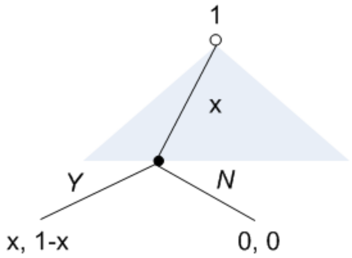
Introduction to Game Theory
Lecture 6: Bargaining

Haifeng Huang
University of California, Merced

Shanghai, Summer 2011

The ultimatum game again

- Recall the ultimatum game: two people need to decide how to divide a dollar. Player 1 proposes to give herself x and give $(1-x)$ to player 2. If player 2 accepts the offer, then they respectively receive x and $1-x$. If 2 rejects the offer, then neither person receives any of the dollar.



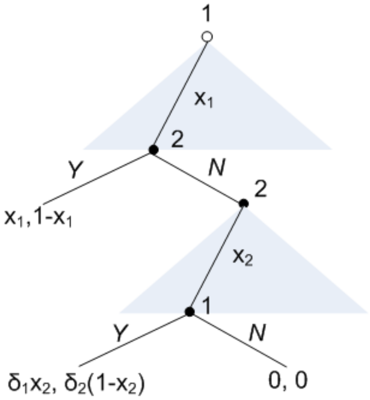
- The stark result of the game is due to the fact that player 2 has no bargaining power.

A finite horizon game with alternating proposals

- Now suppose a player can make a counter proposal after rejecting the other player's proposal; but they have to reach an agreement before or at period $T < \infty$. If the proposal made in period T is rejected, the game ends with both players getting 0.
- Further, each player i discounts the future by a discount factor δ_i; a deal reached in period t that gives player i a share of s_i is equivalent to giving her $\delta_i^t s_i$ today.
- What will be the equilibrium outcome?

$T = 2$

- First, consider the case when $T = 2$, as in the following, where in period t the offer is $(x_t, 1 - x_t)$, i.e., player 1 gets x_t.



- Note the discount factor in the above graph. What is the SPNE? Use backward induction.

Solution for $T = 2$

- The subgame starting after a history in which player 2 rejected the initial offer by player 1 is just the standard ultimatum game, and so we know in a SPNE player 2 will offer zero to the other player and give herself one, which will be accepted by player 1.
- Given this, how will player 1 make the offer in the beginning of the entire game?
- Player 1 has to make an offer $\geq \delta_2$ to player 2 in order for player 2 to accept the offer. Why?
- The unique SPNE is 1) $x_1 = 1 - \delta_2$, 2) player 2 accepts any initial offer that gives her $\geq \delta_2$ and rejects the offer otherwise, 3) if player 2 rejects player 1's offer, player 2 offers zero to player 1 in period 2, and 4) player 1 accepts all offers proposed by player 2.

An arbitrary finite T

- Consider a general T, $T < \infty$. Assume player 2 makes the proposal in period T, i.e., T is even. (What if T odd? Similar logic.)
- In period T player 2 will make the proposal $(0, 1)$.
- Therefore in the penultimate period $(T - 1)$, player 1's proposal is $(1 - \delta_2, \delta_2)$. Therefore in period $T - 2$, player 2's proposal is $(\delta_1(1 - \delta_2), 1 - \delta_1(1 - \delta_2))$ ...
- In the first period, then, player 1 proposes that she gets a share equal to $x_1 = 1 - \delta_2(1 - \delta_1(1 - \delta_2(...)))$ and leave the rest to player 2.

$T < \infty$ cont.

- In equilibrium, player 1 will get

$$\begin{aligned} x_1^* &= 1 - \delta_2(1 - \delta_1(1 - \delta_2(...))) \\ &= 1 - \delta_2 + \delta_1\delta_2 - \delta_1\delta_2^2 + ... + \delta_1^{T/2-1}\delta_2^{T/2-1} - \delta_1^{T/2-1}\delta_2^{T/2} \\ &= \sum_{t=0}^{T/2-1} (1 - \delta_2)(\delta_1\delta_2)^t \\ &= \frac{(1 - \delta_2)[1 - (\delta_1\delta_2)^{T/2}]}{1 - \delta_1\delta_2}. \end{aligned}$$

- And player 2 will get

$$1 - x_1^* = \frac{\delta_2(1 - \delta_1) - (1 - \delta_2)(\delta_1\delta_2)^{T/2}}{1 - \delta_1\delta_2}.$$

Infinite Horizon Bargaining: The Rubinstein Model

- If T is infinity, the game is much harder since we cannot use backward induction.
- Rubinstein (1982), however, shows that the solution has a remarkably simple form.
- **Proposition**: The alternating-proposal bargaining game with $T = \infty$ has a unique SPNE: in any period in which player i makes a proposal, she proposes her own share to be

$$\frac{1-\delta_j}{1-\delta_i\delta_j},$$

and the other player j's share to be

$$\frac{\delta_j(1-\delta_i)}{1-\delta_i\delta_j}.$$

Player j accepts this or any higher offer and rejects any lower offer.

The Rubinstein Model: Proof (1)

- Let $\underline{v}_i$ be the lowest payoff player i receives in any SPNE in a subgame when she makes the initial offer and let $\overline{v}_i$ be her highest SPNE payoff in such a subgame. (Likewise, we have $\underline{v}_j$ and $\overline{v}_j$).
- Consider a subgame in which i makes the initial offer. Player j will not accept an offer that gives her less than $\delta_j \underline{v}_j$, so:

$$\overline{v}_i \leq 1 - \delta_j \underline{v}_j \tag{1}$$

- Similarly, j will accept any offer that gives her at least $\delta_j \overline{v}_j$, so:

$$\underline{v}_i \geq 1 - \delta_j \overline{v}_j \tag{2}$$

- By symmetry, j's payoffs should satisfy

$$\overline{v}_j \leq 1 - \delta_i \underline{v}_i \tag{3}$$

and

$$\underline{v}_j \geq 1 - \delta_i \overline{v}_i \tag{4}$$

The Rubinstein Model: Proof (2)

- Subtracting (2) from (1) and (4) from (3), we have

$$\overline{v}_i - \underline{v}_i \leq \delta_j(\overline{v}_j - \underline{v}_j) \tag{5}$$

and

$$\overline{v}_j - \underline{v}_j \leq \delta_i(\overline{v}_i - \underline{v}_i). \tag{6}$$

- Multipling (6) through by δ_j and combining this with (5) gives

$$\overline{v}_i - \underline{v}_i \leq \delta_i\delta_j(\overline{v}_i - \underline{v}_i).$$

- Since $\delta_i\delta_j < 1$, this implies $\overline{v}_i = \underline{v}_j \equiv v_i$.
- Doing the same thing w.r.t. j's payoffs yields $\overline{v}_j = \underline{v}_j \equiv v_j$. Hence, SPNE payoffs are unique.

The Rubinstein Model: Proof (3)

- Since SPNE payoffs are unique, (1) and (2) become

$$v_i = 1 - \delta_j v_j,$$

and (3) and (4) become

$$v_j = 1 - \delta_i v_i.$$

- Direct substitution then yields

$$v_i^* = \frac{1-\delta_j}{1-\delta_i\delta_j} \text{ and } 1 - v_i^* = \frac{\delta_j(1-\delta_i)}{1-\delta_i\delta_j}.$$

- Since $1 - v_i^*$ equals $\delta_j v_j^*$, j is indifferent and accepts. Q.E.D.

Properties of the SPNE of Rubinstein Model

- Efficiency: player 2 would accept player 1's first proposal, resulting in immediate agreement without delay (which is costly due to discounting).
- The more patient a player is, the better off she will be: a higher δ_j reduces v_i^* and increases $1 - v_i^*$.
- First mover advantage: if $\delta_i = \delta_j = \delta$, then in SPNE whoever makes the initial proposal offers herself a higher share: $\frac{1-\delta}{1-\delta^2} > \frac{\delta(1-\delta)}{1-\delta^2}$.
    - ⋆ When the real time between proposals is shortened toward zero, δ approaches 1 ($\delta^{1/n} \to 1$ as $n \to \infty$), and the first mover advantage disappears. Each player will get 1/2.