag{4}$$

where $U_i\left(p\mid v_i\right)$ is bidder i's expected surplus under truthtelling and obedience,

$$U_1\left(p\mid v_1\right)\equiv \sum_{(v_2,v_3)\in\mathcal{V}_2\times\mathcal{V}_3,b_1,b_2,b_3\in\mathcal{B}} p(b_1,b_2,b_3\mid v_1,v_2,v_3)u_1(b_1,b_2,b_3,v_1)f_2(v_2)f_3(v_3) - \sum_{(v_2,v_3)\in\mathcal{V}_2\times\mathcal{V}_3} m_1\left(v_1,v_2,v_3\right) f_2(v_2)f_3(v_3), \tag{5}$$

$$U_2\left(p\mid v_2\right)\equiv \sum_{(v_1,v_3)\in\mathcal{V}_1\times\mathcal{V}_3,b_1,b_2,b_3\in\mathcal{B}} p(b_1,b_2,b_3\mid v_1,v_2,v_3)u_2(b_1,b_2,b_3,v_2)f_1(v_1)f_3(v_3) - \sum_{(v_1,v_3)\in\mathcal{V}_1\times\mathcal{V}_3} m_2\left(v_1,v_2,v_3\right) f_1(v_1)f_3(v_3), \tag{6}$$

$$U_3\left(p\mid v_3\right)\equiv \sum_{(v_1,v_2)\in\mathcal{V}_1\times\mathcal{V}_2,b_1,b_2,b_3\in\mathcal{B}} p(b_1,b_2,b_3\mid v_1,v_2,v_3)u_3(b_1,b_2,b_3,v_3)f_1(v_1)f_2(v_2) - \sum_{(v_1,v_2)\in\mathcal{V}_1\times\mathcal{V}_2} m_3\left(v_1,v_2,v_3\right) f_1(v_1)f_2(v_2), \tag{7}$$

and $U_i^*\left(p,\delta_i,v_i'\mid v_i\right)$ is bidder i's expected surplus when bidder i's value is v_i, it reports v_i', and it bids according to a deviation function $\delta_i\left(b_i\right)$ that depends on the center's recommendation b_i,

$$U_1^*\left(p,\delta_1,v_1'\mid v_1\right) \equiv \sum_{v_2\in\mathcal{V}_2,v_3\in\mathcal{V}_3,b_1,b_2,b_3\in\mathcal{B}} p(b_1,b_2,b_3\mid v_1',v_2,v_3)u_1(\delta_1(b_1),b_2,b_3,v_1)f_2(v_2)f_3(v_3) - \sum_{v_2\in\mathcal{V}_2,v_3\in\mathcal{V}_3} m_1\left(v_1',v_2,v_3\right) f_2(v_2)f_3(v_3). \tag{8}$$

and

$$U_2^*\left(p,\delta_2,v_2'\mid v_2\right) \equiv \sum_{v_1\in\mathcal{V}_1,v_3\in\mathcal{V}_3,b_1,b_2,b_3\in\mathcal{B}} p(b_1,b_2,b_3\mid v_1,v_2',v_3)u_2(b_1,\delta_2(b_2),b_3,v_2)f_1(v_1)f_3(v_3) - \sum_{v_1\in\mathcal{V}_1,v_3\in\mathcal{V}_3} m_2\left(v_1,v_2',v_3\right) f_1(v_1)f_3(v_3). \tag{9}$$

and

$$U_3^*\left(p,\delta_3,v_3' \mid v_3\right) \equiv \sum_{v_1\in\mathcal{V}_1,v_2\in\mathcal{V}_2,b_1,b_2,b_3\in\mathcal{B}} p(b_1,b_2,b_3 \mid v_1,v_2,v_3')u_3(b_1,b_2,\delta_3(b_3),v_3)f_1(v_1)f_2(v_2) - \sum_{v_1\in\mathcal{V}_1,v_2\in\mathcal{V}_2} m_3\left(v_1,v_2,v_3'\right)f_1(v_1)f_2(v_2). \tag{10}$$

Due to its complexity, the incentive compatibility constraint (4) requires further attention. The notation indicates that the expected utility from deviating (U_i^*) minus the expected surplus from complying (U_i) must be less than or equal to zero. Said differently, the constraint necessitates that a bidder's expected payoff from complying be *at least as great as* his expected payoff from deviating.

As stated earlier, this constraint incorporates both a truth-telling and obedience component. Incentive compatibility for truth-telling implies that ring members have no incentive to misreport their values to the center. This constraint requires that a bidder not benefit from a simple misreporting of its value (denoted v_i'). Incentive compatibility for obedience implies that no bidder has an incentive to deviate from the bid recommendation it receives from the center (regardless of whether their report was truthful or not).

Because bidders must receive their recommended bids from the center prior to submitting their bids at the auction, bidders can potentially make use of inferences from the recommendation regarding the bidding behavior of its rival to adjust their bids. For this reason, allow the deviation function δ_i in (4) to be a function of the bid recommendation bidder i receives from the center. Incentive compatibility requires that given any strategy for reporting, either truthful or not, there is no deviation function δ_i mapping the bidder i's recommended bid from the center to its actual bid that improves upon $\delta_i(b_i) = b_i$.

Although the above formulation accounts for possible inferences made by a bidder regarding his rivals' recommended bids based on the recommendation he receives, the formulation does not explicitly allow a bidder to make inferences regarding its rivals based on the transfer payment required by the center. One might think that a bidder's required payment could be informative as to the rivals' reports because m_1, m_2 and m_3 are functions of the vector of reports.

In light of this, it is possible to define bidder 1's *expected* payment given its own report as

$$M_1(v_1) \equiv \sum_{v_2\in\mathcal{V}_2,v_3\in\mathcal{V}_3} m_1\left(v_1,v_2,v_3\right)f_2(v_2)f_3(v_3),$$

and similarly for bidders 2 and 3.

A critical component of this new formula is that the payment is now only a function of bidder i's *own* value report, whereas before it was a function of all three bidder's valuations. Because only bidder i's *expected* payment given his own report enters the problem above, one can show that any level of total surplus for the cartel that can be achieved with payments $m_i(v_1, v_2, v_3)$ and allowing for possible inferences by the bidders about their rivals can also be achieved with payments $M_i(v_i)$ that, because they depend only on v_i, *do not allow* such inferences. Simply put, the transition from *transfer* payments m_i to *expected* payments M_i eliminates the possibility for a bidder to make inferences regarding his rivals based on the transfer payment required by the center.

In effect, allowing the inferences only introduces additional incentive compatibility constraints, so allowing such inferences is never optimal. More formally, this result follows from Myerson (1982, Proposition 2), which implies that the formulation ignoring possible inferences from the required transfer payments is without loss of generality. In what follows in Section 3.3.2, the set of constraints are simplified by replacing the transfer function $m_i(v_1, v_2, v_3)$ with the expected payment function $M_i(v_i)$.

3.3.2 Refined Linear Program

One difficulty with the program defined in (1)–(10) is that the set of all possible deviation functions δ_i considered in the incentive compatibility constraint (4) is large (exponential to the number of feasible bids). In order to formulate the problem as a tractable linear program, a new optimization problem must be introduced that involves the function $J_i(v_i, v_i', b_i)$, which gives bidder i's expected surplus if it has type v_i, reports v_i', receives recommendation b_i from the center, and bids *optimally*; this is multiplied by $\Pr(b_i \mid v_i')$, which is the probability of receiving recommendation b_i given i's report v_i'. The rewritten incentive compatibility constraint holds for all values v_i and reports v_i', rather than for all values, reports, *and* bid deviation functions as in (4). The possibility of deviations from recommended bids is incorporated in the function $J_i(v_i, v_i', b_i)$.

The rewritten incentive compatibility constraint (15) states that bidder i's expected payoff from reporting truthfully and following the recommendation of the center is greater than or equal to $J_i(v_i, v_i', b_i)$ minus its expected transfer payment from reporting v_i'. The value of $J_i(v_i, v_i', b_i)$ is constrained in (16)–(18) to be at least as great as the expected payoff that bidder i with value v_i could get from submitting any bid, conditional on having reported v_i' and received bid recommendation b_i from the center, thereby eliminating "bad" deviations (in the sense that such deviations would reduce the bidder's expected payoff). Thus, it is possible to conserve on constraints by considering only deviations from obedience that are *optimal* given a bidder's value, submitted report, and observed recommendation.

The rewritten objective function, probability constraints, and ex-ante budget balance condition are analogous to those in (1)–(10). The difference comes in the incentive compatibility constraint as described above.

Lemma 1 *The center's problem can be written as follows:*

$$\max_{\rho, M_1, M_2, M_3, J_1, J_2, J_3} \sum_{i=1}^{3} \sum_{(v_1,v_2,v_3)\in\mathcal{V}, b_1,b_2,b_3\in\mathcal{B}} \rho(b_1,b_2,b_3,v_1,v_2,v_3)u_i\left(b_1,b_2,b_3,v_i\right) f_1(v_1)f_2(v_2)f_3(v_3) \tag{11}$$

subject to ρ representing a conditional probability distribution, $\forall(v_1,v_2,v_3)\in\mathcal{V}$, $\forall(b_1,b_2,b_3)\in\mathcal{B}\times\mathcal{B}\times\mathcal{B}$,

$$\rho(b_1,b_2,b_3,v_1,v_2,v_3)\geq 0, \tag{12}$$

and $\forall(v_1,v_2,v_3)\in\mathcal{V}$,

$$\sum_{b_1,b_2,b_3\in\mathcal{B}} \rho(b_1,b_2,b_3,v_1,v_2,v_3) = 1, \tag{13}$$

ex-ante budget balance,

$$\sum_{(v_1,v_2,v_3)\in\mathcal{V}} \left(M_1(v_1)f_1(v_1)+M_2(v_2)f_2(v_2)+M_3(v_3)f_3(v_3)\right)=0, \tag{14}$$

incentive compatibility, $\forall v_1\in\mathcal{V}_1$, $\forall v_1'\in\mathcal{V}_1$, (and similarly for $\forall v_2\in\mathcal{V}_2$, $\forall v_2'\in\mathcal{V}_2$, and $\forall v_3\in\mathcal{V}_3$, $\forall v_3'\in\mathcal{V}_3$),

$$\begin{aligned} &\sum_{v_2\in\mathcal{V}_2, v_3\in\mathcal{V}_3, b_1,b_2,b_3\in\mathcal{B}} \rho(b_1,b_2,b_3,v_1,v_2,v_3)u_1(b_1,b_2,b_3,v_1)f_2(v_2)f_3(v_3) - M_1\left(v_1\right) \\ \geq\ &\sum_{b_1\in\mathcal{B}} J_1(v_1,v_1',b_1) - M_1(v_1')\ , \end{aligned} \tag{15}$$

and the definition of J, $\forall v_1\in\mathcal{V}_1$, $\forall v_1'\in\mathcal{V}_1$, $\forall b_1\in\mathcal{B}$, $\forall b_1'\in\mathcal{B}$,

$$J_1\left(v_1,v_1',b_1\right)\geq \sum_{v_2\in\mathcal{V}_2, v_3\in\mathcal{V}_3, b_2,b_3,\in\mathcal{B}} \rho(b_1,b_2,b_3,v_1',v_2,v_3)u_1\left(b_1',b_2,b_3,v_1\right)f_2\left(v_2\right)f_3\left(v_3\right), \tag{16}$$

$\forall v_2\in\mathcal{V}_2$, $v_2'\in\mathcal{V}_2$, $b_2\in\mathcal{B}$, $b_2'\in\mathcal{B}$,

$$J_2\left(v_2,v_2',b_2\right)\geq \sum_{v_1\in\mathcal{V}_1, v_3\in\mathcal{V}_3, b_1,b_3\in\mathcal{B}} \rho(b_1,b_2,b_3,v_1,v_2',v_3)u_2\left(b_1,b_2',b_3,v_2\right)f_1\left(v_1\right)f_3\left(v_3\right). \tag{17}$$

$\forall v_3 \in \mathcal{V}_3,\ v_3' \in \mathcal{V}_3,\ b_3 \in \mathcal{B},\ b_3' \in \mathcal{B},$

$$J_3\left(v_3, v_3', b_3\right) \geq \sum_{v_1 \in \mathcal{V}_1, v_2 \in \mathcal{V}_2, b_1, b_2 \in \mathcal{B}} \rho(b_1, b_2, b_3, v_1, v_2, v_3') u_3\left(b_1, b_2, b_3', v_3\right) f_1\left(v_1\right) f_2\left(v_2\right). \quad (18)$$

Proof. *See Appendix B (Section 6.2.2).*

Using Lemma 1, it is possible to apply linear programming techniques to analyze the optimal bid coordination collusive mechanism in the first-price auction. However, the linear program may be made even more tractable through a process of reformulation and further simplification for the specific case of three bidders and two types. This will be completed later in Section 4.3.1.

4 Analytical Results

4.1 Non-cooperative Benchmark with Three Bidders

In order to properly gauge the effectiveness of any collusive mechanism operating in this model, the non-cooperative equilibrium must first be determined for the same auction format. The following proposition describes the non-cooperative equilibrium surplus for a first-price auction in an environment of three independent bidders with two possible valuations, l (low) or h (high).

Proposition 1 *Assume three bidders, each having value l with probability $f(l)$ and value h with probability $f(h) = 1 - f(l)$, where $0 < l < h$. The non-cooperative equilibrium of the first-price auction game is as follows: a bidder with value l bids l; a bidder with value h bids according to the following cumulative distribution:*

$$F(b) = \begin{cases} 0, & \text{if } b < l \\ \frac{f(l)}{f(h)}\left(\sqrt{\frac{h-l}{h-b_i}} - 1\right), & \text{if } b \in [l, b^*] \\ 1, & \text{otherwise,} \end{cases}$$

where $b^* \equiv h - f(l)^2 (h - l)$.

Proof. See Appendix B (Section 6.2.1).

Using Proposition 1, the non-cooperative equilibrium can be characterized as follows.

Corollary 1 *In the environment of Proposition 1, the non-cooperative equilibrium outcome is efficient, expected overall bidder surplus is $3(h-l) f(h) f(l)^2$, and expected seller revenue is $f(l)(1 + f(h)) l + f(h)^2 h$.*

For a numeric example, consider the case of three symmetric bidders, each having a value of 40 or 80 with equal probability. Thus, in the notation above, $l = 40$, $h = 80$, $f(l) = f(h) = \frac{1}{2}$. That is, the bidders will either have an absolutely low valuation ($l = 40$) or an absolutely high valuation ($h = 80$) with no other feasible value increments in-between (note: for the remainder of the paper, 40 and 80 will be the default values for l and h , respectively).

In the non-cooperative equilibrium, a bidder with value l bids l and a bidder with h will mix his bids according to the cumulative distribution laid out in Proposition 1. Thus, the non-cooperative equilibrium bid, $b^* \equiv 80-\ (\frac{1}{2})^2(80-40)$, is 70 and total expected bidder surplus with continuous bids is $3(80-40)(\frac{1}{2})(\frac{1}{2})^2 = 15$, leaving each individual bidder with $(\frac{15}{3}) = 5$. This provides a lower-bound basis by which the profitability of collusive behavior can be compared.

4.2 Result for a Cartel with Three Bidders Using a BSM

Using the non-cooperative outcome from the previous section as a benchmark to measure against, collusive scenarios will now be investigated.

Recall that Marshall & Marx (2007) determined that an all-inclusive cartel using a BCM at a second-price or ascending-bid auction can extract all of the surplus from the seller due to its ability to effectively control the bids of its members. The cartel's surplus is therefore equal to the highest value among the bidders and the seller receives zero (or the reserve price in the case of a positive reserve price). While this result may make sense intuitively, it is possible to strengthen the finding by empirically calculating the expected bidder surplus for an environment with an all-inclusive cartel using a BSM at a first-price auction.

To that end, consider again the case of three bidders, each drawing from the value distribution $\{40, 80\}$ with equal probability on each. A bid submission mechanism would have the highest-valuing ring member bid zero and the other two not bid at all (since it can successfully prevent its members from participating in the legitimate auction). This pure strategy enables the cartel to extract the entire surplus from the seller. The total expected surplus to the cartel would thus follow the formula:

$$80 \times [\Pr(\text{highest of 3 is 80})] + 40 \times [\Pr(\text{highest of 3 is 40})]$$

or

$$80 \times (\tfrac{7}{8}) + 40 \times (\tfrac{1}{8}) = 70 + 5 = 75.$$

Divided evenly, expected surplus to an individual cartel member would be 25. To see where the probabilities come from, note that the highest of 3 is 40 only if all three bidders have value 40, which has probability Pr(all values are 40) $= (\frac{1}{2}) \times (\frac{1}{2}) \times (\frac{1}{2}) = \frac{1}{8}$. The probability of at least one bidder drawing a value of 80 is Pr(one value is 80)+Pr(two values are 80)+Pr(three values are 80)$= [(\frac{1}{2})] + [(\frac{1}{2}) \times (\frac{1}{2})] + [(\frac{1}{2}) \times (\frac{1}{2}) \times (\frac{1}{2})] = (\frac{4}{8}) + (\frac{2}{8}) + (\frac{1}{8}) = \frac{7}{8}$.

4.3 Result for a Cartel with Three Bidders Using a BCM

4.3.1 Optimization Problem Revisited

The previous two sections established a set of duel feasible solutions for expected bidder surplus, with a lower bound set by the non-cooperative equilibrium surplus and an upper bound set by the BSM collusive outcome. This section returns to the refined linear program formulated in Section 3.3.2 and constructs a constrained maximization problem that can be solved numerically. The output generated from linear programming computations will place the outcome for a BCM cartel in the context of the aforementioned bounds.

Bidders are assumed to be symmetric and each bidder has two possible values, l and h, with $h > l > 0$. Notationally, this means that $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{V}_3$ and that for any value $v \in \mathcal{V}_i$ and bids $b_1, b_2, b_3 \in \mathcal{B}$, there exists $f_1(v) = f_2(v) = f_3(v)$ and $u_1(b, \bar{b}, \tilde{b}, v) = u_1(b, \tilde{b}, \bar{b}, v) = u_2\left(\bar{b}, b, \tilde{b}, v\right) = u_2\left(\bar{b}, \tilde{b}, b, v\right) = u_3\left(\tilde{b}, b, \bar{b}, v\right) = u_3\left(\tilde{b}, \bar{b}, b, v\right)$. Therefore, the subscripts on f_i and u_i are suppressed in this section, letting $u(b, \bar{b}, \tilde{b}, v) = u_1(b, \bar{b}, \tilde{b}, v) = u_1(b, \tilde{b}, \bar{b}, v)$.

To take advantage of the assumption of symmetry, it is helpful to simplify the linear program for the optimal collusive mechanism given in (11)–(17). In particular, as later shown in Lemma 2, the formulation may be refined in the following way: let the notation v, $\bar{v}$ and $\tilde{v}$ denote the three bidders' values and b, $\bar{b}$ and $\tilde{b}$ their bids, and let $\widehat{v}$ denote a bidder's report (possibly different from its value) and $\widehat{b}$ a bidder's bid (possibly different from its recommendation from the center).[18] The center's reformulated problem of maximizing expected cartel surplus is now written:

$$\max_{\rho,M,J} \sum_{v,\bar{v},\tilde{v}\in\{l,h\}} \sum_{b,\bar{b},\tilde{b}\in\mathcal{B}} \left(u(b,\bar{b},\tilde{b},v) + u(\bar{b},b,\tilde{b},\bar{v}) + u(\tilde{b},b,\bar{b},\tilde{v})\right) \rho(b,\bar{b},\tilde{b},v,\bar{v},\tilde{v}) f(v) f(\bar{v}) f(\tilde{v}) \tag{19}$$

subject to ρ representing a conditional probability distribution, $\forall v, \bar{v}, \tilde{v} \in \{l, h\}$, $\forall b, \bar{b}, \tilde{b} \in \mathcal{B}$,

$$\rho(b, \bar{b}, \tilde{b}, v, \bar{v}, \tilde{v}) \geq 0, \tag{20}$$

[18] Note that $\widehat{v}$ was formerly denoted v' and $\widehat{b}$ was formerly denoted b'.

and $\forall v, \bar{v}, \tilde{v} \in \{l, h\}$,

$$\sum_{b,\bar{b},\tilde{b} \in \mathcal{B}} \rho(b, \bar{b}, \tilde{b}, v, \bar{v}, \tilde{v}) = 1, \tag{21}$$

ex-ante budget balance,

$$\sum_{v \in \{l,h\}} M(v) f(v) = 0, \tag{22}$$

incentive compatibility, $\forall v, \widehat{v} \in \{l, h\}$,

$$\begin{aligned}
&\sum_{\bar{v},\tilde{v} \in \{l,h\}} \sum_{b,\bar{b},\tilde{b} \in \mathcal{B}} \begin{pmatrix} \rho(b, \bar{b}, \tilde{b}, v, \bar{v}, \tilde{v}) + \rho(b, \tilde{b}, \bar{b}, v, \tilde{v}, \bar{v}) \\ +\rho(\bar{b}, b, \tilde{b}, \bar{v}, v, \tilde{v}) + \rho(\bar{b}, \tilde{b}, b, \bar{v}, \tilde{v}, v) \\ +\rho(\tilde{b}, b, \bar{b}, \tilde{v}, v, \bar{v}) + \rho(\tilde{b}, \bar{b}, b, \tilde{v}, \bar{v}, v) \end{pmatrix} u(b, \bar{b}, \tilde{b}, v) f(\bar{v}) f(\tilde{v}) - M(v) \\
\geq\ & \sum_{b \in \mathcal{B}} J(v, \widehat{v}, b) - M(\widehat{v}),
\end{aligned} \tag{23}$$

and the definition of J, $\forall v, \widehat{v} \in \{l, h\}$, $\forall b, \widehat{b} \in \mathcal{B}$,

$$J(v, \widehat{v}, b) \geq \sum_{\bar{v},\tilde{v} \in \{l,h\}} \sum_{b,\bar{b},\tilde{b} \in \mathcal{B}} \begin{pmatrix} \rho(b, \bar{b}, \tilde{b}, \widehat{v}, \bar{v}, \tilde{v}) + \rho(b, \tilde{b}, \bar{b}, \widehat{v}, \tilde{v}, \bar{v}) \\ +\rho(\bar{b}, b, \tilde{b}, \bar{v}, \widehat{v}, \tilde{v}) + \rho(\bar{b}, \tilde{b}, b, \bar{v}, \tilde{v}, \widehat{v}) \\ +\rho(\tilde{b}, b, \bar{b}, \tilde{v}, \widehat{v}, \bar{v}) + \rho(\tilde{b}, \bar{b}, b, \tilde{v}, \bar{v}, \widehat{v}) \end{pmatrix} u(\widehat{b}, \bar{b}, \tilde{b}, v) f(\bar{v}) f(\tilde{v}). \tag{24}$$

Lemma 2 below establishes that it is sufficient to work with the linear program (19)–(24).

Lemma 2 *The value of the objective function at the optimum is the same in* (19)–(24) *and in* (11)–(17).

Proof. See Appendix B (Section 6.2.3).

In what follows, the bid increment is defined to be $\Delta \equiv \frac{b^* - l}{2B}$ for a given positive integer B. The set of feasible bids can be expressed as

$$\mathcal{B}_B \equiv \{b_j \mid b_j = l + \Delta j \text{ for } j \in \{0, 1, \ldots, nB\}\},$$

in which n is a sufficiently large integer. Therefore, $b_0 = l$ and $b_{2B} = b^*$, and the interval $[l,\ b^*]$ contains $2B + 1$ feasible bids. As the integer B approaches infinity, the bidding increment Δ approaches zero. Assume the lowest feasible bid to be l, which can be enforced by the seller setting a reservation price l. Such an assumption is without loss of generality as the bid increment shrinks to zero, as demonstrated by the following lemma, which states that for any given arbitrarily small bid increment Δ, it is not incentive compatible for the collusive mechanism to recommend any bid below $l - 2\Delta$.

Lemma 3 *A lower bound on the recommended bids in an incentive compatible BCM is $l - 2\Delta$.*

Proof. See Appendix B (Section 6.2.4).

The constrained maximization problem as structured in (19)–(24) may be solved numerically by applying linear programming techniques to analyze the optimal collusive mechanism in a first-price auction.[19] With the results from the linear optimization model, one may attempt to characterize the optimal solution for a BCM cartel. Thus, it will become possible to compare the expected bidder surplus and expected seller revenue in the optimal BCM with that of the optimal BSM and non-cooperative equilibrium, as found in Section 4.2 and 4.1, respectively. In particular, it should become clear whether or not the convergence phenomenon in a two-bidder environment (shown by Figure 3 in Appendix A) will hold for the case of three symmetric bidders.

4.3.2 Key Finding

The best way to illustrate the paper's main result is through the use of a sample hypothetical scenario with three symmetric bidders. In what follows, the optimal BCM is solved using linear programming. By varying the bid increments, it is also possible to numerically demonstrate the speed of convergence.

Assume three symmetric bidders, each with value 40 or 80 with equal probability. Thus, in the notation above, $l = 40$, $h = 80$, $f(l) = f(h) = \frac{1}{2}$. With continuous bids, in the non-cooperative equilibrium a bidder with value l bids l and a bidder with value h bids according to the distribution characterized in Proposition 1 (Section 4.1). Recall from Corollary 1 that the non-cooperative expected bidder surplus with continuous bids is 20.

The output generated from the linear program (19)–(24) confirms the hypothesis that profitable collusion is not possible for a three-bidder cartel using a bid coordination mechanism operating at a first-price sealed-bid auction when the feasible bid set is sufficiently large. This main result is given in Proposition 2 below.

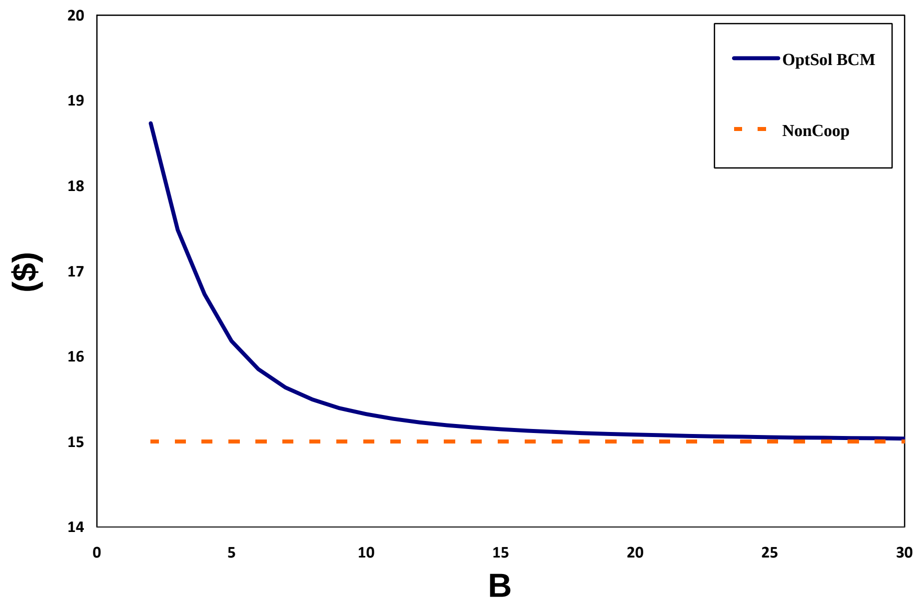
Proposition 2 *Assuming three symmetric bidders and two possible values, in the limit as the bid increment converges to zero, the bidder surplus from the optimal BCM converges to the bidder surplus under non-cooperative bidding.*

Proposition 2 implies that when bidders are symmetric, a first-price auction is robust to collusion using a BCM. Said differently, if the bid increment is sufficiently small, at

[19] For an example of the OPL and Matlab coding used for this program, please see Appendix D, Section 6.4.

least for environments with three bidders and two possible values, profitable collusion is not possible - a cartel can do no better than to bid non-cooperatively. This finding is best expressed by the graph found in Figure 1 below.[20]

Figure 1: Three Bidder BCM Cartel Equilibrium



The optimal collusive bidder surplus ("OptSol BCM") and the non-cooperative equilibrium surplus in the continuous bid case ("NonCoop") as a function of B, which is inversely related to the bid increment. The graph shows the lower bound of the non-cooperative equilibrium surplus as well as the surplus from the optimal BCM based on the linear programming output.

Figure 1 demonstrates that the optimal collusive surplus approaches the non-cooperative surplus as the bid increment approaches zero; this is the same result as the two-bidder scenario found in Lopomo, Marx & Sun (2009). One can see that a BCM cartel is more successful at extracting surplus from the seller when bids are coarse. Intuitively, this is due to the fact that compliance is more profitable than deviation from the center's bid recommendations. However, cheating becomes more attractive for individual bidders as the feasible bid set increases and bid increments shrink. As B approaches infinity and increments are sufficiently low (approaching zero), the central mechanism is forced to recommend the non-cooperative bid ("just bid as you would have, anyway") and cartel surplus is therefore the equivalent to that of the non-collusive equilibrium payoff.

The research output from this investigation is significant in that it proves the validity of the theoretical framework when it is expanded to the case of three bidders. The

[20] Source data for Figure 1 displayed in Chart 2 of Appendix A, Section 6.1.

findings demonstrate that the model successfully holds for an environment with $n = 3$ bidders. It follows logically that it will likewise work for all $n > 3$ as well. Of equal importance, the data generated from the linear program enable a direct comparison of various surplus scenarios between games with two and three bidders; this will be carried out in Section 4.4.

4.4 Comparison of Two vs. Three-Bidder Surplus Scenarios

Chart 1 provides a numerical comparison of equilibrium surplus outcomes for the case of two bidders and three bidders.

Chart 1: Equilibrium Outcomes

	Two Bidder Auction Format			
	BSM	BCM	NonCoop	BSM-BCM
Expected Bidder Surplus (each)	35	10	10	25
Expected Seller Revenue	0	50	50	-50
Expected Total Surplus	70	70	70	0
Total Cartel Surplus (all)	70	20	--	50

	Three Bidder Auction Format			
	BSM	BCM	NonCoop	BSM-BCM
Expected Bidder Surplus (each)	25	5	5	20
Expected Seller Revenue	0	60	60	-60
Expected Total Surplus	75	75	75	0
Total Cartel Surplus (all)	75	15	--	60

It is easiest to begin analysis by pointing out several obvious observations. First, total cartel surplus plus expected seller revenue must sum to 70 for $n = 2$ and 75 for $n = 3$. Second, in equilibrium, the expected surplus for a BCM cartel is equal to that of expected non-cooperative payoff. It is also lower than the expected surplus for a BSM cartel in both two and three-bidder scenarios because the BSM cartel successfully extracts the entire surplus in each case. This is because it is a pure strategy for BSM members to bid up to the reserve price and never higher. Thus, expected seller revenue for a BSM cartel scenario is always zero.[21]

[21]Note: there are two ways of interpreting seller "surplus". If the reserve price is zero, the seller has zero value for the good and so all revenue can be interpreted as their surplus. If the seller's value is sufficiently low relative to the values of the bidders, he does not want to set the reserve price lower than the lower bound of the distribution of the bidder's values (i.e. 39.999) – this ensures that the good is sold. In this case, seller revenue is at least 40 and any winning bid that is higher than 40 results in additional surplus to the seller.

Focusing more intently on the two types of cartel mechanisms, Chart 2 affirms that the expected surplus both for the entire BCM cartel and for each individual BCM cartel member is greater in a two-bidder versus three-bidder auction format (20 vs. 15 and 10 vs. 5, respectively). That is, the "surplus pie" is not only smaller in the three bidder case, but each individual better gets a smaller slice as well. This is likely due to the fact that, since a BCM cartel cannot control the bids of its members, it becomes increasingly harder to prevent deviation and thus more difficult to extract any surplus from the seller when additional bidders are introduced. It is likely that this trend would continue as the cartel grows in size.

Conversely, the incentive to collude using a BSM is stronger when there is a larger number of bidders. This suggests that the incremental value of incorporating a BSM over a BCM is greater with more bidders. To understand the reason for this, note that the expected individual surplus increases by a factor of 3.5 as a result of shifting to a BSM cartel from a BCM cartel with two members. However, there is a fivefold increase in expected individual surplus to be gained from shifting to a BSM cartel in a two-bidder scenario. Thus, both the entire cartel and each individual in the cartel stand to benefit more by switching to a BSM framework when the cartel has more members. While it is more valuable to shift from a BCM to BSM with larger cartels, one caveat to this finding is that it also becomes harder to accomplish since the ring has to control the additional members' bids. This likewise increases the chances of getting caught, displaying a standard risk vs. return tradeoff.

A final observation taken from the data is that the gains from participating in a BCM cartel are initially higher for the case of three bidders in relation to the environment with two bidders. However, these gains decrease more rapidly as the feasible bid set expands (please see Figure 4 in Appendix A for a graphical representation of this comparison). The underlying mechanism driving this convergence toward the non-cooperative outcome is explained in the following section.

4.5 Intuition Behind Convergence Phenomenon

The convergence phenomenon discussed in Section 4.3.2 (and well-illustrated by Figure 1) is hard to explain by way of intuitive reasoning. Here again, a simple example will be used to demonstrate the underlying mechanism driving the paper's main result. A few important considerations need to be made at this juncture.

First, that the BCM cartel's outcome coincides with the non-cooperative equilibrium when bid increments are sufficiently small is related to the central mechanism and not to deviation *per se*. Recall that, since a bid coordination mechanism cannot prevent members from participating in the target auction, the center is therefore presented with the task of adjusting its bid recommendations based on reported values so as to *prevent* any deviation from the suggested bid vector. In order for this to happen, each bidder must be indifferent between all bid recommendations with positive probability.

In a game-theoretic context, bidders randomize their mixed-strategies in the following way. A type l bidder will always bid l and a type h bidder will mix on bids above l according to the cumulative distribution function characterized in Proposition 1.[22] For this reason, a second important consideration is that only cases in which there is *at least one high-valuing member* are worthy of additional attention. These cases are significant because the issue of adverse selection is introduced into the model.

Now consider a simple environment with two symmetric bidders with values 40 or 80 receiving bid recommendations drawn from the feasible bid set $\mathcal{B}_B \equiv \{40, 55, 70\}$. This particular bid set is shown in the first column of Table 1 in Appendix C, Section 6.3. The values and resulting bid vectors for a non-cooperative equilibrium game of complete information,[23] along with an explanation for such bidding strategy is given below.

Non-Cooperative Equilibrium of Complete Information Game

Values		Bids	Expected Surplus Comparison	Intuition
40, 40	→	40, 40	1/2*(80-40) > 0	Never bid higher than value
80, 40	→	55, 40	1*(80-55) > 1/2*(80-40)	
40, 80	→	40, 55	1*(80-55) > 1/2*(80-40)	
80, 80	→	70, 70	1/2*(80-70) > 0	Any surplus better than none

A similar chart for a BCM cartel's equilibrium outcome is below. In this case, the bids are recommendations from the center based on the set of value reports. They are designed so as to maximize total cartel surplus and prevent deviation by individual ring members.

BCM Cartel Recommendations with Bid Set $B_B \equiv \{40, 55, 70\}$

Values		Bids	Expected Surplus Comparison	Intuition
40, 40	→	40, 40	1/2*(80-40) > 0	Never bid higher than value
80, 40	→	55, 40	1*(80-55) > 1/2*(80-40)	
40, 80	→	40, 55	1*(80-55) > 1/2*(80-40)	
80, 80	→	55,55	1/2*(80-55) > 1*(80-70)	Comply > Deviate

[22]Numerically, if a bidder's value is 40, he will bid 40 but if his value is 80, he will randomize his bid on [40, b*].

[23]"Complete" meaning each bidder is aware of the other bidder's value for the good (i.e. public valuations).

Notice that the only difference between the non-cooperative and collusive equilibrium for this feasible bid set occurs when there are two high-valuing bidders. The cartel can instruct its members to both bid lower than they otherwise would have (thus capturing some surplus) and be confident that they will comply with such recommendations, since their expected payoff from doing so is higher than it would be if they deviated to the higher feasible bid (70).

Now introduce a hypothetical feasible bid (56), keeping in mind the cartel's strategy outlined above. Given this new feasible bid set $\mathcal{B}_B \equiv \{40, 55, \mathbf{56}, 70\}$, the cartel's recommendations remain the same as before in cases where there is at least one low-valuing member. However, the center can no longer recommend a bid of (55) when there are *two high-valuing members* since there would be incentive for each member to deviate and bid (56), thereby increasing his expected payoff. This result is shown below.

BCM Cartel Recommendations with Bid Set $B_B \equiv \{40, 55, \mathit{56}, 70\}$

Values		**Bids**	**Expected Surplus Comparison**	**Intuition**
40, 40	→	40, 40	1/2*(80-40) > 0	Never bid higher than value
80, 40	→	55, 40	1*(80-55) > 1/2*(80-40)	
40, 80	→	40, 55	1*(80-55) > 1/2*(80-40)	
80, 80	→	56, 56	1*(80-56) > 1/2*(80-55)	***Deviate > Comply***

From this straightforward example, one can see that when the feasible bid set expands (and bid increments get smaller) it becomes less costly for cartel members to deviate to bids that are higher than the center's recommendation. The center must adjust its recommendations accordingly so as to prevent deviation which, in turn, erodes the total expected surplus. Similar logic indicates the same result when this hypothetical exercise is expanded to the case of three bidders.

5 Conclusion

5.1 Main Result

This paper shows that profitable collusion is not possible at a first-price auction when the collusive mechanism can only suggest non-binding bid recommendations to cartel members. The specific auction environment considered contains three symmetric bidders with two possible values and discrete bids. The theoretical framework resembles that of the one used by Lopomo, Marx & Sun (2009), with the notable introduction of an additional bidder. Analytic results indicate that as the bid increment approaches zero, bidder surplus from the optimal bid coordination mechanism approaches that from non-cooperative bidding. This is due to the fact that any cartel using a BCM which recommends bids that are less than the non-cooperative bids is subject to cheating by its members. While discrete bids may create an environment in which bidders can profitably collude at a first-price auction using a BCM, the benefit from such collusion vanishes as the bid increments become sufficiently small. Therefore, *a first-price auction is robust to collusion using a BCM.*

This result stands in stark contrast to equilibrium outcomes involving alternative auction formats and collusive mechanisms. Marshall & Marx (2007) show that a cartel using a BCM can achieve the first-best collusive outcome for both second-price and open-ascending auction frameworks. McAfee & McMillan (1992) prove a similar result for collusive rings who assume bid submission mechanisms in which the cartel can directly control the members' bids at the auction. In all of these scenarios, the cartel successfully extracts the entire surplus from the seller, leaving the seller with expected revenue of zero. However, the optimal BCM cartel at a first-price auction can do no better than the non-cooperative equilibrium and ultimately shares surplus with the seller. Thus, this research supports the recommendation that sellers concerned about bidder collusion should use a first-price auction format.

5.2 Discussion

5.2.1 Real-World Implications

Auction design is widely regarded as one of the most successful areas of application of economic theory to real-world settings (Alexandrova, 2009). Klemperer (2002) notes that auction theory has been fruitfully applied to fields ranging from political economy and finance to law and labor economics and even industrial organization. More specifically, studies of the internal machinery of bidding rings can be beneficial for gaining a broader understanding of antitrust analysis (Asker, 2009b). One might think that the conditions for this particular research are so intricate and specific that their results cannot be properly generalized for more relaxed and practical environments. However, the game theoretic outcome of this paper has many implications for present and future policy matters.

For instance, the result may inform policymakers within the Federal Communications Commission, who have historically relied upon economic experiments to guide auction design related to the privatization of licenses for electromagnetic spectrum (Connolly & Kwerel, 2007). The FCC recently utilized economic theory and empirical research in their decision about the 700MHz auction framework. Since this research lends additional credibility to the notion that first-price sealed-bid auctions are the most robust framework to collusion, perhaps the FCC will take this finding into consideration for future auction designs.

Such informed application may be similarly beneficial for other government-led activities, such as cap-and-trade regulation, selling mobile-phone licenses, and privatizing companies. In the private sector, the theory and outcome of this research is especially useful for markets prone to collusive behavior, such as the construction, real estate, lumber, antiques, utility procurement, and milk industries (Pesendorfer, 2000). Yet it is important to keep in mind that good auction design is not a "one size fits all" and must rather be "sensitive to the details of the context."[24] Thus, while this paper's theoretical findings contain numerous real-world implications, they must be applied *appropriately* in order to be effective.

5.2.2 Suggestions for Future Research

This paper examined surplus scenarios for a two-bidder and three-bidder auction format. The research paves the way for future analysis into the effects of cartel size on profitability. For instance, one might seek to answer the question, "To what extent are cartels with smaller number of members more effective at extracting surplus from the seller?" This research provides only two data points to support the argument; further exploration into larger cartel sizes could confirm if this result is, indeed, a trend.

Empirical work by Selten (1973), Kwoka (1979), Bresnahan & Reiss (1991) and recent experimental work by Huck et al (2004) suggest that competitiveness is monotonic in the number of participants, although the researchers disagree as to what number of players is sufficient to prevent collusive outcomes. Kovacic et al (2007) analyze tacit and explicit collusion in the Vitamins industry where collusion via market segmentation and price fixing was widespread during the 1990s. Their research examines price data for 30 vitamin products among cartels with two, three, and four conspirators over the course of three periods: prior to, during, and after a period of admitted explicit collusion.[25] Empirical results indicate that in the post-plea period vitamin products with two conspirators continue as if the explicit conspiracy never stopped, while products with three or four conspirators quickly return to pre-conspiracy pricing (Please see Figure 5 in Appendix A, Section 6.1 for a graphical view of this finding). Thus, in the vitamins example, prices after the end of explicit collusion are inversely related to the size of the plea-period

[24] Klemperer (2002) pg. 184.

[25] The research also included a vitamin product that had a one-firm cartel, though the results for this monopolistic producer were less clear.

cartel, which suggests that the number of firms in an industry may affect the potential for profitable and effective tacit collusion as a coordinated effect. Similar research could combine Kovacic et al's (2007) empirical approach along with the theoretical implications from this paper to investigate collusive outcomes for cases of bid rigging with varying numbers of participants (or cartel sizes).

Another study coalescing theory and empirical work may compare the findings from this research with Asker's (2009b) analysis of the bidding behavior of a North American cartel of stamp dealers during the 1990s. This real-world bidding ring operated in a surprisingly similar environment and fashion to the theoretical framework constructed in this paper. Thus, the data from over 1700 auctions in which the stamp cartel was active might provide a good starting point for testing the rigor of this work's model when applied to more realistic settings.

The innovative linear programming approach employed in this paper is potentially useful in developing answers to a number of questions that remain open in technical literature. For example, one may attempt to characterize or construct the optimal collusive mechanism for auction formats and assumptions aside from those considered in this research, such as open-ascending auctions or cartels that are not all-inclusive.

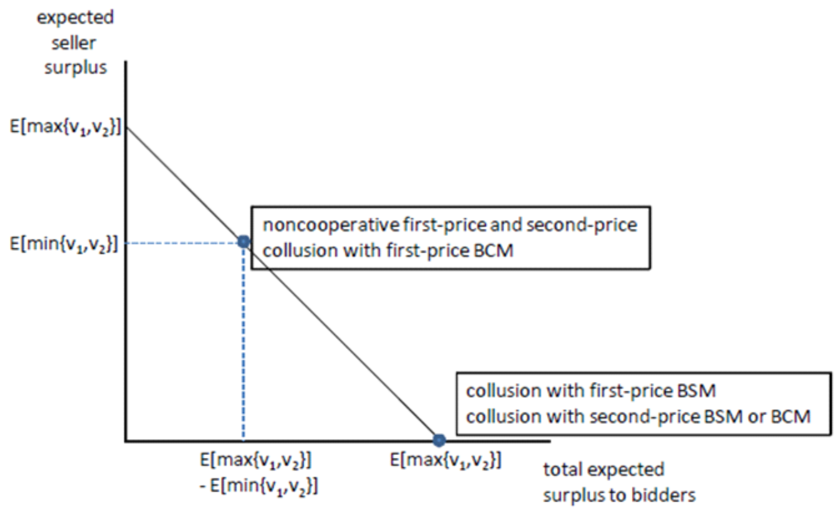
This research introduced an additional bidder to the model developed by Lopomo, Marx, & Sun (2009). In addition to further increasing the number of bidders, the authors' original framework may be adjusted and extended in numerous other ways. Such extensions could incorporate repeated interactions or allow for more general settings such as asymmetric bidders or strategic auctioneers.[26] Assumptions such as risk-neutral bidders, individual rationality, and the ex-ante budget balance constraint could also be modified and retested. Lastly, multiple-cartel and/or multiple-object scenarios present additional research opportunities for questions that remain open in the realm of auction theory. Furthermore, the novel techniques used in this paper may be fruitfully applied to a variety of mechanism design problems beyond the scope of auction theory.

[26] Numerical calculations from this paper suggest that the results likely extend to more general symmetric environments and at least some asymmetric environments.

6 Supplemental Material

6.1 Appendix A: Figures & Charts

Figure 2: Comparison of Non-Cooperative and Collusive Outcomes[27]



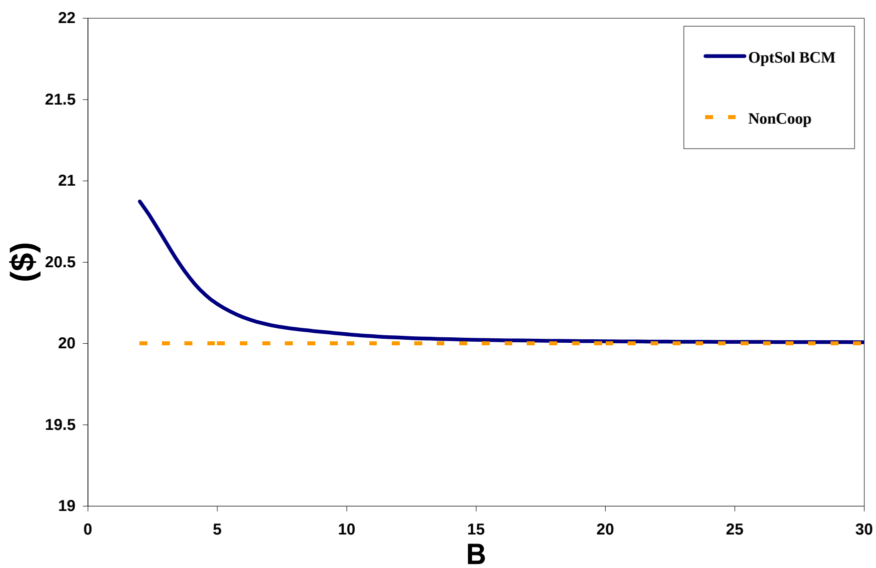
Under the optimal reserve price for a seller facing non-cooperative bidders, the surplus in all cases would be inside the efficient frontier and even further inside for the case of the optimal reserve price for a seller facing colluding bidders. But even with reserve prices of either type, it is still the case that the optimal BCM and BSM at a second-price auction and the optimal BSM at a first-price auction reduces the seller's surplus to zero. In contrast, the optimal BCM at a first-price auction can do no better than the non-cooperative equilibrium, which shares surplus with the seller.

[27] Figure from Lopomo, Marx & Sun (2009).

Chart 2: Three Bidder Surplus Output (Source Data for Figure 1)

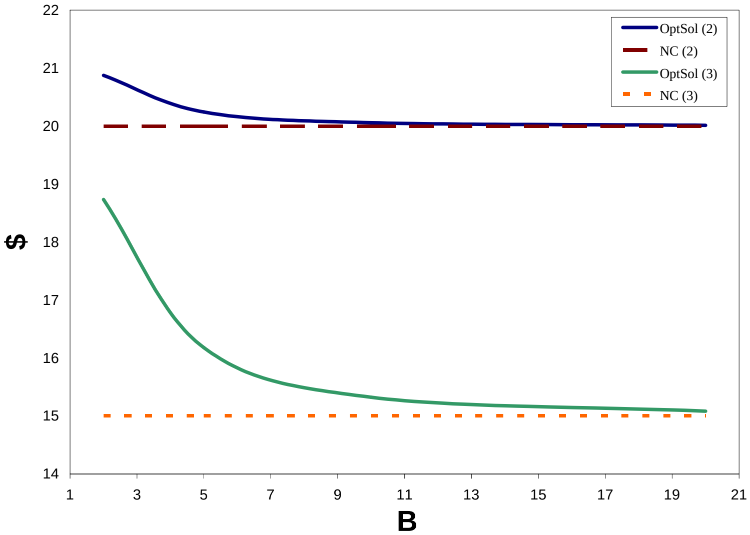
b	OptSol BCM	NonCoop	BSM	b	OptSol BCM	NonCoop	BSM
2	18.7332	15	25	16	15.1286	15	25
3	17.4817	15	25	17	15.1140	15	25
4	16.7287	15	25	18	15.1018	15	25
5	16.1793	15	25	19	15.0914	15	25
6	15.8503	15	25	20	15.0826	15	25
7	15.6358	15	25	21	15.0749	15	25
8	15.4952	15	25	22	15.0683	15	25
9	15.3944	15	25	23	15.0625	15	25
10	15.3228	15	25	24	15.0575	15	25
11	15.2678	15	25	25	15.0530	15	25
12	15.2264	15	25	26	15.0490	15	25
13	15.1934	15	25	27	15.0454	15	25
14	15.1674	15	25	28	15.0423	15	25
15	15.1460	15	25	29	15.0394	15	25
16	15.1286	15	25	30	15.0368	15	25

Figure 3: Two Bidder BCM Cartel Equilibrium



The optimal collusive bidder surplus ("OptSol BCM") and the non-cooperative equilibrium surplus in the continuous bid case ("NonCoop") as a function of B, which is inversely related to the bid increment. Figure 3 shows the lower bound of the non-cooperative equilibrium surplus, as well as the surplus from the optimal BCM, based on the linear programming output. The graph demonstrates that the optimal collusive surplus approaches the non-cooperative surplus as the bid increment approaches zero.

Figure 4: Graphical Comparison of Outcomes (Source Data on Right)



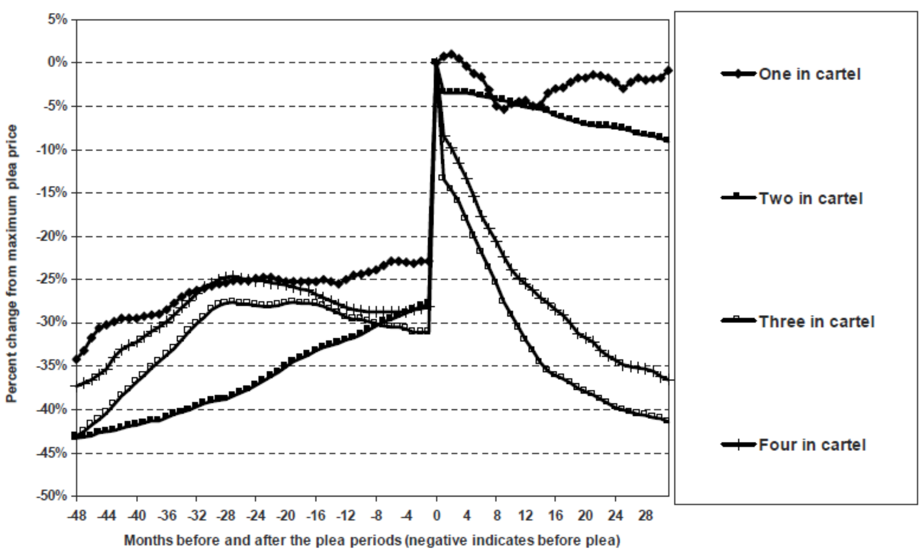
Two Bidder Case

b	OptSol (2)	NC (2)
2	20.8732	20
5	20.2427	20
10	20.0569	20
20	20.0138	20

Three Bidder Case

b	OptSol (3)	NC (3)
2	18.7332	15
5	16.1793	15
10	15.3228	15
20	15.0826	15

Figure 5: Average % Change in Price Relative to the Plea-Period Maxima by Number in Cartel[28]



The figure shows a dramatic difference between the post-plea prices of products with one conspirator or a two-firm cartel versus products with three or four-firm cartels. Two-firm cartels are able to maintain prices significantly above pre-plea levels whereas there is considerable price erosion relative to explicitly collusive levels for three and four-firm cartels.

[28]Figure from Kovacic et. al (2007).

6.2 Appendix B: Proofs

6.2.1 Proof of Proposition 1

It is clear that a bidder with value l can do not better than to bid l. If bidder i has value h, its expected payoff from bid $b_i \in [l, b^*]$ is

$$\begin{aligned}\pi_i &= (1-f(l))^2 \int_l^{b_i} (h-b_i)dF^2(b_{-i}) + 2(1-f(l))f(l) \int_l^{b_i} (h-b_i)dF(b_{-i}) + f(l)^2(h-b_i) \\ &= f(l)^2 (h-l),\end{aligned}$$

which is constant for all $b_i \in [l, b^*]$. A bid below l has expected payoff of zero, and a bid $b_i > b^*$ has expected payoff of $h - b_i < h - b^* = f(l)^2(h-l)$. So the bidding strategy given in the proposition is a best reply. Q.E.D.

6.2.2 Proof of Lemma 1

First, consider a feasible solution (p, m_1, m_2, m_3) of linear program (1)–(10). We construct a solution $(\rho, M_1, M_2, M_3, J_1, J_2, J_3)$ to the linear program (11)–(17) and then show that it is feasible with the same objective function value. Let $\rho(b_1, b_2, b_3, v_1, v_2, v_3) = p(b_1, b_2, b_3 \mid v_1, v_2, v_3)$, let $M_1(v_1) = \sum_{(v_2,v_3)\in\mathcal{V}_2\times\mathcal{V}_3} m_1(v_1, v_2, v_3) f_2(v_2) f_3(v_3)$, and let

$$J_1(v_1, v_2, v_3, b_1) = \max_{b_1'\in\mathcal{B}} \sum_{v_2\in\mathcal{V}_2, v_3\in\mathcal{V}_3, b_2, b_3, \in\mathcal{B}} \rho(b_1, b_2, b_3, v_1, v_2, v_3) u_1(b_1', b_2, b_3, v_1) f_2(v_2) f_3(v_3),$$

with analogous definitions for $M_2(v_2)$, $M_3(v_3)$, $J_2(v_1, v_2, v_3, b_2)$ and $J_3(v_1, v_2, v_3, b_3)$. It is straightforward to show that constraints (13), (14), (16) and (17) are satisfied. Furthermore, consider the deviation function $\delta_1^*(b_1)$ defined by

$$J_1(v_1, v_2, v_3, b_1) = \sum_{v_2\in\mathcal{V}_2, v_3\in\mathcal{V}_3, b_2, b_3, \in\mathcal{B}} p(b_1, b_2, b_3 \mid v_1, v_2, v_3) u_1(\delta_1^*(b_1), b_2, b_3, v_1) f_2(v_2) f_3(v_3),$$

with an analogous definition for $J_2(v_1, v_2, v_3, b_2)$ and $J_3(v_1, v_2, v_3, b_3)$. Constraint (4) implies that,

$$U_1(p \mid v_1) \geq U_1^*(p, \delta_1^*, v_2, v_3 \mid v_1) = \sum_{b_1\in\mathcal{B}} J_1(v_1, v_2, v_3, b_1) - M_1(v_1, v_2) \tag{25}$$

and similarly for U_2 and U_3. Therefore, (15) is satisfied. It can be verified that the corresponding objective values are the same.

Second, consider a feasible solution $(\rho, M_1, M_2, M_3, J_1, J_2, J_3)$ to (11)–(17). Letting $p(b_1, b_2, b_3 \mid v_1, v_2, v_3) = \rho(b_1, b_2, b_3, v_1, v_2, v_3)$ and $m_1(v_1, v_2, v_3) = M_1(v_1)$, then (p, m_1, m_2, m_3) is a feasible solution to (1)–(10) and has the same objective function value. It is obvious that constraints (2) and (3) are satisfied. To see that the incentive compatibility constraints are also satisfied, note that for any $v_1 \in \mathcal{V}_1, v_2 \in \mathcal{V}_2$, $v_3 \in \mathcal{V}_3$, $b_1 \in \mathcal{B}$, and $\delta_1(b_1) \in \mathcal{B}$, (16) and (17) imply

$$J_1(v_1, v_2, v_3, b_1) = \sum_{v_2 \in \mathcal{V}_2, v_3 \in \mathcal{V}_3, b_2, b_3, \in \mathcal{B}} p(b_1, b_2, b_3 \mid v_2, v_3) u_1(\delta_1(b_1), b_2, b_3, v_1) f_2(v_2) f_3(v_3),$$

and analogously for bidder 2. Furthermore, (15) implies (4). Q.E.D.

6.2.3 Proof of Lemma 2

For any feasible solution $(\rho, M_1, M_2, M_3, J_1, J_2, J_3)$ to the linear program (11)–(17), we can construct a feasible solution $(\tilde{\rho}, J, M)$ to the linear program (19)–(24) with the same objective function value as follows: $\tilde{\rho}(b, \bar{b}, \tilde{b}, v, \bar{v}, \tilde{v}) = \frac{1}{6}(\rho(b, \bar{b}, \tilde{b}, v, \bar{v}, \tilde{v}) + \rho(b, \tilde{b}, \bar{b}, v, \tilde{v}, \bar{v}) + \rho(\bar{b}, b, \tilde{b}, \bar{v}, v, \tilde{v}) + \rho(\tilde{b}, b, \bar{b}, \tilde{v}, v, \bar{v}) + \rho(\bar{b}, \tilde{b}, b, \bar{v}, \tilde{v}, v) + \rho(\tilde{b}, \bar{b}, b, \tilde{v}, \bar{v}, v))$, $M = M_1 + M_2 + M_3$, and $J = J_1 + J_2 + J_3$. Similarly, for any feasible solution $(\tilde{\rho}, M, J)$ to the linear program (19)–(24), we can construct a feasible solution $(\rho, M_1, M_2, M_3, J_1, J_2, J_3)$ to (11)–(17) with the same objective function value as follows:

$$\begin{aligned}
\rho(b, \bar{b}, \tilde{b}, v, \bar{v}, \tilde{v}) &= \rho(b, \tilde{b}, \bar{b}, v, \tilde{v}, \bar{v}) = \rho(\bar{b}, b, \tilde{b}, \bar{v}, v, \tilde{v}) \\
&= \rho(\tilde{b}, b, \bar{b}, \tilde{v}, v, \bar{v}) = \rho(\bar{b}, \tilde{b}, b, \bar{v}, \tilde{v}, v) = \rho(\tilde{b}, \bar{b}, b, \tilde{v}, \bar{v}, v) \\
&= \frac{1}{6}\left(\begin{array}{c} \tilde{\rho}(b, \bar{b}, \tilde{b}, v, \bar{v}, \tilde{v}) + \tilde{\rho}(b, \tilde{b}, \bar{b}, v, \tilde{v}, \bar{v}) + \tilde{\rho}(\bar{b}, b, \tilde{b}, \bar{v}, v, \tilde{v}) \\ +\tilde{\rho}(\tilde{b}, b, \bar{b}, \tilde{v}, v, \bar{v}) + \tilde{\rho}(\bar{b}, \tilde{b}, b, \bar{v}, \tilde{v}, v) + \tilde{\rho}(\tilde{b}, \bar{b}, b, \tilde{v}, \bar{v}, v) \end{array}\right),
\end{aligned}$$

$M_1 = M_2 = M_3 = \frac{M}{3}$, and $J_1 = J_2 = J_3 = \frac{J}{3}$. Q.E.D.

6.2.4 Proof of Lemma 3

Let b_{low} be the lowest bid recommended with positive probability in the optimal BCM. Thus, a bid of b_{low} can only win the object through a tie. Assume a bidder with type l has probability p_{low} to be tied at a bid of b_{low}, and therefore receive surplus $(l - b_{low}) p_{low}/2$. Suppose $b_{low} = l - t\Delta < l - 2\Delta$ for some $t > 2$. Whenever the bidder receives the recommended bid $b_{low} = l - t\Delta$ from the center, by bidding $l - (t-1)\Delta$ instead, the bidder wins with probability at least p_{low}, which generates surplus $(t-1)\Delta p_{low}$, which is higher than $t\Delta p_{low}/2 = (l - b_{low})\, p_{low}/2$, which is the surplus from following the recommended bid b_{low}. Similar logic indicates that a type h bidder also has no incentive to bid below $l - 2\Delta$. Q.E.D.

6.3 Appendix C: Sample Feasible Bid Set

Table 1: Index of Feasible Bids

β_B = Feasible Bid Set (by column)

B	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
J																														
0	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00	40.00
1	55.00	47.50	45.00	43.75	43.00	42.50	42.14	41.88	41.67	41.50	41.36	41.25	41.15	41.07	41.00	40.94	40.88	40.83	40.79	40.75	40.71	40.68	40.65	40.63	40.60	40.58	40.56	40.54	40.52	40.50
2	70.00	55.00	50.00	47.50	46.00	45.00	44.29	43.75	43.33	43.00	42.73	42.50	42.31	42.14	42.00	41.88	41.76	41.67	41.58	41.50	41.43	41.36	41.30	41.25	41.20	41.15	41.11	41.07	41.03	41.00
3		62.50	55.00	51.25	49.00	47.50	46.43	45.63	45.00	44.50	44.09	43.75	43.46	43.21	43.00	42.81	42.65	42.50	42.37	42.25	42.14	42.05	41.96	41.88	41.80	41.73	41.67	41.61	41.55	41.50
4		70.00	60.00	55.00	52.00	50.00	48.57	47.50	46.67	46.00	45.45	45.00	44.62	44.29	44.00	43.75	43.53	43.33	43.16	43.00	42.86	42.73	42.61	42.50	42.40	42.31	42.22	42.14	42.07	42.00
5			65.00	58.75	55.00	52.50	50.71	49.38	48.33	47.50	46.82	46.25	45.77	45.36	45.00	44.69	44.41	44.17	43.95	43.75	43.57	43.41	43.26	43.13	43.00	42.88	42.78	42.68	42.59	42.50
6			70.00	62.50	58.00	55.00	52.86	51.25	50.00	49.00	48.18	47.50	46.92	46.43	46.00	45.63	45.29	45.00	44.74	44.50	44.29	44.09	43.91	43.75	43.60	43.46	43.33	43.21	43.10	43.00
7				66.25	61.00	57.50	55.00	53.13	51.67	50.50	49.55	48.75	48.08	47.50	47.00	46.56	46.18	45.83	45.53	45.25	45.00	44.77	44.57	44.38	44.20	44.04	43.89	43.75	43.62	43.50
8				70.00	64.00	60.00	57.14	55.00	53.33	52.00	50.91	50.00	49.23	48.57	48.00	47.50	47.06	46.67	46.32	46.00	45.71	45.45	45.22	45.00	44.80	44.62	44.44	44.29	44.14	44.00
9					67.00	62.50	59.29	56.88	55.00	53.50	52.27	51.25	50.38	49.64	49.00	48.44	47.94	47.50	47.11	46.75	46.43	46.14	45.87	45.63	45.40	45.19	45.00	44.82	44.66	44.50
10					70.00	65.00	61.43	58.75	56.67	55.00	53.64	52.50	51.54	50.71	50.00	49.38	48.82	48.33	47.89	47.50	47.14	46.82	46.52	46.25	46.00	45.77	45.56	45.36	45.17	45.00
11						67.50	63.57	60.63	58.33	56.50	55.00	53.75	52.69	51.79	51.00	50.31	49.71	49.17	48.68	48.25	47.86	47.50	47.17	46.88	46.60	46.35	46.11	45.89	45.69	45.50
12						70.00	65.71	62.50	60.00	58.00	56.36	55.00	53.85	52.86	52.00	51.25	50.59	50.00	49.47	49.00	48.57	48.18	47.83	47.50	47.20	46.92	46.67	46.43	46.21	46.00
13							67.86	64.38	61.67	59.50	57.73	56.25	55.00	53.93	53.00	52.19	51.47	50.83	50.26	49.75	49.29	48.86	48.48	48.13	47.80	47.50	47.22	46.96	46.72	46.50
14							70.00	66.25	63.33	61.00	59.09	57.50	56.15	55.00	54.00	53.13	52.35	51.67	51.05	50.50	50.00	49.55	49.13	48.75	48.40	48.08	47.78	47.50	47.24	47.00
15								68.13	65.00	62.50	60.45	58.75	57.31	56.07	55.00	54.06	53.24	52.50	51.84	51.25	50.71	50.23	49.78	49.38	49.00	48.65	48.33	48.04	47.76	47.50
16								70.00	66.67	64.00	61.82	60.00	58.46	57.14	56.00	55.00	54.12	53.33	52.63	52.00	51.43	50.91	50.43	50.00	49.60	49.23	48.89	48.57	48.28	48.00
17									68.33	65.50	63.18	61.25	59.62	58.21	57.00	55.94	55.00	54.17	53.42	52.75	52.14	51.59	51.09	50.63	50.20	49.81	49.44	49.11	48.79	48.50
18									70.00	67.00	64.55	62.50	60.77	59.29	58.00	56.88	55.88	55.00	54.21	53.50	52.86	52.27	51.74	51.25	50.80	50.38	50.00	49.64	49.31	49.00
19										68.50	65.91	63.75	61.92	60.36	59.00	57.81	56.76	55.83	55.00	54.25	53.57	52.95	52.39	51.88	51.40	50.96	50.56	50.18	49.83	49.50
20										70.00	67.27	65.00	63.08	61.43	60.00	58.75	57.65	56.67	55.79	55.00	54.29	53.64	53.04	52.50	52.00	51.54	51.11	50.71	50.34	50.00
21											68.64	66.25	64.23	62.50	61.00	59.69	58.53	57.50	56.58	55.75	55.00	54.32	53.70	53.13	52.60	52.12	51.67	51.25	50.86	50.50
22											70.00	67.50	65.38	63.57	62.00	60.63	59.41	58.33	57.37	56.50	55.71	55.00	54.35	53.75	53.20	52.69	52.22	51.79	51.38	51.00
23												68.75	66.54	64.64	63.00	61.56	60.29	59.17	58.16	57.25	56.43	55.68	55.00	54.38	53.80	53.27	52.78	52.32	51.90	51.50
24												70.00	67.69	65.71	64.00	62.50	61.18	60.00	58.95	58.00	57.14	56.36	55.65	55.00	54.40	53.85	53.33	52.86	52.41	52.00
25													68.85	66.79	65.00	63.44	62.06	60.83	59.74	58.75	57.86	57.05	56.30	55.63	55.00	54.42	53.89	53.39	52.93	52.50
26													70.00	67.86	66.00	64.38	62.94	61.67	60.53	59.50	58.57	57.73	56.96	56.25	55.60	55.00	54.44	53.93	53.45	53.00
27														68.93	67.00	65.31	63.82	62.50	61.32	60.25	59.29	58.41	57.61	56.88	56.20	55.58	55.00	54.46	53.97	53.50
28														70.00	68.00	66.25	64.71	63.33	62.11	61.00	60.00	59.09	58.26	57.50	56.80	56.15	55.56	55.00	54.48	54.00
29															69.00	67.19	65.59	64.17	62.89	61.75	60.71	59.77	58.91	58.13	57.40	56.73	56.11	55.54	55.00	54.50
30															70.00	68.13	66.47	65.00	63.68	62.50	61.43	60.45	59.57	58.75	58.00	57.31	56.67	56.07	55.52	55.00
31																69.06	67.35	65.83	64.47	63.25	62.14	61.14	60.22	59.38	58.60	57.88	57.22	56.61	56.03	55.50
32																70.00	68.24	66.67	65.26	64.00	62.86	61.82	60.87	60.00	59.20	58.46	57.78	57.14	56.55	56.00
33																	69.12	67.50	66.05	64.75	63.57	62.50	61.52	60.63	59.80	59.04	58.33	57.68	57.07	56.50
34																	70.00	68.33	66.84	65.50	64.29	63.18	62.17	61.25	60.40	59.62	58.89	58.21	57.59	57.00
35																		69.17	67.63	66.25	65.00	63.86	62.83	61.88	61.00	60.19	59.44	58.75	58.10	57.50
36																		70.00	68.42	67.00	65.71	64.55	63.48	62.50	61.60	60.77	60.00	59.29	58.62	58.00
37																			69.21	67.75	66.43	65.23	64.13	63.13	62.20	61.35	60.56	59.82	59.14	58.50
38																			70.00	68.50	67.14	65.91	64.78	63.75	62.80	61.92	61.11	60.36	59.66	59.00
39																				69.25	67.86	66.59	65.43	64.38	63.40	62.50	61.67	60.89	60.17	59.50
40																				70.00	68.57	67.27	66.09	65.00	64.00	63.08	62.22	61.43	60.69	60.00
41																					69.29	67.95	66.74	65.63	64.60	63.65	62.78	61.96	61.21	60.50
42																					70.00	68.64	67.39	66.25	65.20	64.23	63.33	62.50	61.72	61.00
43																						69.32	68.04	66.88	65.80	64.81	63.89	63.04	62.24	61.50
44																						70.00	68.70	67.50	66.40	65.38	64.44	63.57	62.76	62.00
45																							69.35	68.13	67.00	65.96	65.00	64.11	63.28	62.50
46																							70.00	68.75	67.60	66.54	65.56	64.64	63.79	63.00
47																								69.38	68.20	67.12	66.11	65.18	64.31	63.50
48																								70.00	68.80	67.69	66.67	65.71	64.83	64.00
49																									69.40	68.27	67.22	66.25	65.34	64.50
50																									70.00	68.85	67.78	66.79	65.86	65.00
51																										69.42	68.33	67.32	66.38	65.50
52																										70.00	68.89	67.86	66.90	66.00
53																											69.44	68.39	67.41	66.50
54																											70.00	68.93	67.93	67.00
55																												69.46	68.45	67.50
56																												70.00	68.97	68.00
57																													69.48	68.50
58																													70.00	69.00
59																														69.50
60																														70.00

Table 1 lists out the feasible bids for each B_B. Each cell represents a feasible bid b that is a function of B and J. The chart affirms that the non-cooperative equilibrium bid $(b_{2B} = b^* = 70)$ is within the feasible bid set for all B. This forms the upper support limit $MaxJ = 2B$. The lower bound for each bid set is set by the reserve price $(b_0 = l = 40)$ and the interval $[l,\ b^*]$ contains $2B + 1$ feasible bids. One can see that as B expands and bid increments decrease (moving rightward on the feasible bids chart), the highest feasible bid less than the upper support (70) approaches 70.

6.4 Appendix D: Sample Computer Coding

Matlab Coding - Generates data file that works with J_sym_u_primal.mod in OPL

```
numBidder = 2;
numType = 2;
types = [40 80];
B = 2;
numBid = 2*B+1;
bids = zeros(1,numBid);
f = [0.5 0.5];
minBid = types(1);
maxBid = types(2)*f(2) + types(1)*f(1);
delta = (maxBid - minBid)/(2*B);
for b = 1:numBid
    bids(b) = (b-1)*delta + minBid;
end
maxBid

u = zeros(numType, numBid, numBid);
fid = fopen('B2.dat','w');

objval = 2*(types(2)-types(1)) * f(1)*f(2);
fprintf(fid,'// %9.7f\n', objval);
fprintf(fid,'Tup = %d;\nBup = %d;\n\nf = [%6.4f, %6.4f', numType,
numBid, f(1), f(2));

fprintf(fid,'];\n\nv = [');
for i=1:numType
    fprintf(fid, '%6.4f ', types(i));
end
fprintf(fid,'];\n\nb = [');
for i=1:numBid
    fprintf(fid, '%7.5f ', bids(i));
end
fprintf(fid,'];\n\nu = [');
%B2 = 2*(numBid-1);
%expf = (numBidder-2)/(numBidder-1)
for t = 1:numType
    fprintf(fid,'[');
    for b = 1:numBid
        fprintf(fid,'[');
        utmp = (types(t)-bids(b));% * ...
            %((B2+1-f(1).^(numBidder-1)) / ...
            %(B2+b*(1-f(1).^(numBidder-1)))).^expf;
        for bhat = 1:b-1
            u(t,b,bhat) = utmp;
            fprintf(fid,'%9.6f ',u(t,b,bhat));
        end
        u(t,b,b) = utmp/2;
        fprintf(fid,'%9.6f ',u(t,b,b));
        for bhat = b+1:numBid
            fprintf(fid,'0 ',u(t,b,bhat));
        end
        fprintf(fid,']\n');
    end
    fprintf(fid,']\n');
end
fprintf(fid,'];\n\n');

fclose(fid);
```

J_sym_u_primal.mod in OPL

Generates the curver OptSol in Figure 1

```
/*****************************************************
 * This is the SHORT model with the center up:
 * it maximizes the bidders' ex-ante expected social surplus
 * subject to truthtelling and obedience constraints.
 * It displays the stochastic recommendation for any
 * pair of reported types
 * 4/3/07
 *****************************************************/

int Tup = ...;    // highest type of agent 1
int Bup = ...;    // max index for agent 1's bids

range T = 1..Tup;   // Type space of agent 1
range B = 1..Bup;   // Agent 1's bid index set

float v[T] = ...;   // u_1(b)
float b[B] = ...;   // Agent 1's bid

float f[T] = ...;   // prob distribution of u_1(b)
float u[T][B][B][B] = ...; //utility of type t1 bidding b1 and rivals bidding b2 and b3

//float p[t1 in T][t2 in T][t3 in T][b1 in B][b2 in B][b3 in B] =
//  (b[b1]==v[t1]/3 && b[b2]==v[t2]/3 && b[b3]==v[t3]/3)? 1:0;
//float Jo[T][T][T][B] = ...;

dvar float+ p[T][T][T][B][B][B];

dvar float J[T][T][T][B];
dvar float M[T];


//constraint Jdef[T1][T1][B1][B1];

maximize
sum(t1 in T, t2 in T, t3 in T, b1 in B, b2 in B, b3 in B)
    ((u[t1,b1,b2,b3]+u[t2,b1,b2,b3] +u[t3,b1,b2,b3])*p[t1,t2,t3,b1,b2,b3]*f[t1]*f[t2]*f[t3]);

subject to
{
   forall(t1 in T, t2 in T, t3 in T, b1 in B, b2 in B, b3 in B) sym_P:
   p[t1,t2,t3,b1,b2,b3] == p[t1,t3,t2,b1,b3,b2] == p[t2,t1,t3,b2,b1,b3] == p[t2,t3,t1,b2,b3,b1]
== p[t3,t1,t2,b3,b1,b2] == p[t3,t2,t1,b3,b2,b1];

   forall(t1 in T, t2 in T, t3 in T: t1<=t2<=t3) Dist:
   sum(b1 in B, b2 in B, b3 in B)
(p[t1,t2,t3,b1,b2,b3]+p[t1,t3,t2,b1,b3,b2]+p[t2,t1,t3,b2,b1,b3]+p[t2,t3,t1,b2,b3,b1]+p[t3,t1,t2,b
3,b1,b2]+p[t3,t2,t1,b3,b2,b1]) == 2;

   forall(t in T, that in T, b in B, bhat in B)
      Jdef:
   J[that,t,b]
    >=
    sum(tbar in T, ttilde in T, bbar in B, btilde in B)

((p[t,tbar,ttilde,b,bbar,btilde]+p[t,ttilde,tbar,b,btilde,bbar]+p[tbar,t,tilde,bbar,b,btilde]+p[t
bar,ttilde,t,bbar,btilde,b]+p[ttilde,t,tbar,btilde,b,bbar]+p[ttilde,tbar,t,btilde,bbar,b])*u[that
,bhat,bbar,btilde]*f[tbar]*f[ttilde]);

   forall(t in T, that in T) ic:
   sum(tbar in T, ttilde in T, b in B, bbar in B, b tilde in B)

((p[t,tbar,ttilde,b,bbar,btilde]+p[t,ttilde,tbar,b,btilde,bbar]+p[tbar,t,tilde,bbar,b,btilde]+p[t
bar,ttilde,t,bbar,btilde,b]+p[ttilde,t,tbar,btilde,b,bbar]+p[ttilde,tbar,t,btilde,bbar,b])*u[t,b,
bbar,btilde]*f[tbar]*f[ttilde]) - M[t]
   >= sum(b in B) J[t, that, b] -M[that];

   BB: sum(t in T) (M[t]*f[t]) == 0;
}
```

Updated C++ Coding

```
#ifndef COLLUSIONCENTER3BUYERS_H_

#define COLLUSIONCENTER3BUYERS_H_

#include "../common/CplexProgram.h"

#define USE_SYMMETRY

//#undef USE_SYMMETRY

class CollusionCenter3Buyers : public CplexProgram

{

  double unalteredRunObj;

  int Tup;   // highest type of agent 1

  int Bup;   // max index for agent 1's bids

  vector<double> v;  // u_1(b)

  vector<double> b;  // Agent 1's bid

  vector<double> f;  // prob distribution of u_1(b)

  vector< vector< vector< vector<double> > > > u; // [t1][b1][b1hat][b2hat] utility of type t1 bidding b1 and rivals bidding b1hat and b2hat

  vector< vector< vector< vector< vector< IloNumVarArray > > > > > p; // [T][T][T][B][B][B] probability

  vector< vector< IloNumVarArray > > J; // [T][T][B]

  IloNumVarArray M; // [t] payment

  // data..

  void loadData(int argc, char *argv[])

  {

    Tup = 2;

    Bup = 40;

    v.push_back(40.0);

    v.push_back(80.0);

    for (int i = 0; i < Tup; i++)

    {

      cout << v[i] << " ";

    } cout << endl;

    cout << "b..." << endl; // populate B

    for (int i = 0; i < Bup; i++)

    {

      b.push_back(40.0 + ((double) i) * (70.0 - 40.0) / (Bup - 1.0));

      cout << b[i] << " ";      } cout << endl;
```

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