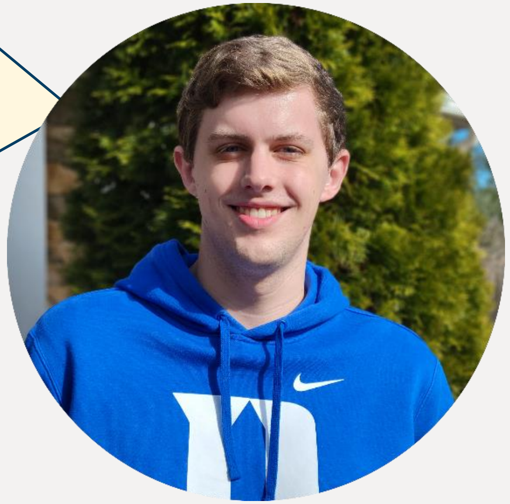
Class starts after this song

Electric Light Orchestra – Livin' Thing (1976)
requested by Erik Dahlberg (TA-of-CM4)

I'm a senior from Asheville, North Carolina going into my fifth semester as a UTA for CS230, and I'm looking forward to graduating at the end of the semester! My hobbies include cooking and playing chess, and I love playing basketball and watching sports of all kinds, especially basketball and hockey.



Duke

Logistic Bulletin Board

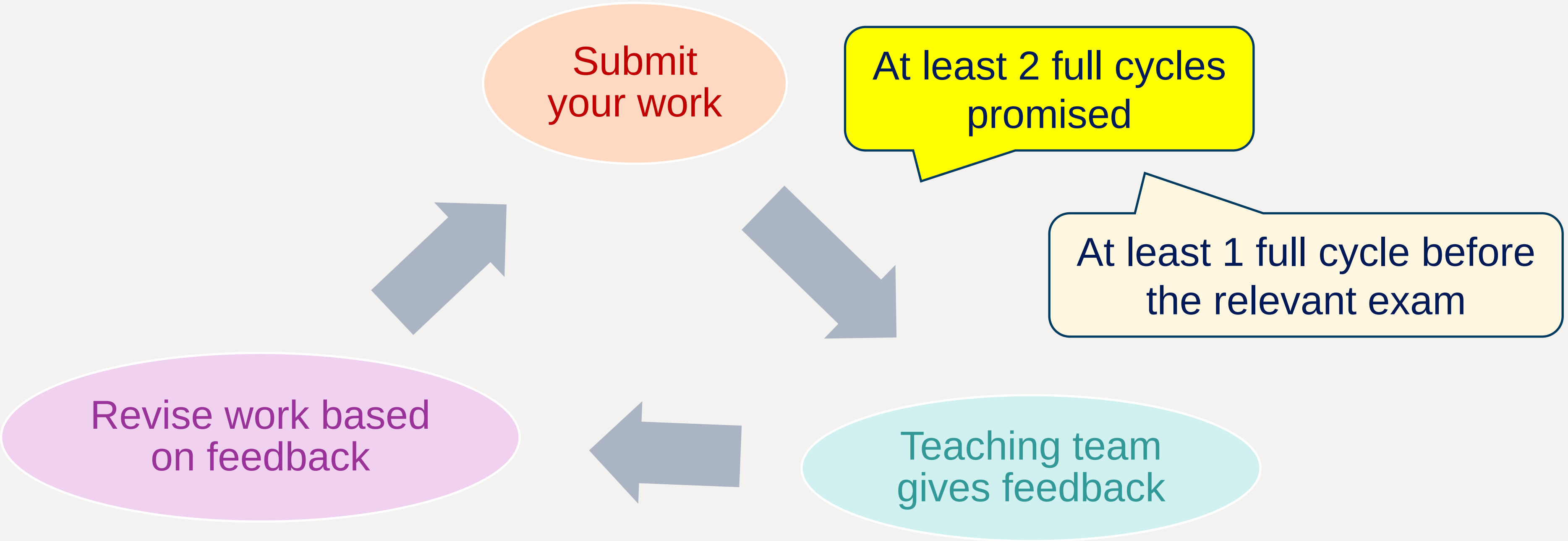
- Exam 1 official reference sheet released
- Practice exam 1 goes out today after class
  - Attempt it yourself (don’t forget about the ref sheet)
  - Your TAs will go over it in recitation on 2/12 (M)
  - Also on 2/12 (M) we will release sample grading results (sample E, S, N answers to each question)

Logistic Bulletin Board

	1st round due	Feedback released	2nd round due	Feedback released
CM1	1/28	2/2	2/7 (tonight)	By 2/12 (expected)
CM2	2/4	2/7 (sometime today)	2/11	(Hopefully) 2/13
CM3	2/11	Late night 2/13	2/18	TBD

Human-graded

What happens for a written assignment



Duke

CS230 Spring 2024
Module 04: Sets, Functions, and Relations

Duke

Why sets/functions/relations?

- It might sound hyperbole but sets are the foundation of math
- Things we take for granted:
  - $\mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}$, 2D space, Euclidean space… how to "define"?
- Set theory also:
  - provides a way for us to "handle" infinity (it is counterintuitive)
  - provides basic language for studying more "immediately useful" things such as counting, probability, and graphs
    - these "immediately useful" things are then useful for us computer scientists because we strive for solving real-world problems

You understand sets/functions/relations

Dictionaries/Maps

- **Dictionaries are another way of organizing data**
- **Dictionaries are sometimes called maps**
- **Keys and Values**
  - Each key maps to a value
  - Some keys can map to the same value
  - Can change the value a key maps to

Dealing with collisions: details

- Bucket is really another list.
- Hash table is really an array of of lists of <key, value> pairs.
- We call this technique for dealing with collisions **chaining**.

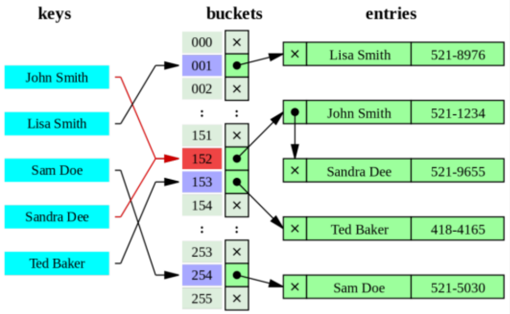


Illustration credit: By Jorge Stolfi - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=6471915

Set and logic operators

Name	*Identity*
commutativity	$A \cup B = B \cup A$ $A \cap B = B \cap A$
associativity	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$
distributive	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
identity	$A \cup \varnothing = A$ $A \cap U = A$
complement	$A \cup \overline{A} = U$ $A \cap \overline{A} = \varnothing$
domination	$A \cup U = U$ $A \cap \varnothing = \varnothing$
idempotent	$A \cup A = A$ $A \cap A = A$
complementation	$\overline{(\overline{A})} = A$
DeMorgan's	$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$
absorption	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$

Table 4.1: Set Identities

Name	*Equivalence*
commutativity	$p \vee q = q \vee p$ $p \wedge q = q \wedge p$
associativity	$p \vee (q \vee r) = (p \vee q) \vee r$ $p \wedge (q \wedge r) = (p \wedge q) \wedge r$
distributive	$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$
identity	$p \vee F = p$ $p \wedge T = p$
negation	$p \vee \neg p = T$ $p \wedge \neg p = F$
domination	$p \vee T = T$ $p \wedge F = F$
idempotent	$p \vee p = p$ $p \wedge p = p$
double negation	$\neg(\neg p) = p$
DeMorgan's	$\neg(p \vee q) = \neg p \wedge \neg q$ $\neg(p \wedge q) = \neg p \vee \neg q$
absorption	$p \vee (p \wedge q) = p$ $p \wedge (p \vee q) = p$

Table 2.3: Common Logical Equivalences

This is not coincidence!

You will establish some of these connections in the recitation.

1 1 point

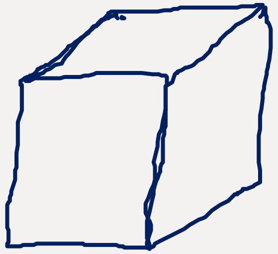
What is $P(\varnothing)$, the power set of $\varnothing$?

- ○ $\{\varnothing, \{\varnothing\}\}$
- ○ $\{\varnothing\}$
- ○ $\varnothing, \{\varnothing\}$
- ○ $\varnothing$

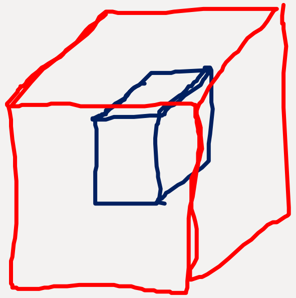
2 1 point

What is $P(\{\varnothing\})$, the power set of $\{\varnothing\}$?

- ○ $\varnothing$
- ○ $\{\varnothing\}$
- ○ $\varnothing, \{\varnothing\}$
- ○ $\{\varnothing, \{\varnothing\}\}$



$\varnothing$



$\{\varnothing\}$

Definition Chaos: Functions

- Relations from domain A to codomain B are defined as subsets of $A \times B$
  - There is no inconsistency/ambiguity about this fact; everyone agrees
  - Sometimes $A = B$ but order still matters
- A relation is functional if no element in the domain is related to multiple elements in the codomain
  - Functions satisfy this, so functions are functional relations

Definition Chaos: Functions

- MCS calls all functional relations functions
  - So the empty relation $\emptyset \subseteq A \times B$ is a function per their terms
- Most of others call all functional total relations functions
  - Total relations map all elements in the domain to something
  - So the empty relation $\emptyset \subseteq A \times B$ is NOT a function per this definition
  - In programming terms: functions cannot/shouldn't "refuse to return"
  - We will use this definition

Definition Chaos: Functions with properties

- Function + injective = injection
- Function + surjective = surjection
- Function + bijective = bijection

Definition Chaos: Functions with properties

- functional + total + injective = injection
- functional + total + surjective = surjection
- functional + total + bijective = bijection

1 1 point

Let the domain be $S := \{s | s \text{ is a CS230 student}\}$ and the codomain be $M := \{m | m \text{ is a CS230 core module}\}$. Then $F := \{(s, m) \in S \times M | s \text{ has completed } m\}$ is... (Select all that apply)

- [x] a relation
- [ ] a function X some student has completed none; some completed multiple
- [ ] an injection
- [ ] a surjection X because it's already not a function
- [ ] a bijection

2 1 point

Let the domain be $S := \{s | s \text{ is a CS230 student}\}$ and the codomain be $R := \{02D/03D, 04D, 05D/06D, 07D, 08D\}$ (i.e., the set of all five CS230 recitation sections). Then $F := \{(s, r) \in S \times R | s \text{ is enrolled in } r\}$ is... (Select all that apply)

- [x] a relation
- [x] a function
- [ ] an injection X many students in each section
- [x] a surjection
- [ ] a bijection

Duke

Relations/functions on infinite sets

- This week (2/7 and 2/9) is mostly about finite sets and relations/functions defined on finite sets
- But $\mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}$ are familiar enough that we can discuss relations/functions defined on them

Relations/functions on infinite sets

- Relations from domain A to codomain B are defined as subsets of $A \times B$
- Functional relations map no element in A to multiple elements in B
- Total relations map all elements in A to some element(s) in B
- Injective relations map different elements in A to different elements in B
- Surjective relations map some element(s) in A to each element in B
- Bijective relations are just injective and surjective relations
- Nothing here says A, B have to be finite

1 1 point

Let the domain be $\mathbb{R}$ and the codomain be $\mathbb{Z}$. Then $f(x) = \lfloor x \rfloor$ is... (Select all that apply)

- [x] a relation
- [x] a function
- [ ] an injection ✗ $\lfloor 1.5 \rfloor = \lfloor 1.2 \rfloor = 1$
- [x] a surjection
- [ ] a bjiection

2 1 point

Let the domain be $\mathbb{Z}$ and the codomain be $\mathbb{R}$. Then $f(x) = \lfloor x \rfloor$ is... (Select all that apply)

- [x] a relation
- [x] a function
- [x] an injection
- [ ] a surjection ✗ No x maps to 1.5
- [ ] a bijection

Proving function/relation properties

- Past CS230 students have this common question:
  "I can tell this function is injective, but how do we prove it?"

- Notice that each property is defined as a logical statement
- Proving that statement is true proves the property

1 | 1 point

From CM4: PI Relation Properties we know that $F := \{(s, r) \in S \times R | s \text{ is enrolled in } r\}$ from the set of CS230 students to the set of CS230 recitation sections is surjective. How can we formally prove it is surjective? (Select all that apply)

- [x] For each recitation section, find a student that is enrolled in that section
- [ ] For each student, find a recitation that the student is enrolled in

2 | 1 point

From CM4: PI What is this again? we know $f(x) = \lfloor x \rfloor$ from $\mathbb{Z}$ to $\mathbb{R}$ is injective. How can we formally prove it is injective? (Select all that apply)

- [x] Prove that $\forall x, y \in \mathbb{Z}[(x \neq y) \rightarrow (\lfloor x \rfloor \neq \lfloor y \rfloor)]$
- [ ] Prove that $\forall x, y \in \mathbb{R}[(x \neq y) \rightarrow (\lfloor x \rfloor \neq \lfloor y \rfloor)]$
- [x] Prove that $\forall x, y \in \mathbb{Z}[(\lfloor x \rfloor = \lfloor y \rfloor) \rightarrow (x = y)]$
- [ ] Prove that $\forall x, y \in \mathbb{R}[(\lfloor x \rfloor = \lfloor y \rfloor) \rightarrow (x = y)]$

Contrapositives of each other

Contrapositives of each other

$R \subseteq A \times A$

	a	b	c	d	e	f	$\dots$
a	X		X		X		
b			X				
c						X	
d				X			
e			X			X	
f		X				X	
$\vdots$							

Duke

Reflexive

	a	b	c	d	e	f	$\dots$
a	X			X			
b	X	X		X	X		
c			X				
d			X	X			
e			X		X		
f		X	X			X	
$\vdots$							$\dots$

Irreflexive

	a	b	c	d	e	f	$\dots$
a				X			
b	X			X	X		
c							
d		X	X			X	
e		X	X			X	
f		X	X				
$\vdots$							$\dots$

Duke

Symmetric

	a	b	c	d	e	f	$\dots$
a			X		X	X	
b					X		
c	X		X	X			
d			X	X			
e	X	X				X	
f	X				X		
$\vdots$							$\ddots$

Anti-Symmetric

	a	b	c	d	e	f	$\ldots$
a			X			X	
b							
c			X	X			
d				X			
e	X	X					
f					X		
$\vdots$							$\ddots$

Duke

Asymmetric = Anti-Symmetric + Irreflexive

	a	b	c	d	e	f	…
a			X			X	
b							
c				X			
d							
e	X	X					
f					X		
⋮							⋱

Transitive

	a	b	c	d	e	f
a		X	X			
b			X			
c						
d						
e						
f						

$$(aRb) \wedge (bRc) \rightarrow (aRc)$$

Duke

Transitive

	a	b	c	d	e	f
a		X	X			
b			X			
c						
d						
e					X	X
f					X	X

$$(aRb) \wedge (bRc) \rightarrow (aRc)$$

$$(eRf) \wedge (fRe) \rightarrow (eRe) \wedge (fRf)$$

Transitive

	a	b	c	d	e	f
a		X	X			
b			X			
c		X				
d		X	X			
e					X	X
f					X	X

$$(aRb) \wedge (bRc) \rightarrow (aRc)$$

$$(eRf) \wedge (fRe) \rightarrow (eRe) \wedge (fRf)$$

$$(dRc) \wedge (cRb) \rightarrow (dRb)$$

Transitive

	a	b	c	d	e	f
a		X	X			
b		X	X			
c		X	X			
d		X	X			
e					X	X
f					X	X

$$(aRb) \land (bRc) \rightarrow (aRc)$$

$$(eRf) \land (fRe) \rightarrow (eRe) \land (fRf)$$

$$(dRc) \land (cRb) \rightarrow (dRb)$$

$$(bRc) \land (cRb) \rightarrow (bRb)$$
$$(cRb) \land (bRc) \rightarrow (cRc)$$

Transitive + Symmetric

	a	b	c	d	e	f
a		X	X			
b	X	X	X	X		
c	X	X	X	X		
d		X	X			
e					X	X
f					X	X

Duke

Transitive + Symmetric

	a	b	c	d	e	f
a	X	X	X	X		
b	X	X	X	X		
c	X	X	X	X		
d	X	X	X	X		
e					X	X
f					X	X

Transitive + Symmetric + Reflexive

	a	b	c	d	e	f
a	X	X	X	X		
b	X	X	X	X		
c	X	X	X	X		
d	X	X	X	X		
e					X	X
f					X	X

Transitive + Symmetric + Reflexive = Equivalence

	a	b	c	d	e	f
a	X	X	X	X		
b	X	X	X	X		
c	X	X	X	X		
d	X	X	X	X		
e					X	X
f					X	X

PI: Equivalenc

	a	b	c	d	e	f
a	X	X	X	X		
b	X	X	X	X		
c	X	X	X	X		
d	X	X	X	X		
e					X	X
f					X	X

1 Multiple Answer 1 point

What is an equivalence class of R? (Select all that apply)

- [ ] a, b, c, d
- [x] $\{a, b, c, d\}$
- [ ] e, f
- [x] $\{e, f\}$
- [ ] $\{\{a, b, c, d\}, \{e, f\}\}$

2 Multiple Answer 1 point

What is a partition induced by R? (Select all that apply)

- [ ] a, b, c, d
- [ ] $\{a, b, c, d\}$
- [ ] e, f
- [ ] $\{e, f\}$
- [x] $\{\{a, b, c, d\}, \{e, f\}\}$

Duke

Inverses of functions and relations

- A function f has an inverse function $g = f^{-1}$ only if it's a bijection
- But the inverse of a relation always exists:
  - Relation $R \subseteq A \times B$
  - Inverse $R^{-1} = \{(b, a) | (a, b) \in R\} \subseteq B \times A$

Inverse relations

	a	b	c	d	e	f	g
a	X		X	X			
b						X	X
c		X		X			

	a	b	c
a	X		
b			X
c	X		
d	X		X
e			
f		X	
g		X	

Class starts after this song

Noah Kahan – Stick Season (2022)

requested by Divyansh Jain (TA-of-CM4)

Sophomore from New Delhi, India studying CS and Math. Likes to play chess, watch crime thrillers and hang out with friends. Fun fact: Went to boarding school in Switzerland!

Logistic Bulletin Board

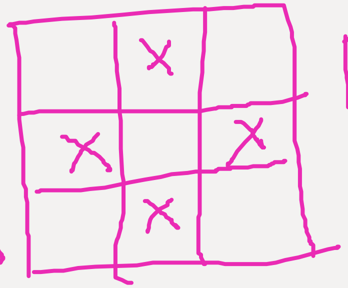
- More hints in CM3 added to the assignment PDF on Canvas

- Read the Canvas announcement/emails on the exam eve for:
  - What to do if you still have N's in your CM1/CM2 assignments
  - What to do if you want a regrade on CM3 Round 1

Post-exam (but pre-exam-grading) reflections

- Exams and assignments are *completely different settings*
  - In assignments, we are harshly/rigorously pushing you to keep improving your work, even if that increases our own workload (by *a lot*)
  - In exams, you get one timed chance to show your learning and there's not a chance for you to revise your own work on the same questions
    - Therefore we don't hold you to the same standard in exams

PI: Inverses



R

Inverse Relation

Let $R \subseteq A \times A$ be a binary relation and $R^{-1} = \{(b,a) | (a,b) \in R\}$ be its inverse relation.

1 1 point

What can we conclude if $R = R^{-1}$? (Select all that apply)

- [x] R is symmetric
- [ ] R is transitive
- [ ] R is reflexive
- [ ] R is irreflexive
- [ ] R is anti-symmetric
- [ ] R is empty, i.e., $R = \varnothing$

$R = \phi$

R=AxA (full relation)

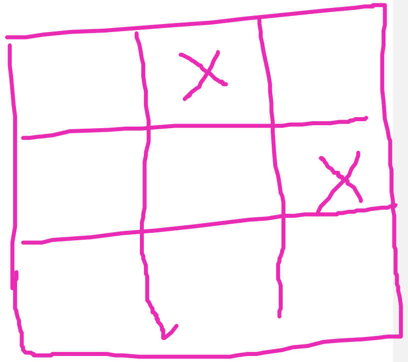
2 1 point

What can we conclude if $R \cap R^{-1} = \varnothing$? (Select all that apply)

- [ ] R is symmetric
- [ ] R is transitive
- [ ] R is reflexive
- [x] R is irreflexive
- [x] R is anti-symmetric
- [ ] R is empty, i.e., $R = \varnothing$

$R = \phi$

R



Function compositions

David Tan, "Coding habits for data scientists", 2019

```
# good example
df = impute_nans(df, categorical_columns=['Embarked'],
                     Continuous_columns =['Fare', 'Age'])
df = add_derived_title(df)
df = encode_title(df)
df = add_is_alone_column(df)
df = add_categorical_columns(df)
X, y = split_features_and_labels(df)

# an even better example. Notice how this reads like a story
prepare_data = compose(impute_nans,
                       add_derived_title,
                       encode_title,
                       add_is_alone_column,
                       add_categorical_columns,
                       split_features_and_labels)

X, y = prepare_data(df)
```

ties

1 1 point

Let $f : \mathbb{R} \to \{1\}$ where $f(x) = 1$ for all x, and $g : \{1\} \to \mathbb{R}$ where $g(1) = 0$. Then $f \circ g$ is... (Select all that apply)

- [x] a relation
- [x] a function
- [x] an injection
- [x] a surjection
- [x] a bijection

$f \circ g : \{1\} \to \{1\}$

g applies first!

2 1 point

On the other hand, $g \circ f$ is... (Select all that apply)

- [x] a relation
- [x] a function
- [ ] an injection
- [ ] a surjection
- [ ] a bijection

$g \circ f : \mathbb{R} \to \mathbb{R}$

f applies first! this composite function maps all real number to 0

Definition Chaos: Partial Orders

- AIDMA defines partial orders as reflexive, anti-symmetric, and transitive
- MCS defines two concepts:
  - Weak partial orders are reflexive, anti-symmetric, and transitive
  - Strict partial orders are irreflexive, anti-symmetric, and transitive
  - Under these definitions, weak and strict are disjoint (for a nonempty domain)
- Other sources (including some past instructors here):
  - (Weak) partial orders are just anti-symmetric, and transitive
  - Strong/strict partial orders are irreflexive, anti-symmetric, and transitive
  - Under these definitions, strict ⊆ weak

Definition Chaos: Partial Orders

- Like whether $0 \subseteq \mathbb{N}$ or not, these definitions all have their pros and cons
- Bottom line:
  - any kind of partial orders are anti-symmetric and transitive
  - Strict partial orders are also irreflexive
- In this class, when regarding (weak) partial orders, we will always explicitly state whether the relation in question needs to be reflexive or not

Infinity

aka where intuitions go wrong

The rest of slides (from this point and on) will not be on any exam/assignment, but one part will reappear in recitations for good practice

Recap: cardinality rules for finite sets

Let A, B be two finite sets. Then: (AIDMA Theorem 4.92 restated)

- $|A| \leq |B| \Leftrightarrow$ there exists an injection from A to B
- $|A| \geq |B| \Leftrightarrow$ there exists a surjection from A to B
- $|A| = |B| \Leftrightarrow$ there exists a bijection from A to B

- Actually proving these theorems need either induction or counting arguments
  - Although they are (hopefully) intuitive

Cardinalities of infinite sets?

- It is okay to really say $|\mathbb{N}| = |\mathbb{Q}| = |\mathbb{R}| = |\mathbb{Z}| = \infty$?
- What does that mean?
  - Does that mean there are as many natural numbers as rational numbers?
  - But isn't $\mathbb{N} \subset \mathbb{Q}$?
  - How can two sets have the same number of elements, yet one strictly contain the other?

Duke

Warmup: $\mathbb{N}$ and $\mathbb{Z}^+$

- They are both infinite sets
- $\mathbb{Z}^+ \subseteq \mathbb{N}$ (let's use the definition that $0 \in \mathbb{N}$)
- Don't know how to reason about their cardinalities…
- But recall that function properties (injective, surjective, bijective) do not rely on the domain/codomain to be finite!

Warmup: $\mathbb{N}$ and $\mathbb{Z}^+$

- There is a bijection between $\mathbb{N}$ and $\mathbb{Z}^+$. Can you find it?

- $f: \mathbb{N} \to \mathbb{Z}^+$ where $f(x) = x + 1$
- $f(0) = 1, f(1) = 2, f(2) = 3, \ldots$

Warmup: $\mathbb{N}$ and $2\mathbb{N}$

- There is a bijection between $\mathbb{N}$ and $2\mathbb{N} = \{2n | n \in \mathbb{N}\}$. Can you find it?

- $f: \mathbb{N} \to 2\mathbb{N}$ where $f(x) = 2x$
- $f(0) = 0, f(1) = 2, f(2) = 4, \ldots$

Warmup: $\mathbb{N}$ and $\mathbb{Z}$

- There is a bijection between $\mathbb{N}$ and $\mathbb{Z}$. Can you find it?

- $f: \mathbb{N} \to \mathbb{Z}$ where $f(x) = \begin{cases} \frac{x}{2}, & x \text{ even} \\ -\frac{x+1}{2}, & x \text{ odd} \end{cases}$
- $f(0) = 0, f(1) = -1, f(2) = 1, f(3) = -2, f(4) = 2, \ldots$

Countable and countably infinite

- An infinite set A is countably infinite if there exists a bijection between $\mathbb{N}$ and A.
  - $\mathbb{Z}^+$, $\mathbb{N}$, $2\mathbb{N}$ are all countably infinite as we just showed
- A set A is countable if there exists a surjection from $\mathbb{N}$ to A.
  - $\{1\}$ is countable because we can define $f: \mathbb{N} \to \{1\}$ where $f(x) = 1$
  - $\{d, \mu, k, \varepsilon\}$ is countable because we can define $f: \mathbb{N} \to \{d, \mu, k, \varepsilon\}$ where $f(1) = d, f(2) = \mu, f(3) = k, f(x) = \varepsilon \; \forall \; x > 3$

Duke

$\mathbb{Q}$ is countable

- We will show $\mathbb{Q}^+$ is countable today
- You will extend it to $\mathbb{Q}$ in the recitations

f(0) = 1/1
f(1) = 1/2
f(2) = 2/1
f(3) = 3/1
...

$\mathbb{Q}^+$ is countable [dovetailing]

- Define the surjection from $\mathbb{N}$ to $\mathbb{Q}^+$:

	1	2	3	4	5	...
1	1/1	1/2	1/3	1/4	1/5	. . .
2	2/1	2/2	2/3	2/4	2/5	
3	3/1	3/2	3/3	3/4	3/5	
4	4/1	4/2	4/3	4/4	4/5	
5	5/1	5/2	5/3	5/4	5/5	
...						

Cartesian product of countable sets

- (Theorem) the Cartesian product of two countable sets is countable.
- The concept is no different than the dovetailing we just did:
  - “List” both sets along the rows/columns
  - Dovetail through the table

Union of countable sets

- (Theorem) the union of countably many countable sets is countable. (wait what?)

- A “proof” is out of reach, needs tools that we haven’t (and won’t) cover
- The “concept” is no different than the dovetailing we just did:
  - “List” every set as a row
  - Dovetail through the rows
  - Need every row to be countable (otherwise “can’t list it as a row”)
  - Need countable number of rows (otherwise “can’t dovetail”)

Okay, so everything is countable?

- Theorem: $\mathbb{R}$ is NOT countable.

- What does this mean?
  - There exists no surjection from $\mathbb{N}$ to $\mathbb{R}$.
  - How do we prove “there exists no…”?

$\mathbb{R}$ is uncountable

- There exist some simple bijections between $[0,1]$ and $\mathbb{R}$
  - For example, $\tan((x - \frac{1}{2}) \times \pi)$
- We will prove there is no surjection from $\mathbb{N}$ to $[0,1]$.
  - Any such surjection composed with $\tan((x - \frac{1}{2}) \times \pi)$ would be a surjection from $\mathbb{N}$ to $\mathbb{R}$

Duke

$\mathbb{R}$ is uncountable (in binary)

- Assume there is a surjection from $\mathbb{N}$ to $[0,1]$.

x	$f(x)$ in binary
0	$.0101001010\ldots$
1	$.1111000001\ldots$
2	$.0101011111\ldots$
3	$.0100000000\ldots$
4	$.1110001111\ldots$
…	

The number $.10111\ldots$ which we obtain by swapping every bit along the diagonal is not in this table

(More detailed explanations can be found in MCS Ch.8)

Duke

$\mathbb{R}$ is uncountable (in base 10 digits)

- Assume there is a surjection from $\mathbb{N}$ to $[0,1]$.

x	$f(x)$ in binary
0	$.324613528\ldots$
1	$.576482261\ldots$
2	$.013899041\ldots$
3	$.000900220\ldots$
4	$.987654321\ldots$
…	

The number $.48406\ldots$ which we obtain by replacing x by $(x+1) \bmod 10$ for every bit along the diagonal is not in this table

(More detailed explanations can be found in MCS Ch.8)

$P(\mathbb{N})$ is uncountable

- Theorem [MCS8.1.12]:
  for any set A, there exist no surjection from A to $\mathrm{P}(A)$.
  - A doesn't need to be infinite!
  - But for finite A this is obvious (cardinality laws)

- Corollary: $P(\mathbb{N})$ is uncountable.
- Corollary: the set of all binary strings is uncountable.
  - This is since we can use binary strings to encode $P(\mathbb{N})$
- Corollary: $\mathbb{R}$ is uncountable again.
  - This is since we can use binary strings to encode $[0,1]$.

Putting this altogether

- finite
- countably infinite = countable + infinite
  - "lowest level of infinity": $\mathbb{N}, \mathbb{Q}, \mathbb{Z}$, their unions, Cartesian products, etc.
- uncountably infinite
  - "not the lowest level of infinity": $\mathbb{R}$, $P(\mathbb{N})$, etc.
  - there are many levels! Think $P(\mathbb{R}), P(P(\mathbb{R}))$ ...

Duke