# Class starts after this song


I'm a Sophomore from Hong Kong. In my free time I enjoy running, coding cool projects and spending time with friends. I'm very excited to be a TA for this course this semester.

***Vincent Capol (TA-of-CM5)***

***The Strokes – Hard to Explain (2001)***
***requested by Sydney Lester (TA-of-CM5)***


I'm a junior computer science major with a minor in environmental policy. At Duke, I'm a WXDU DJ, Chronicle Photographer, and on Club Ski & Board.

# Logistics Bulletin/Recap

- Exam 1 results out
  - 6/6 → E; 4/6 → S, see Canvas announcement
- Assignments:
  - Round 1 and Round 2 on Gradescope
  - Anything post Round 2: come to me (after class or in consulting hours) with a working answer
- Try collaborating with a teammate if you haven't!
  - Megathread on Ed to find partners
  - (From now on and retroapplicable) you can E-out a CM in pairs

# CS230 Spring 2024
# Module 05: Inductions

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# Why inductions?

- It is one (very powerful) proof method.
- *It is the one thing that you should take away from CS230 if you can only take one*

- Inductions are for proving a general statement (or, in logic terms, a quantified *forall* predicate)

# Style Musing: proving $P(n)$ for all $n \geq a$

- "AIDMA style": **preferred**
  - Base Case: $P(a)$
  - Inductive Hypothesis: Let $k \geq a$, and assume $P(k)$ holds (The "n=k" case)
  - Inductive Step: Then… by inductive hypothesis… thus $P(k+1)$ holds
- "MCS style": (not wrong)
  - Base Case: $P(a)$
  - Inductive Case/Step: Assume $P(n)$ holds for $n \geq a$.
    Then… (by inductive hypothesis…) … thus $P(n+1)$ holds

# PI: Induction Step

1   1 point

Consider a (weak/ordinal) induction proof with only one base case $P(a)$.
What among the following could be a valid inductive step for proving $P(n)$ for all $n \geq a$?

- [x] $P(k) \rightarrow P(k+1), \forall k \geq a$
- [ ] $P(k) \rightarrow P(k+1), \forall k > a$
- [x] $P(k-1) \rightarrow P(k), \forall k \geq a$   let k = a+1
- [x] $P(k-1) \rightarrow P(k), \forall k > a$

# $P(k-1) \to P(k)$ or $P(k) \to P(k+1)$?

- As you see in the PI,
  *this is a moot debate* without specifying the range of $k$
- Similar to the style discussion, *there is no one correct approach*
- Sanity check: check the statements can be used to prove
  the first case not covered by the base cases

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# PI: Induction Step Again

1 | 1 point

Consider a (strong) induction proof with base cases $P(a) \wedge P(a+1) \wedge \ldots \wedge P(b)$.
What among the following could be a valid inductive step for proving $P(n)$ for all $n \geq a$?

- [x] $(P(a) \wedge P(a+1) \wedge \ldots \wedge P(k)) \to P(k+1), \forall k \geq b$
- [ ] $(P(a) \wedge P(a+1) \wedge \ldots \wedge P(k)) \to P(k+1), \forall k > b$
- [x] $(P(a) \wedge P(a+1) \wedge \ldots \wedge P(k-1)) \to P(k), \forall k \geq b$
- [x] $(P(a) \wedge P(a+1) \wedge \ldots \wedge P(k-1)) \to P(k), \forall k > b$

let k=b+1

# Induction meets asymptotic notations

- Recall the MergeSort recurrence relation:

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n), T(1) = 1$$

- We have seen in CM3 how to solve to $T(n) = O(n \log n)$ by substitution/tree method/Master theorem.
- We can also prove that by guess-and-~~verify~~ induction

We can write the recurrence this way because we can find such c so that the O(n) term is always bounded from above by cn

- Given: $T(n) \leq 2T\left(\frac{n}{2}\right) + cn, T(1) = 1$
- Prove: $T(n) \leq c' \cdot n \log n, \forall n \geq n_0$ where $n$ is a power of 2
- Let $c' = c + 1$. *Assume base 2 for all logs below.*
- Base case(s): $T(2^1) \leq 2T(1) + c2^1 = 2(c+1) \leq 2c' \cdot 2^1 \log 2^1$
- Inductive Hypothesis: Assume $T(2^k) \leq c' \cdot 2^k \log 2^k$ for $k \geq 1$
- Inductive Step: $T(2^{k+1}) \leq 2T(2^k) + c2^{k+1}$

$$\leq 2c' \cdot 2^k \log 2^k + c'2^{k+1}$$
$$= c' \cdot 2^{k+1} \log 2^k + c'2^{k+1}$$
$$= c' \cdot 2^{k+1}(\log 2^k + 1)$$
$$= c' \cdot 2^{k+1}(\log 2^{k+1})$$

Applying recurrence relation

Applying inductive hypothesis

This step is because we are operating with base 2

Actually, let c' = c could also work, but we did c' = c+1 to explicitly show that the two constants do not necessarily need to be equal; c is the constant for the O(n) term in the rec relation, c' is the constant for the claim that we're proving

Unlike the examples you saw earlier in which we prove a predicate for n=1, n=2, n=3...,
here we are proving a predicate for n=2^1, n=2^2, n=2^3...
and therefore our inductive step goes from n=2^k to n=2^(k+1).

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# Class starts after this song

I am excited to TA CS230 this semester, because it covers some of my favorite topics in all of math and CS. I am a math major, so I'm always happy to talk about obscure proof questions. Outside of math, I'm a film enthusiast, baseball fan, and amateur chef. I also share a birthday with the inventor of the Internet!

***Hari Srinivasan (TA-of-CM5)***

***The Score – Revolution (2017)***
***requested by Elisa Ma (Head TA/TA-of-CM5)***

I am a first-year biostatistics master's student. My research interests include genetics and improving public health with data science. Outside of school, I like to travel, listen to music, and explore new places.

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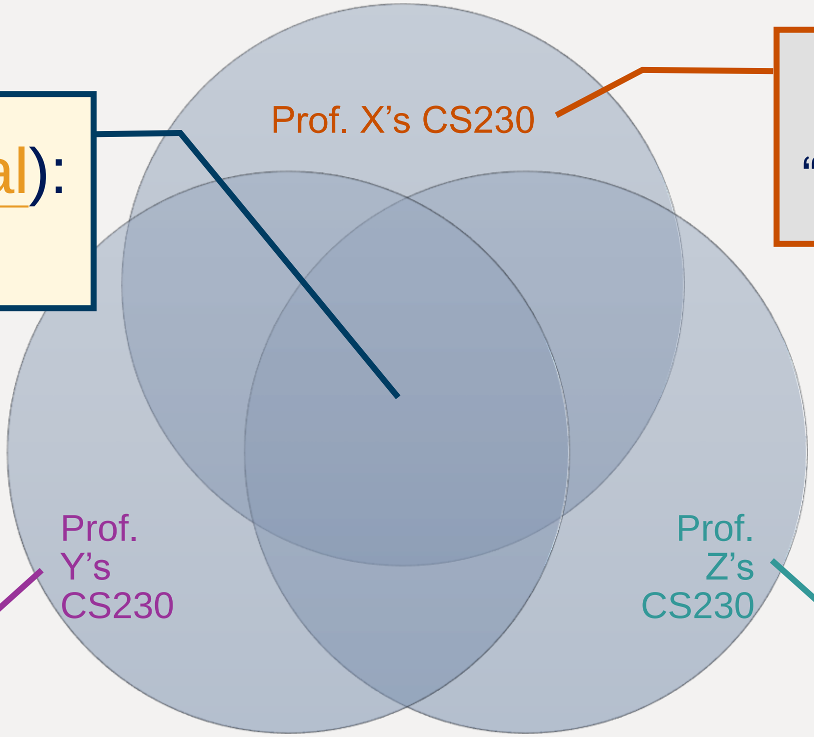
# How do we go about CS230 in Spring 24

Core Modules (CMs, 8 in total):
"What everyone should learn"

Elective Modules (EMs):
"Useful for someone/some field"




Your own CS230

Elective Module

Elective Module

Elective Module

# What happens for an EM (more ad-hoc)

Some sync (in-class), some async (video/reading), some hybrid
~~might be implemented in a recitation time slot~~

Class Material

No need to prepare
CMs already prepared you

PL homework
or Canvas quiz
(autograded)

Assignment
(manually graded)

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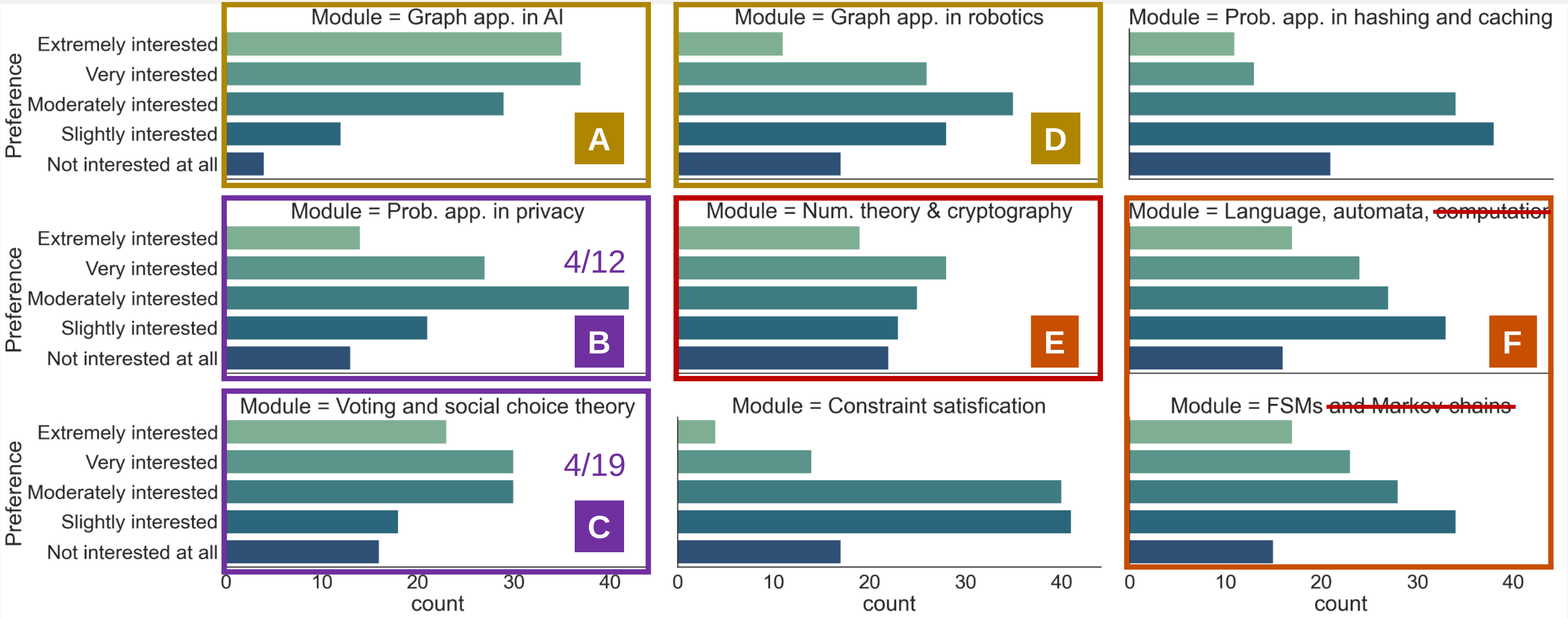
Hybrid (3/8 for sync part)




Sync (see course calendar)

Async (released in Canvas in early March)

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# More EM support/resources

- Some TAs specialized in each EM
  - Post spring break, we will indicate on the consulting hours calendar which TAs' hours you can go to for help on which EM
  - Ed is of course always open

- The last recitation (4/22) is an *EM breakout day*
  - *EM-specialized TAs* will station the recitation classrooms

# Mid-semester Survey

- Released after today’s class
- Due by *end of February (2/29 11:59pm)*
- Required as part of the *Everything outside the modules* module

# PI: Weak vs. Strong Induction

1 | 1 point

What are the difference(s) between weak and strong inductions? (Select all that apply)

- [ ] Weak induction has one base case; strong induction has multiple base cases
- [ ] Weak induction assumes one case in the inductive hypothesis; strong induction assumes more than one cases

# Misconception!

- The *number one misconception in learning inductions* is to think that weak = 1 base case, strong = 2+ base cases.

- That is NOT TRUE.
  Both weak and strong induction
  can have one or more base cases.

# Example: weak induction 1 base case

**P4.1. Induction Proof on Geometric Sum revised**

Drag and drop **all** of the blocks below to create a proof by induction of the following statement.

Claim: for all natural numbers $n$,

$$\sum_{j=0}^{n} 2(-7)^j = \frac{1-(-7)^{n+1}}{4}$$

Drag from here:

Construct your solution here:

Proof by induction on $n$.

Inductive Predicate: $P(n) : \sum_{j=0}^{n} 2(-7)^j = \frac{1-(-7)^{n+1}}{4}$

Base case: At $n = 0$, $\sum_{j=0}^{n} 2(-7)^j = 2$ and $\frac{1-(-7)^{n+1}}{4} = \frac{1-(-7)}{4} = 2$, so the base case, $P(0)$, holds

Inductive Hypothesis: Suppose that $P(n) : \sum_{j=0}^{n} 2(-7)^j = \frac{1-(-7)^{n+1}}{4}$ holds for $n = 0, 1, \ldots, k$.

Inductive Step: We need to show that $P(k+1) : \sum_{j=0}^{k+1} 2(-7)^j = \frac{1-(-7)^{k+2}}{4}$ holds

The left hand side is $\sum_{j=0}^{k+1} 2(-7)^j = \sum_{j=0}^{k} 2(-7)^j + 2(-7)^{k+1}$

By the inductive hypothesis we have $\sum_{j=0}^{k} 2(-7)^j = \frac{1-(-7)^{k+1}}{4}$. So then substituting we get $= \frac{1-(-7)^{k+1}}{4} + 2(-7)^{k+1} = \frac{1-(-7)^{k+1}+8(-7)^{k+1}}{4} = \frac{1+7(-7)^{k+1}}{4}$ which simplifies to $= \frac{1-(-7)^{k+2}}{4}$

So $\sum_{j=0}^{k+1} 2(-7)^j = \frac{1-(-7)^{k+2}}{4}$, which was what we needed to show.

# Example: strong induction 1 base case

**Example 8.24.** Show that every integer $n \geq 2$ can be written as the product of primes.

**Proof:** Let $P(n)$ be the statement "$n$ can be written as the product of primes." We need to show that for all $n \geq 2$, $P(n)$ is true.

Since 2 is clearly prime, it can be written as the product of one prime. Thus $P(2)$ is true.

Assume $[P(2) \wedge P(3) \wedge \cdots \wedge P(k-1)]$ is true for $k > 2$. In other words, assume all of the numbers from 2 to $k-1$ can be written as the product of primes.

We need to show that $P(k)$ is true. If $k$ is prime, clearly $P(k)$ is true. If $k$ is not prime, then we can write $k = a \cdot b$, where $2 \leq a \leq b < k$. By hypothesis, $P(a)$ and $P(b)$ are true, so $a$ and $b$ can be written as the product of primes. Therefore, $k$ can be written as the product of primes, namely the primes from the factorizations of $a$ and $b$. Thus $P(k)$ is true.

Since we proved that $P(2)$ is true, and that $[P(2) \wedge P(3) \wedge \cdots \wedge P(k-1)] \rightarrow P(k)$ if $k > 2$, by the principle of mathematical induction, $P(n)$ is true for all $n \geq 2$. That is, every integers $n \geq 2$ can be written as the product of primes. □

# Example: strong induction 2 base cases

**P4.2. Induction Proof on Recurrence**

Drag and drop **some** of the blocks below to create a proof by induction of the following statement. **Note, not all blocks maybe needed to establish this proof.**

Let function $f : \mathbb{N} \to \mathbb{Z}$ be defined by $f(0) = 2$, $f(1) = 7$, and $f(n) = f(n-1) + 2f(n-2)$, for $n \geq 2$. Prove that $f(n) = 3 \cdot 2^n + (-1)^{n+1}$ for any natural number $n$.

Drag from here:

- The induction hypothesis: Suppose that $f(n) = 3 \cdot 2^n + (-1)^{n+1}$ for $n \geq 0$
- Base case: For $n = 0$ we have, $3 \cdot 2^0 + (-1)^1 = 3 - 1 = 2$ which is equal to $f(0)$. So the claim holds.
- Inductive step: We need to show that $f(k) = 3 \cdot 2^k + (-1)^k$
- So $f(k) = 3 \cdot 2^k + (-1)^k$ which is what we wanted to show

Construct your solution here:

- Proof by induction on $n$.
- Base cases: For $n = 0$ we have, $3 \cdot 2^0 + (-1)^1 = 3 - 1 = 2$ which is equal to $f(0)$. So the claim holds. For $n = 1$, we have $3 \cdot 2^1 + (-1)^2 = 6 + 1 = 7$ which is equal to $f(1)$.So the claim holds.

# Example: weak induction 2 base cases?

# weak induction = strong induction

- We can always simulate a strong induction by a weak induction using a *change-of-predicate technique*:
- If the given strong inductive step is
$$(P(a) \land P(a+1) \land \cdots \land P(k)) \to P(k+1)$$
- Define the predicate $Q(k) \coloneqq (P(a) \land P(a+1) \land \cdots \land P(k))$
- The proof is now a weak induction using the inductive step
$$Q(k) \to Q(k+1)$$

# weak induction = strong induction

- We can always simulate a strong induction by a weak induction using a change-of-predicate technique.
- This means *everything provable by strong induction can be proved by a weak induction* (using the technique we just saw)
- In the other direction: *everything provable by a weak induction can also be proved by strong induction*
(assuming more than needed, which is bad practice but valid)

# Example:
# weak induction
# 2 base cases?

# Example: ~~strong~~ induction 2 base cases

**P4.2. Induction Proof on Recurrence**

Drag and drop **some** of the blocks below to create a proof by induction of the following statement. **Note, not all blocks maybe needed to establish this proof.**

Let function $f: \mathbb{N} \to \mathbb{Z}$ be defined by $f(0) = 2$, $f(1) = 7$, and $f(n) = f(n-1) + 2f(n-2)$, for $n \geq 2$. Prove that $f(n) = 3 \cdot 2^n + (-1)^{n+1}$ for any natural number $n$.

Drag from here:

- The induction hypothesis: Suppose that $f(n) = 3 \cdot 2^n + (-1)^{n+1}$ for $n \geq 0$
- Base case: For $n = 0$ we have, $3 \cdot 2^0 + (-1)^1 = 3 - 1 = 2$ which is equal to $f(0)$. So the claim holds.
- Inductive step: We need to show that $f(k) = 3 \cdot 2^k + (-1)^k$
- So $f(k) = 3 \cdot 2^k + (-1)^k$ which is what we wanted to show

Construct your solution here:

- Proof by induction on $n$.
- Base cases: For $n = 0$ we have, $3 \cdot 2^0 + (-1)^1 = 3 - 1 = 2$ which is equal to $f(0)$. So the claim holds. For $n = 1$, we have $3 \cdot 2^1 + (-1)^2 = 6 + 1 = 7$ which is equal to $f(1)$.So the claim holds.

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# If weak induction = strong induction,

- why do we teach/learn them as separate?

- Equally powerful, equally correct,
  NOT necessarily equally comprehensible
- Certain inductions more readable in weak/strong form

# What can go wrong with inductions?

- "Bogus induction": a direct proof cosplaying as an induction
- Stating a strong hypothesis
  when the inductive step only needs the weak hypothesis
- Stating a weak hypothesis
  when the inductive step needs the strong hypothesis

# “Bogus induction”

- Prove: $1 + 2 + \cdots + n = \frac{n(n+1)}{2}, \forall n \in \mathbb{N}$
- Base case: $1 = \frac{1(1+1)}{2}$
- Inductive hypothesis: let $k \geq 1$ and assume $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$
- Inductive step: $1 + 2 + \cdots + k + (k + 1)$

$$= \frac{[1 + (k+1) + 2 + k + \cdots (k+1) + 1]}{2} = \frac{(k+2)(k+1)}{2}$$

# Stating a strong hypothesis when the inductive step only needs the weak hypothesis

- This is bad practice (assuming more than needed)
- Not technically incorrect; still a valid proof (worth an S)

# Stating a weak hypothesis when the inductive step needs the strong hypothesis

- This is incorrect; the inductive step doesn't go through
- Not a valid proof, will get N

# Induction vs. other proof techniques

- Inductions are NOT mutually exclusive with other proof techniques such as by cases, by contradiction, by construction…
- The inductive step of an induction proof is itself a proof
  - Prove the inductive step by cases
  - Prove the inductive step by contradiction
  - …

# Peer discussion: Strengthening IH

# Failed attempt

- Base case: $1 = 1^2$.
- Inductive hypothesis:
  let $k \geq 1$ and assume $1 + 3 + \cdots + k = m^2$ for some $m \in \mathbb{N}$.
- Inductive step: for $n = k + 2$ (!),

$$1 + 3 + \cdots + k + (k + 2) = m^2 + k + 2 \ldots ?$$

# Successful attempt

- Base case: $1 = 1^2$.
- Strengthened Inductive hypothesis:
  let $k \geq 1$ and assume $1 + 3 + \cdots + k = (\frac{k+1}{2})^2$.
- Inductive step: for $n = k + 2$,
  $1 + 3 + \cdots + k + (k + 2) = (\frac{k+1}{2})^2 + k + 2$ … (some algebra) … $= (\frac{k+3}{2})^2$

# More general inductions?

- Both weak/strong induction (now we know they're "equal") deal with statements of the form $P(n)$ for $n \in \mathbb{N}$ (or some subset of $\mathbb{N}$)
- What about:
  - $P(T)$ for all trees $T$?
  - $P(S)$ for all binary strings $S$?
- We can "induct on" more (discrete) structures than just numbers

# Recursively defined sets

- Now let's go back to a set theory context
- Some sets can be defined recursively
- Example 1. The set of even numbers, $\mathbb{E}$, can be defined as:
  - **Base Case:** $0 \in \mathbb{E}$
  - **Constructor Case:** If $x \in \mathbb{E}$, then $x - 2 \in \mathbb{E}$ and $x + 2 \in \mathbb{E}$.

# Recursively defined sets

- Example 1. The set of even numbers, $\mathbb{E}$, can be defined as:
  - **Base Case:** $2 \in \mathbb{E}$
  - **Constructor Case:** If $x \in \mathbb{E}$, then $x - 2 \in \mathbb{E}$ and $x + 2 \in \mathbb{E}$.

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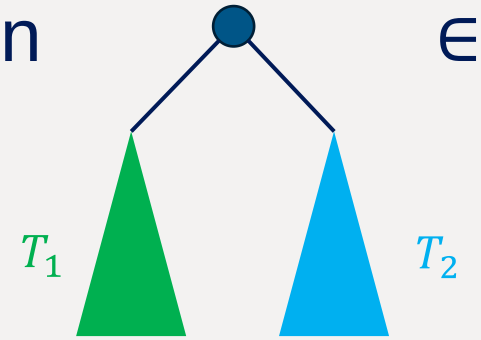
# Recursively defined sets

- One more example on sets
- Example 2. The set of (nonnegative) powers of 2, $P$, can be defined as:
  - **Base Case:** $1 \in P$
  - **Constructor Case:** If $x \in P$, then $2x \in P$

# Recursively defined structures

- Same idea, but the set now contains objects, not just numbers
- Example 3. The set of (undirected) binary trees, $\mathcal{T}$, can be defined as:
  - **Base Case:** $T = (\emptyset, \emptyset) \in \mathcal{T}$ (the "empty tree")
  - **Constructor Case:** If $T_1, T_2 \in \mathcal{T}$, then  $\in \mathcal{T}$

# Recursively defined structures

- Example 4. The set of (undirected) graphs, $\mathcal{G}$, can be defined as:
  - **Base Case:** $G = (\emptyset, \emptyset) \in \mathcal{G}$ (the "empty graph")
  - **Constructor Case:** If $G = (V, E) \in \mathcal{G}$, then:
    - $G' = (V \cup \{v\}, E) \in \mathcal{G}$ ("add a vertex")
    - $G' = (V, E \cup \{e\}) \in \mathcal{G}$ where $e = (u, v), u, v \in V$ ("add an edge")

# Recursively defined structures

- Example 5. The set of balanced parentheses, $\mathcal{B}$, can be defined as:
  - **Base Case:** $\lambda \in \mathcal{B}$ (the “empty string”)
  - **Constructor Case:** If strings $s_1, s_2 \in \mathcal{B}$, then:
    - $(s_1) \in \mathcal{B}$
    - $s_1 s_2 \in \mathcal{B}$