Rebalancing, Conditional Value at Risk, and t-Copula in Asset Allocation

Irvin Di Wang and Perry Xiao Zhan Zheng

Professor Emma Rasiel and Professor Aino Levonmaa, Faculty Advisors

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Duke University

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Abstract

Traditional asset allocation methods for modeling the tradeoff between risk and return do not fully reflect empirical distributions. Thus, recent research has moved away from assumptions of normality to account for risk by looking at "fat tails" and asymmetric distributions. Other studies have also considered multiple period frameworks to include asset rebalancing. We investigate the use of rebalancing with fat tail distributions and optimizing with downside risk as a consideration. Our results verify the underperformance of traditional methods in the single period framework and also demonstrate the underperformance of traditional methods in a multiple period rebalancing framework.

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1 Introduction and Literature Review

The tradeoff between risk and return is the fundamental principle of the investment process. Although choosing individual investment opportunities is important, the top-down analysis of the entire investment portfolio is crucial for maintaining and updating investment goals. The decision of how to allocate capital across asset classes is a critical investment decision that serves as the foundation for building an investment strategy. Traditional mean-variance asset allocation frameworks use a normal distribution to model returns and variance as its risk measure. While these traditional methods for understanding risk and return have come under fire in the recent financial turmoil, research in alternative models is a growing and promising field. Recent research focuses on other risk measures for capturing downside risk and different probability distributions to account for extreme events. We are interested in building upon the recent research in asset allocation by investigating how portfolio rebalancing can be included.

Markowitz (1952) pioneered the use of mean-variance optimization to understand the tradeoff between risk and return for a portfolio of risky assets. Markowitz assumes asset returns are normally distributed, providing a tractable model for the minimization of variance and the maximization of expected returns. The analytical solution of this optimization problem is a set of portfolios along the efficient frontier that represents the best possible returns for each level of variance. Markowitz set the foundation for asset allocation research and understanding the tradeoff between risk and return in a portfolio of risky assets.

In traditional mean-variance portfolio optimization frameworks, it is assumed that returns are normally distributed and joint probabilities are properly captured by the historical, linear correlation matrix. The shortfall of this method is the thinness of the normal probability distribution's tail, implying a very low likelihood for situations where all asset classes are falling significantly in market crash scenarios. Recent research uses a copula fit that can accommodate empirically higher probabilities of joint events (crashes or bubbles). A copula combines marginal cumulative distribution functions (CDF) and historical data of each asset class to create a best fit joint multivariate distribution. A Gaussian copula assumes a multivariate normal distribution whereas a t-copula utilizes a multivariate Student's t-distribution, allowing for fatter tails. The t-copula provides a more accurate representation of real world financial markets and the probability of extreme events (Demarta and McNeil, 2004).

Sheikh and Qiao (2009) provide a real world application and verification of the use of fat tail distributions and CVaR as a risk measure in an asset allocation framework. It is demonstrated that a fat tail distribution better describes the real world market environment than a normal distribution; Student's t-copula more closely fits the historical data, especially the extreme events, than does the normal copula. Furthermore, simulations generated from a t-copula more accurately reflects historical data than simulations generated from a normal copula. The presence of a fat tail in the t-copula allows for a more accurate representation of extreme events for investigating downside risk.

Variance provides a mathematically simple measure of risk for asset allocation, but it is insufficient as a measure of downside risk. Variance symmetrically accounts for both upside risk and downside risk since it is simply a measure of the sample's average squared deviation from the arithmetic mean. When investment managers assess risk, they are typically more concerned with downside risk or losses and view upside risk (which may be referred to as "excess return") favorably. Value-at-risk (VaR) has been incorporated as another risk measure that aims to better account for downside risk than variance. VaR is the minimum expected loss at a certain confidence level (typically 5%) and can be calculated from a probability density function (PDF) of return data (either historical or simulated). However, VaR fails to capture the extent of the possible losses beyond the specified (5%) cutoff. It is possible to have two PDF's with the same VaR (loss associated with the cutoff of 5%) but one with a fatter tail and greater losses to the left of the specified (5%) cutoff.

Conditional value-at-risk (CVaR) attempts to rectify this problem: CVaR is the expected loss given a negative outcome that is greater than the VaR level, and it can be calculated as a weighted average of the worst case losses (Agrawal, 2008). CVaR is a more reliable risk measure than VaR under non-normal and non-continuous probability distributions because it is a "coherent risk measure" and is "sub-additive and convex" (Krohmal, Palmquist, and Uryasev, 2002). This means that the CVaR of a portfolio is always less than or equal to the sum of the CVaR of the weighted individual asset classes; furthermore, it can be extended from continuous probability distributions to discrete scenarios, allowing for optimization using simulations or discrete distributions and eliminating anomalous results such as multiple local minima.

Rockafellar and Uryasev (1999) demonstrate the use of CVaR in an asset allocation framework and the ability to optimize CVaR using linear programming methods. The ability to use linear programming stems from the way CVaR is calculated. Theoretically, CVaR is simply the average value beyond the specified cutoff (5%) calculated using the calculus of integrals. Rockafellar and Uryasev (1999) demonstrate that the use of simulations and a summation to estimate the theoretical CVaR can be exploited to find the minimum CVaR. The minimization of CVaR in most situations also provides a minimum VaR because CVaR is always greater than VaR. Only in cases of extreme skewness does the minimum VaR differ greatly from the minimum CVaR. Rockafellar and Uryasev demonstrate the theoretical and mathematical possibility of optimizing CVaR for portfolio asset allocation.

Sheikh and Qiao (2009) show how the CVaR framework is a useful way to compare normal distributions and fat-tail distributions. The optimal mean-CVaR efficient frontier calculated using simulations generated from a normal copula underestimates the downside risk in comparison to the optimal efficient frontier calculated using simulations generated from a t-copula. Also, the mean-variance optimization has higher downside risk and a more concentrated asset allocation in comparison the the mean-CVaR efficient frontier.

So far, we have discussed methods for single period buy and hold portfolio optimization using CVaR as a risk measure and a t-copula as a joint probability distribution. Realistically, as certain assets outperform others, the asset allocation changes endogenously as well. Recent research considers expanding the optimization to incorporate multiple period rebalancing using the mean-variance framework as well as the mean-CVaR framework.

Master (2003) offers a simple rebalancing strategy in the normal mean-variance framework, which involves setting a target allocation. The investor only rebalances when the allocation deviates beyond a set of trigger points. For instance, if the trigger point is 3% for all asset classes, then an investor only rebalances when the weight for any asset class deviates more than 3% from the target weight. This is also known as a tolerance band rebalancing strategy. According to Master, any rebalancing strategy should incorporate some basic assumptions. The benefit of rebalancing is inversely proportional to an investor's risk preference: if an investor is risk tolerant, meaning he is willing to endure higher risk for higher potential returns, then his tolerance band will be wider, meaning he will rebalance less often. The key benefit of rebalancing is to reduce the tracking error, which is how closely a portfolio follows its target allocation benchmark. The process of rebalancing involves the movement of capital across asset classes and incurs costs. The net benefit of rebalancing is the difference between the benefit of rebalancing and the cost of rebalancing.

Sun (2006) extends the concepts of Master's research to incorporate dynamic programming to minimize the cost of rebalancing. Sun applies the idea to CAPM models, which use mean and variance as the primary portfolio statistics of interest. The optimal strategy is to rebalance only when the expected cost of trading is less than the expected cost of doing nothing. His conclusion is that periodic or tolerance band rebalancing provides suboptimal rebalanced portfolios. However, by treating the rebalancing problem as an optimization problem and solving it using dynamic programming, Sun is able to reduce overall costs of portfolio rebalancing and maximize returns.

Guastaroba (2009) applies rebalancing strategies to a portfolio optimization model that uses Conditional Value at Risk (CVaR) as the risk measure. The model incorporates rebalancing as a set of linear programming constraints. He introduces a decision variable to determine whether to rebalance based on how much the portfolio allocation deviates from a predetermined quantile. Fixed and variable transaction costs are both incorporated in the rebalancing decision. During every rebalancing period, an investor rebalances if net portfolio return, taking into account fixed and proportional transaction costs, is at least the predetermined required return. The paper then uses four in-and-out-of-sample windows to test the model for different levels of minimum required returns (0, 5%, and 10%) and different quantile measures (1%, 5%, 10% and 25%). Based on the back-testing results, he concludes that for a very risk-averse investor the best choice is to rebalance two or three times in six months. On the other hand, a less risk-averse investor should rebalance once or not at all.

Our research goal is to build off of the recent advances in asset allocation research by extending rebalancing to the mean-CVaR framework using a t-copula. We have reproduced the previous single period mean-CVaR model with a t-copula and have shown that the normal mean variance framework underpeforms in both the single period and rebalancing models. Furthermore, we investigated the tradeoff between CVaR and variance and have found a small increase in variance in the optimal mean-CVaR portfolios while obtaining a larger decrease in the downside risk (CVaR) in comparison to the optimal mean-variance portfolios.

In section 2, we describe the indices used as benchmarks for the asset classes, the removal of autocorrelation, and the generation of simulations using a t-copula. In section 3, we describe our methodology and implementation of each component of the mean-CVaR framework. In section 4, we present our results for each efficient frontier analyses along with a discussion. Section 5 concludes.

2 Data and Simulations

2.1 Historical Data

We use the historical monthly return data of publicly available indices representing seven asset classes: Morgan Stanley Capital International (MSCI) All World ($) for equities, Goldman Sachs Commodity index (GSCI) for commodities, a blend of benchmarks for credit[1], Barclays Capital U.S. Treasury Intermediate for interest rate, a blend of benchmarks for inflation sensitive[2], National Association of Real Estate Investment Trusts (NaREIT) for real estate, and Citi 3-month T-bills for cash. All benchmark return data range from 1973 to 2009 and summary statistics are presented in Tables 1 & 4.

The three highest returning asset classes are equities, commodities, and real estate. Commodities has the highest variance followed by real estate and equities. Equities and real estate have a particularly high

[1] 1973-1983 Barclays Intermediate Credit Index (investment grade index); 1983-1999 60% Barclays Intermediate Credit Index, 40% Barclays US High Yield Index; 1999-2009 60% Barclays Intermediate Credit Index, 30% Barclays US High Yield Index, 10% Barclays CMBS Index

[2] 1973-1997 Bridgewater Simulated TIPS; 1997-2009 Barclays TIPS Index

inflation while commodities has a low, positive correlation to those two asset classes. The lowest returning asset class with the lowest variance is cash, while credit, interest rates, and inflation have intermediate returns and variances. Credit has a particularly high historical correlation with interest rates, inflation, and real estate; also, interest rates and inflation are highly correlated. These return and correlation characteristics will be evident in the simulations and impact the efficient frontier optimizations.

2.2 Detecting and Removing Autocorrelation

In order to detect and remove autocorrelation between successive months, or single lag autocorrelation, we follow the methods used by Sheikh and Qiao (2009). We use the Ljung-Box test to detect single lag autocorrelation and the Fisher-Geltner-Webb methodology to unsmooth the data (remove single period autocorrelation). We find and remove autocorrelation in five asset classes, equities, commodities, credit, interest rates, and real estate; we do not test for autocorrelation in inflation and cash due to their low and consistent returns. As can be observed in Tables 1 & 2, removing autocorrelation increases the variance without noticeably affecting the average return.

2.3 t-Copula Fitting

The historical return data of each individual asset class are first fit to independent, marginal Generalized Pareto Distribution; this GPD fit is piecewise so that a different function is fit to the lower tail and upper tail to account for marginal distribution fat tails. Next, independent marginal CDFs are transformed into their inverses, creating a uniform distribution over [0, 1]. The historical returns are then applied to their respective inverse CDF's to determine a marginal probability of each return given the PDF. Using Maximum Likelihood, a multivariate t-distribution fit is then constructed over these inverse CDF historical probabilities to construct a t-copula that is identified by a new correlation matrix and a single degree of freedom.

The resulting t-Copula fit is centered at zero with characteristic fat tails, similar to a standard normal distribution, and is used for generating simulations. The multivariate t probability density function is defined as follows:

$$f(x) = \frac{\Gamma(\frac{v+d}{2})}{\Gamma(\frac{v}{2})\sqrt{(\pi v)^d |\Sigma|}}(1 + \frac{(x+\mu)'\Sigma^{-1}(x-\mu)}{v})^{-\frac{v+d}{2}} \tag{1}$$

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \tag{2}$$

where d is the number of dimensions, v is the degrees of freedom, μ is the mean (zero, in this case), Σ is the correlation matrix, and Γ is the gamma function as defined in Equation 2. The fit to the unsmoothed historical data results in the new correlation table in Table 5 degrees of freedom of 25.25. The degrees of freedom signifies the extent of the fat tails. A degree of freedom of infinite is essentially a normal distribution; previous literature have found t-copulas with a degree of freedom of 30 to be a very close approximation of a normal copula (Sheikh and Qiao, 2009).

2.4 Generating Simulations

In order to simulate a certain number of months into the future, we generate multiple single month returns for all asset classes. A single month's return is simulated for all asset classes by generating multivariate random numbers from the t-Copula fit and applying it to the following formula:

$$\text{Simulated One Month Return} = \mu + \sigma \cdot \vec{x} \tag{3}$$

where μ is a vector of the historical monthly returns for each asset class, σ is a vector of the historical standard deviations for each asset class, and $\vec{x}$ is a vector of the generated multivariate t random numbers. We use the average historical, empirical monthly returns and historical, empirical standard deviations of each asset class in our research for consistency and objectiveness. When simulations are drawn out of a t-copula, the likelihood of a market crash scenario is an order of magnitude higher than that of a Gaussian copula.

We simulate monthly returns of the seven asset classes for three years into the future using a t-copula fit to the unsmoothed monthly return data. We run one thousand separate simulations. As can be observed in Tables 1 & 3, the summary statistics of the t-copula simulations in comparison to the historical data show that the simulations have average returns close to the unsmoothed historical averages but have a wider range of returns exhibited in the higher standard deviations.

2.5 Summary Statistics

The following are the annual summary statistics of the historical, unsmoothed, and simulated data:

Table 1: Annual Statistics: Historical Data

	Equities	Comm.	Credit	IR	Inflation	RE	Cash
μ	9.83%	10.37%	8.49%	7.71%	8.54%	10.21%	5.80%
σ	15.29%	20.64%	5.25%	4.24%	6.13%	18.24%	0.87%

Table 2: Annual Statistics: Unsmoothed Historical Data

	Equity	Comm.	Credit	IR	Inflation	RE	Cash
μ	9.85%	10.25%	8.55%	7.74%	8.56%	10.45%	5.80%
σ	17.38%	24.63%	7.67%	5.08%	6.13%	20.10%	0.87%

Table 3: Annual Statistics: t-Copula Simulation Data

	Equity	Comm.	Credit	IR	Inflation	RE	Cash
μ	10.63%	10.18%	8.74%	7.74%	8.87%	10.50%	5.84%
σ	18.75%	26.28%	8.24%	5.30%	6.39%	21.02%	0.94%

The following are the historical correlations and the correlation of the t-copula fit:

Table 4: Historical Correlation of Monthly Returns

	Equities	Comm.	Credit	IR	Inflation	RE	Cash
Equity	1	0.1044	0.4626	0.0976	0.1470	0.5624	-0.0262
Comm.	0.1044	1	0.0127	-0.0640	0.1630	0.0541	0.0256
Credit	0.4626	0.0127	1	0.6706	0.5785	0.5212	0.0338
IR	0.0976	-0.0640	0.6706	1	0.6643	0.1498	0.1604
Inflation	0.1470	0.1630	0.5785	0.6643	1	0.2275	0.0524
RE	0.5624	0.0541	0.5212	0.1498	0.2275	1	-0.0326
Cash	-0.0262	0.0256	0.0338	0.1604	0.0524	-0.0326	1

Table 5: t-Copula Correlation of Monthly Returns

	Equity	Comm.	Credit	IR	Inflation	RE	Cash
Equity	1	0.1166	0.4349	0.1747	0.1482	0.4985	-0.0207
Comm.	0.1166	1	-0.0021	-0.0099	0.1278	0.0087	0.0170
Credit	0.4349	-0.0021	1	0.7704	0.5686	0.4843	-0.0003
IR	0.1747	-0.0099	0.7704	1	0.6925	0.2356	0.0834
Inflation	0.1482	0.1278	0.5686	0.6925	1	0.2251	0.0163
RE	0.4985	0.0087	0.4843	0.2356	0.2251	1	-0.0432
Cash	-0.0207	0.0170	-0.0003	0.0834	0.0163	-0.0432	1

3 Methodology

3.1 Arithmetic and Geometric Returns

We use two methods for calculating returns for a given time period. The arithmetic average return approach uses the sum of the returns in a simulation, while the geometric average return approach uses the product of one plus each return. Let t be the number of simulated months (36), r_i be the portfolio return for simulated month i given a set of weights, and V_h be the average portfolio return for simulation h given the previous weights; we can express the two approaches as follows:

$$\text{Arithmetic Average Return: } V_h = \frac{1}{t}\sum_{i=1}^{t} r_i \tag{4}$$

$$\text{Geometric Average Return: } V_h = [\prod_{i=1}^{t}(1+r_i)]^{\frac{1}{t}} - 1 \tag{5}$$

The use of arithmetic returns is mathematically and computationally convenient. Our initial analyses utilize the arithmetic average returns before extending the mean-CVaR framework to consider geometric average returns and rebalancing.

3.2 Conditional Value at Risk (CVaR)

In order to calculate the CVaR of a certain portfolio over a set of simulations, we sort the average returns of all the simulations from lowest to highest and take the average of the lowest $\beta * Number of Simulation$ values.

$$\beta\text{-CVaR} = \frac{1}{(\beta \cdot n)} \sum_{i=1}^{\beta \cdot n} R_i \tag{6}$$

where n is the number of simulations (1,000), R_i is the i-th lowest simulated return of the portfolio for a given set of weights, and β is the desired CVaR level (expressed as a percentage, e.g. 5%). We will from now on refer to the 5%-CVaR as simply CVaR.

3.3 Analytical Mean-Variance Optimization

The Markowitz (1952) framework analytically calculates the optimal set of portfolio returns and variances using the historical average return and historical standard deviation of each asset class along with the historical covariance between each pair of asset classes. Let $(w_1, w_2, ..., w_a)$ denote a set of weights for the a number of asset classes (seven), R be the fixed portfolio monthly return rate, and R_i be the average historical monthly return of asset class i. The optimization problem for one level of R can be expressed as follows:

$$\min_{w_1, w_2, \ldots w_n} \sigma^2(w_1, w_2, ..., w_a) \tag{7}$$

where

$$\sigma^2(w_1, w_2, ..., w_a) = \sum_{i=1}^{a} w_i^2 \cdot \sigma_i^2 + 2 \sum_{i=1}^{a} \sum_{j>i}^{a} w_i \cdot w_j \cdot \text{Cov}(i, j) \tag{8}$$

$$\text{Cov}(i, j) = \sigma_i \cdot \sigma_j \cdot \rho_{ij} \tag{9}$$

subject to

$$w_1, w_2, ..., w_a > 0 \tag{10}$$

$$w_1 + w_2 + ... + w_a = 1 \tag{11}$$

$$\sum_{i=1}^{a} w_i \cdot b_i = R \tag{12}$$

Our two linear constraints indicate that the weights must sum to one and be nonnegative (since no borrowing is allowed). The optimization uses multiple levels of R between the lowest and highest single asset class return (where the portfolio is 100% weighted in that single asset class) to determine the set of portfolios along the efficient frontier.

3.4 Single Period Simulated Mean-CVaR Optimization with Arithmetic Returns

The single period optimization of mean-CVaR minimizes CVaR at each given level of return. The optimization calculates the CVaR as described in Equation 6 while satisfying the condition that the average return across all simulations is equal to the given level of return we are optimizing on. Since this is a single period optimization, arithmetic returns are used.

Let R be the fixed portfolio monthly return rate as defined before and $\tilde{R}$ be the calculated portfolio monthly return rate averaged across all simulations. The optimization problem for one level of R can be expressed as follows:

$$\min_{w_1, w_2, \ldots w_n} \text{CVaR}(w_1, w_2, ..., w_a) \tag{13}$$

subject to Equations 10, 11, and

$$\tilde{R}(w_1, w_2, ..., w_a) = R \tag{14}$$

where

$$\tilde{R}(w_1, w_2, ..., w_n) = \frac{1}{n} \sum_{h=1}^{n} V_h \tag{15}$$

$$V_h = \frac{1}{t} \sum_{i=1}^{t} r_i \tag{4$'$}$$

Since we want to optimize on a fixed return, our new constraint is that our average final portfolio return $\tilde{R}$ across all simulations calculated with arithmetic returns must be equivalent to the given fixed portfolio return R.

3.5 Single Period Simulated Mean-CVaR Optimization with Geometric Returns

We adopt the geometrically calculated return approach in the this optimization. The optimization problem for one level of R can be expressed as follows:

$$\min_{w_1, w_2, \ldots w_n} \text{CVaR}(w_1, w_2, ..., w_a) \tag{16}$$

subject to Equations 10, 11, and 14

where Equation 15 holds and

$$V_h = [\prod_{i=1}^{t}(1 + r_i)]^{\frac{1}{t}} - 1 \tag{5'}$$

Since we want to fix return, our new constraint is that our average final portfolio return $\tilde{R}$ across all simulations calculated with geometric returns must be the same as the given fixed portfolio return R.

3.6 Multiple Period Simulated Mean-CVaR Optimization with Rebalancing and Geometric Returns

The implementation is the same as in Section 3.5 with one additional feature: every 12 months, we rebalance the portfolio to the predetermined set of weights. We calculate return of the portfolio for that year and then redistribute the portfolio value according to the predetermined set of weights. Let $\vec{V'_{s+1}}$ be the portfolio vector after rebalancing in year $s + 1$. $\vec{V'_{s+1}}$ contains the value of each asset class at the end of year $s + 1$. We have the following nonlinear programming optimization problem:

$$\min_{w_1, w_2, \ldots w_n} \text{CVaR}(w_1, w_2, ..., w_a) \tag{17}$$

subject to Equations 10, 11, and 14

where Equation 15 holds and

$$V_h = V_f^{1/t} - 1 \tag{18}$$

where V_f is the portfolio after the final rebalancing period. For each rebalancing period, the return of the portfolio is calculated as

$$V_{s+1} = V_s \prod_{j=q}^{p} (1 + r_j) - 1 \tag{19}$$

$$\vec{V'_{s+1}} = V_{s+1} \cdot (w_1, w_2, .., w_n) \tag{20}$$

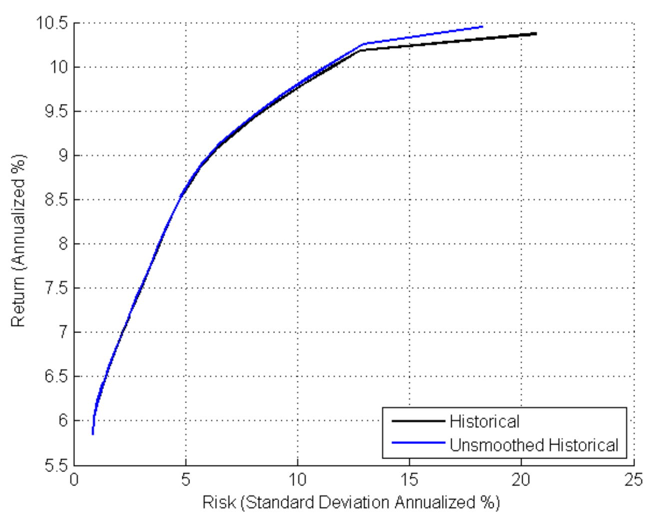
During the rebalancing period $s+1$, between time periods q and p, returns are calculated geometrically. Then, at then end of the rebalancing time period $s+1$, we redistribute the value of the portfolio according to a predetermined vector of weights to obtain a portfolio vector containing the values of each asset class.

4 Results and Discussion

4.1 Analytical Mean-Variance Efficient Frontiers

We create analytical mean-variance efficient frontiers using the average returns and observed variances of the historical data and the unsmoothed historical data as shown in Figure 1. The two efficient frontiers are almost indistinguishable and reflect the similarity of the average returns and variances.

Figure 1: Normal Mean Variance Efficient Frontiers



4.2 Single Period (buy and hold) Mean-CVaR

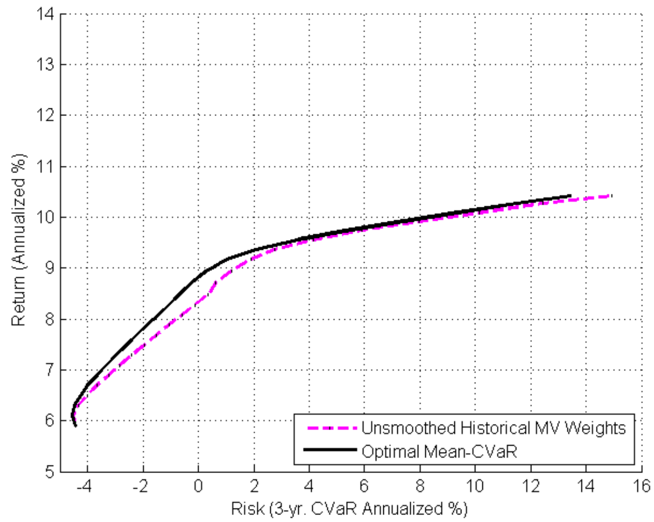
We first use the resulting weights of the analytical, normal mean-variance efficient frontiers to find their corresponding portfolio CVaR's given the simulation data. We use this as a comparison to the actual "optimal" set of portfolios in a mean-CVaR framework. The resulting efficient frontiers are shown in Figure 2.

The results indicate that there exists a set of portfolios that has a lower CVaR (or higher return) than any of the analytical mean-variance portfolios. This is intuitively captured by the fact that optimal mean-CVaR frontier lies to the upper left. The portfolios that maximize return and minimize variance in the previous optimization (Section 4.1) have a greater CVaR than portfolios that minimize CVaR. This suggests the mean-variance optimization does not protect against situations of extreme losses and should not be the only consideration in asset allocation especially if preventing extreme downside risk is a central concern.

The convergence of the efficient frontiers at the end points is due to the fact that only a portfolio that is 100% weighted in a single asset class can achieve the lowest possible return and the highest possible return. The outperformance of the optimal mean-CVaR efficient frontier in comparison to the mean-variance efficient frontier can be seen in the portfolios between the two endpoints.

Unlike the analytical mean-variance framework, the mean-CVaR framework allows for the risk value to be negative. A CVaR less than zero means that the worst 5% performance is a negative loss or a small gain. It is difficult to use an "information ratio" or a "Sharpe ratio" since the intersection of the efficient frontier and the y-axis leads to arbitrarily large ratios. It is interesting to note that for an expected loss of 0%, the optimal mean-CVaR portfolio will return almost 50 basis points more than the mean-variance portfolio.

Figure 2: Single Period Arithmetic Mean-CVaR



4.3 Rebalancing and Geometric Mean-CVaR

We demonstrate the optimal tradeoff between CVaR and return when comparing single period geometric optimization and multiple period (annual) rebalancing optimization for a three year simulation period. The resulting efficient frontiers are shown in Figures 3 & 4.

The mean-variance efficient frontier weights are not the most optimal allocations in both the rebalancing mean-CVaR framework and the geometric mean-CVaR framework. The results indicate that an optimal mean-CVaR portfolio exists in both the geometric and the rebalancing frameworks that significantly outperforms the mean-variance portfolios in the respective frameworks. Significant improvement in higher returns or in lower CVaRs can be obtained over the normal mean-variance efficient frontiers.

The achievement of higher returns and lower CVaR for the geometric portfolios in comparison to the rebalancing portfolios, especially at higher risk and return levels, must be considered in light of the use of a t-copula. This can be visualized by overlaying the geometric mean-CVaR efficient frontier with the rebalancing mean-CVaR efficient frontier as seen in Figure 5. The geometric and rebalancing portfolios at lower risk and lower CVaR are very similar but quickly diverge as the risk and CVaR increase. We believe this is because the t-copula is symmetric, so there exists a fat upper tail as well as a fat lower tail. In this case, geometric may outperform rebalancing by allowing the portfolio to become overweight in high risk-high return assets.

The use of a t-copula to generate simulations appears to play a significant role in comparing the efficient frontiers of geometric and rebalancing in a mean-CVaR framework.

Figure 3: Single Period Geometric Mean-CVaR

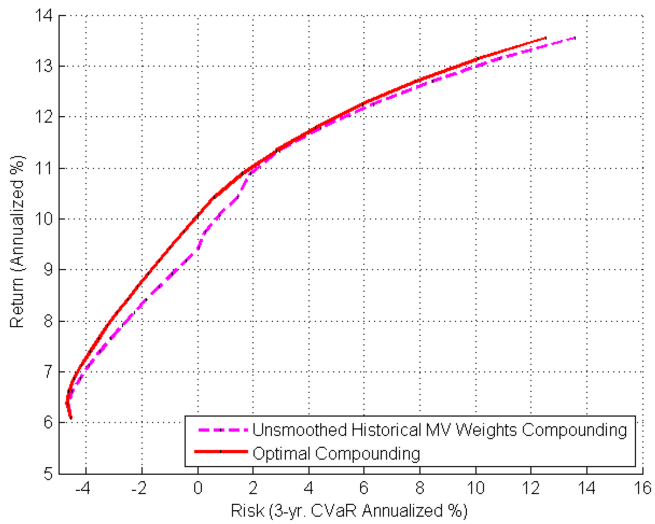


Figure 4: Multiple Period Rebalancing Mean-CVaR

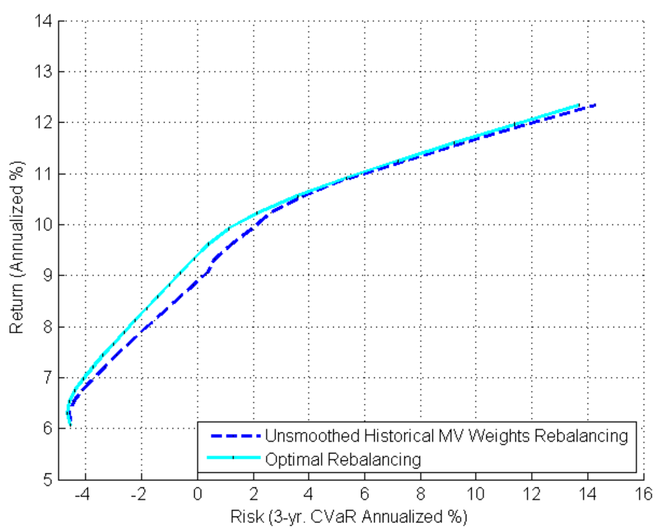
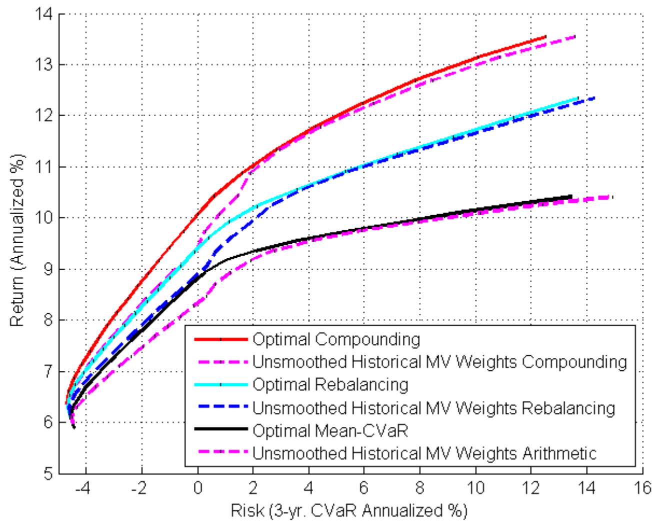


Figure 5: All Three Mean-CVaR Efficient Frontiers



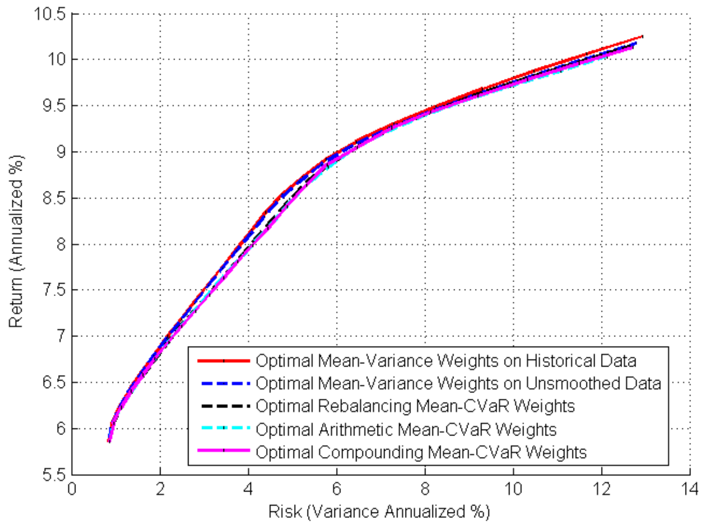
4.4 Optimal Mean-CVaR Efficient Frontiers in the Mean-Variance Framework

We use the weights from the resulting optimal efficient frontiers from the mean-CVaR framework for single period arithmetic (Section 4.1), single period geometric, and multiple period rebalancing (Section 4.3) and graph them in the mean-variance framework. We do this in order to compare how much the optimal mean-CVaR portfolios sacrifice in variance in order to achieve lower downside losses. The resulting efficient frontiers are shown in Figure 6.

Our results indicate that the optimal mean-CVaR portfolios for all three analyses have slightly higher variances than the optimal mean-variance efficient frontier ($< 0.3\%$). Although there is a slightly higher variance in the optimal mean-CVaR portfolios in comparison to the optimal mean-variance portfolios, the optimal mean-CVaR portfolios have much less downside risk as CVaR (0.3 - 1.0%).

This suggests that the cost of protecting against downside risk is only slightly higher variance in the portfolio's returns. This supports our hypothesis that the mean-CVaR framework, both for single period and rebalancing, outperforms the mean-variance portfolio and can be a powerful tool to better understand downside risk.

Figure 6: Corresponding Mean-Variance Efficient Frontiers



4.5 Comparison of the Asset Allocation Weights

The differences in the efficient frontier performance in terms of returns and CVaR stems from the different asset allocations for each portfolio. In order to visualize the differences in the allocation of capital amongst the seven asset classes, we used a 3D-ribbon graph for each of the optimal efficient frontiers (mean-variance and mean-CVaR) in Figures 7, 8, 9, & 10.

Figure 7: Mean Variance Weights

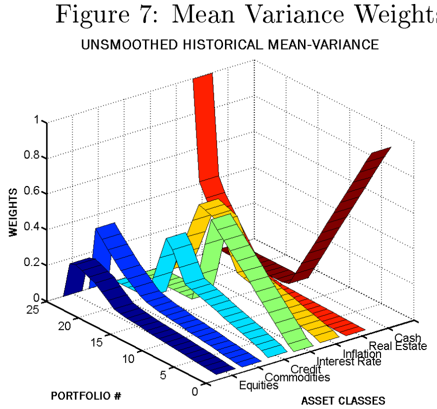


Figure 8: Single Period Arithmetic Mean-CVaR Weights

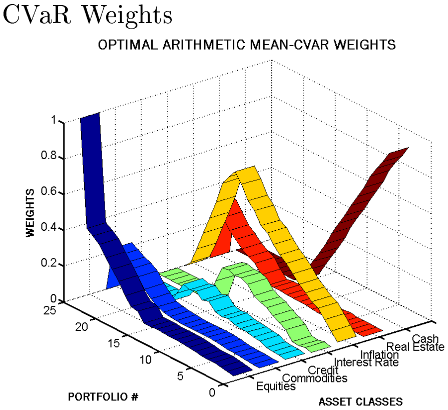


Figure 9: Single Period Geometric Mean-CVaR Weights

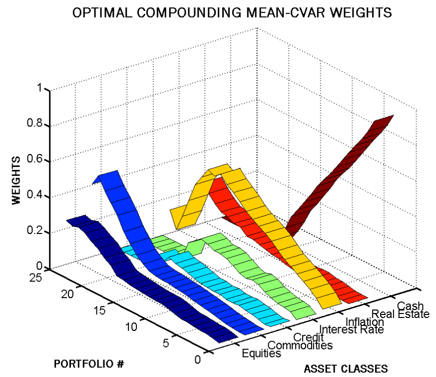
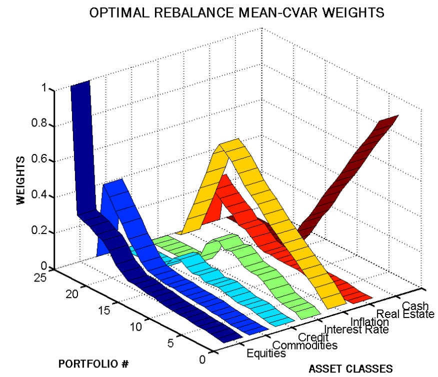


Figure 10: Multiple Period Rebalancing Mean-CVaR Weights



5 Conclusion

We have implemented a set of asset allocation models that utilize a non-normal joint probability distribution, minimize CVaR as the risk measure, and account for portfolio rebalancing. Our results support the previous literature that a mean-variance framework underestimates downside risk. We have shown that variance minimization does not lead to the minimization of downside risk and underperforms the optimal minimum CVaR efficient frontier; however, the optimal minimum CVaR efficient frontier has slightly higher variances than the optimal minimum variances efficient frontier. Our results suggest that CVaR is a viable and important risk measure for investors to consider alongside variance since it better accounts for downside risk. However, in the pursuit of outperformance, investors should carefully consider the tradeoff between higher returns and both risk factors, variance and CVaR.

We have demonstrated that the optimal set of mean-variance portfolios perform poorly when looking at downside risk using CVaR in both a single period and in multiple-period rebalancing. In the single period mean-CVaR framework for both arithmetic and geometric returns, the mean-variance portfolios are suboptimal, since a different set of portfolio weights has a higher return and/or a lower CVaR. Mean-variance especially underperforms in the cases of intermediate returns and medium levels of risk, suggesting that portfolios with low risk/returns and high risk/returns share similar CVaR and variance levels. This may be because the low risk-return and the high risk-return portfolios tend to be more heavily concentrated in one asset class. In the multiple period rebalancing mean-CVaR framework, the mean-variance portfolios again underperform in comparison to the optimal mean-CVaR portfolios. This underperformance is also most pronounced in the medium risk-return region.

Since mean-variance inherently utilizes a normal distribution, the optimal mean-variance efficient frontiers do not account for scenarios of extreme joint losses. Our use of a t-copula allowed us to properly simulate scenarios with a frequency of extreme losses similar to that observed in the historical data. The outperformance of the optimal mean-CVaR efficient frontier in comparison to the mean-variance efficient frontier is evidence that the mean-variance optimization does not account for the extreme losses. Furthermore, the variance of the optimal mean-CVaR portfolios in each of our analyses is marginally higher (< 0.3 %) than the variance of the corresponding optimal mean-variance portfolio. This is smaller than the comparable gain in lower downside risk or CVaR (0.3 - 1.0 %) of the optimal mean-CVaR efficient frontier in comparison to the optimal mean-variance efficient frontier. This is further evidence that the optimal mean-CVaR analyses for both single period and rebalancing offer pertinent insight into the risk of extreme losses in comparison to the traditional mean-variance framework.

We have produced promising results for understanding rebalancing in the mean-CVaR framework with non-normal joint distributions. Our rebalancing framework assumed no transaction costs and enforced a mandatory annual redistribution of capital to the original weights. It was surprising for us to find that the mean-CVaR single period geometric efficient frontier outperforms the corresponding rebalancing efficient frontier. This suggests that the endogenous process of a high risk, high return asset class becoming overweight in a single period portfolio contributes significantly. It may be interesting to investigate a dynamic optimization technique that includes a decision variable whether to rebalance; however, we suspect that the difficulty in predicting future random returns may hinder the value of this process. The real benefit in a dynamic optimization process would be to bring our current rebalancing model a step closer to real world situations with transaction costs.

Future research into the tradeoff between CVaR and variance may prove rewarding since both risk measures could prove relevant in different contexts. A three dimensional optimization that maximizes return while minimizing variance and minimizing CVaR will produce an efficient frontier that is a plane. This would be an interesting framework for understanding the tradeoffs between return, variance, and downside risk and merits future investigation. Also, future research implementing transaction costs and a decision variable to determine when to rebalance may prove valuable as a continuation of the multiple period rebalancing mean-CVaR framework.

The process of asset allocation is critical for understanding the overall behavior of a portfolio of risky assets. Our research has built off of the recent developments in applying fat-tail distributions, downside risk measures, and rebalancing. We have developed a simple rebalancing model that incorporates a fat-tail joint distribution in a mean-CVaR framework and have demonstrated the underperformance of the traditional mean-variance framework.

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