

Problem Sets for the video course “A Mathematics Course for Political and Social Research”

David A. Siegel*

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*Associate Professor, Department of Political Science, Duke University, Durham, NC 27708; e-mail: david.siegel@duke.edu; web: <http://people.duke.edu/~das76>.

1 Lecture 1

1. Identify whether each of the following is a constant or a variable:
 - (a) Support for policy reform.
 - (b) Date of the beginning of World War I.
 - (c) Number of wars between 1900 and 1990.
2. Identify whether each of the following is a variable or a value of a variable:
 - (a) 20-35 years old.
 - (b) Vote choice.
 - (c) Number of Trade Partners.
 - (d) Some College.
3. Identify whether each of the following indicators is measured at a nominal, ordinal, interval, or ratio level. Note also whether each is a discrete or a continuous measure:
 - (a) Education by years of education, using whole numbers for years.
 - (b) Answer to the question: As which ethnic group would you categorize yourself?
 - (c) GDP/capita.
 - (d) Answer to the question: Yes or No, did you vote in the last election?
4. Let $A = (6, 8)$ and $B = [7, 9]$.
 - (a) Is $A \subset B$, $B \subset A$, both, or neither?
 - (b) What is $A \cup B$?
 - (c) What is $A \cap B$?
 - (d) Write an element of the Cartesian product $A \times B$.
5. Simplify into one term or evaluate each of the following:
 - (a) $x \cdot x \cdot x$.
 - (b) $(-a)(-b)^3 - a^2$.
 - (c) $(3 - 5)^2 + 1 - (3 - 1)^2(3 + 2)$.
 - (d) $5!$.
 - (e) $\sqrt[4]{81}$.
 - (f) $\sum_{i=1}^4 \left(\frac{1}{2}\right)^i$.
 - (g) $\prod_{i=1}^4 \left(\frac{1}{2}\right)^i$.
 - (h) $x + 4x(y - 3) + zx$.

6. Represent the following as a ratio, a proportion, and a percentage: Democratic registered voters relative to Republican ones: Democrat 232; Independent 221; Republican 192.
7. If the number of Republican voters were to increase to 210, what would be the percentage change in the number of Republican voters?
8. Simplify this expression, assuming $x \neq -2$: $\frac{7+x^2-2x-15}{7x+14}$.
9. Add these fractions: $\frac{y-10}{5} + \frac{y+5}{4}$.
10. Factor $16\phi^2 + 32\phi - 240$.
11. Solve: $5y + 5 - 2y = 11$.
12. Solve: $x^2 - 20 = 7x - 32$.
13. Solve: $4x^2 - 10x = -5$.
14. Solve: $\theta < \frac{7\theta+1}{5}$.

2 Lecture 2

1. For each pair of ordered sets, state whether it represents a function or a correspondence:
 - (a) $\{2, 5, 1, 6\}, \{3, 1, 2, 7, 4\}$.
 - (b) $\{1, 5, 3, 10, 3, 16, 3\}, \{1, 2, 10, -7, -5, 18, 1\}$.
2. Simplify or evaluate the following expressions, using $f(x) = x^3 - 1$, $g(x) = \ln(x)$, and $h(x) = \frac{1}{x^2}$:
 - (a) $f(g(x))$.
 - (b) $g(h(x))$.
 - (c) $f(x)h(x)$.
 - (d) $g(x)h(x)$.
 - (e) $(h(x))^3$.
 - (f) $f^{-1}(x)$.
 - (g) $h^{-1}(x)$.
3. Either find the following limits or show that they do not exist.
 - (a) $\lim_{x \rightarrow 10} x - 2x + 1$.
 - (b) $\lim_{x \rightarrow \infty} \frac{1}{x^2}$.
 - (c) $\lim_{x \rightarrow 2} \frac{x+2}{x^3-1}$.

- (d) $\lim_{x \rightarrow 2} \frac{x-1}{x^3-8}$.
- (e) $\lim_{x \rightarrow -2} \frac{x+2}{x^2-4}$.
4. For each of the following sets, state whether they are (a) open, closed, both, or neither; (b) bounded; (c) compact; (d) convex:
- (a) $(0, 5]$.
- (b) $[0, 3] \cup [1, 4]$.
- (c) $(0, 3) \cup [4, 7]$.
- (d) $[1, 5] \cap (2, 10]$.
5. For each of the following functions, state whether or not it is continuous on the domain provided.
- (a) $f(x) = \frac{e^{x^2-1}}{x^2-4}$ for $x \in (-2, 2)$.

(b)

$$f(x) = \begin{cases} x - 1 & : x \leq 1, \\ x^2 + 1 & : x > 1, \end{cases}$$

3 Lecture 3

1. Use the definition of the derivative to find the derivative of y with respect to x for the following:
- (a) $y = 10$.
- (b) $y = x^2 + 2x - 1$.
- (c) $y = 2x^3 + 5x^2 - 6$.
- (d) $y = 2x^4 - 3x^2$.
- (e) $y = -5x^4 + 3x^2 + x - 10000$.
2. For each of the following, find the partial derivative with respect to both x and z .
- (a) $f(x, z) = 2z^2 + x^2$.
- (b) $f(x, z) = z^3 - 2$.
- (c) $f(x, z) = xz^2$.
- (d) $f(x, z) = x^3z^2 - x$.

4 Lecture 4

Differentiate the following:

1. $f(x) = x^{-3}$.
2. $f(x) = ax^4 + 10c$, where c is a constant.
3. $f(x) = x^{\frac{2}{3}}$.
4. $f(x) = (x - x^2)(2x^3 + 5x^2 - 7)$.
5. $f(x) = (x + 1)^5$.
6. $f(x) = \left(\frac{-2x^2 + x^3}{x^2 - x}\right)$.
7. $f(x) = (x^3 - 5)^3$.
8. $f(x) = (x^4 - x^2 + 2) \ln(x^2 - 5)$.
9. $f(x) = 3e^{5x^5 - x^2}$.
10. $f(x) = x^2 e^x \ln(x) + 3x^5$.
11. $f(x) = c^{x^3 - 1}$.
12. $f(x) = \frac{2x^3}{\log_c(x)}$.

5 Lecture 5

1. Integrate the following:

- (a) $f(x) = 5x - 3$.
- (b) $f(x) = -x + 5 - 2x^3 + x^5$.
- (c) $f(x) = 2x^{-1} - 2x^{10}$.
- (d) $f(x) = e^{\frac{x}{2}}$.
- (e) $f(x) = (5x^4)e^{x^5+2}$.
- (f) $f(x) = \ln(2x)$.

2. Evaluate the following integrals:

- (a) $\int_1^1 (x^2 - 5x + 2)dx$.
- (b) $\int_1^3 (x^2 - 5x + 2)dx + \int_3^5 (x^2 - 5x + 2)dx$.
- (c) $\int_1^5 (x^2 - 5x + 2)dx$.
- (d) $\int_1^e (\ln(x) - x^{1/2})dx$, where e is the base of the natural logarithm.
- (e) $\int (8x + 12)\sqrt{2x^2 + 6x + 1}dx$.
- (f) $\int (x^2 e^x)dx$.

6 Lecture 6

Find all extrema (local and global) of the following functions on the specified domains, and state whether each extremum is a minimum or maximum and whether each is only local or global on that domain.

1. $f(x) = x^2 + 10x - 5, x \in [-10, 10]$.
2. $f(x) = x + 2, x \in [0, 1]$.
3. $f(x) = \frac{x}{e^x}, x \in [0, \infty)$.
4. $f(x) = 2x - 2x^2 + 10, x \in [0, 10]$.
5. $f(x) = \frac{1}{3}x^3 - 4x + 5, x \in [-3, 3]$.
6. $f(x) = \frac{1}{4}x^4 + \frac{2}{3}x^3 - \frac{1}{2}x^2 - 2x + 5, x \in [-2, 2]$.

7 Lecture 7

1. Identify each of the following as a *classical* (objective), *empirical* (objective), or *subjective* probability claim:
 - (a) Incumbents in the United States Congress are very likely to be reelected.
 - (b) The chance of rolling a seven on two fair dice is one-sixth.
 - (c) The economy will very likely improve next year.
2. Identify the following as simple or compound events:
 - (a) A new government survives an investiture vote.
 - (b) Country A declares war on Country B, leading Country C to enter the conflict by virtue of its alliance with Country B.
 - (c) Several media outlets converge on reporting the same rumor.
 - (d) The latest poll reports 49% support for the president.
3. Characterize the following as independent, mutually exclusive, and/or collectively exhaustive:
 - (a) Years of formal education: 0, 1-5, 6-8, 9-12, 13-14, 15-16, 17+.
 - (b) Female, North American.
 - (c) Always, Never.
4. Let $Pr(a) = 0.45$, $Pr(b) = 0.22$, and $Pr(c) = 0.31$,
 - (a) If a, b, c , and d are mutually exclusive and collectively exhaustive events, what is $Pr(d)$?

- (b) If $a, b, c,$ and d are mutually exclusive and collectively exhaustive events, what is $Pr(a \text{ or } b)$?
- (c) If $a, b, c,$ and d are mutually exclusive and collectively exhaustive events, what is $Pr(a \text{ and } b)$?
- (d) If b and c are independent, what is $Pr(b \text{ and } c)$?
- (e) If $P(a \cap b) = 0.3$, what is $Pr(a \text{ or } b)$?
- (f) Find $Pr(e)$ if $Pr(a \cup e) = .6$ and a and e are independent events.
- (g) Find $Pr(f)$ if $Pr(a \cup f) = .8$ and a and f are independent events.
5. Compute each of the following:
- (a) $\frac{6!}{2!}$.
- (b) $\frac{10!}{8!}$.
- (c) $\frac{9!}{10!}$.
- (d) $\binom{7}{3}$.
- (e) $\binom{8}{6}$.
- (f) Same as (d), but now order matters.
- (g) Same as (e), but now order matters.
6. A committee contains ten legislators with six men and four women. Find the number of ways that a delegation of five:
- (a) Can be chosen.
- (b) With a proportional number of men and women can be chosen.
- (c) With more women than men can be chosen.
7. Assume four independent draws with replacement from a deck of cards. Assuming all cards selected were tens, what is the probability of the ten of clubs being selected exactly one time?
8. Assume a fair pair of dice are rolled. What is the probability of rolling a 2? How about rolling a 9?
9. In one legislature, 20% of the legislators are conservatives, 20% are liberals, and 60% are independents. In a recent vote to expand counterinsurgency operations, 80% of conservatives, 20% of liberals, and 50% of independents voted in favor.
- (a) What is the probability that a legislator voted in favor?
- (b) If the legislator voted in favor, what is the probability that the legislator is conservative?
- (c) Liberal?

- (d) Independent?
10. If $Pr(y) = 0.4$, what are the odds that y occurs?
 11. If the odds of x_1 are 2:1 and the odds of x_2 are 1:4, what is the odds ratio of $x_1 : x_2$?
 12. A study reports that the relative risk ratio of voting for the National Front in a French election is 2.18 for an unemployed person relative to an employed person, and 0.45 for a person with a college degree relative to someone who did not complete high school. Write a sentence that describes the impact of the value of those variables on the probability of voting for the National Front.

8 Lecture 8

1. If the mean number of bills passed in a (horribly gridlocked) legislature is 10 per year, what is the probability that there will be 15 bills in any given year?
2. An experimentalist expects that each treatment has a 20% chance of inducing the subject not to vote in a lab experiment in which voting would surely happen absent the treatment. She observes that 15 out of 25 subjects do not vote after the treatment. If her expectation were true, what would be the probability that this observation would occur?
3. Using the same scenario, if the experimenter were correct in her assessment of the probability of the treatment's behavior (i.e., a 20% chance of inducing non-voting), then what would be the probability that she would have seen 3 non-voters before the second person who voted?
4. Assume the same setting as problem 10 in Chapter 10 of the book. That is, two friends are trying to figure out which concert to go to, Bach or Stravinsky, and friend 1 has the following payoffs for each combination:

		Friend 2	
		Bach	Stravinsky
Friend 1	Bach	12	2
	Stravinsky	-2	6

Friend 1 assigns a probability .2 that Friend 2 is waiting at the Bach concert. Calculate the expected utility of going to the Bach concert and the expected utility of going to the Stravinsky concert. Which should she choose?

5. An oppressed ethnic group (A) considers rebellion against a repressive state (B). A knows B is either strong (S) or weak (W), and that it would crush any rebellion with certainty if it were strong, but would capitulate and accede to A's demands if it were weak. A has a prior belief of 20% that B is strong, and observes maneuvers (M) that a strong B would do 80% of the time but a weak B would do only 10% of the time.

If A gets 1 when the state capitulates, -1 for being crushed, and 0 for not starting a rebellion, should A start a rebellion after observing the maneuvers? (*Hint:* You'll have to use material from Lecture 7 as well.)

9 Lecture 9

1. Compute the third moment about the mean for the uniform distribution on $[0, 1]$
2. Assume a policy outcome, x , depends on a choice of policy, p , and an exogenous economic shock, ϵ , so that $x = p + \epsilon$. Further assume that ϵ is distributed uniformly on $[-1, 1]$.
 - (a) If a legislator's ideal point is $z = 0$, so that her utility is $u = -x^2$, what optimal policy should she pick?
 - (b) Does her optimal policy change if the policy's effect becomes more uncertain, so that ϵ is distributed uniformly on $[-2, 2]$?
 - (c) Does her utility increase, decrease, or stay the same in this more uncertain case?
3. A voter expects to receive a payout from a budget surplus that will be distributed uniformly on $[100, 600]$. Her utility for the amount x is $u = x$. What is her expected utility?
4. If instead the surplus payout was distributed uniformly on $[200, 500]$, how would her utility change, if at all? Explain your answer.
5. Now assume that her utility is $u = \sqrt{x}$. What is her expected utility if the payout is distributed uniformly on $[100, 600]$?
6. If instead the surplus payout was distributed uniformly on $[200, 500]$, how would her utility change, if at all? Explain your answer.

10 Lecture 10

1. Let: $\mathbf{a} = \begin{pmatrix} 4 \\ 1 \\ -5 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 5 \\ 5 \\ 2 \end{pmatrix}$, $\mathbf{c} = (1, 1, 3)$, $\mathbf{d} = (6, 4, 2)$, and $\mathbf{e} = (10, 20, 30, 40)^T$.

Calculate each of the following, indicating that it's not possible if there is a calculation you cannot perform.

- (a) $\mathbf{a} + \mathbf{b}$
- (b) $\mathbf{b} - \mathbf{e}$
- (c) $\mathbf{b} + \mathbf{c}$
- (d) $6\mathbf{a}$

- (e) $\|\mathbf{a} - \mathbf{b}\|$
- (f) $\mathbf{c} \cdot \mathbf{d}$
- (g) $\mathbf{e} \cdot (\mathbf{a} + \mathbf{b})$

2. Identify the following matrices as diagonal, identity, square, symmetric, triangular, or none of the above (note all that apply).

(a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 0 & 5 & 3 \end{bmatrix}.$

(b) $B = \begin{bmatrix} 1 & 2 & 5 & 7 \\ 2 & 3 & 6 & -3 \end{bmatrix}.$

(c) $C = \begin{bmatrix} 10 & 5 & 2 \\ 5 & -3 & 7 \\ 2 & 7 & -9 \end{bmatrix}.$

3. Write down the transpose of matrices A through C from the previous problem.
4. Given the following matrices, perform the calculations below.

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 5 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 4 \\ 2 & 8 \\ -5 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 5 & 0 \\ 3 & 6 \end{bmatrix}$$

- (a) $A + B$
 - (b) B^T .
 - (c) $A - B^T$.
 - (d) $3A$.
 - (e) AC .
 - (f) AB .
 - (g) BA
 - (h) $(AB)^T$.
 - (i) $B^T A^T$.
 - (j) $\text{Trace}(C)$.
5. Find the determinants and, if they exist, the inverses of the following matrices:

(a) $A = \begin{bmatrix} 6 & 8 \\ 2 & 7 \end{bmatrix}.$

(b) $B = \begin{bmatrix} 18 & 6 \\ 3 & 1 \end{bmatrix}.$

$$(c) C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 6 & 4 \\ 1 & 5 & 2 \end{bmatrix}.$$

$$(d) D = \begin{bmatrix} 12 & -7 & 2 \\ 3 & 6 & -5 \end{bmatrix}.$$

11 Lecture 11

1. Let: $\mathbf{a} = \begin{pmatrix} 4 \\ 1 \\ -5 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 5 \\ 5 \\ 2 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 2 \\ -4 \\ -10 \\ 1 \end{pmatrix}$, and $\mathbf{d} = \begin{pmatrix} 4 \\ 10 \\ 10 \\ 4 \end{pmatrix}$. Answer each of the

following questions, saying that it's not possible if there is a calculation you cannot perform.

- (a) Write the most general vector that is a linear combination of \mathbf{a} and \mathbf{b} .
- (b) Are all four of the vectors given in the problem linearly independent? If not, choose independent ones (this is not a unique choice) and write the others in terms of them.
- (c) What dimensional space do the four four-dimensional vectors span?

2. Solve the following systems of equations using substitution or elimination, or both:

(a)

$$\begin{aligned} x + 2y - z &= 7, \\ 5x - 4y + 3z &= 7, \\ -2x + 8y - 5z &= 9. \end{aligned}$$

(b)

$$\begin{aligned} 5x + 4y - 3z &= 4, \\ 2x + 2y - z &= 3, \\ 3x - 2y + 2z &= 5. \end{aligned}$$

(c)

$$\begin{aligned} x + 3y - z &= 9, \\ 5x - y + 3z &= 7, \\ 3x + 9y - 3z &= 27. \end{aligned}$$

(d)

$$\begin{aligned} 2x - 2y - z &= 5, \\ 4x - 4y - 2z &= 9, \\ x + 7y - z &= 7. \end{aligned}$$

3. Let $A = \begin{pmatrix} 5 & 0 \\ 3 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Calculate each of the following, indicating that it's not possible if there is a calculation you cannot perform.
- Rank of A .
 - Rank of B .
 - Solve $A\mathbf{x} = \mathbf{c}$ for \mathbf{x} using matrix inversion.
 - Solve $B\mathbf{x} = \mathbf{c}$ for \mathbf{x} using matrix inversion.
4. Use Cramer's rule to solve the following system of equations:

$$\begin{aligned} 3x - 2y + z &= 9, \\ -2x + y + z &= -5, \\ 2x + y + z &= 11. \end{aligned}$$

12 Lecture 12

- Find the eigenvalues and corresponding eigenvectors for $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Use them to diagonalize A .
- Find the eigenvalues and corresponding eigenvectors for $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.
- Consider the transition matrix $M = \begin{pmatrix} .75 & .8 \\ .25 & .2 \end{pmatrix}$. Is it ergodic? If so, find the steady-state distribution $\boldsymbol{\pi}$ over the two states. If not, explain why not and discuss the consequences for path or history dependence.
- Consider the transition matrix $M = \begin{pmatrix} 1 & .75 & .5 \\ 0 & .25 & 0 \\ 0 & 0 & .5 \end{pmatrix}$. Is it ergodic? If so, find the steady-state distribution $\boldsymbol{\pi}$ over the two states. If not, explain why not and discuss the consequences for path or history dependence.

13 Lecture 13

- Consider the model $y = f(x_1, x_2, x_3) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2^2$. Find the partial derivatives $\frac{\partial f}{\partial x_1}$ and $\frac{\partial f}{\partial x_2}$ and interpret each.
- Find all partial derivatives for the function $f(\mathbf{x}) = 3e^{x_1 x_2} - x_3 \ln x_1^3 + \frac{x_1^2 e^{x_3}}{\ln x_2}$.
- Find all partial derivatives for the function $f(\mathbf{x}) = 2x_1^7 - x_3^{x_1} + x_2 x_3$.
- Find $\int_0^2 \int_1^y x^2 y^2 dx dy$.

14 Lecture 14

1. Let $z = 3 \ln(x) + xe^y$ and $f = x^2z - y$. Find both $\frac{\partial f}{\partial x}$ and $\frac{df}{dx}$.
2. Compute the gradient of $f(x, y, z) = 8x^2e^{yz}$.
3. Compute the Hessian matrix of this same function $f(x, y, z) = 8x^2e^{yz}$. Is f concave, convex, or neither at the point $(1, 1, 1)$?
4. Compute the Jacobian matrix for the vector-valued function $\mathbf{f}(x, y, z) = \begin{pmatrix} 2xz \\ zy - 3 \\ xy^3 \end{pmatrix}$.

15 Lecture 15

1. Find all critical points and determine whether they are minima, maxima, or neither for the function $f = \frac{3}{2}x^2 - 2xy - 13x + 3y^2 + 4y - 7$.
2. Maximize $f = xy + 3y$ subject to the constraint $x + y = 5$.
3. Maximize $f = 2xy + 2y + x$ subject to the constraint $x + 2y = 2$.
4. Maximize $f = xy - x$ subject to the constraints $x \geq 0, y \geq 0, x + y \leq 3$.

16 Lecture 16

1. A candidate's choice of effort (x) depends on the expected marginal benefit for effort (b). After optimizing, the candidate finds that $x^*(b) = b^4 + b^2 - 1$. Find the marginal change in optimal effort with respect to an increase in its marginal benefit.
2. Assume the same scenario, but now the candidate has altered her expectation of effect of effort on success, changing her utility function. Maximizing her new utility function yields $x^*(b) = (\ln(bx^*(b) + 1))^2$. Find the marginal change in optimal effort with respect to its marginal benefit.
3. Assume the utility for person one is $u_1(x_1, x_2; c_1) = x_1x_2 - c_1x_1^2$ and for person two $u_2(x_1, x_2; c_2) = x_1x_2 - c_2x_2^2$. Find the comparative statics of the optimal choices of the x_i with respect to the c_i . In other words, find $J_{\mathbf{x}^*(\mathbf{c})}$.