1 Recitation

1. Prove the theorem: \( \forall n \in \mathbb{N} \ [6 \mid n^3 - n] \) by (weak) induction. Assume \( 0 \in \mathbb{N} \).
   (If the theorem looks familiar, it appeared as a discussion in class back in CM2.)
   - Base case(s):
   - Inductive Hypothesis:
   - Induction Step:

2. Prove by (strong) induction that every positive integer \( n \) can be written as a sum of distinct nonnegative integer powers of 2. Hint: think about our old friend, the trick in CM2 Assignment Q1 and CM3 Assignment Q2.
   - Base case(s):
   - Inductive Hypothesis:
   - Induction Step:
The remaining parts in this week’s recitation are all about tiling checkerboards. Have fun!

(a) A $1 \times 2$ domino.  (b) A tromino.  (c) A $2^3 \times 2^3 = 8 \times 8$ checkerboard.

(d) A $2^4 \times 2^4 = 16 \times 16$ checkerboard, with a central cell occupied (indicated with a cross).

Figure 1: Examples of dominoes, trominoes, and checkerboards.

3. Prove by induction that for every $n \in \mathbb{Z}^+$, the $2^n \times 2^n$ checkerboard (note: not $n \times n$) can be fully tiled by dominoes.

   - Base case(s):
   - Inductive Hypothesis:
   - Induction Step:

4. A central cell of a checkerboard is a cell that touches the center point of the entire board (see Figure 1d for an example). For the $2^1 \times 2^1$ board, every cell is a central cell. Prove by induction that for every $n \in \mathbb{Z}^+$, the $2^n \times 2^n$ checkerboard with a central cell occupied/removed can be fully tiled by trominoes. Hint: consider strengthening the hypothesis.

   - Base case(s):
   - Inductive Hypothesis:
   - Induction Step:

5. Prove that for every $n \in \mathbb{Z}^+$, the $2^n \times 2^n$ checkerboard with any arbitrary cell removed can be fully tiled by trominoes. Explain how to modify the previous proof to easily get this.

6. Use the earlier parts, prove that for every $n \in \mathbb{Z}^+$, the $2^n \times 2^n$ checkerboard with no cells removed can be fully tiled by exactly two dominoes plus an unlimited number of trominoes.
2 PrairieLearn Homework

Complete the CM5 Homework on PrairieLearn individually (There are 3 questions in total there). Note that these proofs use the “MCS style”, i.e., they use the term “inductive case” to include the inductive hypothesis and the inductive step.

3 Gradescope Assignment (Complete outside recitations)

For all parts in this assignment, you are allowed to choose from weak/strong induction. As we saw in class, both technically work, but sometimes one is harder (to write, or to get the details correct) than the other.

1. Denote the Fibonacci numbers by \( \{f_n\}_{n \geq 0} \), where \( f_0 = 0 \), \( f_1 = 1 \), and \( f_n = f_{n-1} + f_{n-2} \) for \( n > 1 \).

(a) Prove by induction that for all nonnegative integer \( n \), 3 divides \( n \) if and only if 2 divides \( f_n \).

(b) Prove by induction that \( f_{n+m} = f_m \times f_{n+1} + f_{m-1} \times f_n \) for all \( n \geq 0 \) and \( m \geq 1 \).

Hint: decide first which variable you are inducting on. Don’t forget to indicate that in your work.

2. (Continued from Exam 1.) Recall that the recurrence relation of the algorithm for Tower of Hanoi is given by

\[
T(1) = 1; \quad T(n) = 2T(n-1) + O(1), \quad \forall n > 1.
\]

Prove by induction that \( T(n) = O(2^n) \). You may assume the \( O(1) \) term is always\(^1\) bounded above by a constant \( c \geq 1 \). Your task is essentially to prove that there exists a \( c' > 0 \) and a \( n_0 \geq 1 \) such that for all \( n \geq n_0 \), \( T(n) \leq c' \cdot 2^n \). Note that \( c \) and \( c' \) are two separate constants that may or may not equal each other.

Hint: consider strengthening the hypothesis.

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\(^1\)We covered why this is okay in class on 2/21.