CS230 Recitation/Assignment 07 - Combinatorics

Recitation 04/01, Assignment Due 04/07

1 Recitation

Introduction of **Notable Women in Computing Cards**.
Each group (of at most 4 people) gets one full suit (13 cards in total). Do not distribute the jokers.
All questions in today's recitation are based on the full suit (13 cards) each group gets.

1. Define the *card values* as follows: A=1, J=11, Q=12, K=13, and the numeric cards are just valued by the number.

   (a) What is the sum of your cards' values?

   (b) Prove that there exist at least 90 different (not necessarily disjoint) subsets of the cards with the same sum (define the sum of no cards to be 0) without explicitly finding these subsets.

   (c) How many *mutually disjoint* subsets of the cards can you find with the same sum, and why?

   (d) Refine the proof in Part 1b to push the constant from 90 to 91.

2. Keep the 13 cards per group and use them to answer the following questions.

   (a) How many ways can we arrange all 13 cards on a line from left to right?

   (b) How many ways can we arrange all 13 cards on a line such that J and K are together (i.e., in consecutive positions)?

   (c) How many ways can we arrange all 13 cards on a line such that the integer powers of 2 are all together (e.g., 1-4-8-2 or 2-8-1-4)?

   (d) How many ways can we arrange all 13 cards on a line such that the integer powers of 2 are all together **and** the integer powers of 3 are all together?

   (e) How many ways can we arrange all 13 cards on a line such that the card values are sorted?

3. Now turn the cards faced down so that we do not know (or care) which card is which.

   (a) How many ways can you distribute the 13 cards among the k people in your group? Note that some group member can get none.

   (b) How many ways can you distribute the 13 cards among the k people in your group such that every group member gets at least one card?

4. Now place the cards on the tables so that each card makes a separate pile. Each group will then play a *card-merging game* as a team. The game proceeds as follows:

   - In each round, pick two separate piles and merge them together. You can merge according to any criteria (e.g., by similarity of the names of the women on the cards, or their occupations, etc.).
   - In each round, if you combine a pile with x cards and a pile with y cards, you get xy points. (So the first round will always get $1 \times 1 = 1$ point.)
   - The game ends when all cards become one pile. Record the toal number of points earned.

   Try finding the best strategy to beat the other groups. Prove why your strategy is the best. The group that beats every other group gets to take away all cards.

2 PrairieLearn Homework

Complete the CM7 Homework on PrairieLearn individually (There are 5 questions in total there).

3 Gradescope Assignment (Complete outside recitations)

You may leave combinatorial notations (e.g., $\binom{n}{k}$), fractions, exponentials, and factorials in your final answers.

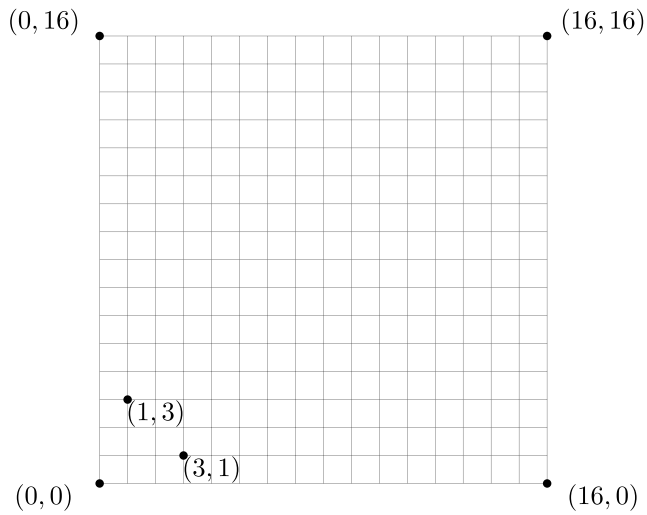
1. (The 1D question.) A caterpillar starts from the origin on a numerical line. At each time step, it either moves one unit to the right, or two units to the right. For all integers $k \geq 1$, define C_k to be the total number of possible ways for the caterpillar to move from the origin to $+k$ unit from the origin.

   (a) Express C_k as a **recursively defined sequence**. That is, define the first few terms explicitly, then express later terms using smaller terms.

   (b) Find another counting argument to show a combinatorial proof that

$$C_k = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k-i}{i}.$$

2. (The 2D question.) A grasshopper named Grace is at $(0,0)$ on a grid graph (see figure below) with $17 \times 17 = 289$ grid vertices (every point (x, y) where $0 \leq x, y \leq 16$ is a grid vertex). Grace hops between grid vertices. Each time she hops, she either (i) moves in a vertical direction for ± 1 units and move in a horizontal direction for ± 3 units, or (ii) moves in a vertical direction for ± 3 units and move in a horizontal direction for ± 1 units. Grace can never leave the grid. In other words, for her first hop from $(0,0)$, the possible positions she can reach are just $(1,3)$ and $(3,1)$.



   (a) What are the grid vertices Grace can never reach? Describe such vertices in a sentence. How many such vertices are there?

   (b) How many steps does Grace need at minimum to reach $(16,16)$, and how many distinct such routes are there?

   (c) How many steps does Grace need at minimum to reach $(16,0)$, and how many distinct such routes are there? **It is allowed to write code (on your own; do not submit code) to help answer this question. The question *can* be answered only using combinatorial techniques within the scope of this module, but coding it up and seeing the actual routes may help tremendously. If you solve this question simply by writing a program to find out all solutions exhaustively, you need to list all routes.**

   (d) How many steps does Grace need at minimum to visit all four corners of the grid counter-clockwisely? This means to first visit $(16,0)$, then visit $(16,16)$, then $(0,16)$, and end at $(0,0)$. How many distinct such routes are there? **To make this part independent, assume the answers to part 2c are p (steps) and q (distinct routes of p steps in part 2c). Express your answer to this part in p and q.**

3. (The 3D question.) **This question may feel relatively boring. It is indeed boring by itself, but it is a warmup for something fun in the next assignment.**

   (a) How many distinct ways are there to rearrange 5 identical apples, 4 identical oranges, and one banana on a line?

   (b) How many distinct ways are there such that the first 5 fruits are two apples and three oranges, and the sixth element is the banana?