# CS230 Recitation/Assignment 05 - Inductions

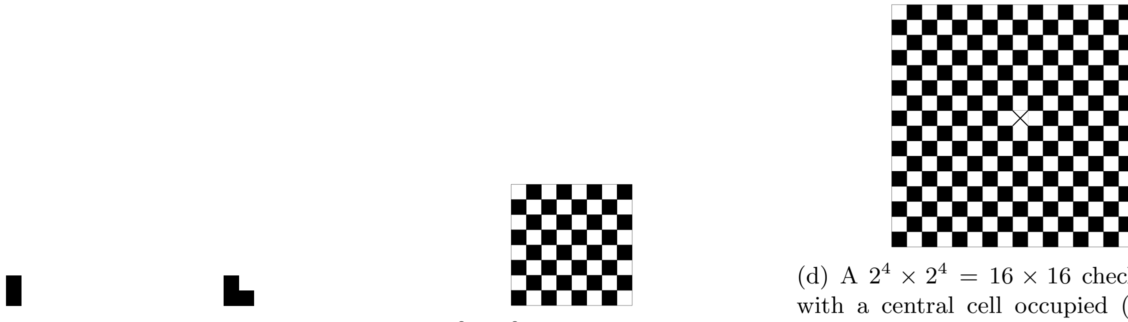
Recitation 02/26, Assignment Due 03/03

## 1 Recitation

1. Prove the theorem: $\forall n \in \mathbb{N} \, [6 \mid n^3 - n]$ by (weak) induction. Assume $0 \in \mathbb{N}$.
   (If the theorem looks familiar, it appeared as a discussion in class back in CM2.)
   - Base case(s):
   - Inductive Hypothesis:
   - Induction Step:

2. Prove by (strong) induction that every positive integer $n$ can be written as a sum of **distinct** nonnegative integer powers of 2. *Hint: think about our old friend, the trick in CM2 Assignment Q1 and CM3 Assignment Q2.*
   - Base case(s):
   - Inductive Hypothesis:
   - Induction Step:

The remaining parts in this week's recitation are all about *tiling checkerboards*. Have fun!



(a) A $1 \times 2$ domino.

(b) A tromino.

(c) A $2^3 \times 2^3 = 8 \times 8$ checkerboard.

(d) A $2^4 \times 2^4 = 16 \times 16$ checkerboard, with a central cell occupied (indicated with a cross).

Figure 1: Examples of dominoes, trominoes, and checkerboards.

3. Prove by induction that for every $n \in \mathbb{Z}^+$, the $2^n \times 2^n$ checkerboard (*note: not* $n \times n$) can be fully tiled by *dominoes*.
   - Base case(s):
   - Inductive Hypothesis:
   - Induction Step:

4. A *central cell* of a checkerboard is a cell that touches the center point of the entire board (see Figure 1d for an example). For the $2^1 \times 2^1$ board, every cell is a central cell. Prove by induction that for every $n \in \mathbb{Z}^+$, the $2^n \times 2^n$ checkerboard *with a central cell occupied/removed* can be fully tiled by *trominoes*. *Hint: consider strengthening the hypothesis.*
   - Base case(s):
   - Inductive Hypothesis:
   - Induction Step:

5. Prove that for every $n \in \mathbb{Z}^+$, the $2^n \times 2^n$ checkerboard with *any arbitrary* cell removed can be fully tiled by trominoes. Explain how to modify the previous proof to easily get this.

6. Use the earlier parts, prove that for every $n \in \mathbb{Z}^+$, the $2^n \times 2^n$ checkerboard *with no cells removed* can be fully tiled by *exactly two dominoes* plus *an unlimited number of trominoes.*

## 2 PrairieLearn Homework

Complete the CM5 Homework on PrairieLearn individually (There are 3 questions in total there). Note that these proofs use the "MCS style", i.e., they use the term "inductive case" to include the inductive hypothesis and the inductive step.

## 3 Gradescope Assignment (Complete outside recitations)

**For all parts in this assignment, you are allowed to choose from weak/strong induction. As we saw in class, both *technically* work, but sometimes one is harder (to write, or to get the details correct) than the other.**

1. Denote the Fibonacci numbers by $\{f_n\}_{n\geq 0}$, where $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n > 1$.

   (a) **Prove by induction** that for all nonnegative integer $n$, 3 divides $n$ **if and only if** 2 divides $f_n$.

   (b) **Prove by induction** that $f_{n+m} = f_m \times f_{n+1} + f_{m-1} \times f_n$ for all $n \geq 0$ and $m \geq 1$.
   *Hint: decide first which variable you are inducting on. Don't forget to indicate that in your work.*

2. (Continued from Exam 1.) Recall that the recurrence relation of the algorithm for `Tower of Hanoi` is given by

$$T(1) = 1; \qquad T(n) = 2T(n-1) + O(1), \quad \forall n > 1.$$

   **Prove by induction** that $T(n) = O(2^n)$. You may assume the $O(1)$ term is *always*[1] bounded above by a constant $c \geq 1$. Your task is essentially to prove that there exists a $c' > 0$ and a $n_0 \geq 1$ such that for all $n \geq n_0$, $T(n) \leq c' \cdot 2^n$. Note that $c$ and $c'$ are two separate constants that may or may not equal each other.
   *Hint: consider strengthening the hypothesis.*

[1] We covered why this is okay in class on 2/21.