CS230 Recitation/Assignment 04 - Sets, Functions, and Relations

Recitation 02/19, Assignment Due 02/25

1 Recitation

1. Prove the set distributive laws:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

using the *logic* distributive laws:

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$$
$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r).$$

Hint. p, q, and r can stand for anything. So this proof is all about finding the appropriate p, q, and r such that plugging them all in directly gives us the set distributive laws.

2. (Continued from class meeting) In the class meeting, we saw a surjection f from $\mathbb{N}$ to $\mathbb{Q}^+$ (the set of all *positive* rational numbers). That technically only proved $\mathbb{Q}^+$ is countably infinite. However, the goal was to prove $\mathbb{Q}$ is countably infinite. Assume $0 \in \mathbb{N}$ (although that really does not matter).

   (a) Use f to define a surjection g from $\mathbb{N}$ to $\mathbb{Q}^-$.

   (b) Use f and g to define a surjection h from $\mathbb{N}$ to $\mathbb{Q} \setminus \{0\} = \mathbb{Q}^+ \cup \mathbb{Q}^-$.

   (c) Finally, use f, g, and h to define a surjection v from $\mathbb{N}$ to $\mathbb{Q}$.

3. (Continued from class meeting) In the class meeting, we defined *strict* partial orders to be relations that are *irreflexive, anti-symmetric,* and *transitive.*

   (a) Justify in one sentence that we could have just defined them as relations that are *asymmetric* and *transitive.* (Note: this should really just be one sentence, and we would not even bother write a proof.)

   (b) Prove that we could have also just defined them as relations that are *irreflexive* and *transitive.* (Note: this is actually how MCS first defines it.)

4. (Partitions, cardinalities, and power sets) Consider the set $A = \{d, \mu, k, \varepsilon\}$ and its power set $P(A)$. Define the relation $\mathbf{R}$ on $P(A)$ such that for two subsets of A, X and Y, $X\mathbf{R}Y$ if and only if $|X| = |Y|$. *Note: if you missed the recitation, no need to submit your work for this question, since it is subsumed by the last question in the assignment part.*

   (a) Write out $P(A)$. (Tedious practice, but you will need them for later steps.)

   (b) Prove $\mathbf{R}$ is an equivalence relation. (There are 3 things to prove.)

   (c) Find all equivalence classes of $\mathbf{R}$. (There are 5 of them.)

   (d) Verify that the set of all equivalence classes is a partition of $P(A)$. (There are 2 things to verify.)

2 PrairieLearn Homework

Complete the CM4 Homework on PrairieLearn individually (There are 4 questions in total there).

3 Gradescope Assignment (Complete outside recitations)

1. (Proving properties of composite functions) Consider arbitrary functions $f : B \to C$ and $g : A \to B$ where A, B, C are arbitrary sets and not necessarily finite. Here are 18 syllogisms:

$$(1)\ \begin{array}{l} f \text{ is injective} \\ g \text{ is injective} \\ \hline \therefore f \circ g \text{ is injective} \end{array} \qquad (2)\ \begin{array}{l} f \text{ is injective} \\ g \text{ is injective} \\ \hline \therefore f \circ g \text{ is surjective} \end{array} \qquad (3)\ \begin{array}{l} f \text{ is injective} \\ g \text{ is surjective} \\ \hline \therefore f \circ g \text{ is injective} \end{array}$$

$$(4)\ \begin{array}{l} f \text{ is injective} \\ g \text{ is surjective} \\ \hline \therefore f \circ g \text{ is surjective} \end{array} \qquad (5)\ \begin{array}{l} f \text{ is injective} \\ g \text{ is bijective} \\ \hline \therefore f \circ g \text{ is injective} \end{array} \qquad (6)\ \begin{array}{l} f \text{ is injective} \\ g \text{ is bijective} \\ \hline \therefore f \circ g \text{ is surjective} \end{array}$$

$$(7)\ \begin{array}{l} f \text{ is surjective} \\ g \text{ is injective} \\ \hline \therefore f \circ g \text{ is injective} \end{array} \qquad (8)\ \begin{array}{l} f \text{ is surjective} \\ g \text{ is injective} \\ \hline \therefore f \circ g \text{ is surjective} \end{array} \qquad (9)\ \begin{array}{l} f \text{ is surjective} \\ g \text{ is surjective} \\ \hline \therefore f \circ g \text{ is injective} \end{array}$$

$$(10)\ \begin{array}{l} f \text{ is surjective} \\ g \text{ is surjective} \\ \hline \therefore f \circ g \text{ is surjective} \end{array} \qquad (11)\ \begin{array}{l} f \text{ is surjective} \\ g \text{ is bijective} \\ \hline \therefore f \circ g \text{ is injective} \end{array} \qquad (12)\ \begin{array}{l} f \text{ is surjective} \\ g \text{ is bijective} \\ \hline \therefore f \circ g \text{ is surjective} \end{array}$$

$$(13)\ \begin{array}{l} f \text{ is bijective} \\ g \text{ is injective} \\ \hline \therefore f \circ g \text{ is injective} \end{array} \qquad (14)\ \begin{array}{l} f \text{ is bijective} \\ g \text{ is injective} \\ \hline \therefore f \circ g \text{ is surjective} \end{array} \qquad (15)\ \begin{array}{l} f \text{ is bijective} \\ g \text{ is surjective} \\ \hline \therefore f \circ g \text{ is injective} \end{array}$$

$$(16)\ \begin{array}{l} f \text{ is bijective} \\ g \text{ is surjective} \\ \hline \therefore f \circ g \text{ is surjective} \end{array} \qquad (17)\ \begin{array}{l} f \text{ is bijective} \\ g \text{ is bijective} \\ \hline \therefore f \circ g \text{ is injective} \end{array} \qquad (18)\ \begin{array}{l} f \text{ is bijective} \\ g \text{ is bijective} \\ \hline \therefore f \circ g \text{ is surjective} \end{array}$$

   (a) Determine **which 8 of the 18 syllogisms** are valid. No need to write any proofs.
   (Hint. Example 4.123 in AIDMA directly gives you syllogism (1) is valid. Now try extrapolating to others.)

   (b) Identify one syllogism among the nonvalid ones, such that any counterexample for this syllogism must also be a counterexample for at least two other nonvalid ones. **Clearly indicate which one implies which other two.**

   (c) Find a counterexample for the set of three nonvalid syllogisms above. Your domains should be either $\{1\}$ or $\{1, 2\}$; in other words, each of A, B, and C can be either $\{1\}$ or $\{1, 2\}$, you decide which is which. Afterwards, define your own $f : B \to C$ and $g : A \to B$. Remember they both are functions, so f needs to map each element in B to an element in C, and g needs to map each element in A to an element in B. No justification needed.

2. (a) Design a binary relation (from the set of real numbers $\mathbb{R}$ to itself) that is *irreflexive*, *transitive*, and *symmetric*. No justifications or elaborations needed; the definition of the relation is sufficient.

   (b) Prove there is no other solution to part (2a) than the one you found.

3. (Continued from recitation) Recitation Part (4) operates on the set $A = \{d, \mu, k, \varepsilon\}$, so we were able to directly write out its power set $P(A)$ explicitly. Now, prove the conclusions in Parts 4b and 4d hold for any arbitrary **finite** set A. We no longer know what A is (except the fact that it is finite), so your reasoning needs to be general enough. The relation $\mathbf{R}$ is still defined from $P(A)$ to $P(A)$ like how it was defined in Recitation Part (4).