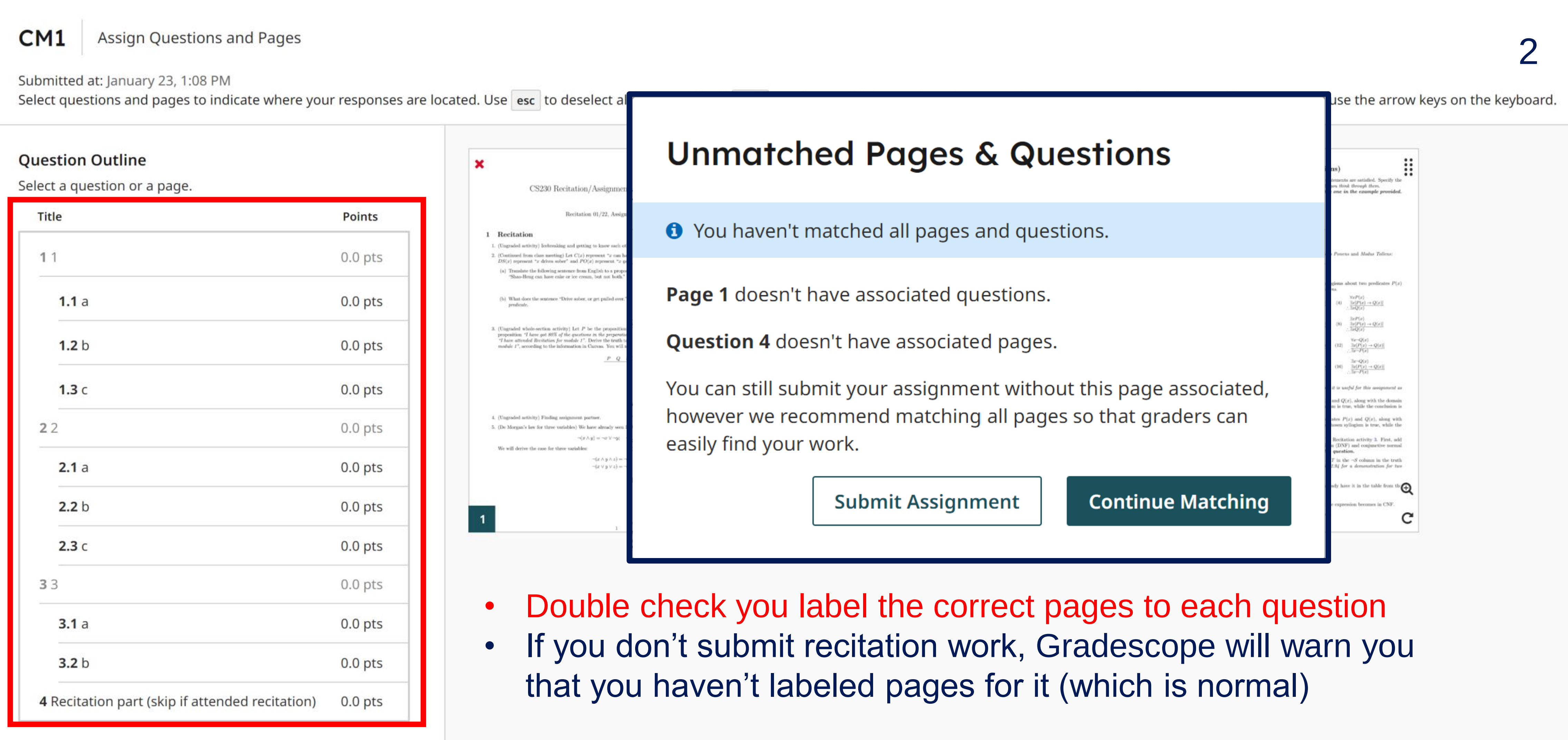
Class starts after this song

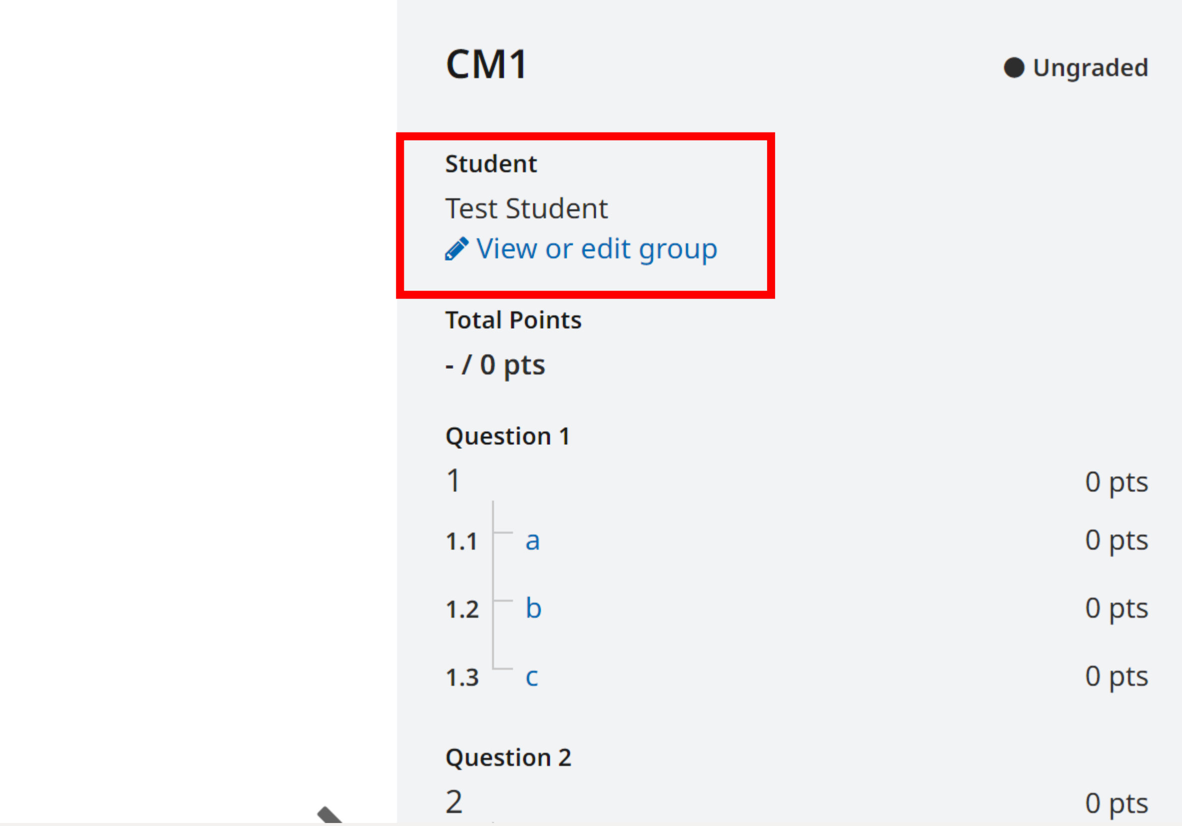
Yiruma – River Flows in You (2001)
requested by Yiyang Shao (Backend head TA)

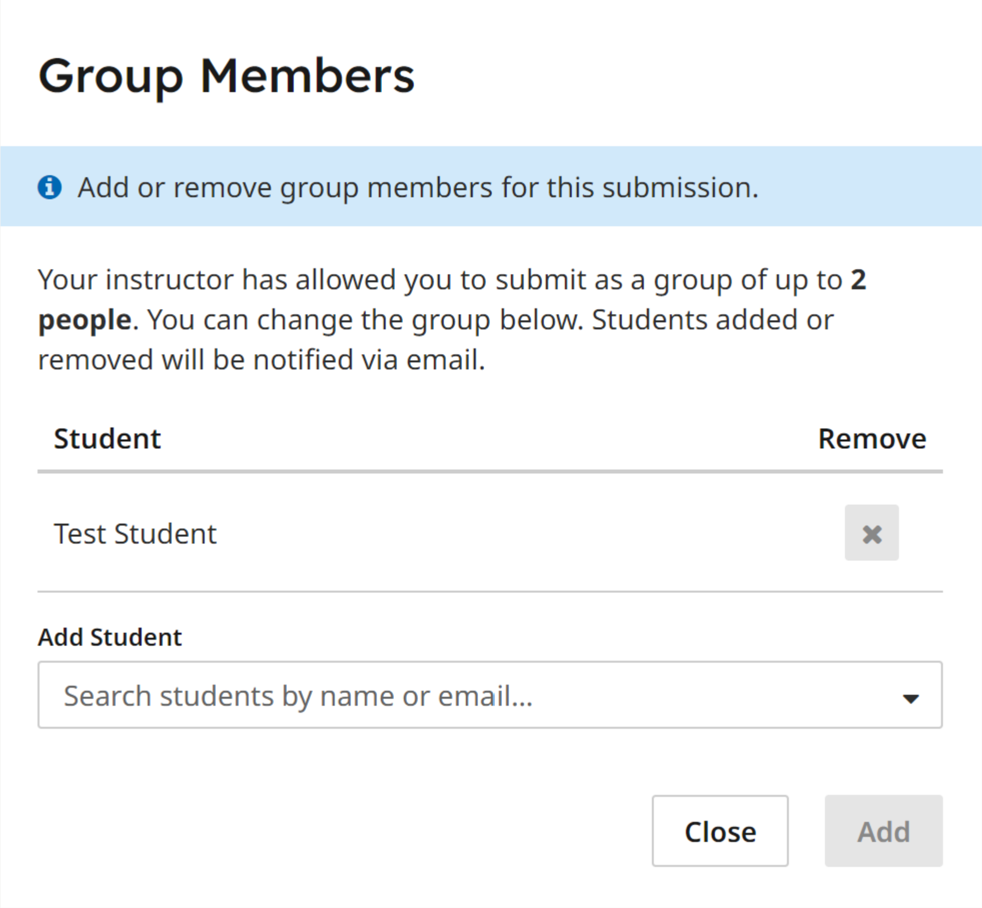
I like playing basketball, volleyball and archery. I have a cat named Lizi(Chestnut).

Duke



- Double check you label the correct pages to each question
- If you don’t submit recitation work, Gradescope will warn you that you haven’t labeled pages for it (which is normal)





- Then add your teammate here; DO NOT SUBMIT SEPARATELY everywhere

“The limit of $f(x)$ exists as x approaches a”

$\forall a \in \mathbb{R}$

$$\exists L \in \mathbb{R} \quad \forall \varepsilon > 0 \quad \exists \delta > 0 \qquad \left[(0 < |x - a| < \delta) \rightarrow (|f(x) - L| < \varepsilon)\right]$$

- We made a mistake in CM1 recitation material here; we forgot to quantify x
- The predicate with this part missing was not “false”; it is still true when the limit exists, but it’s this version with x quantified that semantically translates the limit definition
- In hindsight, this problem still relies on knowing what a limit is; don’t worry about that part and **we were/are just using it for practice on predicate logic**

Duke

CS230 Spring 2024
Module 02: Proof Methods

Duke

Why proof?

- You need to make arguments on how things work as a future computer scientist
- Can you always base your arguments on evidence?
  - “The last 1,000 times I used it, this sorting algorithm actually gives me a sorted list”
- If not, then you sometimes need to base them on logic
  - Proofs are nothing beyond that (but more formal, and more stylistic)

What can be proved?

- Nothing can be proved *without assumptions/axioms/facts given*
  - Very very strictly speaking, a mathematical proof is incomplete without specifying what axioms are used
  - Practical compromise: state all nontrivial assumptions/axioms
- Only propositions can be proved/disproved
  - This is how we “defined” propositions!
  - Not all propositions can be proved/disproved *within propositional logic*

How are $\rightarrow$ and $\Rightarrow$ different?

- $\rightarrow$ (conditional operator): $p \rightarrow q$ can be either true or false
- $\Rightarrow$ (implication): $p \Rightarrow q$ means "$p \rightarrow q$ is a tautology"

Modus Ponens, revisited

$$
\begin{array}{l}
p \to q \\
p \\
\hline
\therefore q
\end{array}
$$

$$((p \to q) \land p) \Rightarrow q$$

Similarly:

- $\leftrightarrow$ (biconditional operator): $p \leftrightarrow q$ means $(p \rightarrow q) \wedge (q \rightarrow p)$
- $\Leftrightarrow, =, \equiv$ (equivalence): $p \Leftrightarrow q$ means $(p \Rightarrow q) \wedge (q \Rightarrow p)$

- When we prove $p \leftrightarrow q$ is true, we proved $p \Leftrightarrow q$ (read: p is equivalent to q)
- "We prove p if and only if q (is true)"

PD: Proof Example 1

- Read this proof individually
- Then turn to your neighbor, discuss *where exactly is the proof wrong*

Proof:

Let $a = 2n + 1$ and $b = 2q + 1$ be two odd integers.

n and q need to be integers

Then $(2n+1)(2q+1) = 4nq + 2n + 1 = 2(2nq + n) + 1$.

missing a $2q$ (algebraic error)

Since $(2nq + n)$ is an integer, $2(2nq + n) + 1$ is an odd integer.

questionable choices of variables (not wrong)

PD: Proof Example 2

- What about this proof?

states more than what is needed (not wrong, but bad practice)

If n is an integer, $n^3 - n = (n-1)n(n+1)$ is the product of three consecutive integers $(n-1)$, n, and $(n+1)$.

Among these three integers, at least one is a multiple of 2 and exactly one is a multiple of 3.

Since 2 and 3 are both primes, $n^3 - n = (n-1)n(n+1)$ is a multiple of $2 \times 3 = 6$.

Proof by construction *[Existential Generalization]*

- To prove a theorem of the form

$$\exists x[P(x)]$$

we merely need to find ("construct") ONE c and show $P(c)$.

$$P(c) \Rightarrow \exists x[P(x)]$$

Proof by construction *[Existential Generalization]*

- Theorem: There exist integers a, b, c such that $a^2 + b^2 = c^2$.
- Proof: let $a = 3, b = 4, c = 5$.

Proof by (complicated) construction

- Theorem: Given two strings $x_1 x_2 \cdots x_n$ and $y_1 y_2 \cdots y_m$, there is an algorithm that runs in $O(mn)$-time that finds the length of their longest common substring, i.e., the largest k for which there exist indices i, j with $x_i x_{(i+1)} \cdots x_{(i+k-1)} = y_j y_{j+1} \cdots y_{(j+k-1)}$.
- Proof:
  - Describe the algorithm
  - Prove the algorithm runs in $O(mn)$-time
  - Prove the algorithm finds the largest such k for all possible input strings

Proof by construction [Existential Generalization]

- To prove a theorem of the form

$$\exists x[P(x)]$$

we merely need to find ("construct") ONE c and show $P(c)$.

$$P(c) \Rightarrow \exists x[P(x)]$$

Disproof by counterexample

- A disproof by counterexample is itself *a proof by construction* of *the negation of the initial proposition*.

- Disproving $\forall x[P(x)]$ is just proving $\neg(\forall x[P(x)])$, which is equivalent to $\exists x[\neg P(x)]$.
- The counterexample is some c that makes $\neg P(c)$ true.

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Class starts after this song

Paramore – Proof (2013)
requested by Luke Lorentzatos (TA-of-CM2)

I’m half Greek and can speak a little bit.
I am a huge Houston Astros fan.

Duke

Visiting Class Today: TAs-of-CM2

I'm always down for poker and table tennis. And I respond to at least 8 pronunciations of my name.

Anirudh Jain

Jessica Chen

Hi, I like taekwondo and reading. I also like skiing and baking but those are harder to do at Duke.

Reminders on Gradescope assignment

- Type your work; no handwriting accepted
- The best way to use the provided LaTeX source code is to directly modify it (as it's meant to be the skeleton/template)
  - Put `macros.tex` in your project folder as a separate file
  - Use `\mathbb{R}` to typeset blackboard bold $\mathbb{R}$
- Label pages after uploading PDF
- Submit just once per group, then add your teammate
- No names necessary in PDF

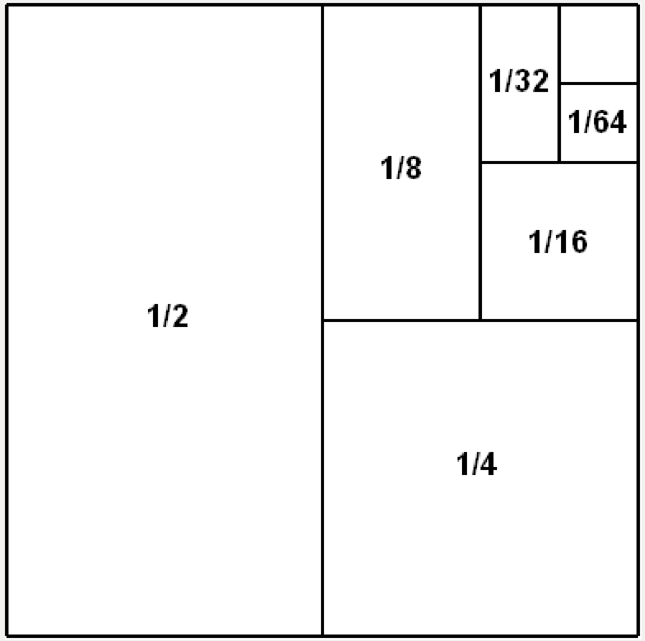
Prove a proposition/theorem/statement

- Direct proof
- Proof by contrapositive
- Proof by contradiction
- Proof by cases
- Proof by construction
- Proof by induction [*big topic itself* - CM5]
- What else?

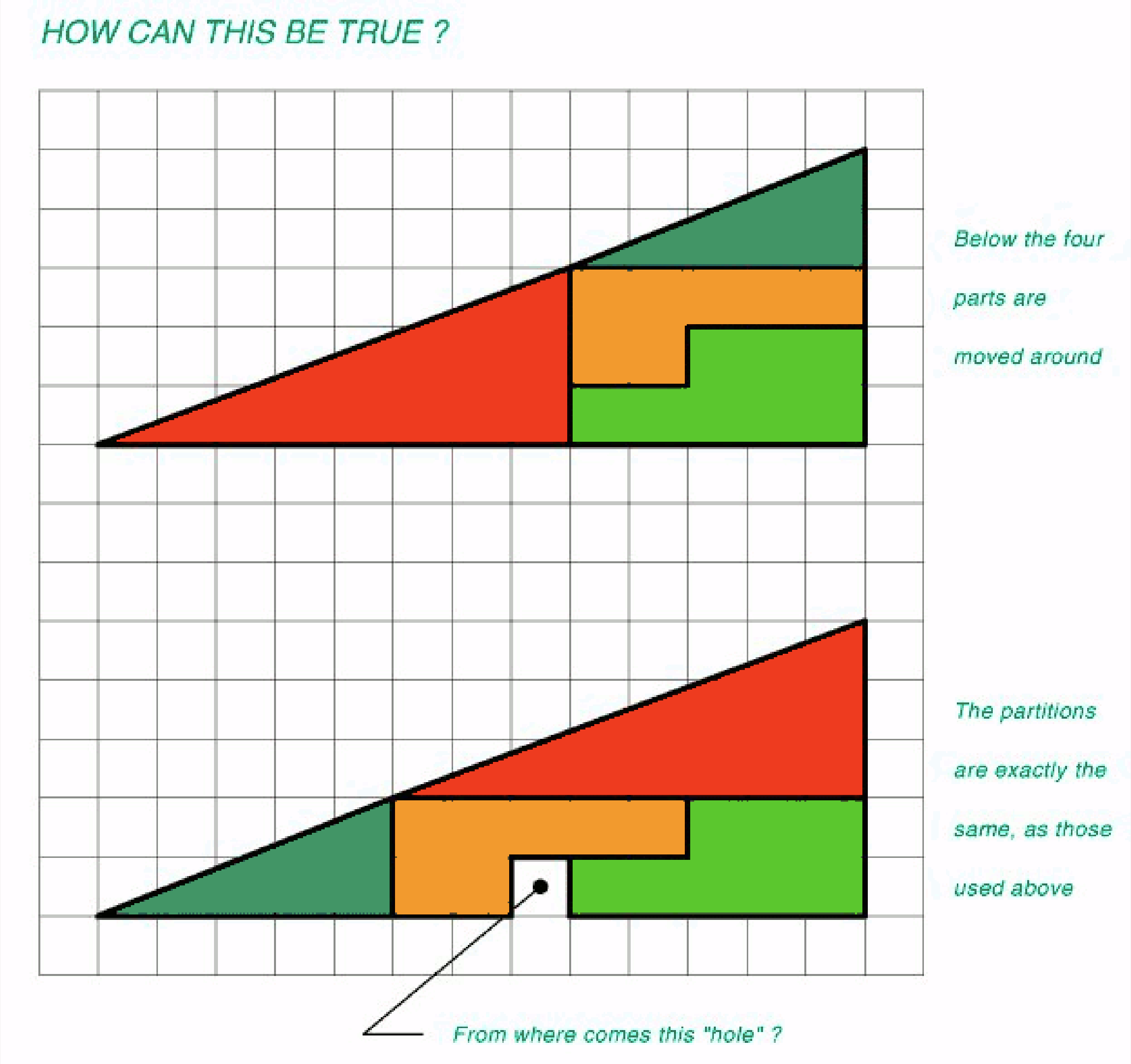
Proof by picture…?

- $\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$

- A picture alone is not a proof



Proof by picture



PI: Biconditional

An abstracted version of the question

- Want to prove: $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$

a) $p \rightarrow q$

b) $q \rightarrow p$

c) $\neg q \rightarrow \neg p$

d) $\neg p \rightarrow \neg q$

In case you did not notice

- The theorem is false
- If exactly two of a, b, c are even and the other is odd, $a^2 + b^2 + c^2$ is also odd (check this yourself)
- That should not matter
  - We were identifying what conditional statements or their contrapositives, that we should prove
  - We were not actually trying to prove them (we would fail)

Proving more than what is needed

- is technically correct
- is usually unnecessary
- sometimes makes the proof easier
    - more examples about this in CM5 (Inductions)

Write good proofs, not just correct proofs

- Correctness of the proof is the *first* priority, not the *only* priority
- Correct proofs are just correct. Good proofs can be *understood*.
  - You want your proofs to be correct and understood
  - What is obvious to yourself, may not be obvious to others

Peer Review (Proof-by-cases)

- Navigate to the Canvas quiz

- Complete two simple proof-by-cases
  - Don’t share accounts; complete the proofs on your own
  - Don’t discuss with anyone else

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Peer Review (Proof-by-cases)

- Navigate to Canvas quiz again
- You are now assigned two anonymous assignments by your classmates
- Read their work, then give feedback
  - Evaluate the technical correctness, readability, and conciseness of the proofs via the rubric provided

Proof. *"For every positive integer n, if $\sqrt{n}$ is rational, then $\sqrt{n}$ is an integer."*.

- We use proof by contradiction: assume $\neg\left(\forall n \in \mathbb{N}\left[\sqrt{n} \in \mathbb{Q} \rightarrow \sqrt{n} \in \mathbb{Z}\right]\right)$
- $\therefore \exists n \in \mathbb{N}[\neg(\sqrt{n} \in \mathbb{Q} \rightarrow \sqrt{n} \in \mathbb{Z})]$
- $\therefore \exists n \in \mathbb{N}[(\sqrt{n} \in \mathbb{Q}) \wedge (\sqrt{n} \notin \mathbb{Z})]$
- $\therefore \exists a \in \mathbb{N}\, \exists b \in \mathbb{N}[(\sqrt{n} = \frac{a}{b}) \wedge (b \nmid a)]$
- Let $c \geq 1 = \mathrm{GCD}(a, b)$.
- $\therefore \exists x \in \mathbb{N}\, \exists y \in \mathbb{N}[(a = cx) \wedge (b = cy) \wedge (y \nmid x)]$
- $\therefore n = \frac{a^2}{b^2} = \frac{c^2 x^2}{c^2 y^2} = \frac{x^2}{y^2}$.
- Since $(y \nmid x) \rightarrow (y^2 \nmid x^2)$, $n = \frac{x^2}{y^2} \notin \mathbb{Z}$.

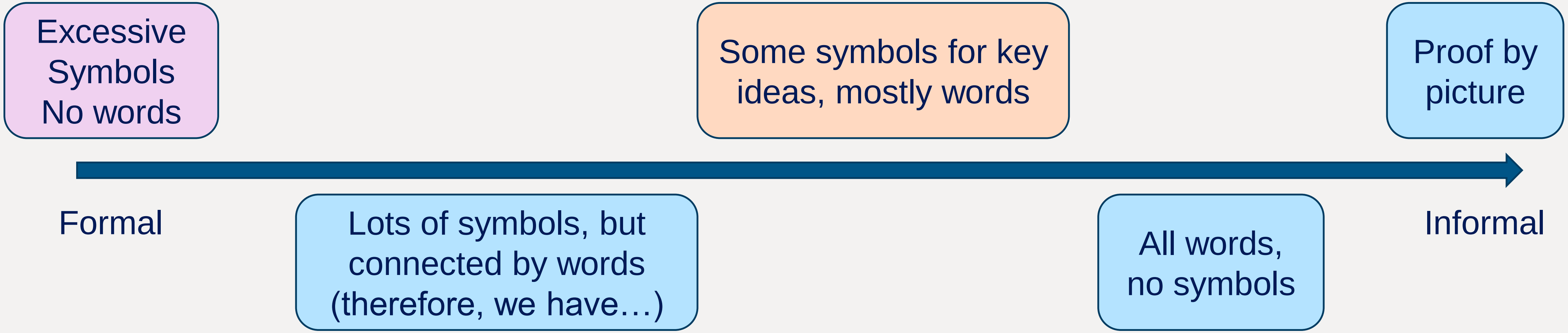
Excessive symbolism

Proof. *"For every positive integer n, if $\sqrt{n}$ is rational, then $\sqrt{n}$ is an integer."*

- We use proof by contradiction: assume the theorem is not true.
- Then there exists a positive integer n such that $\sqrt{n}$ is rational but not an integer.
- Then there exist positive integers a, b such that $\sqrt{n} = \frac{a}{b}$ and $b \nmid a$ (i.e., b does not divide a).
- Let $c \geq 1$ be the greatest common divisor of a and b.
- Then there exist positive integers x, y such that $a = cx$, $b = cy$, and $y \nmid x$. (otherwise $b = cy$ would divide $a = cx$).
- Then $n = \frac{a^2}{b^2} = \frac{c^2 x^2}{c^2 y^2} = \frac{x^2}{y^2}$.
- Since $y \nmid x$ implies $y^2 \nmid x^2$, $n = \frac{x^2}{y^2}$ is not an integer.
- Since we reached a contradiction, the theorem is proved.

Less symbolistic

Level of formality



How to come up with the proof steps?

- Reading proofs (especially simple and beautiful ones) make you feel proofs come “naturally” as if all steps just magically fall into place
  - For most of sufficiently complicated theorems, this is usually not the case
  - A lot of trial-and-error (you don’t have a compiler to tell you there’s an error)
  - Messy thoughts, dead-ends, useful but out-of-order ideas…
- No one expects you to be perfect on the first try
  - Like no one expects you to write 100 lines of code that work immediately