Class starts after this song

Dominic Fike, Weezer – Think Fast (2023)
requested by Nicole Errera (TA-of-CM8)

I'm a third-year majoring in CS with a concentration in AI/ML. I'm from Sacramento, CA and I love being outside.

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UTA applications are open!

- Apply at https://cs.duke.edu/undergraduate/uta

- Apply by 4/22 (soft deadline)
- Can apply to multiple classes; can even work for multiple
- UTAs are hired to help people, not ace the material

CS230 Spring 2024
Module 08: Probability

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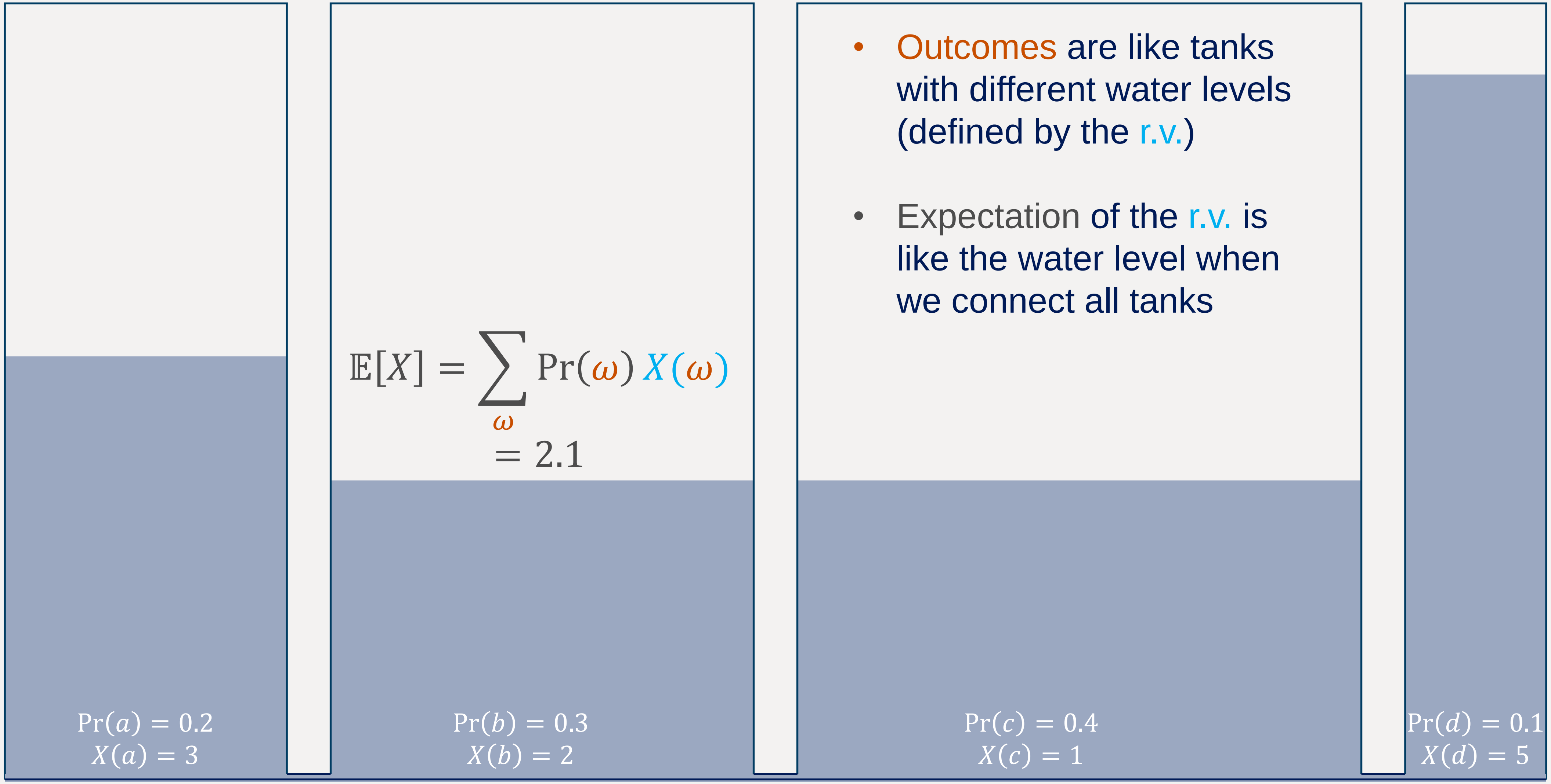
Expectation

- Nothing fancier than a weighted average
- May not “fall on” any existing outcome
  - There are 8 CMs in CS230
  - 3 of them have 3 class meetings; the other 5 have 2 class meetings
  - Weighted average $\frac{19}{8} = 2.375$ class meetings
  - No modules can have 2.375 class meetings and that is okay

- Outcomes are like tanks with different water levels (defined by the r.v.)
- Expectation of the r.v. is like the water level when we connect all tanks

$$\mathbb{E}[X] = \sum_{\omega} \Pr(\omega)\, X(\omega)$$
$$= 2.1$$



WOTO: expectation practice with familiar items

- Consider the Notable Women in Computing cards in CM7 recitation, where the card values are defined as $A = 1, J = 11, Q = 12, K = 13$ for letter cards (assume no jokers)
  - If we draw a card uniformly at random, what is the expected value of the card we draw?
  - If we draw a card such that somehow letter cards (A,J,Q,K) are twice likely to be drawn (as opposed to regular number cards), what is the expected value of the card we draw?

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WOTO: expectation practice with familiar items

- If we draw a card uniformly at random, what is the expected value of the card we draw? 7
- If we draw a card such that somehow letter cards (A,J,Q,K) are twice likely to be drawn (as opposed to regular number cards), what is the expected value of the card we draw?
  - $\Pr(A) = \Pr(J) = \Pr(Q) = \Pr(K) = \frac{2}{17}, \Pr(2) = \cdots = \Pr(10) = \frac{1}{17}$
  - $(1 + 11 + 12 + 13) \times \frac{2}{17} + (2 + \cdots + 10) \times \frac{1}{17} = \frac{128}{17} \approx 7.53 > 7$

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Peer discussion: Expectation

Recall Grace the grasshopper from the CM7 Gradescope assignment. (Try reading this sentence real quick.)

Anyway, as a reminder, at each step, here are the four options Grace can hop:

- +1 unit vertically and +3 units horizontally
- +1 unit vertically and -3 units horizontally
- -1 unit vertically and +3 units horizontally
- -1 unit vertically and -3 units horizontally

Now assume Grace takes each option with equal probability 1/4. **What is Grace's expected movement each step?**

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Peer discussion: Expectation

- In expectation, Grace does not move each step
- In expectation, Grace moves for a distance of $\sqrt{10}$ each step
- Both are correct! The problem lies in we did not define the underlying random variable in an unambiguous way.
- Expectations are defined on random variables. No point in talking about expectations without a concrete random variable.

WOTO: linearity of expectation

- Recall from the recitation the Game of Coin: Alice and Bob are playing a game where they toss a fair coin until there is a sequence of three coin tosses resulting in HHH or THH.
  - Alice wins if the first HHH appears before the first THH.
  - Bob wins if the first THH appears before the first HHH.
- Question: in 100 tosses, how many times in expectation does HHH and THH happen?

WOTO: linearity of expectation

- Question: in 100 tosses, how many times in expectation does HHH and THH happen?
  - Define A_i to be the indicator variable that tosses $i, i+1, i+2$ are HHH
  - We are interested in $\mathbb{E}[\sum_{i=1}^{98} A_i]$
  - By linearity of expectation:

$$\mathbb{E}\left[\sum_{i=1}^{98} A_i\right] = \sum_{i=1}^{98} \mathbb{E}[A_i] = \sum_{i=1}^{98} \Pr(A_i = 1) = \sum_{i=1}^{98} \frac{1}{8} = \frac{98}{8}$$

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WOTO: linearity of expectation

- Question: in 100 tosses, how many times in expectation does HHH and THH happen?
  - Define B_i to be the indicator variable that tosses $i, i+1, i+2$ are THH
  - We are interested in $\mathbb{E}[\sum_{i=1}^{98} B_i]$
  - By linearity of expectation:

$$\mathbb{E}\left[\sum_{i=1}^{98} B_i\right] = \sum_{i=1}^{98} \mathbb{E}[B_i] = \sum_{i=1}^{98} \Pr(B_i = 1) = \sum_{i=1}^{98} \frac{1}{8} = \frac{98}{8}$$

WOTO: linearity of expectation

- Question: in 100 tosses, how many times in expectation does HHH and THH happen?
  - They both happen $\frac{98}{8}$ times in expectation despite THH having an 87.5% chance to appear first!
  - Also note that the indicator variables are not independent.
    But linearity of expectation always holds regardless of independence.

- NOT "at the same scale of X"
- many good properties!
- better for calculation

- "at the same scale of X"
- not many good properties
- better for interpretation

Variance and Standard Deviation

- $\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$
- $\text{Var}(X) = \mathbb{E}[(X - \mu)^2] = \mathbb{E}[X^2] - \mu^2$
- $\sigma_X = \text{std}(X) = \sqrt{\text{Var}(X)}$

- Convention: $\mathbb{E}[X] = \mu$ (meaningful; this conveys μ is a constant and consists of no randomness)

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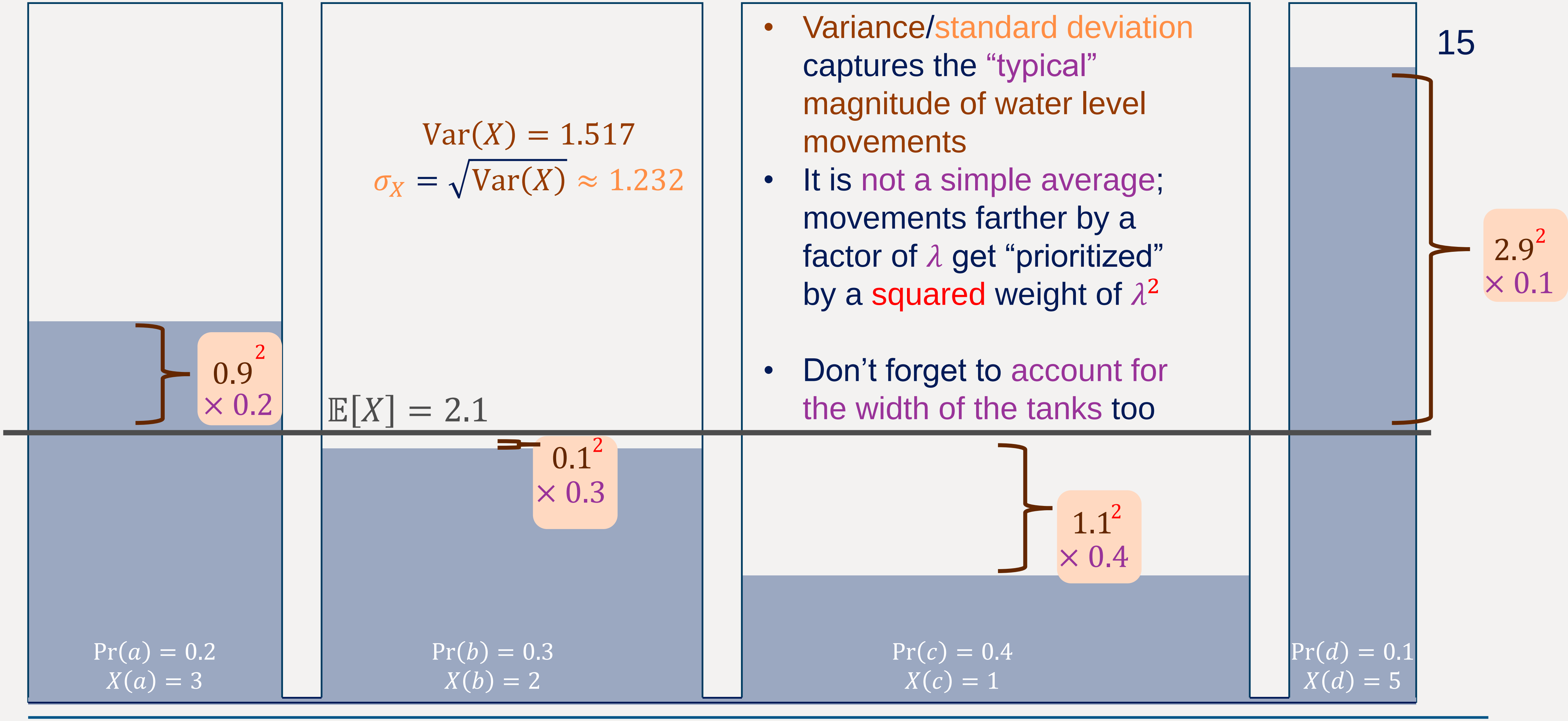
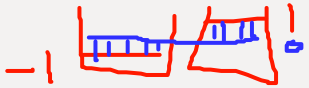
- Variance/standard deviation captures the "typical" magnitude of water level movements
- It is not a simple average; movements farther by a factor of λ get "prioritized" by a squared weight of λ^2
- Don't forget to account for the width of the tanks too

$$\mathrm{Var}(X) = 1.517$$
$$\sigma_X = \sqrt{\mathrm{Var}(X)} \approx 1.232$$





WOTO: Variance and STD Examples

Random variable	X_1	X_2	X_3	$X_4 = X_2 + X_3$
Distribution	$\Pr[X_1 = 0] = 1$	$\Pr[X_2 = -1] = ½$ $\Pr[X_2 = +1] = ½$	$\Pr[X_3 = -10] = ½$ $\Pr[X_3 = +10] = ½$	$\Pr[X_4 = -11] = 1/4$ $\Pr[X_4 = -9] = 1/4$ $\Pr[X_4 = +9] = 1/4$ $\Pr[X_4 = +11] = 1/4$
μ_i	0	0	0	0
$\mathrm{Var}(X_i)$	0	1	100	101
σ_i	0	1	10	$\sqrt{101}$

NOT “at the same scale of X”

“at the same scale of X”

WOTO: Variance and STD Examples

Random variable	X_1	X_2	X_3	X_4
Distribution	$\Pr[X_1 = 0] = 1$	$\Pr[X_2 = -1] = ½$ $\Pr[X_2 = +1] = ½$	$\Pr[X_3 = -10] = ½$ $\Pr[X_3 = +10] = ½$	$\Pr[X_4 = -11] = 1/4$ $\Pr[X_4 = -9] = 1/4$ $\Pr[X_4 = +9] = 1/4$ $\Pr[X_4 = +11] = 1/4$
μ_i	0	0	0	0
$\mathrm{Var}(X_i)$	0	1	100	101
σ_i	0	1	10	$\sqrt{101}$

$\mathrm{Var}(\lambda X) = \lambda^2 \mathrm{Var}(X)$

$\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

Warning: only if independent!

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Expectation, Variance and STD Cheat Sheet

Random variable	Bernoulli(p)	Binomial(n, p)	Geometric(p)
μ	p	np	$1/p$
$\mathrm{Var}(X)$	$p(1-p)$	$np(1-p)$	$\frac{1-p}{p^2}$
σ	$\sqrt{p(1-p)}$	$\sqrt{np(1-p)}$	$\frac{\sqrt{(1-p)}}{p}$

$$\mathbb{E}[X + \cdots + X] = \mathbb{E}[X] + \cdots + \mathbb{E}[X]$$

$$\mathrm{Var}(X + \cdots + X) = \mathrm{Var}(X) + \cdots + \mathrm{Var}(X)$$

Warning: here we know they are independent!

- Really no need to memorize; you can always derive yourself
- Memorizing them does help; just like Java STL

Markov's Inequality/Bound

$$\Pr(X \geq c\mathbb{E}[X]) \leq \frac{1}{c}$$

$$\Pr(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$$

- Intuition:
  given the expectation (connected water level), each tank cannot be too high and too wide at the same time
- Otherwise water in that tank alone is too much
- Therefore, given the height, we have an upper bound for the width
- The equation holds only when all other tanks are empty: $\Pr(X < a) = 0$
- Warning: ***X* needs to be nonnegative** (otherwise the tank analogy does not work)

$c\mathbb{E}[X]$

$\leq \frac{1}{c}$

Chebyshev's Inequality/Bound

$$\Pr(|X - \mathbb{E}[X]| \geq c\sigma_x) \leq \frac{1}{c^2}$$

$$\Pr(|X - \mathbb{E}[X]| \geq \alpha) \leq \frac{\mathrm{Var}(X)}{\alpha^2}$$

$$\Pr((X - \mathbb{E}[X])^2 \geq \beta) \leq \frac{\mathrm{Var}(X)}{\beta}$$

- Intuition: given the variance (water level movements), a tank cannot be too high/low *(compared to the expectation)* and too wide at the same time
- Otherwise water in that tank alone **moves** too much
- Therefore, given the height *(compare to the expectation)*, we again have an upper bound for the width

- Many different variations – they all mean the same thing
- ***X* does not need to be nonnegative**

Proving Chebyshev's Inequality/Bound Using Markov's Inequality/Bound

$\Pr(|X - \mathbb{E}[X]| \geq c\sigma_x) \leq \frac{1}{c^2}$

$=$

$\Pr(|X - \mathbb{E}[X]| \geq \alpha) \leq \frac{\mathrm{Var}(X)}{\alpha^2}$

$=$

$\Pr((X - \mathbb{E}[X])^2 \geq \beta) \leq \frac{\mathrm{Var}(X)}{\beta}$ $=$ $\Pr(Y \geq \beta) \leq \frac{\mathbb{E}[Y]}{\beta}$

Chebyshev's Inequality/Bound — Markov's Inequality/Bound

- Intuition: let's now define the squared water level movement $(X - \mathbb{E}[X])^2$ as a new random variable Y
- What is $\mathbb{E}[Y]$?
- It's just $\mathrm{Var}(X)$
- Apply Markov's Inequality/Bound on Y

PI: Markov's and Chebyshev's Inequalities/Bounds

$$\Pr(X \geq c\mathbb{E}[X]) \leq \frac{1}{c}$$

$$=$$

$$\Pr(X \geq a) \leq \frac{\mathbb{E}[X]}{a}$$

$$\Pr(|X - \mathbb{E}[X]| \geq c\sigma_x) \leq \frac{1}{c^2}$$

$$=$$

$$\Pr(|X - \mathbb{E}[X]| \geq \alpha) \leq \frac{\mathrm{Var}(X)}{\alpha^2}$$

$$=$$

$$\Pr((X - \mathbb{E}[X])^2 \geq \beta) \leq \frac{\mathrm{Var}(X)}{\beta}$$