Introduction to Game Theory
Lecture 6: Bargaining

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The ultimatum game again

- Recall the ultimatum game: two people need to decide how to divide a dollar. Player 1 proposes to give herself $x$ and give $(1 - x)$ to player 2. If player 2 accepts the offer, then they respectively receive $x$ and $1 - x$. If 2 rejects the offer, then neither person receives any of the dollar.

- The stark result of the game is due to the fact that player 2 has no bargaining power.
A finite horizon game with alternating proposals

- Now suppose a player can make a counter proposal after rejecting the other player’s proposal; but they have to reach an agreement before or at period $T < \infty$. If the proposal made in period $T$ is rejected, the game ends with both players getting 0.
- Further, each player $i$ discounts the future by a discount factor $\delta_i$; a deal reached in period $t$ that gives player $i$ a share of $s_i$ is equivalent to giving her $\delta_i^t s_i$ today.
- What will be the equilibrium outcome?
First, consider the case when $T = 2$, as in the following, where in period $t$ the offer is $(x_t, 1 - x_t)$, i.e., player 1 gets $x_t$.

Note the discount factor in the above graph. What is the SPNE? Use backward induction.
Solution for $T = 2$

- The subgame starting after a history in which player 2 rejected the initial offer by player 1 is just the standard ultimatum game, and so we know in a SPNE player 2 will offer zero to the other player and give herself one, which will be accepted by player 1.
- Given this, how will player 1 make the offer in the beginning of the entire game?
- Player 1 has to make an offer $\geq \delta_2$ to player 2 in order for player 2 to accept the offer. Why?
- The unique SPNE is 1) $x_1 = 1 - \delta_2$, 2) player 2 accepts any initial offer that gives her $\geq \delta_2$ and rejects the offer otherwise, 3) if player 2 rejects player 1’s offer, player 2 offers zero to player 1 in period 2, and 4) player 1 accepts all offers proposed by player 2.
An arbitrary finite $T$

- Consider a general $T$, $T < \infty$. Assume player 2 makes the proposal in period $T$, i.e., $T$ is even. (What if $T$ odd? Similar logic.)
- In period $T$ player 2 will make the proposal $(0, 1)$.
- Therefore in the penultimate period $(T - 1)$, player 1’s proposal is $(1 - \delta_2, \delta_2)$. Therefore in period $T - 2$, player 2’s proposal is $(\delta_1(1 - \delta_2), 1 - \delta_1(1 - \delta_2))$ ...
- In the first period, then, player 1 proposes that she gets a share equal to $x_1 = 1 - \delta_2(1 - \delta_1(1 - \delta_2(...)))$ and leave the rest to player 2.
Finite Horizon Bargaining

Infinite Horizon Bargaining: The Rubinstein Model

Application: Baron-Ferejohn Model

\( T < \infty \) cont.

- In equilibrium, player 1 will get

\[
x_1^* = 1 - \delta_2 (1 - \delta_1 (1 - \delta_2 (\ldots)))
= 1 - \delta_2 + \delta_1 \delta_2 - \delta_1 \delta_2^2 + \ldots + \delta_1^{T/2-1} \delta_2^{T/2-1} - \delta_1^{T/2-1} \delta_2^{T/2} \\
= \sum_{t=0}^{T/2-1} (1 - \delta_2)(\delta_1 \delta_2)^t
= (1 - \delta_2)[1 - (\delta_1 \delta_2)^{T/2}] \\
= \frac{(1 - \delta_2)[1 - (\delta_1 \delta_2)^{T/2}]}{1 - \delta_1 \delta_2}.
\]

- And player 2 will get

\[
1 - x_1^* = \frac{\delta_2(1 - \delta_1) - (1 - \delta_2)(\delta_1 \delta_2)^{T/2}}{1 - \delta_1 \delta_2}.
\]
• If $T$ is infinity, the game is much harder since we cannot use backward induction.

• Rubinstein (1982), however, shows that the solution has a remarkably simple form.

• **Proposition:** The alternating-proposal bargaining game with $T = \infty$ has a unique SPNE: in any period in which player $i$ makes a proposal, she proposes her own share to be

$$\frac{1 - \delta_j}{1 - \delta_i \delta_j},$$

and the other player $j$’s share to be

$$\frac{\delta_j(1 - \delta_i)}{1 - \delta_i \delta_j}.$$

Player $j$ accepts this or any higher offer and rejects any lower offer.
The Rubinstein Model: Proof (1)

• Let $v_i$ be the lowest payoff player $i$ receives in any SPNE in a subgame when she makes the initial offer and let $\bar{v}_i$ be her highest SPNE payoff in such a subgame. (Likewise, we have $v_j$ and $\bar{v}_j$).

• Consider a subgame in which $i$ makes the initial offer. Player $j$ will not accept an offer that gives her less than $\delta_j v_j$, so:

\[ \bar{v}_i \leq 1 - \delta_j v_j \]  

(1)

• Similarly, $j$ will accept any offer that gives her at least $\delta_j \bar{v}_j$, so:

\[ v_i \geq 1 - \delta_j \bar{v}_j \]  

(2)

• By symmetry, $j$’s payoffs should satisfy

\[ \bar{v}_j \leq 1 - \delta_i v_i \]  

(3)

and

\[ v_j \geq 1 - \delta_i \bar{v}_i \]  

(4)
The Rubinstein Model: Proof (2)

- Subtracting (2) from (1) and (4) from (3), we have

\[ \bar{v}_i - v_i \leq \delta_j (\bar{v}_j - v_j) \]  
(5)

and

\[ \bar{v}_j - v_j \leq \delta_i (\bar{v}_i - v_i). \]  
(6)

- Multiplying (6) through by \( \delta_j \) and combining this with (5) gives

\[ \bar{v}_i - v_i \leq \delta_i \delta_j (\bar{v}_i - v_i). \]

- Since \( \delta_i \delta_j < 1 \), this implies \( \bar{v}_i = v_j \equiv v_i \).

- Doing the same thing w.r.t. \( j \)'s payoffs yields \( \bar{v}_j = v_j \equiv v_j \).

Hence, SPNE payoffs are unique.
The Rubinstein Model: Proof (3)

- Since SPNE payoffs are unique, (1) and (2) become
  \[ \nu_i = 1 - \delta_j \nu_j, \]
  and (3) and (4) become
  \[ \nu_j = 1 - \delta_i \nu_i. \]
- Direct substitution then yields
  \[ \nu_i^* = \frac{1 - \delta_j}{1 - \delta_i \delta_j} \text{ and } 1 - \nu_i^* = \frac{\delta_j(1 - \delta_i)}{1 - \delta_i \delta_j}. \]
- Since \( 1 - \nu_i^* \) equals \( \delta_j \nu_j^* \), \( j \) is indifferent and accepts. Q.E.D.
Properties of the SPNE of Rubinstein Model

- **Efficiency**: player 2 would accept player 1’s first proposal, resulting in immediate agreement without delay (which is costly due to discounting).

- **The more patient a player is, the better off she will be**: a higher $\delta_j$ reduces $v_i^*$ and increases $1 - v_i^*$.

- **First mover advantage**: if $\delta_i = \delta_j = \delta$, then in SPNE whoever makes the initial proposal offers herself a higher share:

  $$\frac{1 - \delta}{1 - \delta^2} > \frac{\delta(1 - \delta)}{1 - \delta^2}.$$  

  $\star$ When the real time between proposals is shortened toward zero, $\delta$ approaches 1 ($\delta^{1/n} \to 1$ as $n \to \infty$), and the first mover advantage disappears. Each player will get $1/2$. 