CS230 Recitation/Assignment 08 - Probability

Recitation 04/08, Assignment Due 04/14

1 Recitation

1. (Game of coin) Alice and Bob are playing a game where they toss a fair coin until there is a sequence of three coin tosses resulting in HHH or THH. Alice wins if the first HHH appears before the first THH. Bob wins if the first THH appears before the first HHH.

   (a) What exactly makes a "fair coin"? List at least two criteria.

   (b) Fix three coin tosses in the sequence (let's say the i-th, $(i+1)$-th, and the $(i+2)$-th tosses). What's the probability that it's HHH? What's the probability that it's THH?

   (c) What's the probability that Bob wins the game? (for example if the result is HHTHTHH, Bob wins).

2. Consider still the coin problem. Now we toss the coin six times, and record the sequence. You are asked to select one (sub-)sequence from the following choices. You will win 50 dollars if the (sub-)sequence you choose appears.

   - HHTHTT
   - THTT
   - TTTTT

   Which one will you choose?

3. (Hypothetical scenario) A CS230 student either attends the CM8 recitation or not. Every student has a probability of $p = 2/3 \approx 67\%$ to attend the CM8 recitation and a probability of $q = 1 - p = 1/3 \approx 33\%$ to miss it. If a student attends the CM8 recitation, they complete the CM8 assignment with a satisfactory or above with probability $x = 80\%$. If not, the probability that they complete the assignment with a satisfactory or above is $y = 40\%$.

   (a) Given that a student has completed the CM8 assignment with a satisfactory or above, what's the probability that they attended the recitation?

   (b) Now try tweaking all numbers in the question (you can change p, q, x, and y) such that the answer to Part 3a is $3x$. Your scenario should be reasonable in the following:
   - Attending recitation never hurts, so $x \geq y$.
   - The majority of students do attend recitations, so $p \geq \frac{1}{2}$.

4. (*Hypothetical scenario*) **Assume** both the CM8/EMb (probability applications in privacy) Gradescope assignments have infinite feedback loops (i.e., everyone can submit for an unlimited number of times). **Assume** (due to potentially making mistakes) you have a probability of p_1 to get a satisfactory (or above) for each CM8 submission and a probability of p_2 to get a satisfactory (or above) for each EMb submission, and all submissions are mutually independent.[1] **Assume** you will not start working on EMb before getting a satisfactory (or above) for CM8. Also **assume** you will stop submitting one you get a satisfactory (or above) for either assignment.

   (a) For a positive integer $n_1 \geq 1$, what is the probability that it takes you exactly n_1 submissions to get a satisfactory (or above) in the CM8 Gradescope assignment? Your expression should be a one-liner function of n_1 and p_1.

   (b) Similarly, for a positive integer $n_2 \geq 1$, what is the probability that it takes you exactly n_2 submissions to get a satisfactory (or above) in the EMb assignment? Your expression should be a one-liner function of n_2 and p_2.

   (c) What is the probability that it takes you exactly n_1 submissions to get a satisfactory (or above) in the CM8 Gradescope assignment **and** exactly n_2 submissions to get a satisfactory (or above) in the EMb Gradescope assignment? Your expression should be a one-liner function of n_1, p_1, n_2, and p_2 without any summations.

   (d) For a positive integer $N \geq 2$, what is the probability that it takes you **a total of** N submissions to get a satisfactory (or above) in **both** the CM8 Gradescope assignment **and** the EMb Gradescope assignment? Your expression should be a one-liner function of n_1, p_1, n_2, and p_2 *with a summation symbol.* You may also use $q_1 = 1 - p_1$ and $q_2 = 1 - p_2$ to simplify the expression. *Stretch goal: simplify the expression into a close-form expression using p_1, p_2, and n.*

[1] This is not realistic at all but necessary for this question to be sane...

2 PrairieLearn Homework

Complete the CM8 Homework on PrairieLearn individually (There are 6 questions in total there).

3 Gradescope Assignment (Complete outside recitations)

1. (Continued from recitation). Consider again Part 3b of recitation. Can you *only* change x and y (but keep $p = 2/3$ and $q = 1/3$) and *still* make the answer to Part 3a be $3x$? If so, give the values of x and y; if not, prove why you can't. *Note that your answer still needs to be reasonable in that $x \geq y$ (attending recitation never hurts).*

2. (Pachirisu)

Pachirisus are small rodents that own $D\mu k\varepsilon$ university. A wild Pachirisu appears on your way to your C$230 exam. You decide to try catching it before the exam.

In your backpack are 10 items: 5 Regular PokéBalls, 4 Apricorn PokéBalls, and 1 Poke Bowl. You cannot afford to take your eyes off Pachirisu, so you frantically reach into your backpack, grabbing one item out at a time *independently and uniformly at random*, and immediately throw the item towards Pachirisu. You will keep repeating this process until either catching Pachirisu or running out of items.

The catch rate α (i.e., the probability one catches Pachirisu when throwing an item towards it) is defined as follows:[2]

$$\alpha = \left(\frac{3 \times \mathsf{HP}_{\mathsf{max}} - 2 \times \mathsf{HP}_{\mathsf{current}}}{3 \times \mathsf{HP}_{\mathsf{max}}}\right) \times p \times \mathsf{bonus}_{\mathsf{item}},$$

where $\mathsf{HP}_{\mathsf{max}}$ is the maximum number of hit points Pachirisu can have, $\mathsf{HP}_{\mathsf{current}}$ is the current number of hit points it has, $p \in [0, \frac{1}{2}]$ is the base catch rate, and $\mathsf{bonus}_{\mathsf{item}}$ is the bonus multiplier of the item being thrown. The Pachirisu is starved with $\mathsf{HP}_{\mathsf{current}} = 0$. In other words, in the beginning we have $\left(\frac{3\times\mathsf{HP}_{\mathsf{max}}-2\times\mathsf{HP}_{\mathsf{current}}}{3\times\mathsf{HP}_{\mathsf{max}}}\right) = 1$.

A Regular PokéBall has a bonus multiplier of 1, while an Apricorn PokéBall has a bonus multiplier of 2. On the other hand, a Poke Bowl has a bonus multiplier of 0 (it cannot be used to catch anything). What's worse, throwing the delicious Poke Bowl towards Pachirisu will raise its $\mathsf{HP}_{\mathsf{current}}$ to the full level $\mathsf{HP}_{\mathsf{max}}$ (which will make $\left(\frac{3\times\mathsf{HP}_{\mathsf{max}}-2\times\mathsf{HP}_{\mathsf{current}}}{3\times\mathsf{HP}_{\mathsf{max}}}\right) = \frac{1}{3}$.) You have no item that can be used to reduce Pachirisu's HP.

Parts 2a and 2b are ungraded thought experiments that will help you solve Part 2c. *You may skip them altogether if you are confident that you can solve Part 2c.*

(a) **(Not graded)** Fill in the following table of catch rates in all possible scenarios with functions of p.

Status / Item	Regular PokéBall	Apricorn PokéBall	Poke Bowl
Starved ($\mathsf{HP}_{\mathsf{current}} = 0$)			
Full ($\mathsf{HP}_{\mathsf{current}} = \mathsf{HP}_{\mathsf{max}}$)			

(b) **(Not graded)** Now recall there was a 3D question in the CM7 Gradescope assignment. In that question, we treated all apples (and all oranges) as identical. *Now redo that question, but this time treat all apples (and all oranges as well) as different items. In other words, all 10 fruits are distinguishable from others.*

(c) **What is the probability that you successfully catch Pachirisu if $p = \frac{1}{2}$?** Your answer can have summation/product symbols, permutation/combination symbols, fractions, exponents, and any arithmetic operators.
*Hint 1. Note that $p = \frac{1}{2}$ means any Apricorn PokéBall will immediately catch a **starved** Pachirisu.*
Hint 2. The work in Part 2b should be directly useful here.

(d) **[Bonus problem] What is the probability that you successfully catch Pachirisu for a general p?**
You may skip this part altogether. However, solving this alone to an S+ level will earn you a completion for CM8.[3]

[2] Just so you know, this is the actual catch rate formula in Gen IV, the generation that Pachirisu belongs to (with the assumption that the Pachirisu has a normal state). **Each attempt is independent to the others.**

[3] Even if you do crack this part, it is still recommended that you do all the other problems... since they do prepare you for the exam.

3. (Pachirisu, Expectation, Variance, and Probability Bounds)

You did not catch Pachirisu on your way to the C$230 exam, but at least you passed the exam. After wrapping up the exam, you found out that the LDoC class meeting is cancelled and converted into a Pachirisu-catching workshop.[4]

On LDoC, Shao-Heng, Violet, and all students in C$230 (for a total of 140 people)[5] go into $D\mu k\varepsilon$ forest, where a different wild Pachirisu appears in front of everyone (so there are 140 Pachirisus). Each Pachirisu has $\mathsf{HP}_{\mathsf{current}} = 0$.

This time, everyone is equipped with 70 Regular PokéBalls and nothing else (thankfully no one brought any Poke Bowls). Your job, as you have guessed, is to keep throwing Regular PokéBalls until either catching your Pachirisu or running out. Note that each throw's result is independent to any other throw's result. Regular PokéBalls are specific to owners and cannot be shared or reused.

For each part on this page, your answer should be a function of p, but can have summation/product symbols, permutation/combination symbols, fractions, exponents, and any arithmetic operators. Justify each answer by 1-2 sentences, so that if your expression is off by a bit but the justification shows the correct approach, we can still grant you an S.

(a) What is the probability that everyone (all 140 of us) successfully catch our Pachirisu?
(You may want to refer to your table in Question 2a.)

(b) How many Pachirisus *in expectation* will *not* be caught when the workshop concludes?

(c) What is the variance of the number of Pachirisus that are *not* caught when the workshop concludes?

(d) Using **either** *Markov's inequality/bound* **or** *Chebyshev's inequality/bound*, obtain an upper bound of the probability that *none* of the Pachirisus are caught when the workshop concludes. *Then argue why neither is a good bound.*

(e) What is the *expected number of Regular PokéBalls* each person will have when the workshop concludes? Note that one may have zero if they exhaust all their PokéBalls. *Give an estimation if you cannot obtain an exact expression.*

(f) Is it possible that when the workshop concludes, no three people have the same amount of Regular PokéBalls left? Why or why not?

[4] Note: it is not cancelled. We will do something fun on LDoC, but not catching Pachirisu.
[5] Sorry, TAs not invited.