CS230 Recitation/Assignment 06 - Graph Fundamentals

Recitation 03/04, Assignment Due 03/17

1 Recitation

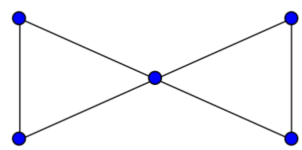

Figure 1: The butterfly graph.

1. (Continued from class meeting.) Recall that the butterfly graph with $(2 \times 2 + 1 = 5)$ vertices is 2-edge connected but not 2-vertex connected. In other words, we can remove any edge and the butterfly graph would still stay connected, but the same is not true if we remove any vertex (as removing the middle vertex would disconnect the graph).

   (a) Find a graph with $(2 \times 3 + 1) = 7$ vertices that is 3-edge connected but not 2-vertex connected.
   (What does 3-edge connected mean?)

   (b) For all $k \geq 2$, find a graph with $(2k + 1)$ vertices that is k-edge connected but not 2-vertex connected.

2. (Continued from class meeting.) Recall from the class meeting that for any graph $G = (V, E)$, $\Delta(G) \coloneqq \max_{v \in V} \deg(v)$ is the maximum vertex degree of G, and every graph G is $(\Delta(G) + 1)$-colorable.

   Find *two* different graphs that are **NOT** $\Delta(G)$-colorable. In other words, we need *exactly* $(\Delta(G)+1)$ colors to properly color such graphs. This shows the theorem is tight. *(Hint: you can find such graphs with any number of vertices.)*

3. Define $\mathcal{B}$ as the set of all undirected simple bipartite graphs and $\mathcal{C}_2$ as the set of all undirected simple graphs that are 2-colorable. Prove $\mathcal{B} = \mathcal{C}_2$. (How do we prove set equivalence again?)

4. An undirected graph is *k-regular* if all its vertices have the same degree k. Prove that every k-regular bipartite graph, as long as $k \geq 1$, has a perfect matching. (*Hint: Hall's theorem.*)

2 PrairieLearn Homework

Complete the CM6 Homework on PrairieLearn individually (There are 4 questions in total there).

3 Gradescope Assignment (Complete outside recitations)

In this assignment, you may reuse all facts/theorems learned in recitation without justification. Some subproblem(s) can be answered using 1-2 sentences if you correctly reuse recitation material.

(a) The claw graph $K_{1,3}$. (b) A claw-free graph with 5 vertices.

Figure 2

1. A simple undirected graph $G = (V, E)$ is said to be *claw-free* if the *claw graph* $K_{1,3}$ (see Fig. 2a) is not an *induced* subgraph of G. Put another way, pick *any* subset of four vertices $\{v_1, v_2, v_3, v_4\}$ of G and consider *all the edges in G among these four vertices*; the subgraph formed by the four vertices and these edges will *not* be the claw graph. For a concrete example, Fig. 2b is a 5-vertex claw-free graph ($K_{1,3}$ is a subgraph but not an *induced* subgraph of Fig. 2b).

   Find a graph G with *exactly* 6 vertices and *at least* 8 edges such that G is *connected*, *planar*, and *claw-free*. *You may directly embed a hand-drawn diagram of the planar graph you find. Please clearly mark the vertices* $\{v_1, v_2, v_3, v_4, v_5, v_6\}$, *so that the graders can effectively communicate an induced claw graph if it exists.*

2. (Continued from recitation.) Define $\mathcal{A}$ as the set of all undirected simple graphs that do not contain odd cycles. Therefore, for every $G \in \mathcal{A}$, either G is acyclic, or every cycle in G has even length. Recall from Recitation Part 3 that $\mathcal{B}$ is the set of all undirected simple bipartite graphs and $\mathcal{C}_2$ is the set of all undirected 2-colorable simple graphs.

   (a) Prove that $\mathcal{C}_2 \subseteq \mathcal{A}$.

   (b) Prove that $\mathcal{B} \subseteq \mathcal{A}$.

3. A **strongly connected component** (SCC)[1] of a directed graph $G = (V, E)$ is a **maximal** subset of vertices $S \subseteq V$ such that for each pair of distinct vertices $u, v \in S$, there exists a directed path from u to v in G.[2] Define the set of all SCCs of a directed graph G as $\mathcal{S}_G$. Assume G is non-empty (it has at least one vertex).

   (a) Prove that if S_1 and S_2 are two SCCs of G, then **either** $S_1 = S_2$ (they are the same SCC) **or** $S_1 \cap S_2 = \varnothing$ (they are disjoint sets).

   (b) Using part (a), prove that $\mathcal{S}_G$ is a partition of V.

   (c) Define $\mathbf{R} : \mathcal{S}_G \to \mathcal{S}_G$ to be the **reachable relation**: $S_1 \mathbf{R} S_2$ if and only if there exists a directed path from some vertex $v_1 \in S_1$ to some vertex $v_2 \in S_2$.[3] Prove $\mathbf{R}$ is a partial order.

   (d) Is $\mathbf{R}$ a strict partial order? Justify in one sentence.

4. A proper k-**edge** coloring of a simple undirected graph $G = (V, E)$ is a function $\mathcal{C} : E \to \{1, 2, \ldots, k\}$ (i.e., mapping **edges** to colors) such that every two edges that share an endpoint have different colors. In other words, for all $e_1 = (u, v_1)$ and $e_2 = (u, v_2)$ such that $v_1 \neq v_2$, we have $\mathcal{C}(e_1) \neq \mathcal{C}(e_2)$. Prove, by **structural** induction, that every simple undirected graph $G = (V, E)$ is $(2\Delta(G))$-edge colorable, where $\Delta(G) = \max_{v \in V} \deg(v)$ is the maximum vertex degree in G. In other words, every G has at least one proper $(2\Delta(G))$-edge coloring.[4]

   (a) State a recursive definition of $\mathcal{G}$ (the set of all simple undirected graphs), including the base case(s) and constructor case(s), that you would use in your structural induction.

   - Base case(s):
   - Constructor case(s):

   (b) Base case(s) of your induction:

   (c) Induction step of your induction:

[1] There exist many algorithms for finding all SCCs of a directed graphs; one of them is just based on DFS. DFS really has a lot of applications.
[2] This is just AIDMA Definition 10.25 in a directed graph context.
[3] Note that a directed path can "start" and "end" at the same vertex and have 0 edges.
[4] The theorem is tight when G has no edges. But otherwise, $(2\Delta(G) - 1)$ colors suffices, which can be proved similarly.