CS230 Spring 2024
Module 07: Combinatorics

Duke

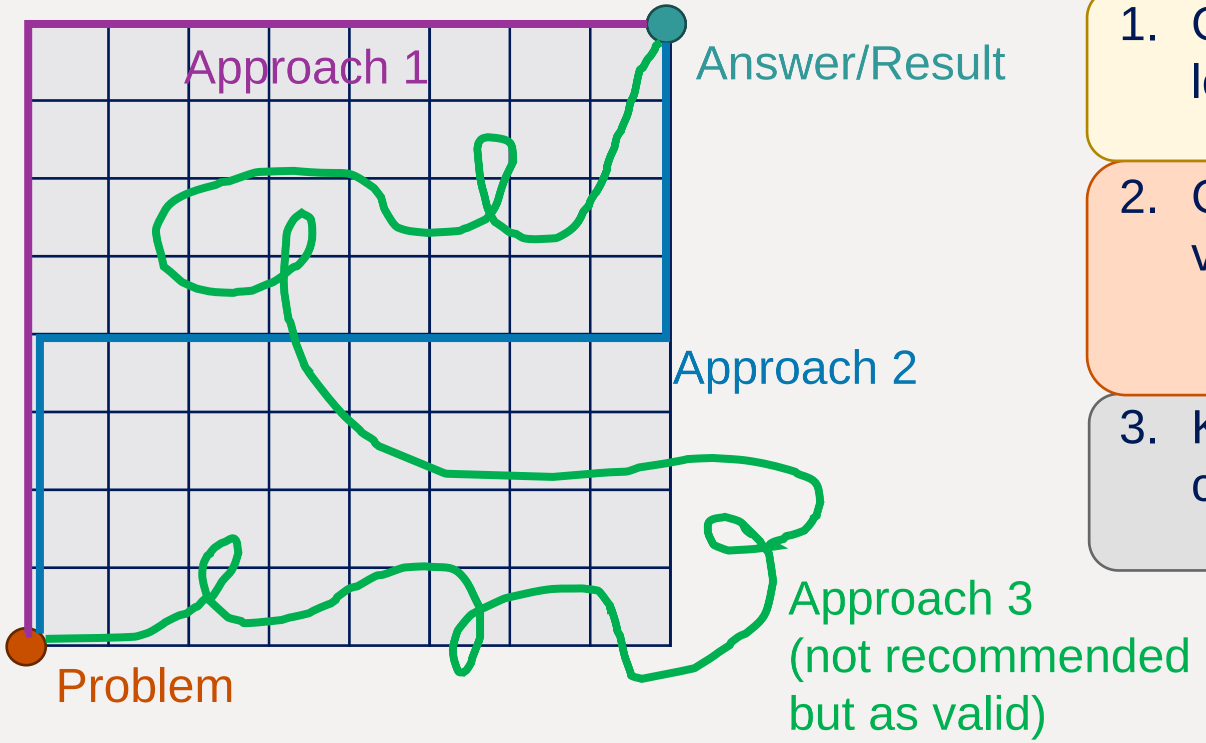
Why combinatorics?

- Before making decisions, know our options
  - Is it reasonable for us to brute-force check all possibilities?
- Reason under limited information
  - “Don’t know how people will self-enroll, but every recitation section will have at least 20 students”
- Provide foundations for discrete probability (CM8)
  - especially useful in uniformly-at-random probability experiments
  - randomized algorithms, hashing (analysis), differential privacy, etc.

Approach

- Focus on the process, not the results
  - Always true for this class
- Focus on one process/approach that makes most sense to you
  - Especially true for CM7: combinatorics and CM8: probability
  - There are usually more than one approaches to solve a problem
  - Different people find different approaches “more natural”
  - All “correct approaches” should lead to the same result(s)

Approach



Learning goals (level of mastery):

1. Given Problem, can find an Approach that leads to Answer/Result
   - Better if elegant, okay if not

S

2. Given someone else's Approach, can verify/understand if it leads to Answer/Result
   - Alternatively, can convince others that your Approach is correct

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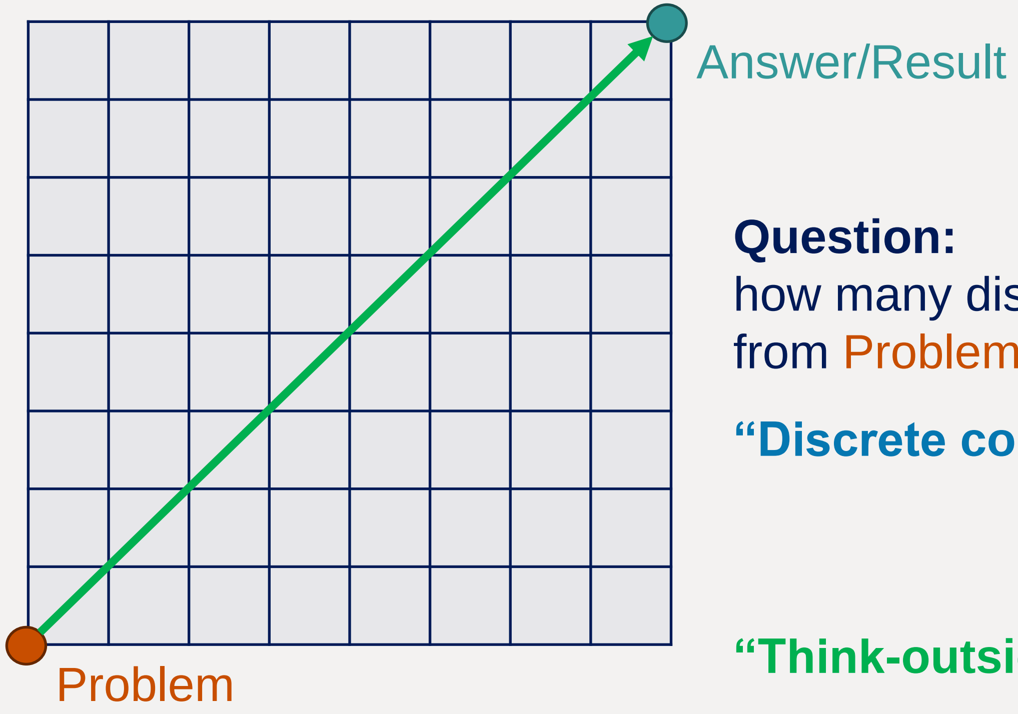
3. Know why they all work, can compare-and-contrast, can guide others to one
   - But not judge

TA

- When in doubt, either ask or state necessary assumptions

Be vigilant of hidden assumptions



Question:
how many distinct shortest paths are there from Problem to Answer/Result?

“Discrete combinatorics answer”:

$$\binom{16}{8}$$

“Think-outside-the-grids” answer:

$$1$$

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PI: Pigeonhole Advanced

- Just because only one of the four choices is the theorem statement does not imply others are wrong

1 — 1 point

We have 128 students and 5 recitation sections in CS230.
As is the case in reality, every student has to be assigned to exactly one recitation section.
Which claim(s) below are always true no matter how we assign the students to the recitation sections? (Select all that apply)

- [x] There will be a recitation section with **at most** $\lceil 128/5 \rceil = 26$ students.
- [x] There will be a recitation section with **at least** $\lceil 128/5 \rceil = 26$ students.
- [x] There will be a recitation section with **at most** $\lfloor 128/5 \rfloor = 25$ students.
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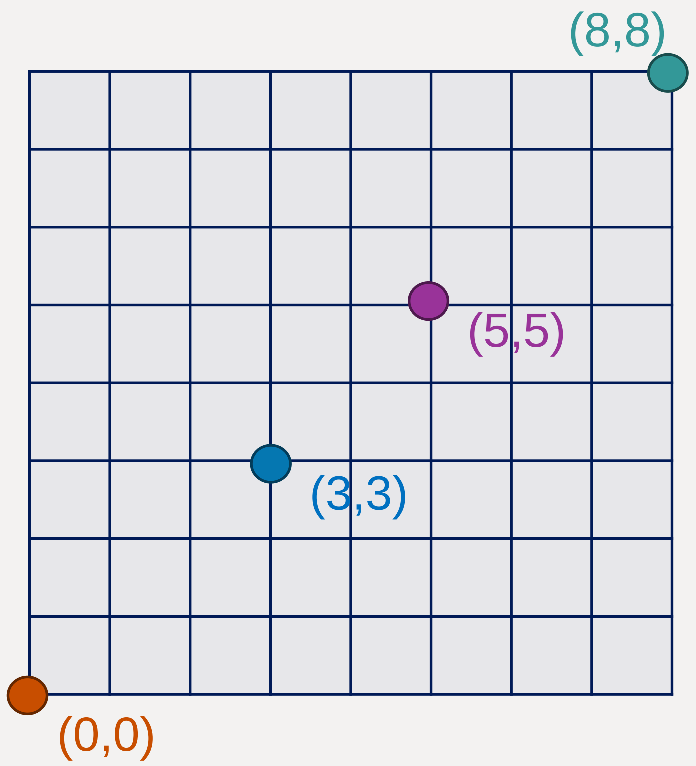
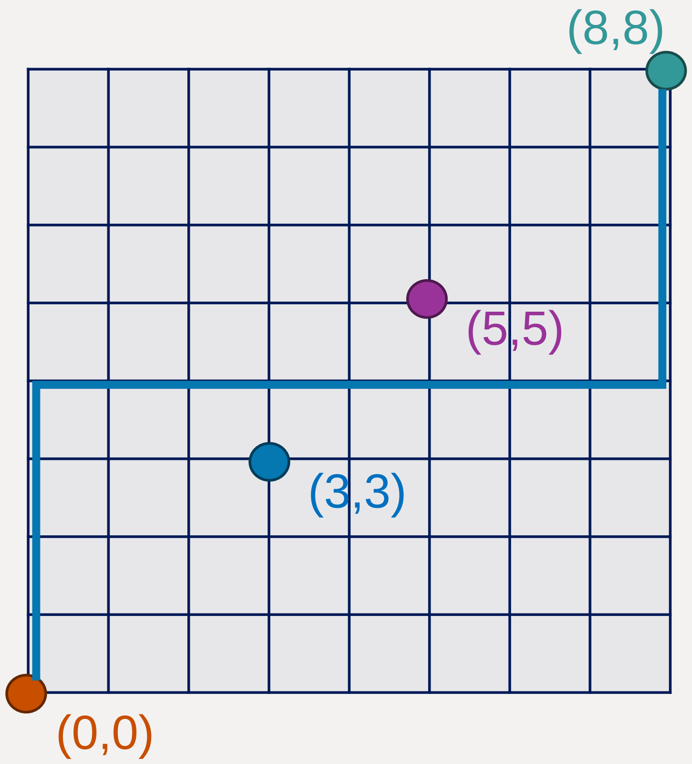
Shortest Paths WOTO



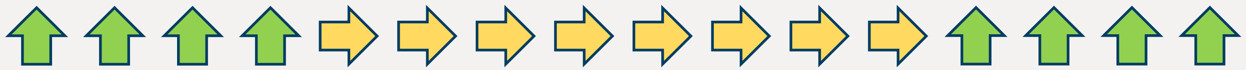
- Suppose we can only move either vertically or horizontally (so “think-outside-the-grids” paths are no longer possible.)
- How many shortest paths from (0,0) to (8,8) are there?
  - You are spoiled that it is $\binom{16}{8}$. So why is it $\binom{16}{8}$?
- How many shortest paths from (0,0) to (8,8) that pass through (3,3) are there?
- How many shortest paths from (0,0) to (8,8) that pass through (5,5) are there?
- How many shortest paths from (0,0) to (8,8) that pass through both (3,3) and (5,5) are there?

Shortest Paths WOTO

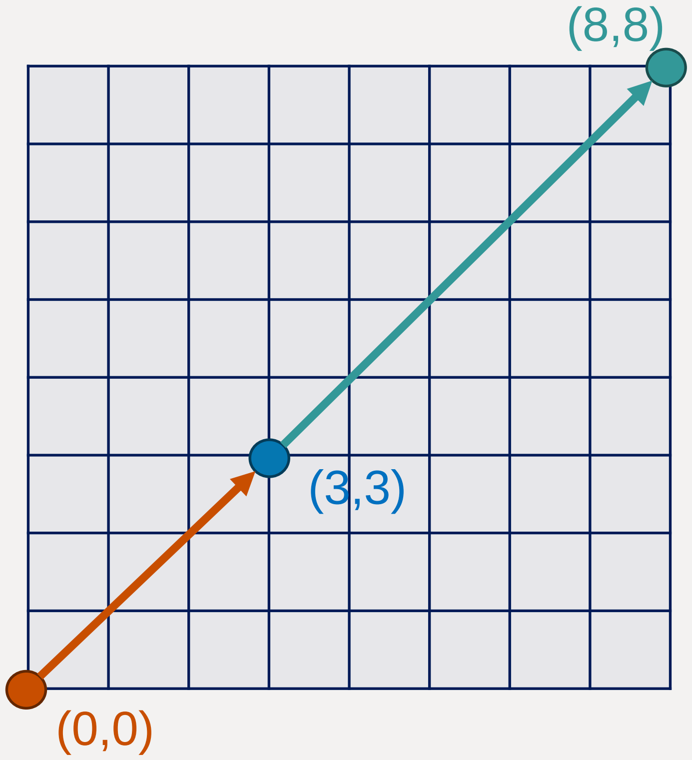


- How many shortest paths from (0,0) to (8,8) are there?
  - You are spoiled that it is $\binom{16}{8}$. So why is it $\binom{16}{8}$?

- Note that any shortest path consists of 8 “right-steps” and 8 “up-steps”
- This is equivalent to rearranging 8 “right-steps” and 8 “up-steps” on a line.



Shortest Paths WOTO

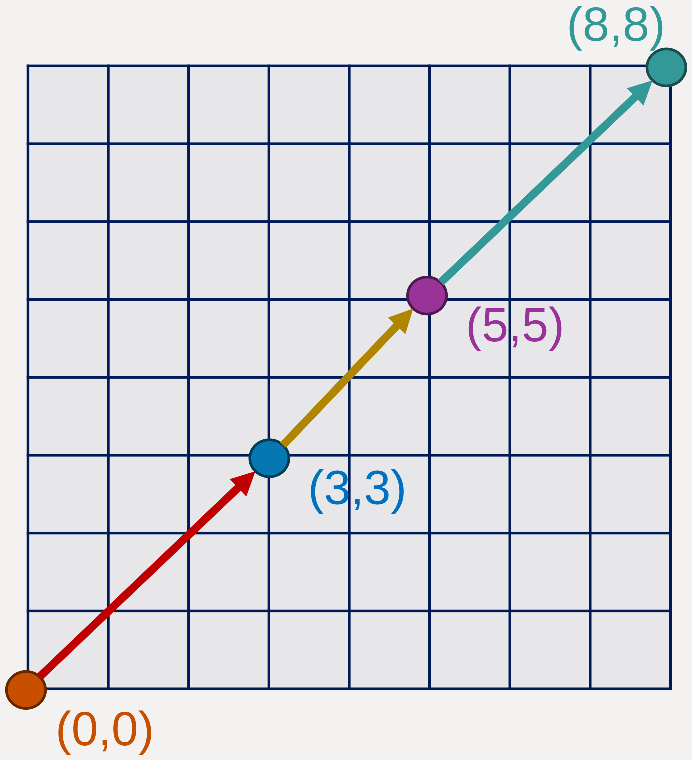


- How many shortest paths from (0,0) to (8,8) that pass through (3,3) are there?
- Any such shortest path is a shortest path from (0,0) to (3,3) concatenated with a shortest path from (3,3) to (8,8)
  - We proved this in the graph theory PrairieLearn HW!
- Shortest paths from (0,0) to (3,3):  $\binom{6}{3}$
- Shortest paths from (3,3) to (8,8):

 $\binom{10}{5}$

- The two parts are independent, so we invoke Product Rule: $\binom{6}{3} \times \binom{10}{5}$

Shortest Paths WOTO

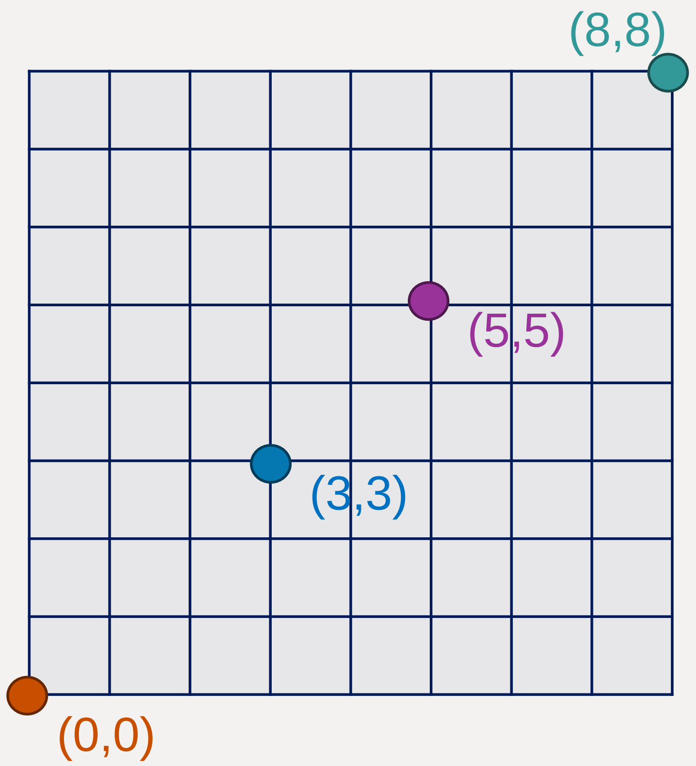


- How many shortest paths from (0,0) to (8,8) that pass through (5,5) are there?
- Symmetric to the previous problem: $\binom{10}{5} \times \binom{6}{3}$

- How many shortest paths from (0,0) to (8,8) that pass through both (3,3) and (5,5) are there?
- Now 3 shortest paths concatenated together:

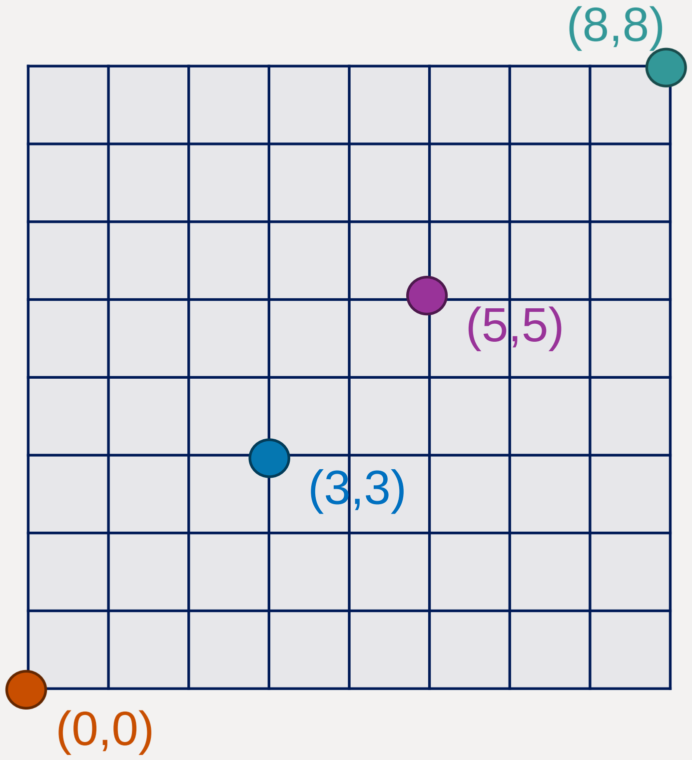
$$\binom{6}{3} \times \binom{4}{2} \times \binom{6}{3}$$

Shortest Paths WOTO(2)

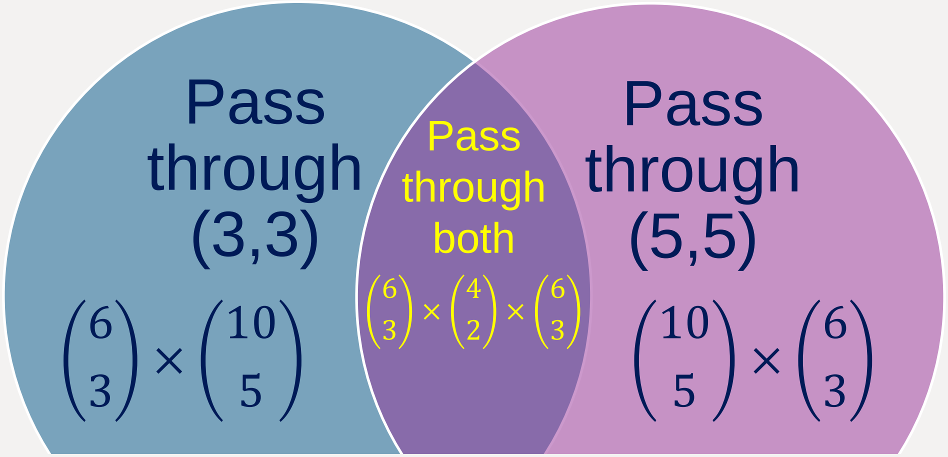


- How many shortest paths from (0,0) to (8,8) that pass through either (3,3) or (5,5) are there?
- How many shortest paths from (0,0) to (8,8) that pass through neither (3,3) nor (5,5) are there?

Shortest Paths WOTO(2)

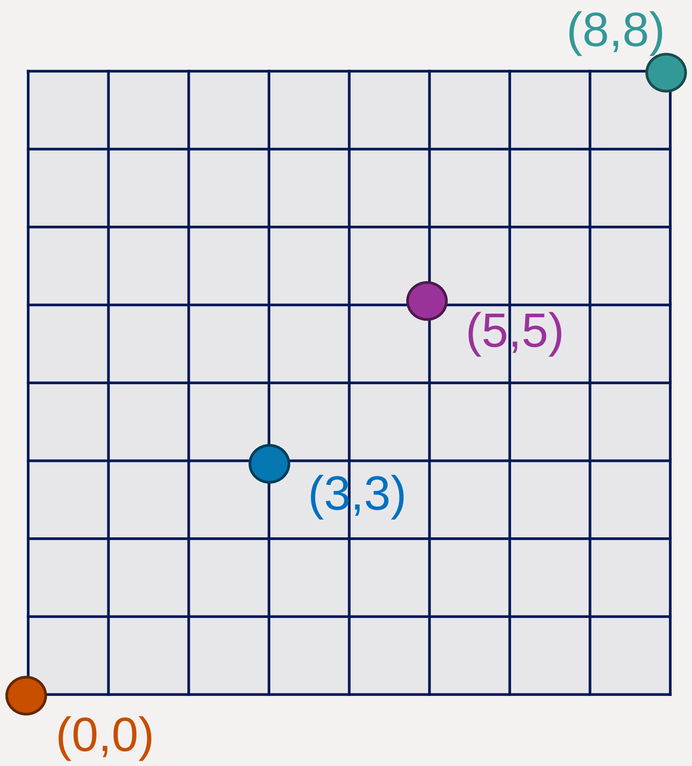


- How many shortest paths from (0,0) to (8,8) that pass through either (3,3) or (5,5) are there?

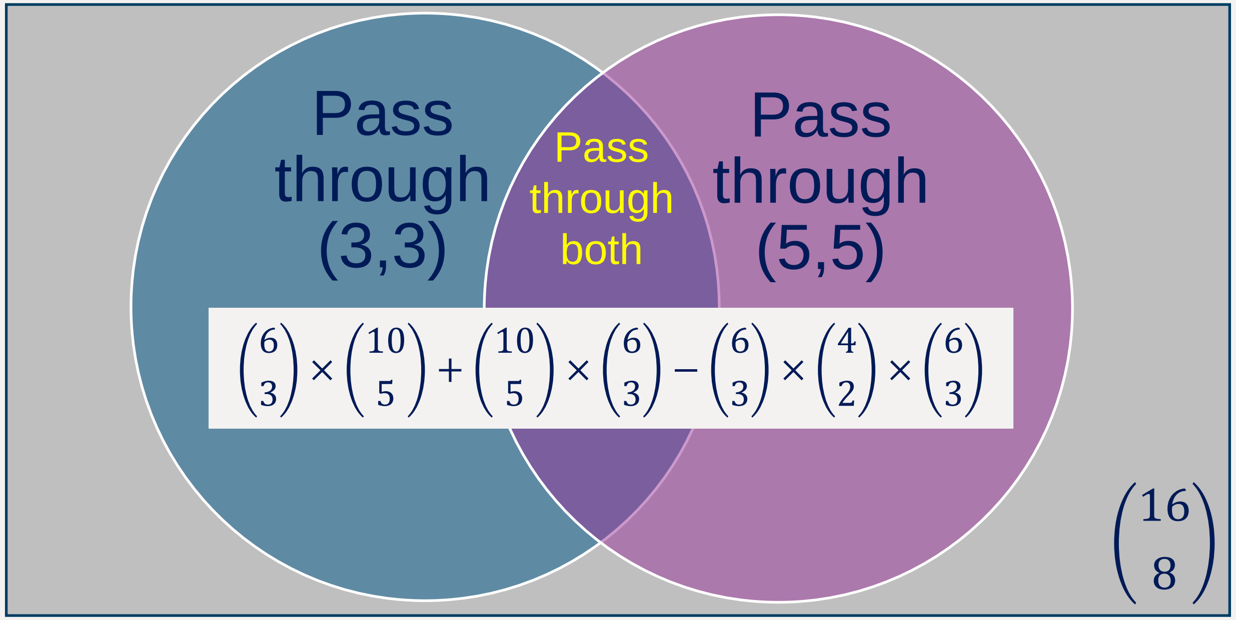


$$\binom{6}{3} \times \binom{10}{5} + \binom{10}{5} \times \binom{6}{3} - \binom{6}{3} \times \binom{4}{2} \times \binom{6}{3}$$

Shortest Paths WOTO(2)



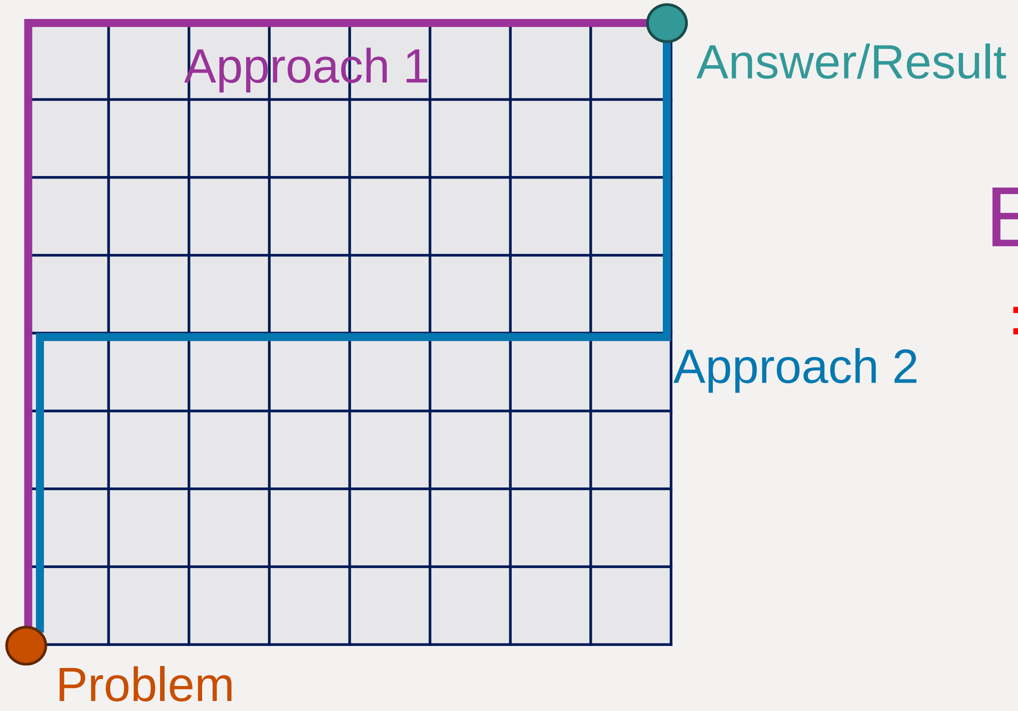
- How many shortest paths from (0,0) to (8,8) that pass through neither (3,3) nor (5,5) are there?



$$\binom{6}{3}\times\binom{10}{5}+\binom{10}{5}\times\binom{6}{3}-\binom{6}{3}\times\binom{4}{2}\times\binom{6}{3}$$

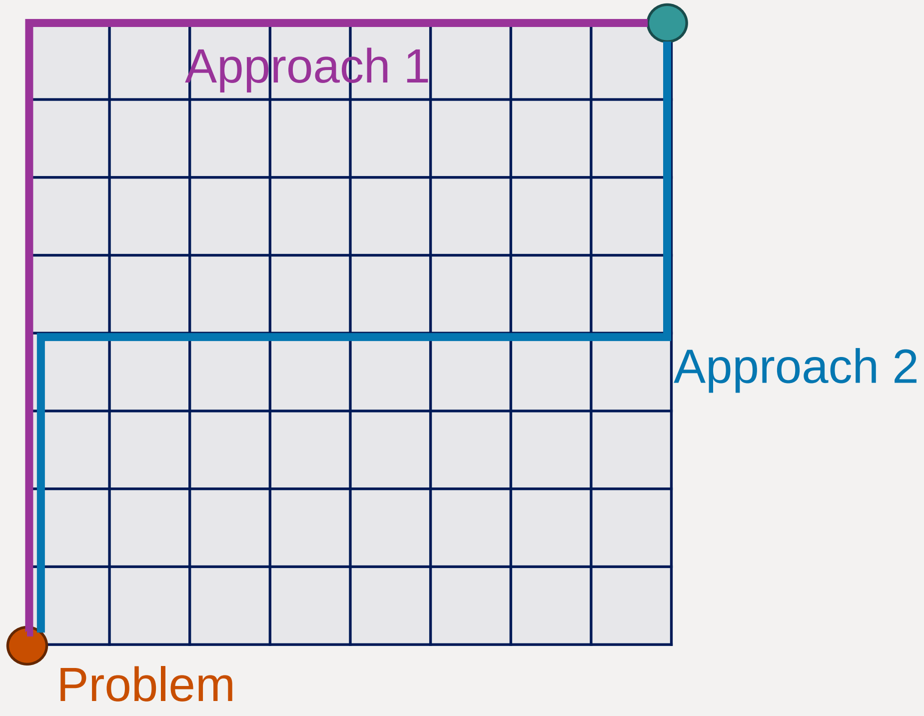
$$\binom{16}{8}$$

Combinatorial Proof



Expression from Approach 1
= Expression from Approach 2

Combinatorial Proof



Problem: for $n > k$, how many ways can we select k people out of n TAs and n students?

- Approach 1: disregard roles:

$$\binom{2n}{k}$$

- Approach 2: select some TAs then some students

$$=$$

$$\sum_{i=0}^{k} \binom{n}{i} \times \binom{n}{k-i}$$

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Combinatorial Proof

Problem: unknown

- Approach 1: unknown
- Approach 2: unknown

$$\binom{2n}{k} = \sum_{i=0}^{k} \binom{n}{i} \times \binom{n}{k-i}$$

Class starts after this song

Green Day – Boulevard of Broken Dreams (2004)

Piano Covered by Frank Tedesco

requested by George Adu (TA-of-CM7)

I like random walks. My favorite manga from last year was Chainsaw Man.

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- Remember *not to discuss* Exam 2 until the entire class has taken it and the grades are announced (ETA Tues/Wed)

- We will start grading EM assignments this weekend; keep an eye on the interactive one (you're about to get a response)

CS230 Spring 2024
Module 07: Combinatorics
Day 2: Counting every module

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Counting in CM1:Logic

- Consider a truth table with n variables.
  - How many rows are there in the table? 2^n
  - How many rows have k T's and $n-k$ F's? $\binom{n}{k}$
  - How many rows have $n-k$ T's and k F's? $\binom{n}{k} = \binom{n}{n-k}$

  - So, which combinatoric proof did we just complete?

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

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Counting in CM3:Math Tools

- We learned about monotonicity in CM3.
- WOTO: How many possible DUID's (7-digit numbers) are:
  - Strictly increasing? (easy)
  - Non-decreasing? (hard)

	Approach 1	Approach 2
$a_n \leq a_{n+1}\ \forall n \in \mathbb{N}$	~~increasing~~	non-decreasing
$a_n < a_{n+1}\ \forall n \in \mathbb{N}$	strictly increasing	~~increasing~~
$a_n \geq a_{n+1}\ \forall n \in \mathbb{N}$	~~decreasing~~	non-increasing
$a_n > a_{n+1}\ \forall n \in \mathbb{N}$	strictly decreasing	~~decreasing~~

Counting in CM3:Math Tools

- How many possible DUID’s are strictly increasing?
  - Choose 7 distinct digits from 0-9
  - There is only one way to sort them in increasing order
  - Therefore, the answer is $\binom{10}{7}$

Counting in CM3:Math Tools

- How many possible DUID's are non-decreasing?
  - Select some distinct digits from 0-9 and decide how many times each digit is repeated
  - Given this information, still only one way to sort them in nondecreasing fashion
  - The task above is to distribute 7 "slots" across 10 possible digits

$$\binom{7+9}{7}$$

Counting in CM4:Sets, Functions, Relations

- Consider a finite set A with $|A| = n$.
  How many distinct relations can we define from A to itself?

	a	b	c	d	e	f	…
a	X		X		X		
b			X				
c						X	
d				X			
e			X			X	
f		X				X	
⋮							

- n^2 cells in the table, each might be a X
- $2^{(n^2)}$ possibilities!

Counting in CM4:Sets, Functions, Relations

- WOTO: Consider a finite set A with $|A| = n$.
- How many functions can we define from A to itself? n^n
- How many injections can we define from A to itself? $n!$
- How many surjections can we define from A to itself? $n!$
- How many bijections can we define from A to itself? $n!$

Counting in CM6:Graph Fundamentals

- We have already counted number of shortest paths
- How many different n-vertex simple undirected graphs are there (vertices are treated as different)?

$$2^{\binom{n}{2}}$$

Counting in CM6:Graph Fundamentals

- How many ways can we color an n-vertex complete graph with n given colors if
  - Vertices are treated as different?

    $n!$

  - Vertices are treated as identical?

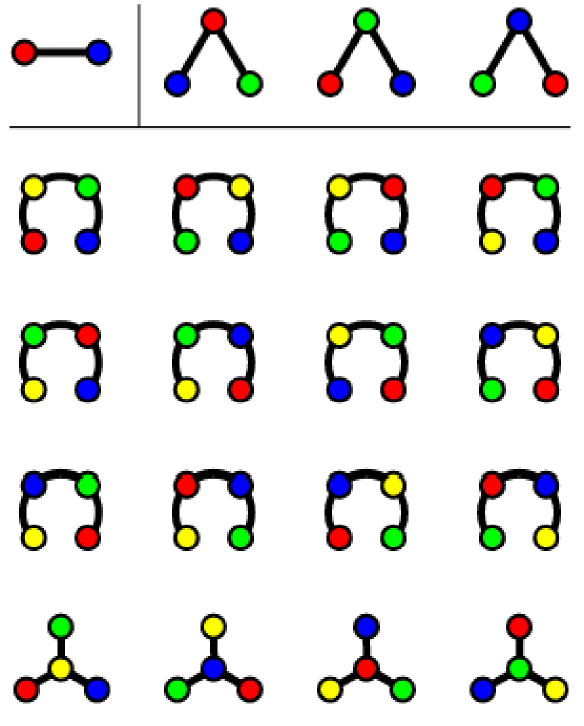
    1

Counting in CM6:Graph Fundamentals

- Consider an unrooted undirected tree with n distinct vertices. There are three such trees with three vertices.
- WOTO: How many trees are there with four vertices?

Counting in CM6:Graph Fundamentals

- How many trees are there with four vertices?
  - Choose 3 edges from all $\binom{4}{2} = 6$ possible edges
  - But 4 of them are cycles (why 4?)
  - $\binom{6}{3} - 4 = 20 - 4 = 16$
  - Read about Cayley's formula for the general case



Accounting for overcounting

- For complicated counting tasks, we can deliberately *overcount* by ignoring/changing the task at hand, then *account for the overcounting* later.

- WOTO: We have n students in the class. How many ways can we WOTO? (In other words, how many ways can we form $n/2$ pairs from n students)?

Accounting for overcounting

- Approach 1.
  - Let's pretend order of pairs matters AND order within pairs matters despite both actually do not matter.
  - Then we simply need to arrange all n students on a line; students $1,2$ make the first pair, students $3,4$ make the second pair, and so on
  - We overcounted the order of all pairs by a factor of $(\frac{n}{2})!$
  - We overcounted the order of students in each pair by a factor of 2
  - Therefore the answer is $\frac{n!}{(n/2)!\times 2^{n/2}}$

Accounting for overcounting

- Approach 2.
  - Let's instead pretend order of pairs doesn't matter BUT order within pairs matters. (Think each pair as a host and a guest.)
  - Then we will first select $\frac{n}{2}$ hosts; then match the remaining $\frac{n}{2}$ guests with the hosts
  - We still overcounted the order of students in each pair by a factor of 2
  - Therefore the answer is $\frac{\binom{n}{n/2} \times (n/2)!}{2^{n/2}}$
  - Looks different than the expression we got from approach 1, but they are equal